博　士　論　文

Divisorial contractions to cDV points
（混合デュバル特異点の因子収縮について）

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1. Introduction

Let $P \in X$ be a germ of a 3-dimensional terminal singularity defined over $\mathbb{C}$. A projective birational morphism $f\colon Y \to X$ is called a *divisorial contraction* if

(i) $-K_Y$ is f-ample,
(ii) Y has only terminal singularities, and
(iii) the exceptional locus E of f is an irreducible divisor.

In this situation, we write $K_Y = f^*K_X + a(E, X)E$ with $a(E, X) \in \mathbb{Q}$. The coefficient $a(E, X)$ is called the *discrepancy* of E over X. When $f(E) = P$, i.e., $f_{Y \setminus E}\colon Y \setminus E \to X \setminus \{P\}$ is an isomorphism, we write $f\colon (Y \supset E) \to (X \ni P)$.

It is a fundamental problem in 3-dimensional birational geometry to find all divisorial contractions $f\colon (Y \supset E) \to (X \ni P)$. In this dissertation, I finished the classification of 3-dimensional divisorial contraction which contracts an irreducible divisor to a cDV point, and therefore we classified all 3-dimensional divisorial contractions to a point. Classification of all divisorial contractions to a point tells us that they are obtained as weighted blow-ups.

Theorem 1.1. *Let $f\colon Y \to X$ be a 3-dimensional divisorial contraction whose exceptional divisor E contracts to a point P. Then f is a weighted blow-up of the singularity $P \in X$ embedded into a cyclic quotient 5-fold.*

Detail version of our main results in Theorem 1.1 shall be given in Section 2, and the detailed list of all divisorial contractions to a cDV point shall be given in Section 5. Classification of all divisorial contractions to a non-Gorenstein point $P \in X$ in Theorem 1.1 has been already settled by [2], [3], [4], [11], [12], and [14]. If $P \in X$ is a Gorenstein point, that is, cDV point, several cases of divisorial contractions to P were already classified. Kawakita showed that f is obtained as a suitable weighted blow-up in the case of non-singular point P in [8], and he classified divisorial contractions to cA_1 point in [9]. He also classified all divisorial contractions to a point into two types, one is ordinary type, and the other is exceptional type in [11]. We know that all divisorial contractions of ordinary type are classified by [11, Theorem 1.2]. I pointed out that his paper [10] has a few mistakes, and he sent me the erratum [13]. The erratum gives us that there is a possibility of divisorial contractions to a cA_2 point with discrepancy 4 which are of type $e1$. Hayakawa classified divisorial contractions to points of type cD, cE with discrepancy 1 in [5], [6].

As a result, the remaining cases in Theorem 1.1 are divisorial contractions of exceptional type with discrepancy greater than 1, which are listed in Table 1. The main aim in this paper is to finish classification of all divisorial contractions listed in Table 1.

Chen, Hayakawa, and Kawakita found several examples of exceptional type listed in Table 1. There are several examples of type $e1$, $e2$, $e3$, and $e9$

Table 1. divisorial contraction of exceptional type

type	terminal P	a	E^3	non-Gorenstein terminal on Y
$e1$	$cA_2{}^{*1)}$, cD	4	$1/r$	$\frac{1}{r}(1,-1,8)$; $r \equiv \pm 3 \pmod 8$$^{*1)}$
	cD	2	$2/r$	$\frac{1}{r}(1,-1,4)$
$e2$	cD, $cE_{6,7}$	2	$1/r$	cA/r or $cD/3$ deforming to
				$2 \times \frac{1}{r}(1,-1,2)$; $cD/3$ for $cE_{6,7}$
$e3$	cA_2, cD, cE_6	3	$1/4$	$cAx/4$ deforming to
				$\frac{1}{2}(1,1,1)$, $\frac{1}{4}(1,3,3)$
$e5$	cE_7	2	$1/7$	$\frac{1}{7}(1,6,6)$
$e9$	$cE_{7,8}$	2	$1/15$	$\frac{1}{3}(1,2,2)$ and $\frac{1}{5}(1,4,4)$

which are weighted blow-ups by [11]. Chen has examples of type $e1$ with P of type cD and discrepancy 4, and there is an example of type $e5$ in [1].

Since we finished the classification of 3-dimensional divisorial contractions to a point, it is possible to classify birational Mori fiber structures of 3-dimensional $\mathbb{Q}$-Fano varieties with terminal singularities which are not quotient. T. Okada told me that he had classify the birational Mori fiber structures of general members of 3-dimensional anticanonically embedded $\mathbb{Q}$-Fano weighted complete intersection of codimension 2 by using the classification of 3-dimensional divisorial contractions.

In this article, we describe divisorial contractions to a Gorenstein point and we show that every divisorial contraction listed in Table 1 is obtained as a weighted blow-up if it exists. Our method of the classification is to study the structure of the graded ring $\bigoplus_j f_*\mathcal{O}(-jE)/f_*\mathcal{O}(-(j+1)E)$. We find local coordinates at P to meet this structure and verify that f should be a certain weighted blow-up. In certain cases, there are some choices of local coordinates unlike the non-Gorenstein cases. So we should compute weighted blow-up in detail, and in several cases, there is no suitable local coordinate. There is no divisorial contraction of type $e1$ with P of type cA_2 and discrepancy 4, type $e2$ with type cE_7, and type $e3$ with type cE_6.

We shall give the results in Section 2, and their proofs shall be given in Section 4. We explain terminal singularity, weighted blow-up, and singular Riemann-Roch theorem in Section 3. In Section 5, we give the detailed list of all divisorial contractions to a cDV point.

Acknowledgments. I was motivated to write this dissertation by some questions of Professor T. Hayakawa. I am grateful to him for his useful advise and helpful comments. I would like to thank Professor M. Kawakita for answering to my question earnestly and sending the erratum [13] to me.

$^{*1)}$The new case and the condition given by the erratum [13].

2. Main results

We consider divisorial contractions $f\colon (Y \supset E) \to (X \ni P)$ listed in Table 1. Our main results show that such contractions are obtained as weighted blow-ups embedded into $\mathbb{C}^4$ or $\mathbb{C}^5$ if they exist. The following is a detailed version of our main results. Proofs shall be given in Section 4.

Theorem 2.1. *There is no divisorial contraction of type* e1 *which contracts to a cA_2 point with discrepancy* 4.

Theorem 2.2. *Suppose that f is a divisorial contraction of type* e1 *which contracts to a cD point with discrepancy* 4. *Then f is the weighted blow-up with* $\mathrm{wt}(x_1, x_2, x_3, x_4, x_5) = (\frac{r+1}{2}, \frac{r-1}{2}, 4, 1, r)$ *with $r \geq 7$, $r \equiv \pm 3 \pmod 8$ after an identification*

$$P \in X \simeq o \in \begin{pmatrix} x_1^2 + \lambda x_2 x_3^k + x_4 x_5 + p(x_3, x_4) = 0, \\ x_2^2 + 2x_1 q_1(x_3, x_4) + q_2(x_3, x_4) + x_5 = 0 \end{pmatrix} \subset \mathbb{C}^5_{x_1x_2x_3x_4x_5}.$$

Moreover the equations defining X satisfy the following conditions:

(i) $\lambda \in \mathbb{C}$, $k > \frac{r+3}{8}$, $\mathrm{wt}\, p \geq r+1$, $\mathrm{wt}\, q_1 = \frac{r-3}{2}$, $\mathrm{wt}\, q_2 = r-1$, *and q_1, q_2 are weighted homogeneous for the weights distributed above.*

(ii) *q_2 is not square if $q_1 = 0$.*

(iii) *If $r \equiv 3 \pmod 8$ (resp. $r \equiv -3 \pmod 8$), then $x_3^{\frac{r+1}{4}} \in p$ (resp. $x_3^{\frac{r-1}{4}} \in q_2$).*

Theorem 2.3. *Suppose that f is a divisorial contraction of type* e1 *which contracts to a cD point with discrepancy* 2. *Then f is the weighted blow-up with* $\mathrm{wt}(x_1, x_2, x_3, x_4, x_5) = (\frac{r+1}{2}, \frac{r-1}{2}, 2, 1, r)$ *with $r \geq 5$ after an identification*

$$P \in X \simeq o \in \begin{pmatrix} x_1^2 + \lambda x_2 x_3^k + x_4 x_5 + p(x_3, x_4) = 0, \\ x_2^2 + 2x_1 q_1(x_3, x_4) + q_2(x_3, x_4) + x_5 = 0 \end{pmatrix} \subset \mathbb{C}^5_{x_1x_2x_3x_4x_5}.$$

Moreover the equations defining X satisfy the following conditions:

(i) $\lambda \in \mathbb{C}$, $k > \frac{r+1}{4}$, $\mathrm{wt}\, p \geq r+1$, $\mathrm{wt}\, q_1 = \frac{r-3}{2}$, $\mathrm{wt}\, q_2 = r-1$, *and q_1, q_2 are weighted homogeneous for the weights distributed above.*

(ii) *q_2 is not square if $q_1 = 0$.*

(iii) $x_3^{\frac{r+1}{2}} \in p$.

Theorem 2.4. *Suppose that f is a divisorial contraction of type* e2 *which contracts to a cD point with discrepancy* 2. *Then one of the following holds:*

(i) *f is the weighted blow-up with* $\mathrm{wt}(x_1, x_2, x_3, x_4) = (r, r, 2, 1)$ *after an identification of $P \in X$ with*

$$o \in \left(x_1^2 + x_2^2 x_4 + 2x_2 x_4 p(x_3, x_4) + \lambda x_2 x_3^k + q(x_3, x_4) = 0\right) \subset \mathbb{C}^4_{x_1x_2x_3x_4}.$$

Moreover the equation defining X satisfies the following conditions:

(1) $\lambda \in \mathbb{C}$, $k > \frac{r}{2}$, $\mathrm{wt}\, q \geq 2r$, *and p is weighted homogeneous of weight $r-1$ for the weights distributed above.*

(2) $p \neq 0$ *or* $q_{wt=2r} \neq 0$*, and* $q_{\mathrm{wt}=2r}$ *is not square if* $p = 0$.
(3) $x_3^r \in q$.

The non-Gorenstein singularity of Y *is of type* cA/r.

(ii) f *is the weighted blow-up with* $\mathrm{wt}(x_1, x_2, x_3, x_4) = (3, 3, 1, 2)$ *after an identification of* $P \in X$ *with*

$$o \in \Big(x_1^2 + x_2^2 x_4 + 2x_2 x_4 p(x_3, x_4) + x_2 x_3^3 + q(x_3, x_4) = 0\Big) \subset \mathbb{C}^4_{x_1x_2x_3x_4}.$$

Moreover the equation defining X *satisfies the following conditions:*

(1) $\mathrm{wt}\, q \geq 6$*, and* p *is weighted homogeneous of weight* 2 *for the weights distributed above.*
(2) $x_4^3 \in q$.

The non-Gorenstein singularity of Y *is of type* $cD/3$*, and* P *is of type* cD_4.

Theorem 2.5. *Suppose that* f *is a divisorial contraction of type* e2 *which contracts to a* cE_6 *point with discrepancy* 2. *Then* f *is the weighted blow-up with* $\mathrm{wt}(x_1, x_2, x_3, x_4) = (3, 3, 2, 1)$ *after an identification of* $P \in X$ *with*

$$o \in \Big(x_1^2 + \{x_2 - p(x_3, x_4)\}^3 + x_2 g(x_3, x_4) + h(x_3, x_4) = 0\Big) \subset \mathbb{C}^4_{x_1x_2x_3x_4}.$$

Moreover the equation defining X *satisfies the following conditions:*

(i) $\mathrm{wt}\, g \geq 3$, $\mathrm{wt}\, h \geq 6$*, and* p *is weighted homogeneous of weight* 2 *for the weights distributed above.*
(ii) $\deg g \geq 3$ *and* $\deg h \geq 4$.
(iii) $x_3 \in p$ *and* $x_4^3 \in g$.

There is no divisorial contraction of type e2 *which contracts to a* cE_7 *point with discrepancy* 2.

Theorem 2.6. *Suppose that* f *is a divisorial contraction of type* e3 *which contracts to a* cA_2 *point with discrepancy* 3. *Then* f *is the weighted blow-up with* $\mathrm{wt}(x_1, x_2, x_3, x_4) = (4, 3, 2, 1)$ *after an identification of* $P \in X$ *with*

$$o \in \left(\begin{array}{l} x_1^2 + x_2^2 + 2cx_1x_2 + 2x_1 p(x_3, x_4) \\ \qquad\qquad + 2cx_2 p_{\mathrm{wt}=3}(x_3, x_4) + x_3^3 + g(x_3, x_4) = 0 \end{array}\right) \subset \mathbb{C}^4_{x_1x_2x_3x_4}.$$

Moreover the equation defining X *satisfies the following conditions:*

(i) $c \neq \pm 1$, $\mathrm{wt}\, g \geq 6$*, and* p *contains only monomials with weight* 2 *and* 3 *for the weights distributed above.*
(ii) $x_4^2 \in p$ *and* $\deg g(x_3, 1) \leq 2$.

Theorem 2.7. *Suppose that* f *is a divisorial contraction of type* e3 *which contracts to a* cD_4 *point with discrepancy* 3. *Then* f *is the weighted blow-up with* $\mathrm{wt}(x_1, x_2, x_3, x_4) = (3, 4, 2, 1)$ *after an identification of* $P \in X$ *with*

$$o \in \big(x_1^2 + x_2^2 x_4 + 2x_2 x_4 p(x_3, x_4) + \lambda x_2 x_3^k + q(x_3, x_4) = 0\big) \subset \mathbb{C}^4_{x_1x_2x_3x_4}.$$

Moreover the equation defining X *satisfies the following conditions:*

(i) $\lambda \in \mathbb{C}$, $k > 2$, $\mathrm{wt}\, q \geq 6$*, and* p *contains only monomials with weight* ≤ 3 *for the weights distributed above.*

(ii) $x_4 \in p$ *and* $x_3^3 \in q$.

For any $n > 5$, *there is no divisorial contraction of type* e3 *which contracts to a* cD_n *point with discrepancy* 3.

Theorem 2.8. *There is no divisorial contraction of type* e3 *which contracts to a* cE_6 *point with discrepancy* 3.

Theorem 2.9. *Suppose that* f *is a divisorial contraction of type* e5 *which contracts to a* cE_7 *point with discrepancy* 2. *Then* f *is the weighted blow-up with* $\operatorname{wt}(x_1, x_2, x_3, x_4, x_5) = (5, 3, 2, 2, 7)$ *after an identification*

$$P \in X \simeq o \in \begin{pmatrix} x_1^2 + x_2x_5 + p(x_3, x_4) = 0, \\ x_2^2 + q(x_3, x_4) + x_5 = 0 \end{pmatrix} \subset \mathbb{C}^5_{x_1x_2x_3x_4x_5}.$$

Moreover the equations defining X *satisfy the following conditions:*

(i) $\operatorname{wt} p \geq 10$, $\operatorname{wt} q \geq 6$ *for the weights distributed above.*
(ii) $\gcd(p_5, q_3) = 1$.

Theorem 2.10. *Suppose that* f *is a divisorial contraction of type* e9 *which contracts to a* $cE_{7,8}$ *point with discrepancy* 2. *Then* f *is the weighted blow-up with* $\operatorname{wt}(x_1, x_2, x_3, x_4) = (7, 5, 3, 2)$ *after an identification of* $P \in X$ *with*

$$o \in \left(x_1^2 + x_2^3 + \lambda x_2^2x_4^2 + x_2g(x_3, x_4) + h(x_3, x_4) = 0\right) \subset \mathbb{C}^4_{x_1x_2x_3x_4}.$$

Moreover the equation defining X *satisfies the following conditions:*

(i) $\lambda \in \mathbb{C}$ *and* $\operatorname{wt} g \geq 9$, $\operatorname{wt} h \geq 14$ *for the weights distributed above.*
(ii) *If* P *is of type* cE_7 *(resp.* cE_8*), then* $x_3^3 \in g$ *(resp.* x_3^5 *or* $x_3^4x_4 \in h$*).*
(iii) $x_4^7 \in h$.

We can show that every 3-dimensional divisorial contraction to a Gorenstein point is obtained as a weighted blow-up by [4], [5], [6], [8], [9], [10], and the above theorems. Therefore we can prove Theorem 1.1 by [12].

Proofs of these theorems shall be given in Section 4, and the list of all divisorial contractions to a *cDV* point shall be given in Section 5.

Notation. (i) We denote $\mathbb{C}^n$ with coordinates $x_1, \ldots, x_n$ by $\mathbb{C}^n_{x_1\ldots x_n}$.

(ii) We define the action of a cyclic group μ_m of order m on $\mathbb{C}^n_{x_1\ldots x_n}$ by

$$(x_1, \ldots, x_n) \mapsto (\zeta^{a_1}x_1, \ldots, \zeta^{a_n}x_n),$$

where ζ is a primitive m-th root of unity. The quotient space is denoted by $\mathbb{C}^n_{x_1\ldots x_n}/\frac{1}{m}(a_1, \ldots, a_n)$, $\mathbb{C}^n/\frac{1}{m}(a_1, \ldots, a_n)$ or simply $\frac{1}{m}(a_1, \ldots, a_n)$.

(iii) For $\operatorname{wt}(x_3, x_4) = (a, b)$ and $g(x_3, x_4) = \sum p_{ij}x_3^ix_4^j \in \mathbb{C}\{x_3, x_4\}$, we define

$$\operatorname{wt}(g(x_3, x_4)) = \inf\{ai + bj \mid p_{ij} \neq 0\}.$$

For a positive integer n, we define

$$g_{\mathrm{wt}=n}(x_3,x_4)=\sum_{ai+bj=n}p_{ij}x_3^ix_4^j$$
$$\text{and } g_{\mathrm{wt}\geq n}(x_3,x_4)=\sum_{ai+bj\geq n}p_{ij}x_3^ix_4^j.$$

(iv) Let $\mathbb{C}\{x_1,\dots,x_n\}$ be the ring of convergent power series in variable $x_1,\cdots,x_n$. For $f\in\mathbb{C}\{x_1,\dots,x_n\}$, we denote by f_m the homogeneous part of degree m of f.

(v) We say that "a monomial, e.g. x^n, appears in a power series f" or "f contains x^n" if there exists a monomial x^n with non-zero coefficient in the power series expansion of f, and denote it by $x^n\in f$.

3. Preliminaries

3.1. **Classification of terminal singularities.** It is known that a 3-dimensional Gorenstein terminal singularity is an isolated cDV hypersurface singularity, i.e., a singularity with local equation of the form

$$f(x_1,x_2,x_3)+x_4g(x_1,x_2,x_3,x_4)=0$$

for some $f(x_1,x_2,x_3)$ defining a Du Val (equivalently rational double point) singularity. If $P\in X$ is a 3-dimensional Gorenstein terminal singularity, then according to the type of $f(x_1,x_2,x_3)$, we have that $P\in X\simeq o\in(\varphi=0)\subset\mathbb{C}^4$ for some φ belongs to one of the following:

(i) type cA: $(x_1x_2+g(x_3,x_4)=0)\subset\mathbb{C}^4$ with $g(x_3,x_4)\in\mathfrak{m}^2$.

(ii) type cD: $(x_1^2+x_2^2x_4+\lambda x_2x_3^l+g(x_3,x_4)=0)\subset\mathbb{C}^4$ with $\lambda\in\mathbb{C}$, $l\geq 2$, $g(x_3,x_4)\in\mathfrak{m}^3$.

(iii) type cE: $(x_1^2+x_2^3+x_2g(x_3,x_4)+h(x_3,x_4)=0)\subset\mathbb{C}^4$ with $g(x_3,x_4)\in\mathfrak{m}^3$, $h(x_3,x_4)\in\mathfrak{m}^4$,

where $\mathfrak{m}$ denotes the maximal ideal of $o\in\mathbb{C}^4$. In the cE case, it is of type cE_6 (resp. cE_7, cE_8) if $h_4\neq 0$ (resp. $h_4=0$ and $g_3\neq 0$, $h_4=g_3=0$ and $h_5\neq 0$).

To prove Theorem 2.1 and Theorem 2.6, we need to construct a standard identification.

Lemma 3.1. *Let $P\in X$ be a germ of a 3-dimensional Gorenstein terminal singularity. If P is of type cA_2, then there is an identification*

$$P\in X\simeq o\in\big(x_1x_2+x_3^3+g(x_3,x_4)=0\big)\subset\mathbb{C}^4_{x_1x_2x_3x_4}$$
$$\simeq o\in\big(x_1^2+x_2^2+x_3^3+g(x_3,x_4)=0\big)\subset\mathbb{C}^4_{x_1x_2x_3x_4},$$

where $\deg g(x_3,1)\leq 2$.

Proof. By definition, there is an identification

$$P\in X\simeq o\in\big(x_1^2+x_2^2+x_3^3+x_4F(x_1,x_2,x_3,x_4)=0\big)\subset\mathbb{C}^4_{x_1x_2x_3x_4}$$

for some $F(x_1, x_2, x_3, x_4) \in \mathfrak{m}^2$. By using Weierstrass preparation theorem and completing a square, we may assume that

$$P \in X \simeq o \in \big(x_1^2 + x_2^2 + x_3^3 + x_4 F'(x_3, x_4) = 0\big)$$

for $F'(x_3, x_4) \in \mathfrak{m}^2$. We may assume that $\deg F'(x_3, 1) \leq 2$ by Weierstrass preparation for x_3. Thus we get the desired forms by the automorphism $x_1 + ix_2 \mapsto x_1$ and $x_1 - ix_2 \mapsto x_2$ if necessary. □

Mori classified that a 3-dimensional terminal singularity $P \in X$ with index $r > 1$ is isomorphic to a cyclic quotient of an isolated cDV singularity (see [16]), and Kollár and Shepherd-Barron showed that these isolated cDV's quotient are terminal singularities in [15].

