STRONG UNIQUE CONTINUATION FOR CRITICAL POINTS OF SOME VARIATIONAL QUANTITY

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学位論文要旨

Abstract

Abstract　The unique continuation property is one of the most important one for analytic functions. There are two types of that property in the strong and weak sence. In 1988 J. L. Kazdan [3] gave the next question : *Does the unique continuation property hold for critcal points of the variational quantity* $J_1(v)$ *for* $p>1$? The definitions of $J_i(v)$, $1\leqq i\leqq 3$, are given at pages 2-3. Recently, K. Hayasida [1] proved that the weak unique continuation property holds for the critical point u of another variational quantity $J_2(v)$, which is equivalent to $J_1(v)$. Here we will construct the third variational quantity $J_3(v)$ equivalent to $J_1(v)$, whose critical points have the strong unique continuation property.

Let L be either a linear or nonlinear operator in a domain Ω of R^N. Let u be a solution of $Lu=0$ in Ω. If the following is satisfied, we say that the unique continuation property holds for u:

If $u=0$ *in a subdomain* Ω' *of* Ω, *then* u *vanishes identically in* Ω.

Without loss of generality we may assume that the domain Ω contains the origin. If the following is satisfied, we say that the strong unique continuation property holds for u:

The solution u *vanishes identically in* Ω, *if* $(D^\alpha u)(x)=o(|x|^n)$ $(x\to O, |\alpha|\leqq k)$, *for any* n, *where* k *depends only on the order of* L.

For example the harmonic function has the strong unique continuation property, because it is analytic.

Now let $0<R<1$ and B_R be the open sphere in R^N with center O and radius R. Let v be any function in $C_0^2(B_R)$. We assume that $v(x=o(|x|^n)$ as $x\to O$ for any n. Then by E. Heinz [2] the inequality holds :

$$\int_{B_R} |x|^{-2n}(v^2+|\nabla v|^2)dx \leqq C\int_{B_R} |x|^{-2n}|\Delta v|^2 dx, \tag{1}$$

where C is a constant independent of R, n and v. Using this inequality we immediately see that a function $u \in C^2(\Omega)$ has the strong unique continuation property, if it satisfies the differerntial inequality

$$|\Delta u| \leqq K(u|+|\nabla u|) \quad \text{locally in } \Omega. \tag{2}$$

In particular let L be the operator such that

$$L = \Delta + \sum_{i=1}^{N} b_i(x)\partial_{x_i} + c(x),$$

where b_i and c are locally bounded in Ω. Then the solution u of $Lu=0$ satisfies (2). So the strong unique continuation property holds for u.

On the other hand we define the weak unique continuation property. Let Γ be aN$-$1-dimensional open hypersurface of class C^1 in $\partial\Omega$. We assume that Γ contains O and it is strongly convex at O. Let u be a solution of $Lu=0$ in Ω. If the following holds, we say that the weak unique continuation property holds for u:

If $D^\alpha u=0$ on Γ $(|\alpha| \leqq k)$ then u vanishes identically in a neighborhood of Γ.

Let the x_N-axis be perpendicular to the hyperplane, which is tangent to Γ at O. We assume that $\Omega \subset \{x_N > 0\}$. Let $\Omega_R = \Omega \cap B_R$. For sufficiently small $R > 0$ let v be in $C_0^2(\Omega_R)$. Then the inequality holds

$$\int_{\Omega_R} e^{2\lambda x_N}(v^2+|\nabla v|^2)dx \leqq CR^2\int_{\Omega_R} e^{2\lambda x_N}|\Delta v|^2 dx, \tag{3}$$

where λ is any negative number and C is a constant independent of λ, R and v. This is due to L. Nirenberg [8]. Now let u be in $C^2(\Omega_R)$ and satisfy

$$|\Delta u| \leqq K(|u|+|\nabla u|) \quad \text{in } \Omega_R.$$

Using (3), we see that $u=0$ in $\Omega_{R/4}$, if $u=\nabla u=0$ on $\Gamma \cap B_R$.

From the view point of geometry J. L. Kazdan [3] discussed the strong unique continuation property for the mapping $u : M^N \to R^S$, where u satisfies some elliptic system in a N-dimensional Riemannian manifold M^N. Only semilinear cases were treated in [3]. But he gave the next question in the final section of [3] :

Does the unique contiunation property hold for critical points of the variational quantity

$$J_1(v)=\int_\Omega |\nabla v|^p dx$$

for $p>1$? The critical point of $J_1(v)$ satisfies the degenerate quasilinear elliptic equation

$$\nabla\cdot(|\nabla u|^{p-2}\nabla u)=0 \quad \text{in } \Omega. \tag{4}$$

If $N=2$ in particular, the unique continuation property holds for solutions of (4) by the result of J. J. Manfredi [5]. In fact writing $z=x+iy$ and $F=u_x-iu_y$, we see that F is K-quasiregular for $K=\frac{1}{2}\left((p-1)+\frac{1}{p-1}\right)\geqq 1$ and it is μ-quasiconformal. More precisely, $F_{\bar z}=\mu F_z$, in Ω, where the essential maximum of μ is not greater than $\left(\frac{K-1}{K+1}\right)^{1/2}$. So the unique cotinuation property holds for u, because the unique continuation property holds for the μ-quasiconformal mapping (see [4]).