Theorem 3.2. *There exists an identification*

$$P \in X \simeq o \in (\varphi = 0) \subset \mathbb{C}^4_{x_1x_2x_3x_4}/\mu_r,$$

where μ_r denotes the cyclic group of order r and x_1, x_2, x_3, x_4, φ are μ_r-semi-invariant. Furthermore, φ and the action of μ_r have one of the following forms:

(i) *type cA/r:* $(x_1x_2 + g(x_3^r, x_4) = 0) \subset \mathbb{C}^4/\frac{1}{r}(a, -a, 1, 0)$ *with* $g(x_3, x_4) \in \mathfrak{m}^2$, $\gcd(a, r) = 1$.
(ii) *type $cAx/2$:* $(x_1^2 + x_2^2 + g(x_3, x_4) = 0) \subset \mathbb{C}^4/\frac{1}{2}(0, 1, 1, 1)$ *with* $g(x_3, x_4) \in \mathfrak{m}^3$.
(iii) *type $cAx/4$:* $(x_1^2 + x_2^2 + g(x_3, x_4) = 0) \subset \mathbb{C}^4/\frac{1}{4}(1, 3, 1, 2)$ *with* $g(x_3, x_4) \in \mathfrak{m}^3$.
(iv) *type $cD/3$:* $(\varphi = 0) \subset \mathbb{C}^4/\frac{1}{3}(0, 2, 1, 1)$, *where φ has one of the following forms:*
  (1) $x_1^2 + x_2^3 + x_3^3 + x_4^3$.
  (2) $x_1^2 + x_2^3 + x_3^2 x_4 + x_2 g(x_3, x_4) + h(x_3, x_4)$ *with* $g \in \mathfrak{m}^4$, $h \in \mathfrak{m}^6$.
  (3) $x_1^2 + x_2^3 + x_3^3 + x_2 g(x_3, x_4) + h(x_3, x_4)$ *with* $g \in \mathfrak{m}^4$, $h \in \mathfrak{m}^6$.
(v) *type $cD/2$:* $(\varphi = 0) \subset \mathbb{C}^4/\frac{1}{2}(1, 0, 1, 1)$, *where φ has one of the following forms:*
  (1) $x_1^2 + x_2^3 + x_2x_3x_4 + g(x_3, x_4)$ *with* $g \in \mathfrak{m}^4$.
  (2) $x_1^2 + x_2x_3x_4 + x_2^n + g(x_3, x_4)$ *with* $n \leq 4$, $g \in \mathfrak{m}^4$.
  (3) $x_1^2 + x_2x_3^2 + x_2^n + g(x_3, x_4)$ *with* $n \leq 3$, $g \in \mathfrak{m}^4$.
(vi) *type $cE/2$:* $(x_1^2 + x_2^3 + x_2 g(x_3, x_4) + h(x_3, x_4) = 0) \subset \mathbb{C}^4/\frac{1}{2}(1, 0, 1, 1)$ *with* g, $h \in \mathfrak{m}^4$, $h_4 \neq 0$.

Conversely, if φ as above defines an isolated singularity and the action of μ_r on $\varphi = 0$ is free outside the origin, then P is a terminal singularity.

3.2. **Weighted blow-up.** We recall the construction of weighted blow-ups by using the toric language.

Let $N = \mathbb{Z}^d$ be a free abelian group, called *lattice*, of rank d with standard basis $\{e_1, \ldots, e_d\}$. Let M be the dual lattice of N. Let σ be the cone in $N \otimes \mathbb{R}$ generated by the standard basis $e_1, \ldots, e_d$ and Δ be the fan which consists of σ and all the faces of σ. We consider

$$T_N(\Delta) := \operatorname{Spec} \mathbb{C}[\sigma^\vee \cap M] = \mathbb{C}^d.$$

Let $v = (a_1, \dots, a_d)$ be a primitive vector in N, i.e., the vector which has no element in N between 0 and v. We assume that $a_i \in \mathbb{Z}_{\geq 0}$ and $\gcd(a_1, \dots, a_d) = 1$. For any i with $a_i > 0$, let σ_i be the cone generated by $\{e_1, \dots, e_{i-1}, v, e_{i+1}, \dots, e_d\}$ and $\Delta(v)$ be the fan consisting of all σ_i and their all faces. $\Delta(v)$ is called the *star shaped decomposition* for v. Then

$$T_N(\Delta(v)) = \textstyle\bigcup_{a_i>0} \operatorname{Spec} \mathbb{C}[\sigma_i^\vee \cap M].$$

If $a_i > 0$ for all i, the natural map $\pi\colon T_N(\Delta(v)) \to T_N(\Delta)$ is called the *weighted blow-up* over $o \in T_N(\Delta)$ with weight $v = (a_1, \dots, a_d)$. In each affine chart $\mathcal{U}_i := \operatorname{Spec} \mathbb{C}[\sigma_i^\vee \cap M]$, the natural map $\mathcal{U}_i \to T_N(\Delta)$ is given by

$$\begin{cases} x_j \mapsto x_j x_i^{a_j}, & \text{if } j \neq i; \\ x_i \mapsto x_i^{a_i}. \end{cases}$$

The exceptional divisor $\mathcal{E}$ of π is isomorphic to $\mathbb{P}(a_1, \dots, a_d)$.

Let $X := (\varphi(x_1, \dots, x_d) = 0) \subset T_N(\Delta)$ be a hypersurface, and Y be the birational transform on $T_N(\Delta(v))$ of X. We also call the induced map $\pi'\colon Y \to X$ the *weighted blow-up* of X with weight v. The affine chart $U_i := \mathcal{U}_i \cap Y$ can be expressed as

$$\left(\varphi(x_1 x_i^{a_1}, \dots, x_{i-1} x_i^{a_{i-1}}, x_i^{a_i}, x_{i+1} x_i^{a_{i+1}}, \dots, x_d x_i^{a_d}) x_i^{-\operatorname{wt}\varphi} = 0\right) \subset \mathcal{U}_i$$

for each i. The exceptional divisor of π' is denoted by $E := \mathcal{E} \cap Y$. If E is irreducible and reduced, and we have $\dim(T_N(\Delta(v)) \cap Y) \leq 1$, then we have the adjunction formula

$$K_Y = \pi'^* K_X + (\textstyle\sum_i a_i - \operatorname{wt}\varphi - 1)E.$$

We define weighted blow-ups of the complete intersection similarly.

3.3. **The singular Riemann-Roch formula.** As we shall use the method in [11] and [12], we recall the singular Riemann-Roch formula.

Theorem 3.3 ([17, Theorem 10.2])**.** *Let X be a projective 3-fold with canonical singularities and D a divisor on X such that $D \sim e_P K_X$ with $e_P \in \mathbb{Z}$ at each $P \in X$.*

(i) *There is a formula of the form*

$$\chi(\mathcal{O}_X(D)) = \chi(\mathcal{O}_X) + \frac{1}{12} D(D - K_X)(2D - K_X) + \frac{1}{12} D \cdot c_2(X) + \sum_P c_P(D),$$

where the summation takes place over the singularities on X , and $c_P(D) \in \mathbb{Q}$ is a contribution due to the singularity at P, depending only on the local analytic type of P and $\mathcal{O}_X(D)$.

(ii) *If $P \in X$ is a terminal cyclic quotient singularity of type $\frac{1}{r_P}(1, -1, b_P)$, then*

$$c_P(D) = -\overline{i_P}\frac{r_P^2 - 1}{12 r_P} + \sum_{l=1}^{\overline{i_P}-1} \frac{\overline{lb_P}(r_P - \overline{lb_P})}{2r_P},$$

where $\bar{i} = i - \lfloor \frac{i}{r_P} \rfloor r_P$ denotes the residue of i modulo r_P (the sum $\sum_{l=1}^{\overline{i_P}-1}$ is zero by convention if $\overline{i_P} = 0$ or 1).

(iii) *For an arbitrary terminal singularity P,*

$$c_P(D) = \sum_Q c_Q(D_Q)$$

where $\{(Q, D_Q)\}$ is a flat deformation of (P, D) to the basket of terminal cyclic quotient singularities Q.

4. Proofs of main results

In this section we prove the main theorem by using the method in [11] and [12]. Our strategy for the classification is to determine the exceptional divisor in the sense of valuation by applying Lemma 4.1 or Lemma 4.2 (see [10, Lemma 6.1], [11, Lemma 6.1]).

Lemma 4.1. *Let $f\colon (Y \supset E) \to (X \ni P)$ be a germ of a 3-dimensional divisorial contraction to a cDV point P. We identify $P \in X$ with*

$$P \in X \simeq o \in (\varphi = 0) \subset \bar{X} := \mathbb{C}^4_{x_1x_2x_3x_4}.$$

Let a denote the discrepancy of f and m_i denote the multiplicity of x_i along E, that is, the largest integer such that $x_i \in f_*\mathcal{O}_Y(-m_iE)$. Suppose that (m_1, m_2, m_3, m_4) is primitive in $\mathbb{Z}^4$. Let d denote the weighted order of φ with respect to weights $\mathrm{wt}(x_1, x_2, x_3, x_4) = (m_1, m_2, m_3, m_4)$, and decompose φ as

$$\varphi = \varphi_d(x_1, x_2, x_3, x_4) + \varphi_{>d}(x_1, x_2, x_3, x_4),$$

where φ_d is the weighted homogeneous part of weight d and $\varphi_{>d}$ is the part of weight greater than d. Set $c := m_1 + m_2 + m_3 + m_4 - 1 - d$. Let $\bar{g}\colon (\bar{Z} \supset \bar{F}) \to (\bar{X} \ni o)$ be the weighted blow-up with weights $\mathrm{wt}(x_1, x_2, x_3, x_4) = (m_1, m_2, m_3, m_4)$, $\bar{F}$ its exceptional divisor. Let Z denote the birational transform on $\bar{Z}$ of X and $g\colon Z \to X$ the induced morphism. If we have four conditions:

(i) *$\bar{F} \cap Z$ defines an irreducible and reduced 2-cycle F,*
(ii) *Z is smooth at the generic point of F,*
(iii) *$\dim(\mathrm{Sing}\, \bar{Z} \cap Z) \le 1$, and*
(iv) *$c = a$,*

then we have $f \simeq g$ over X.

We shall apply the following extension of Lemma 4.1 to several cases.

Lemma 4.2. *Let $f\colon (Y \supset E) \to (X \ni P)$ be a germ of a 3-dimensional divisorial contraction to a cDV point P. We identify $P \in X$ with*

$$P \in X \simeq o \in \begin{pmatrix} \varphi = 0 \\ \psi = 0 \end{pmatrix} \subset \bar{X} := \mathbb{C}^5_{x_1x_2x_3x_4x_5}.$$

Let a denote the discrepancy of f and m_i denote the multiplicity of x_i along E. Suppose that $(m_1, m_2, m_3, m_4, m_5)$ is primitive in $\mathbb{Z}^5$. Let d (resp. e) denote the weighted order of φ (resp. ψ) with respect to weights $\mathrm{wt}(x_1, x_2, x_3, x_4, x_5) = (m_1, m_2, m_3, m_4, m_5)$, and decompose φ and ψ as

$$\varphi = \varphi_d(x_1, x_2, x_3, x_4, x_5) + \varphi_{>d}(x_1, x_2, x_3, x_4, x_5),$$
$$\psi = \psi_e(x_1, x_2, x_3, x_4, x_5) + \psi_{>e}(x_1, x_2, x_3, x_4, x_5),$$

where φ_d (resp. ψ_e) is the weighted homogeneous part of weight d (resp. e) and $\varphi_{>d}$ (resp. $\psi_{>e}$) is the part of weight greater than d (resp. e). Set $c := m_1 + m_2 + m_3 + m_4 + m_5 - 1 - d - e$. Let $\bar{g}\colon (\bar{Z} \supset \bar{F}) \to (\bar{X} \ni o)$ be the weighted blow-up with weights $\mathrm{wt}(x_1, x_2, x_3, x_4, x_5) = (m_1, m_2, m_3, m_4, m_5)$, $\bar{F}$ its exceptional divisor. Let Z denote the birational transform on $\bar{Z}$ of X and $g\colon Z \to X$ the induced morphism. If we have four conditions:

(i) *$\bar{F} \cap Z$ defines an irreducible and reduced 2-cycle F,*
(ii) *Z is smooth at the generic point of F,*
(iii) $\dim(\mathrm{Sing}\,\bar{Z} \cap Z) \leq 1$, *and*
(iv) *$c = a$,*

then we have $f \simeq g$ over X.

Now we study 3-dimensional divisorial contractions to cDV points. We let

$$f\colon (Y \supset E) \to (X \ni P)$$

be a germ of a 3-dimensional divisorial contraction whose exceptional divisor E contracts to a singular point P of index 1, and a denote its discrepancy. Let $I_0 := \{Q, \text{of type}(1/r_Q)(1, -1, b_Q)\}$ denote the basket of fictitious singularities on Y, and let e_Q for $Q \in I_0$ be the smallest positive integer such that $E \sim e_Q K_Y$ at Q. By replacing b_Q with $r_Q - b_Q$ if necessary, we may assume that $v_Q := \overline{e_Q b_Q} \leq r_Q/2$, where $\bar{}$ denotes the residue modulo r_Q. We set $I := \{Q \in I_0 \mid v_Q \neq 0\}$ and $J := \{(r_Q, v_Q)\}_{Q \in I}$. We can compute J for each case in Table 1, and we give its results in Table 2.

TABLE 2

type	J	type	J
$e1$	$(r, 2)$	$e5$	$(7, 3)$
$e2$	$(r, 1)$, $(r, 1)$	$e9$	$(5, 2)$, $(3, 1)$
$e3$	$(2, 1)$, $(4, 1)$		

We shall prove the main results as follows:

Step1. For an integer j, we compute the dimension of the vector space

$$V_j := f_*\mathcal{O}_Y(-jE)/f_*\mathcal{O}_Y(-(j+1)E).$$

This space is regarded as the space of functions on X vanishing with multiplicity j along E. For a function h on X, we let $\operatorname{mult}_E h$ denote the multiplicity of h along E.

Step2. We find basis of V_j starting with an arbitrary identification

$$P \in X \simeq o \in (\varphi = 0) \subset \mathbb{C}^4_{x_1x_2x_3x_4}, \tag{1}$$

and we compute the favorite weights $\operatorname{wt}(x_1, x_2, x_3, x_4)$.

Step3. In order to apply Lemma 4.1 or Lemma 4.2, we follow these procedures:

(i) Determine $\operatorname{wt}(x_1, x_2, x_3, x_4)$, and rewrite φ.

(ii) Let $f' \colon Z \to X$ be the weighted blow-up with $\operatorname{wt} x_i = \operatorname{mult} x_i$. Find the condition that the exceptional locus of f' is irreducible and reduced.

(iii) Verify the assumption of Lemma 4.1, and find the condition that every singular point in Z is terminal.

Step4. Then we can apply Lemma 4.1 or Lemma 4.2, and show that f coincide with f'.

We note that $\dim V_j$ and basis of V_j are dependent only of the type of f but not of the type of P. So we shall show the main theorems according to the type of f.

We compute $\dim V_j$ by using the singular Riemann-Roch formula. For each j, there is a natural exact sequence

$$0 \to \mathcal{O}_Y(-(j+1)E) \to \mathcal{O}_Y(-jE) \to \mathcal{O}_E(-jE|_E) \to 0.$$

So we have a long exact sequence

$$\begin{aligned} 0 &\to f_*\mathcal{O}_Y(-(j+1)E) \to f_*\mathcal{O}_Y(-jE) \to f_*\mathcal{O}_E(-jE|_E) \\ &\to R^1f_*\mathcal{O}_Y(-(j+1)E) \to R^1f_*\mathcal{O}_Y(-jE) \to R^1f_*\mathcal{O}_E(-jE|_E) \\ &\to \cdots . \end{aligned}$$

Since P is terminal, we have $R^if_*\mathcal{O}_Y(-(j+1)E) = 0$ and $R^if_*\mathcal{O}_Y(-jE) = 0$ for any $i \geq 1$, j by Kawamata-Viehweg theorem and $R^if_*\mathcal{O}_E(-jE|_E) = H^i(E, \mathcal{O}_E(-jE|_E))$ for any i, j. Then

$$\begin{aligned} \dim_{\mathbb{C}} V_j &= \dim_{\mathbb{C}} f_*\mathcal{O}_E(-jE|_E) \\ &= \dim_{\mathbb{C}} H^0(E, \mathcal{O}_E(-jE|_E)) = \chi(\mathcal{O}_E(-jE|_E)) \\ &= \chi(\mathcal{O}_Y(-jE)) - \chi(\mathcal{O}_Y(-(j+1)E)). \end{aligned}$$

Applying the singular Riemann-Roch formula, we have

$$\begin{aligned} \dim V_j = \frac{1}{12}\big(6j(j+a+1) + (a+1)(a+2)\big)E^3 \\ + \frac{1}{12}E \cdot c_2(Y) + A_j - A_{j+1}. \end{aligned} \tag{$*$}$$

Here the contribution term A_j is given by $A_j := \sum_{Q\in I} A_Q(\overline{-je_Q})$, where

$$(**)\qquad A_Q(k) := -k\frac{r_Q^2-1}{12r_Q} + \sum_{l=1}^{k-1}\frac{\overline{lb_Q}(r_Q-\overline{lb_Q})}{2r_Q}.$$

For $j<0$, we have $V_j=0$. Now we compute $\dim V_j$ explicitly and show that f is a weighted blow-up in each case. Since we shall use similar procedures in each case, we start with easy cases and proceed to complicated cases.

4.1. **Case $e9$ with discrepancy 2.** In this subsection, we suppose that $f\colon (Y\supset E)\to(X\ni P)$ is of type $e9$, and its discrepancy a is 2. In this case, Y has two non-Gorenstein singular points. One point Q_1 is of type $\frac{1}{3}(1,2,2)$ and another point Q_2 is of type $\frac{1}{5}(1,4,4)$. Set $N_j := \{(l_1,l_2,l_3,l_4)\in\mathbb{Z}^4_{\geq 0} \mid 7l_1+5l_2+3l_3+2l_4=j,\ l_1\leq 1\}$.

Lemma 4.3. *$\dim V_j = \#N_j$.*

Proof. By Table 1 and Table 2, we see that $(r_{Q_1},b_{Q_1},v_{Q_1})=(3,2,1)$, $(r_{Q_2},b_{Q_2},v_{Q_2})=(5,4,2)$ and $E^3=1/15$. We also have $e_{Q_1}=2$, $e_{Q_2}=3$. So

$$\begin{aligned}\dim V_j =&\frac{1}{30}j(j+3)+\frac{1}{15}+\frac{1}{12}E\cdot c_2(Y)\\ &-(\bar{j}-\overline{j+1})\frac{2}{9}+\left(\sum_{l=1}^{\bar{j}-1}-\sum_{l=1}^{\overline{j+1}-1}\right)\frac{\overline{2l}(3-\overline{2l})}{6}\\ &-(\overline{2j}'-\overline{2(j+1)}')\frac{2}{5}+\left(\sum_{l=1}^{\overline{2j}'-1}-\sum_{l=1}^{\overline{2(j+1)}'-1}\right)\frac{\overline{4l}'(5-\overline{4l}')}{10}.\end{aligned}$$

Here $\bar{\ }$ denotes the residue modulo 3 and $\bar{\ }'$ the residue modulo 5. Since $\dim V_0=1$, we have

$$\frac{1}{15}+\frac{1}{12}E\cdot c_2(Y)=\frac{17}{45}.$$

Now we consider

$$\begin{aligned}\dim V_j-\dim V_{j-5}=&\frac{1}{3}(j-1)-\frac{2}{9}(\bar{j}-2\overline{j+1}+\overline{j+2})\\ &+\left(\sum_{l=1}^{\bar{j}-1}-2\sum_{l=1}^{\overline{j+1}-1}+\sum_{l=1}^{\overline{j+2}-1}\right)\frac{\overline{2l}(3-\overline{2l})}{6}\end{aligned}$$

for any $j\geq 5$. We have

$$\dim V_j-\dim V_{j-5}=\begin{cases} j/3 & \text{if } k\equiv 0 \pmod 3\\ (j-1)/3 & \text{if } k\equiv 1 \pmod 3\\ (j-2)/3 & \text{if } k\equiv 2 \pmod 3).\end{cases}$$

On the other hand, we have a decomposition

$$N_j = \{(l_1, 0, l_3, l_4) \in N_j\} \sqcup \{\vec{l} + (0,1,0,0) \mid \vec{l} \in N_{j-5}\}.$$

Hence for any $j \geq 5$,

$$\#N_j - \#N_{j-5} = \#\{(l_1, 0, l_3, l_4) \in N_j\}.$$

So we have

$$\#N_j - \#N_{j-5} = \begin{cases} j/3 & \text{if } k \equiv 0 \pmod 3 \\ (j-1)/3 & \text{if } k \equiv 1 \pmod 3 \\ (j-2)/3 & \text{if } k \equiv 2 \pmod 3). \end{cases}$$

Therefore we have $\dim V_j - \dim V_{j-5} = \#N_j - \#N_{j-5}$ for any $j \geq 5$. We can compute $\dim V_j = \#N_j$ for $j \leq 4$. Then we have $\dim V_j = \#N_j$ for any j. □

Lemma 4.4.

(i) *There exists some* $1 \leq k,\, l \leq 4$ *with* $\operatorname{mult}_E x_k = 2$ *and* $\operatorname{mult}_E x_l = 3$. *By permutation, we may assume that* $x_k = x_4$, $x_l = x_3$. *Moreover* $\operatorname{mult}_E x_k \geq 4$ *for* $k = 1, 2$.

(ii) *If* $j < 5$, *the monomials* $x_3^{l_3} x_4^{l_4}$ *for* $(0, 0, l_3, l_4) \in N_j$ *form a basis of* V_j. *In particular, for* $k = 1, 2$, $\operatorname{mult}_E \bar{x}_k \geq 5$ *for* $\bar{x}_k := x_k + \sum c_{kl_3l_4} x_3^{l_3} x_4^{l_4}$ *with some* $c_{kl_3l_4} \in \mathbb{C}$ *and summation over* $(0, 0, l_3, l_4) \in \cup_{j<5} N_j$.

(iii) *There exists some* $k = 1, 2$ *with* $\operatorname{mult}_E \bar{x}_k = 5$ *such that the monomials* $\bar{x}_k$ *and* $x_3^{l_3} x_4^{l_4}$ *for* $(0, 0, l_3, l_4) \in N_5$ *form a basis of* V_5. *By permutation, we may assume that* $\bar{x}_k = \bar{x}_2$.

(iv) *The monomials* $\bar{x}_2^{l_2} x_3^{l_3} x_4^{l_4}$ *for* $(0, l_2, l_3, l_4) \in N_6$ *form a basis of* V_6, *and we have* $\operatorname{mult} \hat{x}_1 \geq 7$ *for* $\hat{x}_1 := \bar{x}_1 + \sum c_{l_2l_3l_4} \bar{x}_2^{l_2} x_3^{l_3} x_4^{l_4}$ *with some* $c_{l_2l_3l_4} \in \mathbb{C}$ *and summation over* $(0, l_2, l_3, l_4) \in N_6$.

(v) *We have* $\operatorname{mult}_E \hat{x}_1 = 7$, *and for* $j < 14$, *the monomials* $\hat{x}_1^{l_1} \bar{x}_2^{l_2} x_3^{l_3} x_4^{l_4}$ *for* $(l_1, l_2, l_3, l_4) \in N_j$ *form a basis of* V_j.

(vi) *Set* $\tilde{N}_j = \{(l_1, l_2, l_3, l_4) \in \mathbb{Z}^4_{\geq 0} | 7l_1 + 5l_2 + 3l_3 + 2l_4 = j\}$. *The monomials* $\hat{x}_1^{l_1} \bar{x}_2^{l_2} x_3^{l_3} x_4^{l_4}$ *for* $(l_1, l_2, l_3, l_4) \in N_{14}$ *have one non-trivial relation, say* ψ, *in* V_{14}. *The natural exact sequence*

$$0 \to \mathbb{C}\psi \to \bigoplus_{(l_1,l_2,l_3,l_4) \in \tilde{N}_{14}} \mathbb{C}\hat{x}_1^{l_1} \bar{x}_2^{l_2} x_3^{l_3} x_4^{l_4} \to V_{14} \to 0$$

is exact.

Proof. We have $\dim V_1 = 0$, $\dim V_2 = \dim V_3 = 1$ by Lemma 4.3. This implies (i). By permutation, we may assume that $\operatorname{mult}_E x_4 = 2$, $\operatorname{mult}_E x_3 = 3$. To prove (ii), we shall show that the monomials $x_3^{l_3} x_4^{l_4}$ for $(0, 0, l_3, l_4) \in N_j$ are linearly independent in V_j for any j. Suppose

$$0 = \sum_{(0,0,l_3,l_4) \in N_j} c_{l_3l_4} x_3^{l_3} x_4^{l_4} \in V_j, \ \ (c_{l_3l_4} \in \mathbb{C}).$$

We shall show that $c_{l_3 l_4} = 0$ for any $(0,0,l_3,l_4) \in N_j$. We set $j = 6k + \alpha$, where $0 \le k \in \mathbb{Z}$ and $0 \le \alpha \le 5$. We study the case $j = 6k$ for $0 \le k \in \mathbb{Z}$. So we can write

$$\sum_{(0,0,l_3,l_4)\in N_j} c_{l_3 l_4} x_3^{l_3} x_4^{l_4} = \sum_{l=0}^{k} c_l x_3^{2l} x_4^{3(k-l)}$$

for $c_l \in \mathbb{C}$. Since $\mathbb{C}$ is an algebraically closed field, we factorize

$$\sum_{l=0}^{k} c_l x_3^{2l} x_4^{3(k-l)} = \left(d_1 x_3^2 + d_2 x_4^3\right)\left(\sum_{l=1}^{k} c'_l x_3^{2(l-1)} x_4^{3(k-l)}\right)$$

for c'_l, d_1, $d_2 \in \mathbb{C}$. Hence we have $c_l = 0$ for all $0 \le l \le k$ by induction on k. We can show that $c_{l_3 l_4} = 0$ for any other case similarly. We set $W(j) := \left\langle x_3^{l_3} x_4^{l_4} \middle| (0,0,l_3,l_4) \in N_j \right\rangle \subset V_j$ for each j. Then $\dim W(j) = \# N_j$ for $j < 5$, and thus we obtain (ii) by Lemma 4.3. Since $\dim V_5 = \dim W(5) + 1$ by Lemma 4.3, we obtain (iii). By permutation, we may assume that $\bar{x}_2$ forms a basis of $V_5/W(5) \simeq \mathbb{C}$. Since the monomials $x_3^{l_3} x_4^{l_4}$ for $(0,0,l_3,l_4) \in N_j$ are linearly independent in V_j for any j, and $\dim V_7 = W(7) + 2$ by Lemma 4.3, we obtain (iv) and $\operatorname{mult}_E \hat{x}_1 = 7$. For any $j < 14$, we have $\dim V_j = \# \tilde{N}_j$ by Lemma 4.3. This implies (v). Since $\dim V_{14} = \# N_{14} = \# \tilde{N}_{14} - 1$, we have a non trivial relation, say ψ in V_{14}, and we obtain the natural exact sequence in (vi). □

Corollary 4.5. *We distribute weights* $\operatorname{wt}(\hat{x}_1, \bar{x}_2, x_3, x_4) = (7,5,3,2)$ *to the coordinates* $\hat{x}_1$,$\bar{x}_2$,x_3,x_4 *obtained in* Lemma 4.4. *Then* φ *is of form*

$$\varphi = c\psi + \varphi_{>14}(\hat{x}_1, \bar{x}_2, x_3, x_4)$$

with $c \in \mathbb{C}$ *and a function* $\varphi_{>14}$ *of weighted order* > 14, *where* ψ *in* (1) *is the one in* Lemma 4.4(vi).

Proof. Decompose $\varphi = \varphi_{\le 14} + \varphi_{>14}$ into the part $\varphi_{\le 14}$ of weighted order ≤ 14 and $\varphi_{>14}$ of weighted order > 14. Then $\operatorname{mult}_E \varphi_{\le 14} = \operatorname{mult}_E \varphi_{>14} > 14$, so $\varphi_{\le 14}$ is mapped to zero by the natural homomorphism

$$\bigoplus_{(l_1,l_2,l_3,l_4)\in \bigcup_{j\le 14} \tilde{N}_j} \mathbb{C}\hat{x}_1^{l_1} \bar{x}_2^{l_2} x_3^{l_3} x_4^{l_4} \to \mathcal{O}_X / f_* \mathcal{O}_Y(-15E),$$

whose kernel is $\mathbb{C}\psi$ by Lemma 4.4(v), (vi). □

Proof of Theorem 2.10. The $cE_{7,8}$ point $P \in X$ has an identification such that

$$\varphi = x_1^2 + x_2^3 + x_2 g(x_3, x_4) + h(x_3, x_4) = 0,$$

where $g \in \mathfrak{m}^3$ and $h \in \mathfrak{m}^4$. If P is of type cE_7 (resp. cE_8), then $g_3 \neq 0$ (resp. $g_3 = 0$, $h_5 \neq 0$).

(i) We shall show that we distribute weight $\mathrm{wt}(x_1,x_2,x_3,x_4)=(7,5,3,2)$, and that φ can write

$$\varphi = x_1^2 + x_2^3 + \lambda x_2^2 x_4^2 + x_2 g(x_3,x_4) + h(x_3,x_4),$$

with $\lambda \in \mathbb{C}$, $g \in \mathfrak{m}^3$, and $h \in \mathfrak{m}^4$.
By Corollary 4.5, we have $\mathrm{wt}\,\varphi = 14$. So we can show that we distribute weight $\mathrm{wt}(x_1,x_2,x_3,x_4)=(7,5,3,2)$ easily. We obtain a quartuple $(\hat{x}_1,\bar{x}_2,x_3,x_4)$ by $\hat{x}_1 = x_1 + c\bar{x}_2 + p(x_3,x_4)$, $\bar{x}_2 = x_2 + q(x_3,x_4)$, where $c \in \mathbb{C}$, p, and q as in Lemma 4.4, that is, p (resp. q) contains only monomials with weight ≤ 6 (resp. ≤ 4).

Then we rewrite φ as

$$\begin{aligned}\varphi &= (\hat{x}_1 - c\bar{x}_2 - p)^2 + (\bar{x}_2 - q)^3 + (\bar{x}_2 - q)g + h\\ &= \hat{x}_1^2 - 2p\hat{x}_1 - 2c\hat{x}_1\bar{x}_2 + \bar{x}_2^3 + (c^2 - 3q)\bar{x}_2^2\\ &\quad + (2cp + 3q^2 + g)\bar{x}_2 + (p^2 - q^3 - qg + h).\end{aligned}$$

Since $\mathrm{wt}\,\varphi = 14$, we can show that $c = p = 0$, $\mathrm{wt}\,q = 4$, $\mathrm{wt}(3q^2 + g) \geq 9$, and $\mathrm{wt}(-q^3 - qg + h) \geq 14$. We also have $q = \lambda x_4^2$ with $\lambda \in \mathbb{C}$. Moreover if P is of type cE_7 (resp. cE_8), then we have $x_3^3 \in g$ (resp. x_3^5 or $x_3^4 x_4 \in h$). Replacing $3q^2 + g$ with g and $-q^3 - qg + h$ with h, and replacing variables, we have the desired expression in (i).

(ii) Let $f'\colon Z \to X$ be the weighted blow-up with $\mathrm{wt}\,x_i = \mathrm{mult}_E\,x_i$. If P is of type cE_7, it is obvious that the exceptional locus F of f' is irreducible and reduced. If P is of type cE_8, we need the condition that $x_3x_4^3 \in g$ or $x_4^7 \in h$ if $\lambda = 0$ and $x_3^4 x_4 \notin h$, which is equivalent that F is irreducible and reduced.

(iii) We shall show that φ has the condition $x_4^7 \in h$ if and only if every singular point in Z is terminal.
The x_4-chart U_4 of the weighted blow-up f' can be expressed as

$$U_4 = \left(\begin{aligned} & x_1'^2 + x_2'^3 x_4' + \lambda x_2'^2 \\ &\quad + x_2' \frac{1}{x_4'^9} g(x_3' x_4'^3, x_4'^2) + \frac{1}{x_4'^{14}} h(x_3' x_4'^3, x_4'^2) = 0 \end{aligned} \right) \Big/ \frac{1}{2}(1,1,1,1).$$

If the origin o is contained in U_4, then this point is not terminal since this equation has only even degree terms. So we need the condition $o \notin U_4$, which is equivalent to the condition $x_4^7 \in h$. Hence Z is covered by U_1, U_2, and U_3. We study U_2 and U_3.

$$U_2 = \left(\begin{aligned} x_1'^2 + x_2' + \lambda x_4'^2 + \frac{1}{x_2'^9} g(x_2'^3 x_3', x_2'^2 x_4') \\ + \frac{1}{x_2'^{14}} h(x_2'^3 x_3', x_2'^2 x_4') = 0 \end{aligned} \right) \Big/ \frac{1}{5}(4,3,1,4),$$

$$U_3 = \left(\begin{aligned} x_1'^2 + x_2'^3 x_3' + \lambda x_2'^2 x_4'^2 + x_2' \frac{1}{x_3'^9} g(x_3'^3, x_3'^2 x_4') \\ + \frac{1}{x_3'^{14}} h(x_3'^3, x_3'^2 x_4') = 0 \end{aligned} \right) \Big/ \frac{1}{3}(1,2,2,2).$$

The origin of U_2 is of type $\frac{1}{5}(1,4,4)$ and the origin of U_3 is of type $\frac{1}{3}(1,2,2)$. We shall check that U_3 has only isolated singularities. Every singular point in U_3 lies only on the hyperplane $(x_3' = 0)$ since F is contracted to P by f'. So it is enough to study terms of degree ≤ 1 with respect to x_3'.

$$\text{terms of degree 0: } x_1'^2 + x_2' g_{\mathrm{wt}=9}(1, x_4') + h_{\mathrm{wt}=14}(1, x_4').$$
$$\text{terms of degree 1: } x_2'^3 + x_2' g_{\mathrm{wt}=10}(1, x_4') + h_{\mathrm{wt}=15}(1, x_4').$$

Therefore we can check that U_3 has only isolated singularities. Similarly we can check that U_1 and U_2 have only isolated singularities. Thus the proof of (iii) is finished.

Therefore we can apply Lemma 4.1, and f should coincide with f'. The proof of Theorem 2.10 is completed. □

4.2. **Case $e2$ with discrepancy 2.** In this subsection, we suppose that $f\colon (Y \supset E) \to (X \ni P)$ is of type $e2$, and its discrepancy a is 2. In this case, Y has one non-Gorenstein singular point. This point deforms to two points Q_1 and Q_2 which are of type $\frac{1}{r}(1,-1,2)$. Set $N_j := \{(l_1,l_2,l_3,l_4) \in \mathbb{Z}^4_{\geq 0} \mid rl_1 + rl_2 + 2l_3 + l_4 = j,\ l_1 l_2 = 0\}$.

Lemma 4.6. $\dim V_j = \# N_j$.

Proof. By Table 1 and Table 2, we see that $(r_{Q_i}, b_{Q_i}, v_{Q_i}) = (r,2,1)$ for $i = 1, 2$ and $E^3 = 1/r$. We also have $e_{Q_i} = (r+1)/2$. So

$$\begin{aligned} \dim V_j = {} & \frac{1}{2r} j(j+3) + \frac{1}{r} + \frac{1}{12} E \cdot c_2(Y) \\ & - \Big(\overline{j\frac{r-1}{2}} - \overline{(j+1)\frac{r-1}{2}} \Big) \frac{r^2-1}{12r} + \Big(\sum_{l=1}^{\overline{j\frac{r-1}{2}}-1} - \sum_{l=1}^{\overline{(j+1)\frac{r-1}{2}}-1} \Big) \frac{\overline{2l}(r-\overline{2l})}{2r}. \end{aligned}$$

Here $\bar{\ }$ denotes the residue modulo r. Since $\dim V_0 = 1$, we have

$$\frac{1}{r} + \frac{1}{12} E \cdot c_2(Y) = 1 - \frac{r-1}{2} \cdot \frac{r^2-1}{12r} + \sum_{l=1}^{\frac{r-1}{2}-1} \frac{\overline{2l}(r-\overline{2l})}{2r}.$$

Now we can compute

$$\dim V_j - \dim V_{j-2} = \frac{1}{r}(2j+1) + \frac{\overline{j+1}(r-\overline{j+1}) - \bar{j}(r-\bar{j})}{2r}$$

for any $j \geq 2$. We can show $\dim V_j - \dim V_{j-2} = \#N_j - \#N_{j-2}$ as Lemma 4.3. □

Lemma 4.7.

(i) *There exists some* $1 \leq k,\, l \leq 4$ *with* $\mathrm{mult}_E\, x_k = 1$ *and* $\mathrm{mult}_E\, x_l = 2$. *By permutation, we may assume that* $x_k = x_4$, $x_l = x_3$. *Moreover* $\mathrm{mult}_E\, x_k \geq 3$ *for* $k = 1,\, 2$.

(ii) *If* $j < r$, *the monomials* $x_3^{l_3} x_4^{l_4}$ *for* $(0,0,l_3,l_4) \in N_j$ *form a basis of* V_j. *In particular, for* $k = 1,\, 2$, $\mathrm{mult}_E\, \bar{x}_k \geq r$ *for* $\bar{x}_k := x_k + \sum c_{kl_3l_4} x_3^{l_3} x_4^{l_4}$ *with some* $c_{kl_3l_4} \in \mathbb{C}$ *and summation over* $(0,0,l_3,l_4) \in \cup_{j<r} N_j$.

(iii) *We have* $\mathrm{mult}_E\, \bar{x}_k = r$ *for* $k = 1,\, 2$, *and if* $j < 2r$, *the monomials* $\bar{x}_1^{l_1} \bar{x}_2^{l_2} x_3^{l_3} x_4^{l_4}$ *for* $(l_1,l_2,l_3,l_4) \in N_j$ *form a basis of* V_j.

(iv) *Set* $\tilde{N}_j = \{(l_1,l_2,l_3,l_4) \in \mathbb{Z}^4_{\geq 0} \mid rl_1 + rl_2 + 2l_3 + l_4 = j\}$. *The monomials* $\bar{x}_1^{l_1} \bar{x}_2^{l_2} x_3^{l_3} x_4^{l_4}$ *for* $(l_1,l_2,l_3,l_4) \in N_{2r}$ *have one non-trivial relation, say* ψ, *in* V_{2r}. *The natural exact sequence*

$$0 \to \mathbb{C}\psi \to \bigoplus_{(l_1,l_2,l_3,l_4)\in \tilde{N}_{2r}} \mathbb{C}\bar{x}_1^{l_1} \bar{x}_2^{l_2} x_3^{l_3} x_4^{l_4} \to V_{2r} \to 0$$

is exact.

Proof. We follow the proof of Lemma 4.4, with using the computation of Lemma 4.6. (i) follows from $\dim V_1 = 1$ and $\dim V_2 = 2$. By permutation, we may assume that $\mathrm{mult}_E\, x_4 = 1$, $\mathrm{mult}_E\, x_3 = 2$. To prove (ii), we shall show that the monomials $x_3^{l_3} x_4^{l_4}$ for $(0,0,l_3,l_4) \in N_j$ are linearly independent in V_j for any j. Suppose $0 = \sum_{(0,0,l_3,l_4)\in N_j} c_{l_3l_4} x_3^{l_3} x_4^{l_4} \in V_j$, $(c_{l_3l_4} \in \mathbb{C})$. We shall show that $c_{l_3l_4} = 0$ for any $(0,0,l_3,l_4) \in N_j$. We study the case $j = 2k$ for $0 \leq k \in \mathbb{Z}$. So we can write

$$\sum_{(0,0,l_3,l_4)\in N_j} c_{l_3l_4} x_3^{l_3} x_4^{l_4} = \sum_{l=0}^{k} c_l x_3^l x_4^{2(k-l)}$$

for $c_l \in \mathbb{C}$. We factorize

$$\sum_{l=0}^{k} c_l x_3^l x_4^{2(k-l)} = (d_1 x_3 + d_2 x_4^2)(\sum_{l=1}^{k} c_l' x_3^{l-1} x_4^{2(k-l)})$$

for c_l', d_1, $d_2 \in \mathbb{C}$. Hence we have $c_l = 0$ for all $0 \leq l \leq k$ by induction on k. We can show that $c_{l_3l_4} = 0$ for the case j is odd similarly. We set $W(j) := \left\langle x_3^{l_3} x_4^{l_4} \middle| (0,0,l_3,l_4) \in N_j \right\rangle \subset V_j$ for each j. Then $\dim W(j) = \#N_j$ for $j < r$, and thus we obtain (ii). Since $\dim V_r = \dim W(r) + 2$, by permutation, we may assume that $\bar{x}_2$ and $\bar{x}_1$ forms a basis of $V_r/W(r) \simeq \mathbb{C}^2$, and we have $\mathrm{mult}_E\,\bar{x}_1 = \mathrm{mult}_E\,\bar{x}_2 = r$. Since the monomials $x_3^{l_3}x_4^{l_4}$ for $(0,0,l_3,l_4) \in N_j$ are linearly independent in V_j for any j, and we have $\dim V_j = \dim W(j) + 2\#N_{j-r} = \#\tilde{N}_j$ for any $j < 2r$. This implies (iii). Since $\dim V_{2r} = \#N_{2r} = \#\tilde{N}_{2r} - 1$, we have a non trivial relation, say ψ in V_{2r}, and we obtain the natural exact sequence in (iv). □

Corollary 4.8. *We distribute weights* $\mathrm{wt}(\bar{x}_1, \bar{x}_2, x_3, x_4) = (r, r, 2, 1)$ *to the coordinates* $\bar{x}_1$,$\bar{x}_2$,x_3,x_4 *obtained in* Lemma 4.7. *Then* φ *is of form*

$$\varphi = c\psi + \varphi_{>2r}(\bar{x}_1, \bar{x}_2, x_3, x_4)$$

with $c \in \mathbb{C}$ *and a function* $\varphi_{>2r}$ *of weighted order* $> 2r$*, where* ψ *in* (1) *is the one in* Lemma 4.7(iv). □

Proof of Theorem 2.4. The cD point $P \in X$ has an identification such that

$$\varphi = x_1^2 + x_2^2 x_4 + \lambda x_2 x_3^k + g(x_3, x_4) = 0,$$

where $g \in \mathfrak{m}^3$, $\lambda \in \mathbb{C}$, and $k \geq 2$.