Up to now Kazdan's above question is not yet solved. In 1988, O. Martio [6] gave a counterexample for the unique continuation property. He constructed a function u and a vector function $A(x,\xi): R^N\times R^N\to R^N$ such that $|A(x,\xi)|\leqq C|\xi|^{N-1}$, $A(x,\xi)\cdot\xi\geqq c|\xi|^N (c>0)$ and $u(x)=0$ for $x_N\leqq 0$, $u(x)\neq 0$ for $x_N>0$.

Now we define another variational quantity

$$J_2(v)=\sum_{i=1}^N\int_\Omega |\partial x_i v|^p d\mathrm{x},\quad p>1,$$

whose critical points satisfy

$$\sum_{i=1}^N \partial_{x_i}(|\partial x_i u|^{p-2}\partial x_i u)=0 \quad \text{in } \Omega. \tag{5}$$

We see that $J_1(v)$ and $J_2(v)$ are equivalent. Recently, K. Hayasida [1] proved that the weak unique continuation property holds for u satisfying (5).

Our aim is to give a variational quantity equivalent to $J_1(v)$ such that its critical points have the strong unique continuation property. We use hyperspherical coordinates $r, \theta_1,\cdots \theta_{N-1}$ such that

$\mathrm{x}_1=r\cos\theta_1$,

$\mathrm{x}_j=r\sin\theta_1\ r\sin\theta_2\cdots\sin\theta_{j-1}\cos\theta_j,\quad 1<j<N$

$\mathrm{x}_N=r\sin\theta_1\ r\sin\theta_2\cdots\sin\theta_{N-1}$,

where $r\geqq 0$, $0\leqq\theta_1,\cdots,\theta_{N-2}\leqq\pi$ and $0\leqq\theta_{N-1}\leqq 2\pi$,

Taking the mapping $\tau=-\log r$. For sufficiently small $R>0$ we define

$$J_3(v)=\int_0^{2\pi}\int_0^{\pi}\cdots\int_0^{\pi}\int_0^{\infty}\left[|\partial_\tau v|^p+|\partial_{\theta_1}v|^p+\sum_{j=1}^{N-2}|\sin\theta_1\cdots\sin\theta_j|^{-p}|\partial_{\theta_{j+1}}v|^p\right]$$
$$\cdot e^{(p-N)\tau}(\sin\theta_1)^{N-2}(\sin\theta_2)^{N-3}\cdots\sin\theta_{N-2}\,dr\,d\theta_1\cdots d\theta_{N-1}.$$

Then we see that $J_3(v)$ is equivalent to $J_1(v)$. Our result is the following

[(**Theorem**)]. Let $p \geqq 2$ and u belong to $C^2(B_1)$. Let u be a critical point of $J_3(v)$. If u, $|\nabla_x u| = o(|x|^n)$ as $x \to O$ for any $n > 0$. Then u vanishes identically in B_1.

[(**Remark**)]. Part of the hypothesis of this theorem, u, $|\nabla_x u| = o(|\mathrm{x}|^n)$ as $\mathrm{x} \to O$ can be weakened by the following, $\int_{B_R} |\nabla u|^p dx = o(|R|^n)$ *as* $R \to 0$ *and* $u(O) = 0$.

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学位論文の審査結果の要旨

平成8年1月30日に第1回学位論文審査委員会を開き，平成8年2月5日に口頭発表並びに審査委員会を開催し，慎重に検討した結果以下の様に判定した。

本論文は楕円型偏微分方程式の解の一意接続性に関するものである。2階線形の場合の解の一意接続性は，1957年 N. Aronszajn によって証明された。この一意接続性は強い意味におけるそれ(強一意接続性）である。その後，高階線形楕円型方程式の解の一意接続性について多くの結果が出され，今日に至っている。

他方，非線形の場合一意接続性の研究は未だ手探りの状況にある。1988年，J.L. Kazdan 氏は，p乗 Dirichlet 積分を変分量としたときのその危点が一意接続性を持つか否かの問題提起を行った。その危点はその変分量に対応する Euler-Lagrange 方程式の解になっているので，そのことはとりもなおさず，2階退化型準線形楕円型方程式の解が一意接続性を持つかどうかということである。

本論文では，p乗 Dirichlet 積分と同等な変分量が極座標空間の中で巧みにそして独自の方法で構成され，その変分量の危点が強一意接続性をみたすことが示されている。証明の中心部分は Carlemann 評価式を導くことである。この結果は非線形楕円型，特に2階退化型準線形楕円型偏微分方程式の解の一意接続性について明るい展望を示唆している。その意義は大きく，博士学位論文としての価値があるものと判定する。