(i) By Corollary 4.8, we have $\mathrm{wt}\,\varphi = 2r$. So we have $\mathrm{wt}\,x_1$, $\mathrm{wt}\,x_2 = r$. We obtain a quartuple $(\bar{x}_1, \bar{x}_2, x_3, x_4)$ by $\bar{x}_1 = x_1 + p(x_3, x_4)$, $\bar{x}_2 = x_2 + q(x_3, x_4)$, where p, q as in Lemma 4.7. Then we rewrite φ as

$$\begin{aligned}\varphi &= (\bar{x}_1 - p)^2 + (\bar{x}_2 - q)^2 x_4 + \lambda(\bar{x}_2 - q)x_3^k + g\\ &= (\bar{x}_1 - p)^2 + \bar{x}_2^2 x_4 - 2\bar{x}_2 x_4 q + \lambda \bar{x}_2 x_3^k + (q^2 x_4 - \lambda q x_3^k + g).\end{aligned}$$

Since $\mathrm{wt}\,\varphi = 2r$, we can show that $p = 0$, $\mathrm{wt}(q^2 x_4 - \lambda q x_3^k + g) \geq 2r$, and q contains only monomial with weight $r-2$ and $r-1$. So by replacing variables, we can rewrite φ as

$$\varphi = x_1^2 + x_2^2 x_4 + 2x_2 x_4 p(x_3, x_4) + \lambda x_2 x_3^k + q(x_3, x_4),$$

with $\lambda \in \mathbb{C}$, $k \geq 2$, $\mathrm{wt}\,q \geq 2r$, and p contains only monomial with weight $r-2$ and $r-1$.

• Suppose that $\mathrm{wt}(x_1, x_2, x_3, x_4) = (r, r, 2, 1)$.

In this case, we have $k > r/2$, and p is weighted homogeneous of weight $r-1$ for the weights distributed above. Let $f'\colon Z \to X$ be the weighted blow-up with $\mathrm{wt}(x_1, x_2, x_3, x_4) = (r, r, 2, 1)$.

(ii) We have two conditions below if and only if the exceptional locus F of f' is irreducible and reduced.

(1) $p \neq 0$ or $q_{\mathrm{wt}=2r} \neq 0$.
(2) $q_{\mathrm{wt}=2r}$ is not square if $p = 0$.

If $x_3^r \in q$, then either (1) or (2) holds. (iii) We shall show that φ has the condition $x_3^r \in q$ if and only if every singular point in Z is terminal.

The x_3-chart U_3 of the weighted blow-up f' can be expressed as

$$\Big(x_1'^2 + x_2'^2 x_3' x_4' + 2x_2' x_4' p + \lambda x_2' x_3'^{2k-r} + \frac{1}{x_3'^{2r}} q(x_3'^2, x_3' x_4') = 0\Big)/\frac{1}{2}(1,1,1,1).$$

If the origin o is contained in U_3, then this point is not terminal since this equation has only even degree terms. So we need the condition $o \notin U_3$, which is equivalent to the condition $x_3^r \in q$. Hence Z is covered by U_1, U_2, and U_4. The origin of U_2 is of type cA/r. We can check that Z has only isolated singularities as the proof of Theorem 2.10.

Therefore we can apply Lemma 4.1, and f should coincide with f'.

• Suppose that $\mathrm{wt}(x_1,x_2,x_3,x_4)=(r,r,1,2)$.

In this case, we have $k \geq r$. Let $f'\colon Z \to X$ be the weighted blow-up with $\mathrm{wt}(x_1,x_2,x_3,x_4)=(r,r,1,2)$.

We shall show that $r=3$, $\lambda \neq 0$ and $k=3$. The x_2-chart U_2 of weighted blow-up f' can be expressed as

$$\left(\begin{array}{r} x_1'^2 + x_2'^2 x_4' + 2x_4' \dfrac{1}{x_2'^{r-2}} p(x_2'x_3', x_2'^2 x_4') \\ + \lambda x_2'^{k-r} x_3'^k + \dfrac{1}{x_2'^{2r}} q(x_2'x_3', x_2'^2 x_4') = 0 \end{array}\right) \Big/ \frac{1}{r}\left(0, \tfrac{r-1}{2}, -\tfrac{r-1}{2}, 1\right).$$

It is impossible that the origin of U_2 is of type cA/r. So it is necessary that the origin is of type $cD/3$, and we need $r=3$, $\lambda \neq 0$ and $k=3$. Moreover we have $\mathrm{wt}\, p = 2$.

Replacing variables, we can rewrite φ as

$$\varphi = x_1^2 + x_2^2 x_4 + 2x_2 x_4 p(x_3,x_4) + x_2 x_3^3 + q(x_3,x_4),$$

where $\mathrm{wt}\, q \geq 6$ and p is weighted homogeneous of weight 2.

(ii$'$) The exceptional locus F of f' is irreducible and reduced if and only if $q_{\mathrm{wt}=6}$ is not square.

(iii$'$) We shall show that φ has the condition $x_4^3 \in q$ if and only if every singular point in Z is terminal.

The x_4-chart U_4 of the weighted blow-up f' can be expressed as

$$\left(\begin{array}{r} x_1'^2 + x_2'^2 x_4'^2 + 2x_2' \dfrac{1}{x_4'} p(x_3'x_4', x_4'^2) \\ + x_2' x_3'^3 + \dfrac{1}{x_4'^6} q(x_3'x_4', x_4'^2) = 0 \end{array}\right) \Big/ \frac{1}{2}(1,1,1,1).$$

If the origin o is contained in U_4, then this point is not terminal since this equation has only even degree terms. So we have the condition $o \notin U_4$, which is equivalent to the condition $x_4^3 \in q$. Hence Z is covered by U_1, U_2, and U_3. The origin of U_2 is of type $cD/3$. We can check that Z has only isolated singularities as the proof of Theorem 2.10.

Therefore we can apply Lemma 4.1, and f should coincide with f'. The proof of Theorem 2.4 is completed. □

Proof of Theorem 2.5. The $cE_{6,7}$ point $P \in X$ has an identification such that

$$\varphi = x_1^2 + x_2^3 + x_2 g(x_3,x_4) + h(x_3,x_4) = 0,$$

where $g \in \mathfrak{m}^3$ and $h \in \mathfrak{m}^4$. If P is of type cE_6 (resp. cE_7), then $h_4 \neq 0$ (resp. $h_4 = 0$, $g_3 \neq 0$).

(i) We shall show that we distribute weight $\mathrm{wt}(x_1, x_2, x_3, x_4) = (3, 3, 2, 1)$, and that φ can be written as

$$\varphi = x_1^2 + \{x_2 - p(x_3, x_4)\}^3 + x_2 g(x_3, x_4) + h(x_3, x_4),$$

where $g \in \mathfrak{m}^3$, $h \in \mathfrak{m}^4$, and p is weighted homogeneous of weight 2 for the weights distributed above.
By Table 1, Y has $cD/3$ at which E is not Cartier, so we have $r = 3$. By Corollary 4.8, we have $\mathrm{wt}\,\varphi = 6$. So we can distribute weight $\mathrm{wt}(x_1, x_2, x_3, x_4) = (3, 3, 2, 1)$. We obtain a quartuple $(\bar{x}_1, \bar{x}_2, x_3, x_4)$ by $\bar{x}_1 = x_1 + p(x_3, x_4)$, $\bar{x}_2 = x_2 + q(x_3, x_4)$, where p and q as in Lemma 4.7. Then we rewrite φ as

$$\begin{aligned}\varphi &= (\bar{x}_1 - p)^2 + (\bar{x}_2 - q)^3 + (\bar{x}_2 - q)g(x_3, x_4) + h(x_3, x_4)\\ &= (\bar{x}_1 - p)^2 + (\bar{x}_2 - q)^3 + \bar{x}_2 g + (-qg + h).\end{aligned}$$

Since $\mathrm{wt}\,\varphi = 6$, we can show that $p = 0$, $\mathrm{wt}\,g \geq 3$, $\mathrm{wt}(-qg + h) \geq 6$, and q is weighted homogeneous of weight 2. Replacing $\bar{x}_1$, $\bar{x}_2$, q, and h, we have the desired expression in (i).

(ii) Let $f' \colon Z \to X$ be the weighted blow-up with $\mathrm{wt}\,x_i = \mathrm{mult}_E\, x_i$. We can show that the exceptional locus F of f' is irreducible and reduced in (iii).

(iii) We shall show that φ has the condition $x_4^3 \in g$ and $x_3 \in p$ if and only if every singular point in Z is terminal.
The x_2-chart U_2 of the weighted blow-up f' can be expressed as

$$\left(\begin{aligned} &x_1'^2 + \{x_2' - p(x_3', x_4')\}^3 \\ &\quad + \frac{1}{x_2'^3} g(x_2'^2 x_3', x_2' x_4') + \frac{1}{x_2'^6} h(x_2'^2 x_3', x_2' x_4') = 0 \end{aligned}\right) \Big/ \frac{1}{3}(0, 1, 1, 2).$$

It is necessary that the origin is of type $cD/3$. So we need $x_4^3 \in g$. Moreover we show that the exceptional locus F of f' is irreducible and reduced. The x_3-chart U_3 of the weighted blow-up f' can be expressed as

$$\left(\begin{aligned} &x_1'^2 + \{x_2' x_3' - p(1, x_4')\}^3 \\ &\quad + \frac{x_2'}{x_3'^3} g(x_3'^2, x_3' x_4') + \frac{1}{x_3'^6} h(x_3'^2, x_3' x_4') = 0 \end{aligned}\right) \Big/ \frac{1}{2}(1, 1, 1, 1).$$

If the origin o is contained in U_3, then this point is not terminal since this equation has only even degree terms. So we have the condition $o \notin U_3$, which is equivalent to the condition $x_3 \in p$. We can check that Z has only isolated singularities as the proof of Theorem 2.10.

Therefore we can apply Lemma 4.1, and f should coincide with f'.

Let $\bar{x}_2 = x_2 - p$. Then we have

$$\varphi = x_1^2 + \bar{x}_2^3 + \bar{x}_2 g(x_3, x_4) + (p(x_3, x_4) g(x_3, x_4) + h(x_3, x_4)).$$

If P is of type cE_7, then h should contain $x_3 x_4^3$ since $x_3 \in p$ and $x_4^3 \in g$. This is a contradiction to $\mathrm{wt}\,h \geq 6$. So P is of type cE_6. Therefore the proof of Theorem 2.5 is completed. $\square$

4.3. **Case** $e5$ **with discrepancy** 2. In this subsection, we suppose that $f\colon (Y \supset E) \to (X \ni P)$ is of type $e5$, and its discrepancy a is 2. In this case, Y has one non-Gorenstein singular point. This point Q is of type $\frac{1}{7}(1,6,6)$. Set $N_j := \{(l_1,l_2,l_3,l_4,l_5) \in \mathbb{Z}^5_{\geq 0} \mid 5l_1+3l_2+2l_3+2l_4+7l_5 = j,\ l_1,l_2 \leq 1\}$.

Lemma 4.9. $\dim V_j = \#N_j$.

Proof. By Table 1 and Table 2, we see that $(r_Q, b_Q, v_Q) = (7,3,6)$ and $E^3 = 1/7$. We also have $e_Q = 4$. So

$$\begin{aligned}\dim V_j = {} & \frac{1}{14}j(j+3) + \frac{1}{7} + \frac{1}{12}E\cdot c_2(Y) \\ & - (\overline{3j} - \overline{3(j+1)})\frac{4}{7} + \Big(\sum_{l=1}^{\overline{3j}-1} - \sum_{l=1}^{\overline{3(j+1)}-1}\Big)\frac{\overline{6l}(7-\overline{6l})}{14}.\end{aligned}$$

Here $^-$ denotes the residue modulo 7. Since $\dim V_0 = 1$, we have

$$\frac{1}{7} + \frac{1}{12}E\cdot c_2(Y) = \frac{3}{7}.$$

Now we consider

$$\dim V_j - \dim V_{j-7} = j-2$$

for any $j \geq 7$. We can show $\dim V_j - \dim V_{j-7} = \#N_j - \#N_{j-7}$ as Lemma 4.3. ∎

Lemma 4.10.

(i) *There exists some* $1 \leq k,\ l \leq 4$ *with* $\mathrm{mult}_E\, x_k = \mathrm{mult}_E\, x_l = 2$. *By permutation, we may assume that* $x_k = x_4$, $x_l = x_3$. *Moreover there exists some* $k = 1,\ 2$ *with* $\mathrm{mult}_E\, x_k = 3$. *By permutation, we may assume that* $x_k = x_2$.

(ii) *If* $j < 5$, *the monomials* $x_2^{l_2}x_3^{l_3}x_4^{l_4}$ *for* $(0,l_2,l_3,l_4,0) \in N_j$ *form a basis of* V_j. *In particular,* $\mathrm{mult}_E\, \bar{x}_1 \geq 5$ *for* $\bar{x}_1 := x_1 + \sum c_{l_2l_3l_4}x_2^{l_2}x_3^{l_3}x_4^{l_4}$ *with some* $c_{l_2l_3l_4} \in \mathbb{C}$ *and summation over* $(0,l_2,l_3,l_4,0) \in \cup_{j<5}N_j$.

(iii) $\mathrm{mult}_E\, \bar{x}_1 = 5$, *and the monomials* $\bar{x}_1^{l_1}x_2^{l_2}x_3^{l_3}x_4^{l_4}$ *for* $(l_1,l_2,l_3,l_4,0) \in N_5$ *form a basis of* V_5.

(iv) *Set* $\tilde{N}_j = \{(l_1,l_2,l_3,l_4,l_5) \in \mathbb{Z}^5_{\geq 0} \mid 5l_1+3l_2+2l_3+2l_4+7l_5 = j\}$. *The monomials* $\bar{x}_1^{l_1}x_2^{l_2}x_3^{l_3}x_4^{l_4}$ *for* $(l_1,l_2,l_3,l_4,0) \in N_6$ *have one non-trivial relation, say* ψ, *in* V_6. *The natural exact sequence*

$$0 \to \mathbb{C}\psi \to \bigoplus_{(l_1,l_2,l_3,l_4,0)\in\tilde{N}_6} \mathbb{C}\bar{x}_1^{l_1}x_2^{l_2}x_3^{l_3}x_4^{l_4} \to V_6 \to 0$$

is exact.

(v) *We have* $\operatorname{mult}_E \psi = 7$. *The natural exact sequences*

$$0 \to \mathbb{C}x_3\psi \oplus \mathbb{C}x_4\psi \to \bigoplus_{(l_1,l_2,l_3,l_4,l_5)\in\tilde{N}_8} \mathbb{C}\bar{x}_1^{l_1}x_2^{l_2}x_3^{l_3}x_4^{l_4}\psi^{l_5} \to V_8 \to 0,$$

$$0 \to \mathbb{C}x_2\psi \to \bigoplus_{(l_1,l_2,l_3,l_4,l_5)\in\tilde{N}_9} \mathbb{C}\bar{x}_1^{l_1}x_2^{l_2}x_3^{l_3}x_4^{l_4}\psi^{l_5} \to V_9 \to 0$$

are exact.

Proof. We follow the proof of Lemma 4.4, with using the computation of Lemma 4.9. (i) follows from $\dim V_1 = 0$ and $\dim V_2 = 2$. Now (ii) to (iv) follow from the same argument as in Lemma 4.4. Since $\psi = 0$ in $V_6 = f_*\mathcal{O}_Y(-6E)/f_*\mathcal{O}_Y(-7E)$, we have $\operatorname{mult}_E \psi = 7$. We also obtain the sequences in (v), which are exact possibly except for the middle. Their exactness is verified by comparing dimensions. □

Corollary 4.11. *We distribute weights* $\operatorname{wt}(\bar{x}_1, x_2, x_3, x_4) = (5, 3, 2, 2)$ *to the coordinates* $\bar{x}_1$,x_2,x_3,x_4 *obtained in* Lemma 4.10. *Then* φ *is of form*

$$\varphi = cx_2\psi + \varphi_{>9}(\bar{x}_1, x_2, x_3, x_4)$$

with $c \in \mathbb{C}$ *and a function* $\varphi_{>9}$ *of weighted order* > 9, *where* ψ *in* (1) *is the one in* Lemma 4.10(iv). □

Proof of Theorem 2.9. The cE_7 point $P \in X$ has an identification such that

$$\varphi = x_1^2 + x_2^3 + x_2 g(x_3, x_4) + h(x_3, x_4) = 0,$$

where $g \in \mathfrak{m}^3$, $h \in \mathfrak{m}^5$, and $g_3 \neq 0$.

(i) We shall show that we distribute weight $\operatorname{wt}(x_1, x_2, x_3, x_4) = (5, 3, 2, 2)$, and that φ and ψ can write

$$\varphi = x_1^2 + x_2^3 + x_2 g(x_3, x_4) + h(x_3, x_4),$$
$$\psi = x_2^2 + g_{\mathrm{wt}=6}(x_3, x_4),$$

where $\operatorname{wt} g \geq 6$ and $\operatorname{wt} h \geq 10$.
By Corollary 4.11, we have $\operatorname{wt} \varphi = 9$. So we show that we distribute weight $\operatorname{wt}(x_1, x_2, x_3, x_4) = (5, 3, 2, 2)$. We obtain a quartuple $(\bar{x}_1, x_2, x_3, x_4)$ by $\bar{x}_1 = x_1 + cx_2 + p(x_3, x_4)$, where $c \in \mathbb{C}$ and p as in Lemma 4.10. Then we rewrite φ as

$$\varphi = (\bar{x}_1 - cx_2 - p)^2 + x_2^3 + x_2 g + h.$$

Since $\operatorname{wt} \varphi = 9$, we can show that $c = p = 0$, $\operatorname{wt} g \geq 6$, and $\operatorname{wt} h \geq 10$. By Corollary 4.11, we have $\psi = x_2^2 + g_{\mathrm{wt}=6}(x_3, x_4)$. Replacing $\bar{x}_1$ with x_1, we have the desired expression in (i).

By setting $x_5 := -(\psi + g_{\mathrm{wt}\geq 7})$ and replacing $x_2 \mapsto -x_2$, we rewrite φ as

$$\begin{cases} \varphi = x_1^2 + x_2 x_5 + p(x_3, x_4) = 0 \\ x_2^2 + q(x_3, x_4) + x_5 = 0, \end{cases}$$

with $\operatorname{wt} p \geq 10$ and $\operatorname{wt} q \geq 6$.

(ii) Let $f'\colon Z \to X$ be the weighted blow-up with $\mathrm{wt}(x_1, x_2, x_3, x_4, x_5) = (5, 3, 2, 2, 7)$. It is obvious that the exceptional locus F of f' is irreducible and reduced.

(iii) We shall show that we have the condition that $\gcd(p_5, q_3) = 1$ if and only if every singular point in Z is terminal.
The x_3-chart U_3 of the weighted blow-up f' can be expressed as

$$\text{(A)} \qquad U_3 = \begin{pmatrix} x_1'^2 + x_2'x_5' + \frac{1}{x_3'^{10}}p(x_3'^2, x_3'^2x_4') = 0, \\ x_2'^2 + \frac{1}{x_3'^6}q(x_3'^2, x_3'^2x_4') + x_3'x_5' = 0 \end{pmatrix} / \frac{1}{2}(1,1,1,0,1).$$

the origin o is contained in U_3, then this point is not terminal since U_3 is not embedded in 4-dimensional quotient space. So we need the condition $o \notin U_3$, which is equivalent to the condition $x_3^5 \in p$ or $x_3^3 \in q$. Moreover the action on equations (A) is free outside the points $(0, 0, 0, x_4', 0)$, which satisfy the equations

$$\text{(B)} \qquad \begin{cases} p_{\mathrm{wt}=10}(1, x_4') = 0 \\ q_{\mathrm{wt}=6}(1, x_4') = 0. \end{cases}$$

Since such points are of type $\frac{1}{2}(1, 1, 1, 1)$, there is no solution on (B). Similarly we have the condition $x_4^5 \in p$ or $x_4^3 \in q$, and there is no solution on

$$\text{(C)} \qquad \begin{cases} p_{\mathrm{wt}=10}(x_3', 1) = 0 \\ q_{\mathrm{wt}=6}(x_3', 1) = 0. \end{cases}$$

It is easy to show that these four conditions:

- $x_3^5 \in p$ or $x_3^3 \in q$,
- there is no solution on (B),
- $x_4^5 \in p$ or $x_4^3 \in q$, and
- there is no solution on (C),

are equivalent to the condition $\gcd(p_5, q_3) = 1$. We can check that Z has only isolated singularities by using Jacobian criterion. Thus the proof of (iii) is finished.

Therefore we can apply Lemma 4.2, and f should coincide with f'. The proof of Theorem 2.9 is completed. □

4.4. **Case** $e1$ **with discrepancy** 2. In this subsection, we suppose that $f\colon (Y \supset E) \to (X \ni P)$ is of type $e1$, and its discrepancy a is 2. In this case, Y has one non-Gorenstein singular point. This point Q is of type $\frac{1}{r}(1, -1, 4)$. Set $N_j := \{(l_1, l_2, l_3, l_4, l_5) \in \mathbb{Z}^5_{\geq 0} \mid \frac{r+1}{2}l_1 + \frac{r-1}{2}l_2 + 2l_3 + l_4 + rl_5 = j,\ l_1, l_2 \leq 1\}$ and $M_j := \{(l_1, l_2, l_3, l_4) \in \mathbb{Z}^4_{\geq 0} \mid 3l_1 + 2l_2 + l_3 + l_4 = j,\ l_2 \leq 1\}$.

Lemma 4.12. $\dim V_j = \begin{cases} \#N_j & \text{if } r \geq 5, \\ \#M_j & \text{if } r = 3. \end{cases}$

Proof. By Table 1 and Table 2, we see that $(r_Q, b_Q, v_Q) = (r, 4, 2)$ and $E^3 = 2/r$. We also have $e_Q = (r+1)/2$. So

$$\dim V_j = \frac{1}{r} j(j+3) + \frac{2}{r} + \frac{1}{12} E \cdot c_2(Y) - (\overline{j\frac{r-1}{2}} - \overline{(j+1)\frac{r-1}{2}}) \frac{r^2-1}{12r} + \Big(\sum_{l=1}^{\overline{j\frac{r-1}{2}}-1} - \sum_{l=1}^{\overline{(j+1)\frac{r-1}{2}}-1} \Big) \frac{\overline{4l}(r-\overline{4l})}{2r}.$$

Here $^-$ denotes the residue modulo r. Since $\dim V_0 = 1$, we have

$$\frac{2}{r} + \frac{1}{12} E \cdot c_2(Y) = 1 - \frac{r-1}{2} \cdot \frac{r^2-1}{12r} - \sum_{l=1}^{\frac{r-1}{2}-1} \frac{\overline{4l}(r-\overline{4l})}{2r}.$$

If $r \geq 5$, we consider

$$\dim V_j - \dim V_{j-2} = \frac{2}{r}(2j+1) + \frac{\overline{2(j+1)}(r - \overline{2(j+1)}) - \overline{2j}(r-\overline{2j})}{2r}$$

for any $j \geq 2$. We can show $\dim V_j - \dim V_{j-2} = \# N_j - \# N_{j-2}$ as Lemma 4.3. If $r = 3$, we consider

$$\dim V_j - \dim V_{j-3} = 2j$$

for any $j \geq 3$. We can show $\dim V_j - \dim V_{j-3} = \# M_j - \# M_{j-3}$ as Lemma 4.3. $\square$

Lemma 4.13. *If $r \geq 5$, then we have the following condition:*

(i) *There exists some $1 \leq k,\ l \leq 4$ with $\operatorname{mult}_E x_k = 1$, $\operatorname{mult}_E x_l = 2$. By permutation, we may assume that $x_k = x_4$, $x_l = x_3$.*

(ii) *If $j < \frac{r-1}{2}$, the monomials $x_3^{l_3} x_4^{l_4}$ for $(0,0,l_3,l_4,0) \in N_j$ form a basis of V_j. In particular for $k = 1, 2$, $\operatorname{mult}_E \bar{x}_k \geq \frac{r-1}{2}$ for $\bar{x}_k := x_k + \sum c_{kl_3l_4} x_3^{l_3} x_4^{l_4}$ with some $c_{kl_3l_4} \in \mathbb{C}$ and summation over $(0,0,l_3,l_4,0) \in \cup_{j<\frac{r-1}{2}} N_j$.*

(iii) *There exists some $k = 1, 2$ with $\operatorname{mult}_E \bar{x}_k = \frac{r-1}{2}$ such that the monomials $\bar{x}_k$ and $x_3^{l_3} x_4^{l_4}$ for $(0,0,l_3,l_4) \in N_{\frac{r-1}{2}}$ form a basis of $V_{\frac{r-1}{2}}$. By permutation, we may assume that $\bar{x}_k = \bar{x}_2$, then $\operatorname{mult} \hat{x}_1 \geq \frac{r+1}{2}$ for $\hat{x}_1 := \bar{x}_1 + \sum c_{l_2l_3l_4} \bar{x}_2^{l_2} x_3^{l_3} x_4^{l_4}$ with some $c_{l_2l_3l_4} \in \mathbb{C}$ and summation over $(0,l_2,l_3,l_4) \in N_{\frac{r-1}{2}}$.*

(iv) *We have $\operatorname{mult}_E \hat{x}_1 = \frac{r+1}{2}$, and if $j < r-1$, the monomials $\hat{x}_1^{l_1} \bar{x}_2^{l_2} x_3^{l_3} x_4^{l_4}$ for $(l_1,l_2,l_3,l_4) \in N_j$ form a basis of V_j.*

(v) *Set $\tilde{N}_j = \{(l_1,l_2,l_3,l_4,l_5) \in \mathbb{Z}^5_{\geq 0} \mid \frac{r+1}{2} l_1 + \frac{r-1}{2} l_2 + 2l_3 + l_4 + rl_5 = j\}$. The monomials $\hat{x}_1^{l_1} \bar{x}_2^{l_2} x_3^{l_3} x_4^{l_4}$ for $(l_1,l_2,l_3,l_4,0) \in N_{r-1}$ have one non-trivial relation, say ψ, in V_{r-1}. The natural exact sequence*

$$0 \to \mathbb{C}\psi \to \bigoplus_{(l_1,l_2,l_3,l_4,0)\in\tilde{N}_{r-1}} \mathbb{C}\hat{x}_1^{l_1} \bar{x}_2^{l_2} x_3^{l_3} x_4^{l_4} \to V_{r-1} \to 0$$

is exact.

(vi) $\operatorname{mult}_E \psi = r$. *The natural exact sequence*

$$0 \to \mathbb{C}x_4\psi \to \bigoplus_{(l_1,l_2,l_3,l_4,l_5)\in\tilde{N}_r} \mathbb{C}\hat{x}_1^{l_1}\bar{x}_2^{l_2}x_3^{l_3}x_4^{l_4}\psi^{l_5} \to V_r \to 0$$

is exact.

Proof. We follow the proof of Lemma 4.10, with using the computation of Lemma 4.12. (i) follows from $\dim V_1 = 1$ and $\dim V_2 = 2$. Now (ii) to (vi) follow from the same argument as in Lemma 4.10. □

Corollary 4.14. *We distribute weights* $\operatorname{wt}(\hat{x}_1, \bar{x}_2, x_3, x_4) = (\frac{r+1}{2}, \frac{r-1}{2}, 2, 1)$ *to the coordinates* $\hat{x}_1$,$\bar{x}_2$,x_3,x_4 *obtained in* Lemma 4.13. *Then* φ *is of form*

$$\varphi = cx_4\psi + \varphi_{>r}(\hat{x}_1, \bar{x}_2, x_3, x_4)$$

with $c \in \mathbb{C}$ *and a function* $\varphi_{>r}$ *of weighted order* $> r$*, where* ψ *in* (1) *is the one in* Lemma 4.13(v). □

Lemma 4.15. *If* $r = 3$*, then we have the following condition.*

(i) *There exists some* $1 \le k,\ l \le 4$ *with* $\operatorname{mult}_E x_k = \operatorname{mult}_E x_l = 1$. *By permutation, we may assume that* $x_k = x_4$, $x_l = x_3$. *Moreover there exists some* $k = 1,\ 2$ *with* $\operatorname{mult}_E x_k = 2$. *By permutation, we may assume that* $x_k = x_2$.

(ii) *The monomials* $x_2^{l_2}x_3^{l_3}x_4^{l_4}$ *for* $(0, l_2, l_3, l_4, 0) \in N_2$ *form a basis of* V_2. *In particular,* $\operatorname{mult}_E \bar{x}_1 \ge 3$ *for* $\bar{x}_1 := x_1 + \sum c_{l_2l_3l_4}x_2^{l_2}x_3^{l_3}x_4^{l_4}$ *with some* $c_{l_2l_3l_4} \in \mathbb{C}$ *and summation over* $(0, l_2, l_3, l_4, 0) \in \cup_{j<2}N_j$.

(iii) $\operatorname{mult}_E \bar{x}_1 = 3$, *and the monomials* $\bar{x}_1^{l_1}x_2^{l_2}x_3^{l_3}x_4^{l_4}$ *for* $(l_1, l_2, l_3, l_4, 0) \in N_3$ *form a basis of* V_3.

(iv) *Set* $\tilde{N}_j = \{(l_1, l_2, l_3, l_4) \in \mathbb{Z}^4_{\ge 0} \mid 3l_1 + 2l_2 + l_3 + l_4 = j\}$. *The monomials* $\bar{x}_1^{l_1}x_2^{l_2}x_3^{l_3}x_4^{l_4}$ *for* $(l_1, l_2, l_3, l_4) \in N_4$ *have one non-trivial relation, say* ψ*, in* V_4. *The natural exact sequence*

$$0 \to \mathbb{C}\psi \to \bigoplus_{(l_1,l_2,l_3,l_4)\in\tilde{N}_4} \mathbb{C}\bar{x}_1^{l_1}x_2^{l_2}x_3^{l_3}x_4^{l_4} \to V_4 \to 0$$

is exact. □

Corollary 4.16. *We distribute weights* $\operatorname{wt}(\bar{x}_1, x_2, x_3, x_4) = (4, 3, 2, 1)$ *to the coordinates* $\bar{x}_1$,x_2,x_3,x_4 *obtained in* Lemma 4.15. *Then* φ *is of form*

$$\varphi = c\psi + \varphi_{>4}(\bar{x}_1, x_2, x_3, x_4)$$

with $c \in \mathbb{C}$ *and a function* $\varphi_{>4}$ *of weighted order* > 4*, where* ψ *in* (1) *is the one in* Lemma 4.15(iv). □

Proof of Theorem 2.3. The cD point $P \in X$ has an identification such that

$$\varphi = x_1^2 + x_2^2x_4 + \lambda x_2x_3^k + g(x_3, x_4) = 0,$$

where $g \in \mathfrak{m}^3$, $\lambda \in \mathbb{C}$, and $k \ge 2$.

We shall show that $r \geq 5$. Suppose $r = 3$. By Corollary 4.16, we have $\operatorname{wt}\varphi = 4$. So it is possible to distribute weight $\operatorname{wt}(x_1, x_2, x_3, x_4) = (3, 2, 1, 1)$, $(3, 1, 1, 2)$, $(2, 3, 1, 1)$, or $(2, 1, 1, 3)$.

We suppose $\operatorname{wt}(x_1, x_2, x_3, x_4) = (3, 2, 1, 1)$. Then We obtain a quartuple $(\bar{x}_1, x_2, x_3, x_4)$ by $\bar{x}_1 = x_1 + cx_2 + p(x_3, x_4)$, where $c \in \mathbb{C}$ and p as in Lemma 4.15. Thus we rewrite φ as

$$\varphi = (\bar{x}_1 - cx_2 - p)^2 + x_2^2 x_4 + \lambda x_2 x_3^k + g.$$

We replace $\bar{x}_1$ with x_1. Let $f' \colon Z \to X$ be the weighted blow-up with $\operatorname{wt}(x_1, x_2, x_3, x_4) = (3, 2, 1, 1)$. The x_1-chart U_1 of the weighted blow-up f' can be expressed as

$$\left(\begin{aligned} &(x'_1 - cx'_2 - \frac{1}{x_1'^2} p(x'_1x'_3, x'_1x'_4))^2 + x'_1 x_2'^2 x'_4 \\ &\qquad + \lambda x_1'^{k-2} x'_2 x_3'^k + \frac{1}{x_1'^2} g(x'_1x'_3, x'_1x'_4) = 0 \end{aligned} \right) \Big/ \frac{1}{3}(1, 1, 2, 2).$$

It is necessary that $o \in U_1$ is of type $\frac{1}{3}(1, 1, -1)$, but it is impossible. So we have a contradiction. Similarly we have a contraction in any other case. Therefore we have $r \geq 5$.

(i) We shall show that we distribute $\operatorname{wt}(x_1, x_2, x_3, x_4) = (\frac{r+1}{2}, \frac{r-1}{2}, 2, 1)$, and that φ can write

$$\begin{aligned} \varphi &= x_1^2 + \lambda x_2 x_3^k + x_4 \psi + p(x_3, x_4), \\ \psi &= x_2^2 + 2x_1 q_1(x_3, x_4) + q_2(x_3, x_4), \end{aligned}$$

where $\lambda \in \mathbb{C}$, $k > \frac{r+1}{4}$, $\operatorname{wt} p \geq r + 1$, $\operatorname{wt} q_1 = \frac{r-3}{2}$, $\operatorname{wt} q_2 = r - 1$, and q_1, q_2 are weighted homogeneous for the weights distributed above.

By Corollary 4.14, we have $\operatorname{wt}\varphi = r$. So we can distribute weight $\operatorname{wt}(x_1, x_2, x_3, x_4) = (\frac{r+1}{2}, \frac{r-1}{2}, 2, 1)$. We obtain a quartuple $(\hat{x}_1, \bar{x}_2, x_3, x_4)$ by $\hat{x}_1 = x_1 + c\bar{x}_2 + p(x_3, x_4)$, $\bar{x}_2 = x_2 + q(x_3, x_4)$, where $c \in \mathbb{C}$, p, and q as in Lemma 4.13. Then we rewrite φ as

$$\varphi = (\hat{x}_1 - c\bar{x}_2 - p)^2 + (\bar{x}_2 - q)^2 x_4 + \lambda(\bar{x}_2 - q) x_3^k + g.$$

Since $\operatorname{wt}\varphi = r$, we can show that $c = 0$, $k > \frac{r+1}{4}$, $q = 0$, $\operatorname{wt}(p^2 + g) \geq r$, and p is weighted homogeneous of weight $\frac{r-1}{2}$. So by replacing variables, we can rewrite φ as

$$\varphi = x_1^2 + 2x_1 p(x_3, x_4) + x_2^2 x_4 + \lambda x_2 x_3^k + g(x_3, x_4),$$

where $\lambda \in \mathbb{C}$, $k > \frac{r+1}{4}$, $\operatorname{wt} g \geq r$, and p is weighted homogeneous of weight $\frac{r-1}{2}$. We can write ψ as

$$\psi = x_2^2 + 2x_1 \frac{1}{x_4} p(x_3, x_4) + \frac{1}{x_4} g_{\mathrm{wt}=r}(x_3, x_4).$$

Therefore we have the desired expression in (i).

(ii) Set $x_5 = \psi$. Let $f' \colon Z \to X$ be the weighted blow-up with $\operatorname{wt} x_i = \operatorname{mult} x_i$. We have the condition that q_2 is not square if $q_1 = 0$, which is equivalent to the condition that the exceptional locus F of f' is irreducible and reduced.

(iii) We shall show that φ has the condition $x_3^{\frac{r+1}{2}} \in p$ if and only if every singular point in Z is terminal.
The x_3-chart U_3 of the weighted blow-up f' can be expressed as

$$\begin{pmatrix} x_1'^2 + \lambda x_2' x_3'^{2k-\frac{r+3}{2}} + x_4' x_5' + \frac{1}{x_3'^{r+1}} p(x_3'^2, x_3' x_4') = 0, \\ x_2'^2 + 2x_1' q_1(1, x_4') + q_2(1, x_4') + x_3' x_5' = 0 \end{pmatrix} / \frac{1}{2}\left(-\tfrac{r-3}{2}, \tfrac{r-5}{2}, 1, 1, 1\right).$$

the origin o is contained in U_3, then this point is not terminal since U_3 is not embedded in 4-dimensional quotient space. So we need the condition $o \notin U_3$, which is equivalent to the condition $x_3^{\frac{r+1}{2}} \in p$. Hence Z is covered by U_1, U_2, U_4, and U_5. The origin of U_5 is of type $\frac{1}{r}(1, -1, 4)$. We can check that Z has only isolated singularities as the proof of Theorem 2.9.

Therefore we can apply Lemma 4.2, and f should coincide with f'. The proof of Theorem 2.3 is completed. □

4.5. **Case** $e1$ **with discrepancy** 4**.** In this subsection, we suppose that $f\colon (Y \supset E) \to (X \ni P)$ is of type $e1$, and its discrepancy a is 4. In this case, Y has one non-Gorenstein singular point. This point Q is of type $\frac{1}{r}(1, -1, 8)$. Set $N_j := \{(l_1, l_2, l_3, l_4, l_5) \in \mathbb{Z}^5_{\geq 0} \mid \frac{r+1}{2} l_1 + \frac{r-1}{2} l_2 + 4l_3 + l_4 + rl_5 = j,\ l_1, l_2 \leq 1\}$, $M_j := \{(l_1, l_2, l_3, l_4) \in \mathbb{Z}^4_{\geq 0} \mid 5l_1 + 3l_2 + 2l_3 + l_4 = j,\ l_2 \leq 1\}$, and $L_j := \{(l_1, l_2, l_3) \in \mathbb{Z}^3_{\geq 0} \mid 3l_1 + l_2 + l_3 = j\}$.

Lemma 4.17.

$$\dim V_j = \begin{cases} \# N_j & \text{if } r > 5, \\ \# M_j & \text{if } r = 5, \\ \# L_j & \text{if } r = 3. \end{cases}$$

Proof. By Table 1 and Table 2, we see that $(r_Q, b_Q, v_Q) = (r, 8, 2)$ and $E^3 = 1/r$. We also have $e_Q = (r+1)/4$ (resp. $e_Q = (3r+1)/4$) if $r \equiv 3 \pmod 8$ (resp. $r \equiv -3 \pmod 8$). So

$$\begin{aligned} \dim V_j =& \frac{1}{2r} j(j+5) + \frac{5}{2r} + \frac{1}{12} E \cdot c_2(Y) \\ & - (\overline{-je_Q} - \overline{-(j+1)e_Q}) \frac{r^2-1}{12r} + \Big(\sum_{l=1}^{\overline{-je_Q}-1} - \sum_{l=1}^{\overline{-(j+1)e_Q}-1} \Big) \frac{\overline{8l}(r - \overline{8l})}{2r}. \end{aligned}$$

Here $\bar{\ }$ denotes the residue modulo r. Since $\dim V_0 = 1$, we have

$$\frac{5}{2r} + \frac{1}{12} E \cdot c_2(Y) = 1 - \overline{-e_Q} \cdot \frac{r^2-1}{12r} + \sum_{l=1}^{\overline{-e_Q}-1} \frac{\overline{8l}(r - \overline{8l})}{2r}.$$

If $r > 5$, we consider

$$\dim V_j - \dim V_{j-4} = \frac{2}{r}(2j+1) - (\overline{-je_Q} - \overline{-(j+1)e_Q} - \overline{-(j-4)e_Q} + \overline{(j-3)e_Q}) + \sum \frac{\overline{8l}(r-\overline{8l})}{2r}$$

for any $j \geq 4$. We can show $\dim V_j - \dim V_{j-4} = \#N_j - \#N_{j-4}$ as Lemma 4.3. If $r = 5$ (resp. $r = 3$), we consider

$$\dim V_j - \dim V_{j-5} = j \text{ (resp. } \dim V_j - \dim V_{j-3} = j+1)$$

for any $j \geq 5$ (resp. $j \geq 3$). We can show $\dim V_j = \#M_j$ (resp. $\dim V_j = \#L_j$) as Lemma 4.3. □

Lemma 4.18. *If $r > 5$, then we have the following condition:*

(i) *There exists some $1 \leq k,\ l \leq 4$ with $\operatorname{mult}_E x_k = 1$, $\operatorname{mult}_E x_l = 4$. By permutation, we may assume that $x_k = x_4$, $x_l = x_3$.*

(ii) *If $j < \frac{r-1}{2}$, the monomials $x_3^{l_3}x_4^{l_4}$ for $(0,0,l_3,l_4,0) \in N_j$ form a basis of V_j. In particular for $k = 1,\ 2$, $\operatorname{mult}_E \bar{x}_k \geq \frac{r-1}{2}$ for $\bar{x}_k := x_k + \sum c_{kl_3l_4}x_3^{l_3}x_4^{l_4}$ with some $c_{kl_3l_4} \in \mathbb{C}$ and summation over $(0,0,l_3,l_4,0) \in \cup_{j<\frac{r-1}{2}} N_j$.*

(iii) *There exists some $k = 1,\ 2$ with $\operatorname{mult}_E \bar{x}_k = \frac{r-1}{2}$ such that the monomials $\bar{x}_k$ and $x_3^{l_3}x_4^{l_4}$ for $(0,0,l_3,l_4) \in N_{\frac{r-1}{2}}$ form a basis of $V_{\frac{r-1}{2}}$. By permutation, we may assume that $\bar{x}_k = \bar{x}_2$, then $\operatorname{mult} \hat{x}_1 \geq \frac{r+1}{2}$ for $\hat{x}_1 := \bar{x}_1 + \sum c_{l_2l_3l_4}\bar{x}_2^{l_2}x_3^{l_3}x_4^{l_4}$ with some $c_{l_2l_3l_4} \in \mathbb{C}$ and summation over $(0,l_2,l_3,l_4) \in N_{\frac{r-1}{2}}$.*

(iv) *We have $\operatorname{mult}_E \hat{x}_1 = \frac{r+1}{2}$, and if $j < r-1$, the monomials $\hat{x}_1^{l_1}\bar{x}_2^{l_2}x_3^{l_3}x_4^{l_4}$ for $(l_1,l_2,l_3,l_4) \in N_j$ form a basis of V_5.*

(v) *Set $\tilde{N}_j = \{(l_1,l_2,l_3,l_4,l_5) \in \mathbb{Z}^5_{\geq 0} \mid \frac{r+1}{2}l_1 + \frac{r-1}{2}l_2 + 4l_3 + l_4 + rl_5 = j\}$. The monomials $\hat{x}_1^{l_1}\bar{x}_2^{l_2}x_3^{l_3}x_4^{l_4}$ for $(l_1,l_2,l_3,l_4,0) \in N_{r-1}$ have one non-trivial relation, say ψ, in V_{r-1}. The natural exact sequence*

$$0 \to \mathbb{C}\psi \to \bigoplus_{(l_1,l_2,l_3,l_4,0)\in\tilde{N}_{r-1}} \mathbb{C}\hat{x}_1^{l_1}\bar{x}_2^{l_2}x_3^{l_3}x_4^{l_4} \to V_{r-1} \to 0$$

is exact.

(vi) *We have $\operatorname{mult}_E \psi = r$. The natural exact sequence*

$$0 \to \mathbb{C}x_4\psi \to \bigoplus_{(l_1,l_2,l_3,l_4,l_5)\in\tilde{N}_r} \mathbb{C}\hat{x}_1^{l_1}\bar{x}_2^{l_2}x_3^{l_3}x_4^{l_4}\psi^{l_5} \to V_r \to 0$$

is exact. □

Corollary 4.19. *We distribute weights $\operatorname{wt}(\hat{x}_1,\bar{x}_2,x_3,x_4) = (\frac{r+1}{2},\frac{r-1}{2},4,1)$ to the coordinates $\hat{x}_1$,$\bar{x}_2$,x_3,x_4 obtained in* Lemma 4.18. *Then φ is of form*

$$\varphi = cx_4\psi + \varphi_{>r}(\hat{x}_1,\bar{x}_2,x_3,x_4)$$

with $c \in \mathbb{C}$ and a function $\varphi_{>r}$ of weighted order $> r$, where ψ in (1) is the one in Lemma 4.18(v). $\square$

Lemma 4.20. *If $r = 5$, then we have the following condition.*

(i) *There exists some $1 \le k,\ l \le 4$ with $\operatorname{mult}_E x_k = 1$ and $\operatorname{mult}_E x_l = 2$. By permutation, we may assume that $x_k = x_4$, $x_l = x_3$. the monomials $x_3^{l_3}x_4^{l_4}$ for $(0,0,l_3,l_4) \in M_2$ form a basis of V_2. In particular, for $k = 1,\ 2$, $\operatorname{mult}_E \bar{x}_k \ge 3$ for $\bar{x}_k := x_k + \sum c_{kl_3l_4}x_3^{l_3}x_4^{l_4}$ with some $c_{kl_3l_4} \in \mathbb{C}$ and summation over $(0,0,l_3,l_4) \in \cup_{j<3}M_j$.*

(ii) *There exists some $k = 1,\ 2$ with $\operatorname{mult}_E \bar{x}_k = 3$ such that the monomials $\bar{x}_k^{l_2}x_3^{l_3}x_4^{l_4}$ for $(0,l_2,l_3,l_4) \in M_j$ form a basis of V_j if $j < 5$. By permutation, we assume that $\bar{x}_k = \bar{x}_2$. Then $\operatorname{mult}\hat{x}_1 \ge 5$ for $\hat{x}_1 := \bar{x}_1 + \sum c_{l_2l_3l_4}\bar{x}_2^{l_2}x_3^{l_3}x_4^{l_4}$ with some $c_{l_2l_3l_4} \in \mathbb{C}$ and summation over $(0,l_2,l_3,l_4) \in \cup_{j<5}M_j$.*

(iii) $\operatorname{mult}_E \hat{x}_1 = 5$*, and the monomials $\hat{x}_1^{l_1}\bar{x}_2^{l_2}x_3^{l_3}x_4^{l_4}$ for $(l_1,l_2,l_3,l_4) \in M_5$ form a basis of V_5.*

(iv) *Set $\tilde{M}_j = \{(l_1,l_2,l_3,l_4) \in \mathbb{Z}^4_{\ge 0} \mid 5l_1 + 3l_2 + 2l_3 + l_4 = j\}$. The monomials $\hat{x}_1^{l_1}\bar{x}_2^{l_2}x_3^{l_3}x_4^{l_4}$ for $(l_1,l_2,l_3,l_4) \in M_6$ have one non-trivial relation, say ψ, in V_6. The natural exact sequence*

$$0 \to \mathbb{C}\psi \to \bigoplus_{(l_1,l_2,l_3,l_4)\in\tilde{M}_6} \mathbb{C}\hat{x}_1^{l_1}\bar{x}_2^{l_2}x_3^{l_3}x_4^{l_4} \to V_6 \to 0.$$

is exact. $\square$

Corollary 4.21. *We distribute weights $\operatorname{wt}(\hat{x}_1,\bar{x}_2,x_3,x_4) = (5,3,2,1)$ to the coordinates $\hat{x}_1$,$\bar{x}_2$,x_3,x_4 obtained in* Lemma 4.20. *Then φ is of form*

$$\varphi = c\psi + \varphi_{>6}(\hat{x}_1,\bar{x}_2,x_3,x_4)$$

with $c \in \mathbb{C}$ and a function $\varphi_{>6}$ of weighted order > 6, where ψ in (1) is the one in Lemma 4.20(iv). $\square$

If $r = 3$, we have the following condition.

(i) There exists some $1 \le k,\ l \le 4$ with $\operatorname{mult}_E x_k = \operatorname{mult}_E x_l = 1$. By permutation, we may assume that $x_k = x_2$, $x_l = x_3$. the monomials $x_2^{l_2}x_3^{l_3}$ for $(0,l_2,l_3) \in L_2$ form a basis of V_2. In particular, for $k = 1$, 4, $\operatorname{mult}_E \bar{x}_k \ge 3$ for $\bar{x}_k := x_k + \sum c_{kl_2l_3}x_2^{l_2}x_3^{l_3}$ with some $c_{kl_2l_3} \in \mathbb{C}$ and summation over $(0,l_2,l_3) \in \cup_{j<3}L_j$.

(ii) There exists some $k = 1$, 4 with $\operatorname{mult}_E \bar{x}_k = 3$ such that the monomials $\bar{x}_k^{l_1}x_2^{l_2}x_3^{l_3}$ for $(l_1,l_2,l_3) \in L_j$ form a basis of V_j for any j. By permutation, we assume that $\bar{x}_k = \bar{x}_1$.

So we have $\bigoplus_{(l_1,l_2,l_3)\in L_j} \mathbb{C}\bar{x}_1^{l_1}x_2^{l_2}x_3^{l_3} \simeq V_j$ for any j. This means that $\varphi \in \mathbb{C}\{x_1,x_2,x_3\}$. This is a contradiction that P is *cDV*. Therefore we have $r \ge 5$.

Proof of Theorem 2.1. The cA_2 point $P \in X$ has an identification such that

$$\varphi = x_1^2 + x_2^2 + x_3^3 + g(x_3, x_4) = 0 \text{ or} \tag{2}$$

$$\varphi = x_1x_2 + x_3^3 + g(x_3, x_4) = 0, \tag{3}$$

where $g \in \mathfrak{m}^2$ and $\deg g(x_3, 1) \leq 2$. We shall show that there is no suitable weight $\mathrm{wt}(x_1, x_2, x_3, x_4)$ in each cases.

Case (2). If $r = 5$, we can show that $\mathrm{wt}(x_1, x_2, x_3, x_4) = (5, 3, 2, 1)$ by Corollary 4.21. We obtain a quartuple $(\hat{x}_1, \bar{x}_2, x_3, x_4)$ by $\hat{x}_1 = x_1 + c\bar{x}_2 + p(x_3, x_4)$, $\bar{x}_2 = x_2 + q(x_3, x_4)$, where $c \in \mathbb{C}$, p and q as in Lemma 4.20. Then we rewrite φ as

$$\varphi = (\hat{x}_1 - c\bar{x}_2 - p)^2 + (\bar{x}_2 - q)^2 + x_3^3 + g(x_3, x_4).$$

By replacing variables, we rewrite φ as

$$\begin{aligned}\varphi = x_1^2 &+ 2cx_1x_2 + (c^2+1)x_2^2 \\ &+ 2x_1p(x_3, x_4) + 2cx_2p(x_3, x_4) + x_3^3 + q(x_3, x_4),\end{aligned}$$

where $c \in \mathbb{C}$, $\mathrm{wt}\, q \geq 6$, and p contains only monomials with weight 3 and 4.

Let $f' \colon Z \to X$ be the weighted blow-up with $\mathrm{wt}\, x_i = \mathrm{mult}_E\, x_i$. Then the x_1-chart U_1 of the weighted blow-up f' can be expressed as

$$\left(\begin{aligned} &x_1'^4 + 2cx_1'^2x_2' + (c^2+1)x_2'^2 + 2\frac{1}{x_1'}p(x_1'^2x_3', x_1'x_4') \\ &+ 2c\frac{x_2'}{x_1'^3}p(x_1'^2x_3', x_1'x_4') + x_3'^3 + \frac{1}{x_1'^6}q(x_1'^2x_3', x_1'x_4') = 0 \end{aligned} \right) \Big/ \frac{1}{5}(1, -3, 3, -1)$$

The origin is a non-hidden singularity which is not of type $\frac{1}{5}(1, -1, 3)$. It is a contradiction by Table 1.

If $r > 5$, there is no suitable weight $\mathrm{wt}(x_1, x_2, x_3, x_4)$ by Corollary 4.19.

Case (3). If $r = 5$, we can distribute weights $\mathrm{wt}(x_1, x_2, x_3, x_4) = (5, 2, 3, 1)$, $(5, 3, 2, 1)$. Let $f' \colon Z \to X$ be the weighted blow-up with $\mathrm{wt}\, x_i = \mathrm{mult}_E\, x_i$. As the proof of case (2), The origin of the x_1-chart U_1 of the weighted blow-up f' is not a non-hidden singularity which is not of type $\frac{1}{5}(1, -1, 3)$. It is a contradiction.

If $r > 5$, by Lemma 4.18, we show that $r = 11$ and $\mathrm{wt}(x_1, x_2, x_3, x_4) = (6, 5, 4, 1)$. However since $\mathrm{wt}(x_1x_2) = 11$, it is impossible that φ forms as Corollary 4.19.

Therefore there is no divisorial contraction of type $e1$ which contracts to a cA_2 point with discrepancy 4. The proof of Theorem 2.1 is completed. □

Proof of Theorem 2.2. The cD point $P \in X$ has an identification such that

$$\varphi = x_1^2 + x_2^2x_4 + \lambda x_2x_3^k + g(x_3, x_4) = 0,$$

where $g \in \mathfrak{m}^3$, $\lambda \in \mathbb{C}$, and $k \geq 2$.

We can show that $r \neq 5$ as the proof of Theorem 2.3.

(i) As the proof of Theorem 2.3, we can show that $\mathrm{wt}(x_1, x_2, x_3, x_4) = (\frac{r+1}{2}, \frac{r-1}{2}, 4, 1)$, and that φ can write

$$\varphi = x_1^2 + \lambda x_2 x_3^k + x_4\psi + p(x_3, x_4),$$
$$\psi = x_2^2 + 2x_1 q_1(x_3, x_4) + q_2(x_3, x_4),$$

where $\lambda \in \mathbb{C}$, $k > \frac{r+3}{4}$, $\mathrm{wt}\, p \geq r+1$, $\mathrm{wt}\, q_1 = \frac{r-3}{2}$, $\mathrm{wt}\, q_2 = r-1$, and q_1, q_2 are weighted homogeneous for the weights distributed above.

(ii) Set $x_5 = -\psi$ and replace x_4 with $-x_4$. Let $f' \colon Z \to X$ be the weighted blow-up with $\mathrm{wt}\, x_i = \mathrm{mult}\, x_i$. We have the condition that q_2 is not square if $q_1 = 0$, which is equivalent to the condition that the exceptional locus F of f' is irreducible and reduced.

(iii) We shall show that the condition below if and only if every singular point in Z is terminal:

- $x_3^{\frac{r+1}{4}} \in p$ if $r \equiv 3 \pmod 8$,
- $x_3^{\frac{r-1}{4}} \in q_2$ if $r \equiv -3 \pmod 8$.

The x_3-chart U_3 of the weighted blow-up f' can be expressed as

$$\left(\begin{matrix} x_1'^2 + \lambda x_2' x_3'^{4k - \frac{r+3}{2}} + x_4' x_5' + \frac{1}{x_3'^{r+1}} p(x_3'^4, x_3' x_4') = 0, \\ x_2'^2 + 2x_1' q_1(1, x_4') + q_2(1, x_4') + x_3' x_5' = 0 \end{matrix} \right) \Big/ \frac{1}{4}\left(\tfrac{7-r}{2}, \tfrac{9-r}{2}, 1, 3, 4-r\right).$$

If $o \in U_3$, the origin is not terminal since U_3 is not embedded in 4-dimensional quotient space. So we have the condition $o \notin U_3$, which is equivalent to the condition $x_3^{\frac{r+1}{4}} \in p$ (resp. $x_3^{\frac{r-1}{4}} \in q_2$) if $r \equiv 3 \pmod 8$ (resp. $r \equiv 5 \pmod 8$). Hence Z is covered by U_1, U_2, U_4, and U_5. The origin of U_5 is of type $\frac{1}{r}(1, -1, 8)$. We can check that Z has only isolated singularities as the proof of Theorem 2.9.

Therefore we can apply Lemma 4.2, and f should coincide with f'. The proof of Theorem 2.2 is completed. □

4.6. **Case $e3$ with discrepancy 3.** In this subsection, we suppose that $f \colon (Y \supset E) \to (X \ni P)$ is of type $e3$, and its discrepancy a is 3. In this case, Y has one non-Gorenstein singular point. This point deforms to two points Q_1 of type $\frac{1}{2}(1,1,1)$ and Q_2 of type $\frac{1}{4}(1,3,3)$. Set $N_j := \{(l_1, l_2, l_3, l_4) \in \mathbb{Z}^4_{\geq 0} \mid 4l_1 + 3l_2 + 2l_3 + l_4 = j,\ l_1 l_3 = 0\}$.

Lemma 4.22. $\dim V_j = \# N_j$.

Proof. By Table 1 and Table 2, we can see that $(r_{Q_1}, b_{Q_1}, v_{Q_1}) = (2,1,1)$, $(r_{Q_2}, b_{Q_2}, v_{Q_2}) = (4,3,1)$, and $E^3 = 1/4$. We also have $e_{Q_1} = 1$, $e_{Q_2} = 3$. So

$$\begin{aligned} \dim V_j =& \frac{1}{8} j(j+4) + \frac{5}{12} + \frac{1}{12} E \cdot c_2(Y) \\ & - (\bar{j} - \overline{j+1})\frac{1}{8} - (\bar{j}' - \overline{j+1}')\frac{5}{16} + \Big(\sum_{l=1}^{\bar{j}'-1} - \sum_{l=1}^{\overline{j+1}'-1} \Big) \frac{\overline{3l}'(4 - \overline{3l}')}{8}. \end{aligned}$$

Here $^-$ denotes the residue modulo 2 and $^{-\prime}$ the residue modulo 4. Since $\dim V_0 = 1$, we have

$$\frac{5}{12} + \frac{1}{12} E \cdot c_2(Y) = \frac{9}{16}.$$

Now we consider

$$\begin{aligned}\dim V_j - \dim V_{j-3} =& \frac{3}{8}(2j+1) - \frac{1}{4}(\bar{j} - \overline{j+1}) \\ & - \frac{5}{16}(\bar{j}' - 2\overline{j+1}' + \overline{j+2}') + \sum \frac{\overline{3l}'(4 - \overline{3l}')}{8}\end{aligned}$$

for any $j \geq 3$. We can show $\dim V_j - \dim V_{j-3} = \#N_j - \#N_{j-3}$ as Lemma 4.3. □

Lemma 4.23.

(i) *There exists some* $1 \leq k,\, l \leq 4$ *with* $\mathrm{mult}_E\, x_k = 1$ *and* $\mathrm{mult}_E\, x_l = 2$. *By permutation, we may assume that* $x_k = x_4$, $x_l = x_3$. *the monomials* $x_3^{l_3} x_4^{l_4}$ *for* $(0, 0, l_3, l_4) \in N_2$ *form a basis of* V_2. *In particular, for* $k = 1,\ 2$, $\mathrm{mult}_E\, \bar{x}_k \geq 3$ *for* $\bar{x}_k := x_k + \sum c_{kl_3l_4} x_3^{l_3} x_4^{l_4}$ *with some* $c_{kl_3l_4} \in \mathbb{C}$ *and summation over* $(0, 0, l_3, l_4) \in \cup_{j<3} N_j$.

(ii) *There exists some* $k = 1,\ 2$ *with* $\mathrm{mult}_E\, \bar{x}_k = 3$ *such that the monomials* $\bar{x}_k$ *and* $x_3^{l_3} x_4^{l_4}$ *for* $(0, 0, l_3, l_4) \in N_3$ *form a basis of* V_3. *By permutation,* $\bar{x}_k = \bar{x}_2$. *Then* $\mathrm{mult}\, \hat{x}_1 \geq 4$ *for* $\hat{x}_1 := \bar{x}_1 + \sum c_{l_2l_3l_4} \bar{x}_2^{l_2} x_3^{l_3} x_4^{l_4}$ *with some* $c_{l_2l_3l_4} \in \mathbb{C}$ *and summation over* $(0, l_2, l_3, l_4) \in N_4$.

(iii) *We have* $\mathrm{mult}_E\, \hat{x}_1 = 4$. *If* $j < 6$, *the monomials* $\hat{x}_1^{l_1} \bar{x}_2^{l_2} x_3^{l_3} x_4^{l_4}$ *for* $(l_1, l_2, l_3, l_4) \in N_j$ *form a basis of* V_j.

(iv) *Set* $\tilde{N}_j = \{(l_1, l_2, l_3, l_4) \in \mathbb{Z}^4_{\geq 0} \mid 4l_1 + 3l_2 + 2l_3 + l_4 = j\}$. *The monomials* $\hat{x}_1^{l_1} \bar{x}_2^{l_2} x_3^{l_3} x_4^{l_4}$ *for* $(l_1, l_2, l_3, l_4) \in N_6$ *have one non-trivial relation, say* ψ, *in* V_6. *The natural exact sequence*

$$0 \to \mathbb{C}\psi \to \bigoplus_{(l_1,l_2,l_3,l_4) \in \tilde{N}_6} \mathbb{C}\hat{x}_1^{l_1} \bar{x}_2^{l_2} x_3^{l_3} x_4^{l_4} \to V_6 \to 0$$

is exact. □

Corollary 4.24. *We distribute weights* $\mathrm{wt}(\hat{x}_1, \bar{x}_2, x_3, x_4) = (4, 3, 2, 1)$ *to the coordinates* $\hat{x}_1$,$\bar{x}_2$,x_3,x_4 *obtained in* Lemma 4.23. *Then* φ *is of form*

$$\varphi = c\psi + \varphi_{>6}(\hat{x}_1, \bar{x}_2, x_3, x_4)$$

with $c \in \mathbb{C}$ *and a function* $\varphi_{>6}$ *of weighted order* > 6, *where* ψ *in* (1) *is the one in* Lemma 4.23(iv). □

Proof of Theorem 2.6. The cA_2 point $P \in X$ has an identification such that

$$\varphi = x_1^2 + x_2^2 + x_3^3 + g(x_3, x_4) = 0,$$

where $g \in \mathfrak{m}^2$ and $\deg g(x_3, 1) \leq 2$.

(i) We shall show that we distribute weight $\mathrm{wt}(x_1, x_2, x_3, x_4) = (4, 3, 2, 1)$, and that φ can write

$$\begin{aligned}\varphi = x_1^2 + x_2^2 + 2cx_1x_2 + 2x_1p(x_3, x_4)\\ + 2cx_2p_{\mathrm{wt}=3}(x_3, x_4) + x_3^3 + g(x_3, x_4) = 0,\end{aligned}$$

where $c \neq \pm 1$, $2 \leq \mathrm{wt}\, p \leq 3$, $\mathrm{wt}\, g \geq 6$, and $\deg g(x_3, 1) \leq 2$.
By Corollary 4.24, we can distribute weight $\mathrm{wt}(x_1, x_2, x_3, x_4) = (4, 3, 2, 1)$. We obtain a quartuple $(\hat{x}_1, \bar{x}_2, x_3, x_4)$ by $\hat{x}_1 = x_1 + c\bar{x}_2 + p(x_3, x_4)$, $\bar{x}_2 = x_2 + q(x_3, x_4)$, where $c \in \mathbb{C}$, p, and q as in Lemma 4.23. Then we rewrite φ as

$$\varphi = (\hat{x}_1 - c\bar{x}_2 - p)^2 + (\bar{x}_2 - q)^2 + x_3^3 + g.$$

Since $\mathrm{wt}\, \varphi = 6$, we have $cp_{\mathrm{wt}\leq 2} = -q$, and p contains only monomials with weight 2 and 3. Moreover since $P \in X$ is of type cA_2, we have $c^2 + 1 \neq 0$. So by replacing variables, we have the desired expression in (i).

(ii) Let $f' \colon Z \to X$ be the weighted blow-up with $\mathrm{wt}\, x_i = \mathrm{mult}_E\, x_i$. We have the condition that g is not square if $p_{\mathrm{wt}=2} = 0$, which is equivalent to the condition that the exceptional locus F of f' is irreducible and reduced.

(iii) We shall show that φ needs the condition $x_4^2 \in p$, and that every singular point in Z is terminal.
The x_1-chart U_1 of the weighted blow-up f' can be expressed as

$$\left(\begin{aligned} x_1'^2 + x_2'^2 + 2cx_1'x_2' + 2\frac{1}{x_1'^2}p(x_1'^2x_3', x_1'x_4')\\ + 2cx_2'p_{\mathrm{wt}=3}(x_3', x_4') + x_3'^3 + \frac{1}{x_1'^6}g(x_1'^2x_3', x_1'x_4') = 0\end{aligned}\right) \Big/ \frac{1}{4}(1, 1, 2, 3).$$

It is necessary that the origin is of type $cAx/4$. So we have the condition $x_4^2 \in p$. We can check that Z has only isolated singularities as the proof of Theorem 2.10.

Therefore we can apply Lemma 4.1, and f should coincide with f'. The proof of Theorem 2.6 is completed. □

Proof of Theorem 2.7. The cD point $P \in X$ has an identification such that

$$\varphi = x_1^2 + x_2^2x_4 + \lambda x_2x_3^k + g(x_3, x_4) = 0,$$

where $g \in \mathfrak{m}^3$, $\lambda \in \mathbb{C}$, and $k \geq 2$. Since $\mathrm{wt}\, \varphi = 6$, we can distribute weight $\mathrm{wt}(x_1, x_2, x_3, x_4) = (4, 3, 2, 1)$, $(4, 3, 1, 2)$, $(4, 2, 1, 3)$, $(3, 4, 2, 1)$, $(3, 4, 1, 2)$, $(3, 2, 1, 4)$, or $(3, 1, 2, 4)$.

• At first, we suppose $\mathrm{wt}(x_1, x_2, x_3, x_4) = (3, 4, 2, 1)$.

(i) We shall show that φ can write

$$\begin{aligned}\varphi = x_1^2 + x_2^2x_4 + 2x_2x_4p(x_3, x_4) + c^2x_1^2x_4 + \lambda x_2x_3^k\\ + c(2x_1x_2x_4 + 2x_1x_4p(x_3, x_4) + \lambda x_1x_3^k) + g(x_3, x_4) = 0,\end{aligned}$$

where c, $\lambda \in \mathbb{C}$, $k > 2$, $\mathrm{wt}\, g \geq 6$, and p contains only monomials with weight ≤ 3.

We obtain quartuple $(\bar{x}_1, \hat{x}_2, x_3, x_4)$ by $\bar{x}_1 = x_1 + p(x_3, x_4)$, $\hat{x}_2 = x_2 + c\bar{x}_1 + q(x_3, x_4)$, where $c \in \mathbb{C}$, p and q as in Lemma 4.23. Then we rewrite φ as

$$\varphi = (\bar{x}_1 - q)^2 + (\hat{x}_2 - \bar{x}_1 - p)^2 x_4 + \lambda(\hat{x}_2 - c\bar{x}_1 - p)x_3^k + g(x_3, x_4).$$

Since $\operatorname{wt}\varphi = 6$, we can assume $q = 0$. Moreover we have $\operatorname{wt}(p^2 x_4 - \lambda p x_3^k + g) \geq 6$ and p contains only monomials with weight ≤ 3. So replacing variables, we have the desired expression in (i).

(ii) Let $f' \colon Z \to X$ be the weighted blow-up with $\operatorname{wt} x_i = \operatorname{mult}_E x_i$. We have the condition that g is not square if $p_{\mathrm{wt}=1} = 0$, which is equivalent to the condition that the exceptional locus F of f' is irreducible and reduced. If $x_4 \in p$, then F is irreducible and reduced.

(iii) We shall show that φ has the conditions $c = 0$, $x_4 \in p$, and $x_3^3 \in g$ if and only if every singular point in Z is terminal and Z has a non-hidden terminal of type $cAx/4$.

The x_2-chart U_2 of the weighted blow-up f' can be expressed as

$$\left(\begin{array}{l} x_1'^2 + x_2'^3 x_4' + 2\dfrac{x_4'}{x_2'} p(x_2'^2 x_3', x_2' x_4') + c^2 x_1'^2 x_2' x_4' \\ \quad + c\left(2x_1' x_2'^2 x_4' + 2x_1' x_4' \dfrac{1}{x_2'^2} p(x_2'^2 x_3, x_2' x_4) + \lambda x_1' x_2'^{2k-3} x_3'^k\right) \\ \quad + \lambda x_2'^{2k-2} x_3'^k + \dfrac{1}{x_2'^6} g(x_2'^2 x_3', x_2' x_4') = 0 \end{array}\right) \Big/ \frac{1}{4}(1,1,2,3).$$

The origin of U_2 is of type $cAx/4$. So we have the conditions $x_4 \in p$ and $c = 0$. Moreover since the equation is free outside the origin, we have $g_{\mathrm{wt}=6}(x_3, 0) \neq 0$, which is equivalent to the condition $x_3^3 \in g$. Thus φ can be written as

$$\varphi = x_1^2 + x_2^2 x_4 + 2x_2 x_4 p(x_3, x_4) + \lambda x_2 x_3^k + g(x_3, x_4),$$

and P is of type cD_4. We can check that Z has only isolated singularities as the proof of Theorem 2.10.

Therefore we can apply Lemma 4.1, and f should coincide with f' if $P \in X$ is cD_4.

• Next, we shall show that there is no weighted blow-up of type $e3$ which contracts to a cD point with $\operatorname{wt} x_1 = 4$.

We select $\operatorname{wt}(x_1, x_2, x_3, x_4) = (4, 3, 2, 1)$. We obtain quartuple $(\hat{x}_1, \bar{x}_2, x_3, x_4)$ by $\hat{x}_1 = x_1 + c\bar{x}_2 + p(x_3, x_4)$, $\bar{x}_2 = x_2 + q(x_3, x_4)$, where $c \in \mathbb{C}$, p and q as in Lemma 4.23. Then we rewrite φ as

$$\varphi = (\hat{x}_1 - c\bar{x}_2 - p)^2 + (\bar{x}_2 - q)^2 x_4 + \lambda(\bar{x}_2 - q)x_3^k + g(x_3, x_4).$$

We replace $\hat{x}_1 \mapsto x_1$ and $\bar{x}_2 \mapsto x_2$. Let $f' \colon Z \to X$ be the weighted blow-up with $\operatorname{wt} x_i = \operatorname{mult}_E x_i$. Then the x_1-chart U_1 of the weighted blow-up f' can be expressed as

$$\left(\begin{array}{l}\Big(x_1'-cx_2'-\dfrac{1}{x_1'^3}p(x_1'^2x_3',x_1'x_4')\Big)^2\\ +\Big(x_2'-\dfrac{1}{x_1'^3}q(x_1'^2x_3',x_1'x_4')\Big)^2x_1'x_4'\\ +\lambda\Big(x_2'-\dfrac{1}{x_1'^3}q(x_1'^2x_3',x_1'x_4')\Big)x_1'^{2k-3}x_3'^k\\ +\dfrac{1}{x_1'^6}g(x_1'^2x_3',x_1'x_4')=0\end{array}\right)\Big/\frac{1}{4}(1,1,2,3).$$

It is necessary that the origin is of type $cAx/4$. So we have $x_4^2\in p$, and moreover $c=0$. Now the x_2-chart U_2 of the weighted blow-up f' can be expressed as

$$\left(\begin{array}{l}\Big(x_1'x_2'-\dfrac{1}{x_2'^3}p(x_2'^2x_3',x_2'x_4')\Big)^2\\ +\Big(1-\dfrac{1}{x_2'^3}q(x_2'^2x_3',x_2'x_4')\Big)^2x_2'x_4'\\ +\lambda\Big(1-\dfrac{1}{x_2'^3}q(x_2'^2x_3',x_2'x_4')\Big)x_2'^{2k-3}x_3'^k\\ +\dfrac{1}{x_2'^6}g(x_2'^2x_3',x_2'x_4')=0\end{array}\right)\Big/\frac{1}{3}(2,1,1,2).$$

The origin is a non-hidden singularity. It is a contradiction by Table 1. Similarly We have a contradiction in any other case. Therefore there is no weighted blow-up of type $e2$ which contracts to a cD point with $\operatorname{wt} x_1=4$.

• Finally, we shall show that there is no weighted blow-up of type $e3$ which contracts to a cD_n point with $\operatorname{wt} x_1=3$ for any $n\geq 5$.

We can show that P is of type cD_4 with the weight $\operatorname{wt}(x_1,x_2,x_3,x_4)=(3,4,1,2)$ as the proof with the weight $\operatorname{wt}(x_1,x_2,x_3,x_4)=(3,4,2,1)$. We select $\operatorname{wt}(x_1,x_2,x_3,x_4)=(3,2,1,4)$, or $(3,1,2,4)$. We obtain quartuple $(\bar{x}_1,x_2,x_3,\hat{x}_4)$ by $\hat{x}_4=x_4+c\bar{x}_1+p(x_2,x_3)$, $\bar{x}_1=x_1+q(x_2,x_3)$, where $c\in\mathbb{C}$, p and q as in Lemma 4.23. Then we rewrite φ as

$$\varphi=(\bar{x}_1-q)^2+x_2^2(\hat{x}_4-c\bar{x}_1-p)+\lambda x_2x_3^k+g(x_3,\hat{x}_4-c\bar{x}_1-p).$$

Replacing variable, we can rewrite φ as

$$\varphi=x_1^2+x_2^2\big(x_4+cx_1+p(x_2,x_3)\big)+\lambda x_2x_3^k+g(x_1,x_2,x_3,x_4),$$

where $c\in\mathbb{C}$, $k\geq 2$ $\operatorname{wt} g\geq 6$, and p contains only monomials with weight ≤ 3. Let $f'\colon Z\to X$ be the weighted blow-up with $\operatorname{wt} x_i=\operatorname{mult}_E x_i$.

If $\operatorname{wt}(x_1,x_2,x_3,x_4)=(3,2,1,4)$, the x_4-chart U_4 of the weighted blow-up f' can be expressed as

$$\begin{pmatrix} x_1'^2+x_2'^2(x_4'^2+cx_1'x_4'+\dfrac{1}{x_4'^2}p(x_2'x_4'^2,x_3'x_4')) \\ +\lambda x_2'x_3'^k x_4'^{k-4}+\dfrac{1}{x_4'^6}g(x_1'x_4'^3,x_2'x_4'^2,x_3'x_4',x_4'^4)=0 \end{pmatrix}\Big/\frac{1}{4}(1,2,3,1).$$

It is necessary that the origin is of type $cAx/4$. So we have the condition $x_3^2x_4\in g$. This means that P is of type cD_4.

If $\operatorname{wt}(x_1,x_2,x_3,x_4)=(3,1,2,4)$, we have $c=0$, and we can assume $p=0$ by replacing g if necessary. The x_3-chart U_3 of the weighted blow-up f' can be expressed as

$$\left(x_1'^2+x_2'^2x_4'+\lambda x_2'x_3'^{2k-5}+\frac{1}{x_3'^6}g(x_1'x_3'^3,x_2'x_3',x_3'^2,x_3'^4x_4')=0\right)\Big/\frac{1}{2}(1,1,1,0).$$

We need the condition $o\notin U_3$, which is equivalent to the condition $x_3^3\in g$. Then P is of type cD_4. Therefore there is no divisorial contraction of type $e3$ which contracts cD_n point with discrepancy 3 for any $n\geq 5$. The proof of Theorem 2.7 is completed. □

Proof of Theorem 2.8. The cE_6 point $P\in X$ has an identification such that

$$\varphi=x_1^2+x_2^3+x_2g(x_3,x_4)+h(x_3,x_4)=0,$$

where $g\in\mathfrak{m}^3$, $h\in\mathfrak{m}^4$, and $h_4\neq 0$. By Corollary 4.24, we have $\operatorname{wt}\varphi=6$. So we can distribute weights $\operatorname{wt}(x_1,x_2,x_3,x_4)=(4,3,2,1)$, $(4,2,3,1)$, $(3,4,2,1)$, or $(3,2,4,1)$. Suppose $\operatorname{wt}(x_1,x_2,x_3,x_4)=(4,3,2,1)$. Then we obtain a quartuple $(\hat{x}_1,\bar{x}_2,x_3,x_4)$ by $\hat{x}_1=x_1+c\bar{x}_2+p(x_3,x_4)$, $\bar{x}_2=x_2+q(x_3,x_4)$, where $c\in\mathbb{C}$, p, and q as in Lemma 4.23. We rewrite φ as

$$\varphi=(\hat{x}_1-c\bar{x}_2-p)^2+(\bar{x}_2-q)^3+(\bar{x}_2-q)g+h.$$

We replace $\hat{x}_1$ with x_1 and $\bar{x}_2$ with x_2. Since $\operatorname{wt}\varphi=6$, we can rewrite φ as

$$\begin{aligned}\varphi=x_1^2+x_2^3+p(x_3,x_4)x_2^2+2cx_1x_2&\\ +2q(x_3,x_4)x_1+x_2g(x_3,x_4)+h(x_3,x_4)&=0\end{aligned}$$

where $g\in\mathfrak{m}^3$, $h\in\mathfrak{m}^4$, $h_4\neq 0$, $c\in\mathbb{C}$, and p (resp. q) contains only monomials with weight 1 and 2 (resp. 2 and 3).

Let $f'\colon Z\to X$ be the weighted blow-up with $\operatorname{wt}x_i=\operatorname{mult}_E x_i$. The x_1-chart U_1 of the weighted blow-up f' can be expressed as

$$\begin{pmatrix} x_1'^2+x_1'^3x_2'^3+p(x_1'^2x_3',x_1'x_4')x_2'^2 \\ +2cx_1'x_2'+2\dfrac{1}{x_1'^2}q(x_1'^2x_3',x_1'x_4') \\ +x_2'\dfrac{1}{x_1'^3}g(x_1'^2x_3',x_1'x_4')+\dfrac{1}{x_1'^6}h(x_1'^2x_3',x_1'x_4')=0 \end{pmatrix}\Big/\frac{1}{4}(1,1,2,3).$$

It is necessary that the origin is of type $cAx/4$. So we need $x_4^2\in q$ and $x_3\notin q$. Moreover we need that the action is free outside the origin, which is equivalent to the condition that $x_3^3 \in h$. This is a contradiction. Similarly we have a contradiction in any other case. Therefore there is no divisorial contraction of type $e3$ which contracts to a cE_6 point with discrepancy 3. The proof of Theorem 2.8 is completed. $\square$

5. Appendix

In this section, We give the detailed list of all 3-dimensional divisorial contractions to cDV points.

Theorem 5.1. *Let $f\colon Y \to X$ be a 3-dimensional divisorial contraction whose exceptional divisor E contracts to a cDV point P. Then one of the following holds:*

I. *The case P is non-singular*

(1) *f is the weighted blow-up with* $\mathrm{wt}(x_1, x_2, x_3) = (1, a, b)$ *where (x_1, x_2, x_3) is local coordinates at P, and where a and b are coprime positive integers.*

II. *The case P is of type cA*

•*Ordinary and minimal discrepancy case*

(1) *f is the weighted blow-up with* $\mathrm{wt}(x_1, x_2, x_3, x_4) = (r_1, r_2, a, 1)$ *after an identification*

$$P \in X \simeq o \in \Big(x_1x_2 + g(x_3, x_4) = 0\Big) \subset \mathbb{C}^4_{x_1x_2x_3x_4}.$$

Moreover there are following conditions:

(i) $a \mid (r_1 + r_2)$, $\gcd(a, r_1) = 1$, *and* $\gcd(a, r_2) = 1$.

(ii) $\mathrm{wt}\, g = r_1 + r_2$ *and* $x_3^{\frac{r_1+r_2}{a}} \in g$.

•*Exceptional case*

(2) *f is the weighted blow-up with* $\mathrm{wt}(x_1, x_2, x_3, x_4) = (5, 1, 3, 2)$ *after an identification*

$$P \in X \simeq o \in \Big(x_1x_2 + x_3^2 + x_4^3 = 0\Big) \subset \mathbb{C}^4_{x_1x_2x_3x_4}.$$

$P \in X$ is of type cA_1.

(3) *f is the weighted blow-up with* $\mathrm{wt}(x_1, x_2, x_3, x_4) = (4, 3, 2, 1)$ *after an identification of $P \in X$ with*

$$o \in \begin{pmatrix} x_1^2 + x_2^2 + 2cx_1x_2 + 2x_1p(x_3, x_4) \qquad\qquad\qquad \\ \qquad\qquad + 2cx_2p_{\mathrm{wt}=3}(x_3, x_4) + x_3^3 + g(x_3, x_4) = 0 \end{pmatrix} \subset \mathbb{C}^4_{x_1x_2x_3x_4}.$$

Moreover the equation defining X satisfies the following conditions:

(i) $c \neq \pm 1$, $\mathrm{wt}\, g \geq 6$, *and p contains only monomials with weight 2 and 3 for the weights distributed above.*

(ii) $x_4^2 \in p$ *and* $\deg g(x_3, 1) \leq 2$.

$P \in X$ is of type cA_2.

III. *The case P is of type cD*

•*Minimal discrepancy case*

(1) *f is the weighted blow-up with* $\mathrm{wt}(x_1,x_2,x_3,x_4)=(r+1,r,1,2)$ *with* $r=\min\{k-1,\lfloor\frac{\mathrm{wt}\,g}{2}\rfloor\}-1$ *after an identification*

$$P\in X\simeq o\in\Big(x_1^2+x_2^2x_4+\lambda x_2x_3^k+g(x_3,x_4)=0\Big)\subset\mathbb{C}^4_{x_1x_2x_3x_4}.$$

Moreover the equation defining X satisfies the conditions $\lambda\in\mathbb{C}$, $k\geq 3$, and $\mathrm{wt}\,g\geq 4$.

(2) *f is the weighted blow-up with* $\mathrm{wt}(x_1,x_2,x_3,x_4)=(r,r,1,1)$ *with* $r=\min\{k,l\}$ *after an identification of $P\in X$ with*

$$o\in\Big(x_1^2+x_2^2x_4+2x_2x_4p(x_3,x_4)+\lambda x_2x_3^k+q(x_3,x_4)=0\Big)\subset\mathbb{C}^4_{x_1x_2x_3x_4}.$$

Moreover the equation defining X satisfies the following conditions:

(i) *$\lambda\in\mathbb{C}$, $k\geq 2$, $\mathrm{wt}\,p=l-1$, $\mathrm{wt}\,q\geq 2l$, and p is weighted homogeneous for the weights distributed above.*

(ii) *$q_{\mathrm{wt}=2l}$ is not square if $p=0$.*

(iii) *Either* (a) *$l\geq k$,* (b) *$k=l+1$, x_3^{2l} or $x_3^{2l-1}x_4\in q$, and $x_3^{2l+1}\in q$, or* (c) *$k>l+1\geq 2$, $q_{\mathrm{wt}=2l}\neq 0$, and $x_3^{l-1}\in p$.*

(3) *f is the weighted blow-up with* $\mathrm{wt}(x_1,x_2,x_3,x_4,x_5)=(\frac{r+1}{2},\frac{r-1}{2},1,1,r)$ *with $r\geq 3$ after an identification*

$$P\in X\simeq o\in\begin{pmatrix}x_1^2+\lambda x_2x_3^k+x_4x_5+p(x_3,x_4)=0,\\ x_2^2+2x_1q_1(x_3,x_4)+q_2(x_3,x_4)+x_5=0\end{pmatrix}\subset\mathbb{C}^5_{x_1x_2x_3x_4x_5}.$$

Moreover the equations defining X satisfy the following conditions:

(i) *$\lambda\in\mathbb{C}$, $k\geq\frac{r+3}{2}$, $p_{\mathrm{wt}=r+1}\neq 0$, $\mathrm{wt}\,p\geq r+1$, $\mathrm{wt}\,q_1=\frac{r-3}{2}$, $\mathrm{wt}\,q_2=r-1$, and q_1, q_2 are weighted homogeneous for the weights distributed above.*

(ii) *q_2 is not square if $q_1=0$.*

(iii) *Either* (a) *$k=\frac{r+3}{2}$, x_3^{r+1} or $x_3^rx_4\in p$, $x_3^{r+2}\in p$, and $x_3^{\frac{r-3}{2}}\in q_1$, or* (b) *$x_3^{r-1}$ or $x_3^{r-2}x_4\in q_2$.*

(4) *f is the weighted blow-up with* $\mathrm{wt}(x_1,x_2,x_3,x_4)=(r+1,r,1,1)$ *after an identification of $P\in X$ with*

$$o\in\Big(x_1^2+2x_1p(x_3,x_4)+x_2^2x_4+\lambda x_2x_3^k+q(x_3,x_4)=0\Big)\subset\mathbb{C}^4_{x_1x_2x_3x_4}.$$

Moreover the equation defining X satisfies the following conditions:

(i) *$\lambda\in\mathbb{C}$, $k\geq r+1$, $q_{\mathrm{wt}=2r+1}\neq 0$, $\mathrm{wt}\,p=r$, $\mathrm{wt}\,q\geq 2r+1$, and p is weighted homogeneous for the weights distributed above.*

(ii) *If $k=r+1$, then $p\neq 0$ or there is no $s(x_3,x_4)$ which satisfies $q_{\mathrm{wt}=2r+1}=\lambda x_3^{r+1}s-x_4s^2$. Otherwise $x_3^r\in p$ or $x_3^{2r+1}\in q$.*

(5) *f is the weighted blow-up with* $\mathrm{wt}(x_1,x_2,x_3,x_4,x_5)=(r+1,r,1,1,r+2)$ *after an identification*

$$P\in X\simeq o\in\begin{pmatrix}x_1^2+x_2x_5+p(x_3,x_4)=0,\\ x_2x_4+\lambda x_3^{r+1}+2x_4q(x_3,x_4)+x_5=0\end{pmatrix}\subset\mathbb{C}^5_{x_1x_2x_3x_4x_5}.$$

Moreover the equations defining X satisfy the following conditions:

(i) $\lambda \in \mathbb{C}$, $\operatorname{wt} p \geq 2r+2$, $\operatorname{wt} q = r$, *and* q *is weighted homogeneous for the weights distributed above.*

•*Ordinary case*

(6) f *is the weighted blow-up with* $\operatorname{wt}(x_1, x_2, x_3, x_4) = (r+1, r, a, 1)$ *after an identification of* $P \in X$ *with*

$$o \in \Big(x_1^2 + 2x_1 p(x_3, x_4) + x_2^2 x_4 + \lambda x_2 x_3^2 + \mu x_3^3 + q(x_2, x_3, x_4) = 0\Big) \subset \mathbb{C}^4_{x_1x_2x_3x_4}.$$

Moreover there are following conditions:

(i) $a \mid (2r+1)$, $a \neq 2r+1$, *and* a *is odd.*

(ii) λ, $\mu \in \mathbb{C}$, $\operatorname{wt} p \geq r$, $\operatorname{wt} q \geq 2r+1$, *and* $q \in \mathfrak{m}^4$.

(iii) *The weighted homogeneous part of weight* $2r+1$ *is irreducible, and* $p_{\mathrm{wt}=r} \neq 0$ *unless* $p = 0$.

(7) f *is the weighted blow-up with* $\operatorname{wt}(x_1, x_2, x_3, x_4, x_5) = (r+1, r, a, 1, r+2)$ *after an identification*

$$P \in X \simeq o \in \begin{pmatrix} x_1^2 + x_2 x_5 + p(x_2, x_3, x_4) = 0, \\ x_2 x_4 + x_3^{\frac{r+1}{a}} + 2x_4 q(x_3, x_4) + x_5 = 0 \end{pmatrix} \subset \mathbb{C}^5_{x_1x_2x_3x_4x_5}.$$

Moreover there are following conditions:

(i) $a \mid (r+1)$ *and* $a \neq r+1$.

(ii) $p \in \mathfrak{m}^4$, $\operatorname{wt} p \geq 2r+2$, $\operatorname{wt} q = r$, *and* q *is weighted homogeneous for the weights distributed above.*

•*Exceptional case*

(8) f *is the weighted blow-up with* $\operatorname{wt}(x_1, x_2, x_3, x_4, x_5) = (\frac{r+1}{2}, \frac{r-1}{2}, 4, 1, r)$ *with* $r \geq 7$, $r \equiv \pm 3 \pmod 8$ *after an identification*

$$P \in X \simeq o \in \begin{pmatrix} x_1^2 + \lambda x_2 x_3^k + x_4 x_5 + p(x_3, x_4) = 0, \\ x_2^2 + 2x_1 q_1(x_3, x_4) + q_2(x_3, x_4) + x_5 = 0 \end{pmatrix} \subset \mathbb{C}^5_{x_1x_2x_3x_4x_5}.$$

Moreover the equations defining X satisfy the following conditions:

(i) $\lambda \in \mathbb{C}$, $k > \frac{r+3}{8}$, $\operatorname{wt} p \geq r+1$, $\operatorname{wt} q_1 = \frac{r-3}{2}$, $\operatorname{wt} q_2 = r-1$, *and* q_1, q_2 *are weighted homogeneous for the weights distributed above.*

(ii) q_2 *is not square if* $q_1 = 0$.

(iii) *If* $r \equiv 3 \pmod 8$ (*resp.* $r \equiv -3 \pmod 8$), *then* $x_3^{\frac{r+1}{4}} \in p$ (*resp.* $x_3^{\frac{r-1}{4}} \in q_2$).

(9) f *is the weighted blow-up with* $\operatorname{wt}(x_1, x_2, x_3, x_4, x_5) = (\frac{r+1}{2}, \frac{r-1}{2}, 2, 1, r)$ *with* $r \geq 5$ *after an identification*

$$P \in X \simeq o \in \begin{pmatrix} x_1^2 + \lambda x_2 x_3^k + x_4 x_5 + p(x_3, x_4) = 0, \\ x_2^2 + 2x_1 q_1(x_3, x_4) + q_2(x_3, x_4) + x_5 = 0 \end{pmatrix} \subset \mathbb{C}^5_{x_1x_2x_3x_4x_5}.$$

Moreover the equations defining X satisfy the following conditions:

(i) $\lambda \in \mathbb{C}$, $k > \frac{r+1}{4}$, $\mathrm{wt}\, p \geq r+1$, $\mathrm{wt}\, q_1 = \frac{r-3}{2}$, $\mathrm{wt}\, q_2 = r-1$, *and* q_1, q_2 *are weighted homogeneous for the weights distributed above.*
(ii) q_2 *is not square if* $q_1 = 0$.
(iii) $x_3^{\frac{r+1}{2}} \in p$.

(10) f *is the weighted blow-up with* $\mathrm{wt}(x_1, x_2, x_3, x_4) = (r, r, 2, 1)$ *after an identification of* $P \in X$ *with*

$$o \in \Big(x_1^2 + x_2^2 x_4 + 2x_2 x_4 p(x_3, x_4) + \lambda x_2 x_3^k + q(x_3, x_4) = 0\Big) \subset \mathbb{C}^4_{x_1x_2x_3x_4}.$$

Moreover the equation defining X *satisfies the following conditions:*
(i) $\lambda \in \mathbb{C}$, $k > \frac{r}{2}$, $\mathrm{wt}\, q \geq 2r$, *and* p *is weighted homogeneous of weight* $r-1$ *for the weights distributed above.*
(ii) $p \neq 0$ *or* $q_{wt=2r} \neq 0$, *and* $q_{\mathrm{wt}=2r}$ *is not square if* $p = 0$.
(iii) $x_3^r \in q$.

(11) f *is the weighted blow-up with* $\mathrm{wt}(x_1, x_2, x_3, x_4) = (3, 3, 1, 2)$ *after an identification of* $P \in X$ *with*

$$o \in \Big(x_1^2 + x_2^2 x_4 + 2x_2 x_4 p(x_3, x_4) + x_2 x_3^3 + q(x_3, x_4) = 0\Big) \subset \mathbb{C}^4_{x_1x_2x_3x_4}.$$

Moreover the equation defining X *satisfies the following conditions:*
(i) $\mathrm{wt}\, q \geq 6$, *and* p *is weighted homogeneous of weight* 2 *for the weights distributed above.*
(ii) $x_4^3 \in q$.

P *is of type* cD_4.

(12) f *is the weighted blow-up with* $\mathrm{wt}(x_1, x_2, x_3, x_4) = (3, 4, 2, 1)$ *after an identification of* $P \in X$ *with*

$$o \in \big(x_1^2 + x_2^2 x_4 + 2x_2 x_4 p(x_3, x_4) + \lambda x_2 x_3^k + q(x_3, x_4) = 0\big) \subset \mathbb{C}^4_{x_1x_2x_3x_4}.$$

Moreover the equation defining X *satisfies the following conditions:*
(i) $\lambda \in \mathbb{C}$, $k > 2$, $\mathrm{wt}\, q \geq 6$, *and* p *contains only monomials with weight* ≤ 3 *for the weights distributed above.*
(ii) $x_4 \in p$ *and* $x_3^3 \in q$.

P *is of type* cD_4.

IV. *The case* P *is of type* cE

•*Minimal discrepancy case*

(1) f *is the weighted blow-up with* $\mathrm{wt}(x_1, x_2, x_3, x_4) = (2, 2, 1, 1)$ *after an identification*

$$P \in X \simeq o \in \Big(x_1^2 + x_2^3 + x_2 g(x_3, x_4) + h(x_3, x_4) = 0\Big) \subset \mathbb{C}^4_{x_1x_2x_3x_4}.$$

Moreover the equation defining X *satisfies the following conditions:*
(i) $h_{\mathrm{wt}=4} \neq 0$ *is not square, and* $\mathrm{wt}\, g \geq 3$ *and* $\mathrm{wt}\, h \geq 4$ *for the weights distributed above.*
(ii) *There is no linear form* $l(x_3, x_4)$ *which satisfies* $l^2 \mid h_{\mathrm{wt}=4}$, $l \mid g_{\mathrm{wt}=3}$, *and* $l \mid h_{\mathrm{wt}=5}$.

P is of type cE_6.

(2) *f is the weighted blow-up with $\mathrm{wt}(x_1, x_2, x_3, x_4) = (3, 2, 1, 1)$ after an identification of $P \in X$ with*

$$o \in \left(x_1^2 + 2x_1 p(x_3, x_4) + x_2^3 + x_2 g(x_3, x_4) + h(x_3, x_4) = 0\right) \subset \mathbb{C}^4_{x_1x_2x_3x_4}.$$

Moreover the equation defining X satisfies the following conditions:

(i) $\mathrm{wt}\, p = 2$, $\mathrm{wt}\, g \geq 3$, $\mathrm{wt}\, h \geq 5$, *and p is weighted homogeneous for the weights distributed above.*
(ii) $\gcd(p, g_{\mathrm{wt}=3}, h_{\mathrm{wt}=5}) = 1$.

P is of type cE_6 or cE_7.

(3) *f is the weighted blow-up with $\mathrm{wt}(x_1, x_2, x_3, x_4) = (3, 2, 2, 1)$ after an identification*

$$P \in X \simeq o \in \left(x_1^2 + x_2^3 + x_2 g(x_3, x_4) + h(x_3, x_4) = 0\right) \subset \mathbb{C}^4_{x_1x_2x_3x_4}.$$

Moreover the equation defining X satisfies the following conditions:

(i) $\mathrm{wt}\, g \geq 4$ *and* $\mathrm{wt}\, h \geq 6$ *for the weights distributed above.*
(ii) $x_3^2 \mid h_4$, $x_3 \mid g_3$, *and* $x_3 \mid h_5$.
(iii) *There is no $\mu \in \mathbb{C}$ which satisfies*

$$x_2^3 + x_2 g_{\mathrm{wt}=4} + h_{\mathrm{wt}=6} = (x_2 - \mu x_4^2)^2 (x_2 + 2\mu x_4^2),$$
$$\text{and } \mu x_4^2 g_{\mathrm{wt}=5} + h_{\mathrm{wt}=7} = 0.$$

(4) *f is the weighted blow-up with $\mathrm{wt}(x_1, x_2, x_3, x_4) = (4, 3, 2, 1)$ after an identification of $P \in X$ with*

$$o \in \left(x_1^2 + x_2^3 + 3\lambda x_2^2 x_4^2 + x_2 g(x_3, x_4) + h(x_3, x_4) = 0\right) \subset \mathbb{C}^4_{x_1x_2x_3x_4}.$$

Moreover the equation defining X satisfies the following conditions:

(i) $\lambda \in \mathbb{C}$, $\mathrm{wt}\, g \geq 5$, *and* $\mathrm{wt}\, h \geq 8$ *for the weights distributed above.*
(ii) $x_3^4 \mid h_4$, $x_3^2 \mid g_3$, *and* $x_3^3 \mid h_5$.
(iii) $3\lambda x_2^2 x_4^2 + x_2 g_{\mathrm{wt}=5} + h_{\mathrm{wt}=8}$ *is not square.*

(5) *f is the weighted blow-up with $\mathrm{wt}(x_1, x_2, x_3, x_4) = (5, 3, 2, 1)$ after an identification of $P \in X$ with*

$$o \in \left(x_1^2 + 2x_1 p(x_2, x_3, x_4) + x_2^3 + x_2 g(x_3, x_4) + h(x_3, x_4) = 0\right) \subset \mathbb{C}^4_{x_1x_2x_3x_4}.$$

Moreover the equation defining X satisfies the following conditions:

(i) $\mathrm{wt}\, p = 4$, $\mathrm{wt}\, g \geq 6$, $\mathrm{wt}\, h \geq 9$, *and p is weighted homogeneous for the weights distributed above.*
(ii) $p \neq 0$ *or* $x_2^3 + x_2 g_{\mathrm{wt}=6} + h_{\mathrm{wt}=9}$ *is irreducible.*
(iii) *If P is of type cE_6 (resp. cE_7, cE_8), then $x_3^2 \in p$ (resp. $x_3^3 \mid g_3$, $x_3^4 \mid h_5$).*

(iv) $x_2x_4 \in p$ *or there is no* $\mu \in \mathbb{C}$ *which satisfies*

$$(x_3 - \mu x_4^2)^2 \mid p,$$
$$(x_3 - \mu x_4^2)^i \mid g_{\mathrm{wt}=8-i} \text{ for } i = 1,\ 2,$$
$$\text{and } (x_3 - \mu x_4^2)^j \mid h_{\mathrm{wt}=12-j} \text{ for } j = 1,\ 2,\ 3.$$

(6) *f is the weighted blow-up with* $\mathrm{wt}(x_1, x_2, x_3, x_4) = (5, 4, 2, 1)$ *after an identification of* $P \in X$ *with*

$$o \in \Big(x_1^2 + x_2^3 + 3x_2^2 p(x_3, x_4) + x_2 g(x_3, x_4) + h(x_3, x_4) = 0\Big) \subset \mathbb{C}^4_{x_1x_2x_3x_4}.$$

Moreover the equation defining X satisfies the following conditions:

(i) $\mathrm{wt}\, p = 3$, $\mathrm{wt}\, g \geq 6$, $\mathrm{wt}\, h \geq 10$, *and p is weighted homogeneous for the weights distributed above.*
(ii) *If P is of type cE_7 (resp. cE_8), then $x_3^3 \in g$ (resp. $x_3^5 \in h$).*
(iii) *There is no* $\mu \in \mathbb{C}$ *which satisfies*

$$(x_3 - \mu x_4^2) \mid p,$$
$$(x_3 - \mu x_4^2)^i \mid g_{\mathrm{wt}=8-i} \text{ for } i = 1,\ 2,$$
$$\text{and } (x_3 - \mu x_4^2)^j \mid h_{\mathrm{wt}=12-j} \text{ for } j = 1,\ 2.$$

P is of type cE_7 or cE_8.

(7) *f is the weighted blow-up with* $\mathrm{wt}(x_1, x_2, x_3, x_4) = (6, 4, 3, 1)$ *after an identification*

$$P \in X \simeq o \in \Big(x_1^2 + x_2^3 + x_2 g(x_3, x_4) + h(x_3, x_4) = 0\Big) \subset \mathbb{C}^4_{x_1x_2x_3x_4}.$$

Moreover the equation defining X satisfies the following conditions:

(i) $\mathrm{wt}\, g \geq 8$ *and* $\mathrm{wt}\, h \geq 12$ *for the weights distributed above.*
(ii) *If P is of type cE_6 (resp. cE_7, cE_8), then $x_3^4 \in h$ (resp. $x_3^3 \in g$, $x_3^4 \mid h_5$).*
(iii) *There is no $s(x_3, x_4)$ which satisfies*

$$x_2^3 + x_2 g_{\mathrm{wt}=8} + h_{\mathrm{wt}=12} = (x_2 - s)^2(x_2 + 2s),$$
$$\text{and } s g_{\mathrm{wt}=9} + h_{\mathrm{wt}=13} = 0.$$

(8) *f is the weighted blow-up with* $\mathrm{wt}(x_1, x_2, x_3, x_4) = (7, 5, 3, 1)$ *after an identification of* $P \in X$ *with*

$$o \in \Big(x_1^2 + x_2^3 + 3x_2^2 p(x_3, x_4) + x_2 g(x_3, x_4) + h(x_3, x_4) = 0\Big) \subset \mathbb{C}^4_{x_1x_2x_3x_4}.$$

Moreover the equation defining X satisfies the following conditions:

(i) $\mathrm{wt}\, p = 4$, $\mathrm{wt}\, g \geq 9$, $\mathrm{wt}\, h \geq 14$, *and p is weighted homogeneous for the weights distributed above.*
(ii) *If P is of type cE_7 (resp. cE_8), then $x_3^3 \in g$ (resp. $x_3^5 \in h$).*

(iii) *$p \neq 0$ or there is no $\mu \in \mathbb{C}$ which satisfies*

$$(x_3 - \mu x_4^3)^i \mid g_{\mathrm{wt}=12-i} \text{ for } i = 1,\ 2,\ 3,$$
$$\text{and } (x_3 - \mu x_4^3)^j \mid h_{\mathrm{wt}=18-j} \text{ for } j = 1,\ 2,\ 3,\ 4.$$

P is of type cE_7 or cE_8.

(9) *f is the weighted blow-up with* $\mathrm{wt}(x_1, x_2, x_3, x_4) = (8, 5, 3, 1)$ *after an identification of $P \in X$ with*

$$o \in \left(x_1^2 + 2x_1x_4p(x_2, x_3, x_4) + x_2^3 + x_2g(x_3, x_4) + h(x_3, x_4) = 0\right) \subset \mathbb{C}^4_{x_1x_2x_3x_4}.$$

Moreover the equation defining X satisfies the following conditions:

(i) $\mathrm{wt}\, p = 6$*,* $\mathrm{wt}\, g \geq 10$*,* $\mathrm{wt}\, h \geq 15$*, and p is weighted homogeneous for the weights distributed above.*

(ii) *$x_3^5 \in h$.*

(iii) *$x_2x_4 \in p$ or there is no $\mu \in \mathbb{C}$ which satisfies*

$$(x_3 - \mu x_4^3)^2 \mid p,$$
$$(x_3 - \mu x_4^3)^i \mid g_{\mathrm{wt}=12-i} \text{ for } i = 1,\ 2,$$
$$\text{and } (x_3 - \mu x_4^3)^j \mid h_{\mathrm{wt}=18-j} \text{ for } j = 1,\ 2,\ 3.$$

P is of type cE_8.

(10) *f is the weighted blow-up with* $\mathrm{wt}(x_1, x_2, x_3, x_4) = (9, 6, 4, 1)$ *after an identification*

$$P \in X \simeq o \in \left(x_1^2 + x_2^3 + x_2g(x_3, x_4) + h(x_3, x_4) = 0\right) \subset \mathbb{C}^4_{x_1x_2x_3x_4}.$$

Moreover the equation defining X satisfies the following conditions:

(i) $\mathrm{wt}\, g \geq 12$ *and* $\mathrm{wt}\, h \geq 18$ *for the weights distributed above.*

(ii) *If P is of type cE_7 (resp. cE_8), then $x_3^3 \in g$ (resp. $x_3^5 \in h$).*

(iii) *There is no $s(x_3, x_4)$ which satisfies*

$$x_2^3 + x_2g_{\mathrm{wt}=12} + h_{\mathrm{wt}=18} = (x_2 - s)^2(x_2 + 2s),$$
$$\text{and } sg_{\mathrm{wt}=13} + h_{\mathrm{wt}=19} = 0.$$

P is of type cE_7 or cE_8.

(11) *f is the weighted blow-up with* $\mathrm{wt}(x_1, x_2, x_3, x_4) = (10, 7, 4, 1)$ *after an identification of $P \in X$ with*

$$o \in \left(x_1^2 + x_2^3 + 3x_2^2p(x_3, x_4) + x_2g(x_3, x_4) + h(x_3, x_4) = 0\right) \subset \mathbb{C}^4_{x_1x_2x_3x_4}.$$

Moreover the equation defining X satisfies the following conditions:

(i) $\mathrm{wt}\, p = 6$*,* $\mathrm{wt}\, g \geq 13$*,* $\mathrm{wt}\, h \geq 20$*, and p is weighted homogeneous for the weights distributed above.*

(ii) *$x_3^5 \in h$.*

(iii) *$p \neq 0$ or there is no $\mu \in \mathbb{C}$ which satisfies*

$$(x_3 - \mu x_4^4)^i \mid g_{\mathrm{wt}=16-i} \text{ for } i = 1,\ 2,\ 3,$$
$$\text{and } (x_3 - \mu x_4^4)^j \mid h_{\mathrm{wt}=24-j} \text{ for } j = 1,\ 2,\ 3,\ 4.$$

P is of type cE_8.

(12) *f is the weighted blow-up with* $\mathrm{wt}(x_1, x_2, x_3, x_4) = (12, 8, 5, 1)$ *after an identification*

$$P \in X \simeq o \in \left(x_1^2 + x_2^3 + x_2 g(x_3, x_4) + h(x_3, x_4) = 0\right) \subset \mathbb{C}^4_{x_1x_2x_3x_4}.$$

Moreover the equation defining X satisfies the following conditions:

(i) $\mathrm{wt}\, g \geq 16$ *and* $\mathrm{wt}\, h \geq 24$ *for the weights distributed above.*
(ii) $x_3^5 \in h$.
(iii) *There is no $\mu \in \mathbb{C}$ which satisfies*

$$(x_3 - \mu x_4^5)^i \mid g_{\mathrm{wt}=20-i} \text{ for } i = 1, \cdots, 4,$$
$$\text{and } (x_3 - \mu x_4^5)^j \mid h_{\mathrm{wt}=30-j} \text{ for } j = 1, \cdots, 6.$$

P is of type cE_8.

(13) *f is the weighted blow-up with* $\mathrm{wt}(x_1, x_2, x_3, x_4) = (15, 10, 6, 1)$ *after an identification*

$$P \in X \simeq o \in \left(x_1^2 + x_2^3 + x_2 g(x_3, x_4) + h(x_3, x_4) = 0\right) \subset \mathbb{C}^4_{x_1x_2x_3x_4}.$$

Moreover the equation defining X satisfies the following conditions:

(i) $\mathrm{wt}\, g \geq 20$ *and* $\mathrm{wt}\, h \geq 30$ *for the weights distributed above.*
(ii) $x_3^5 \in h$.

P is of type cE_8.

(14) *f is the weighted blow-up with* $\mathrm{wt}(x_1, x_2, x_3, x_4, x_5) = (3, 2, 1, 1, 5)$ *after an identification of $P \in X$ with*

$$o \in \begin{pmatrix} x_1^2 + x_2^3 + x_5 p(x_3, x_4) + x_2 g(x_3, x_4) + h(x_3, x_4) = 0, \\ 2x_1 q_1(x_3, x_4) + x_2 q_2(x_3, x_4) + q_3(x_3, x_4) + x_5 = 0 \end{pmatrix} \subset \mathbb{C}^5_{x_1x_2x_3x_4x_5}.$$

Moreover the equations defining X satisfy the following conditions:

(i) $\mathrm{wt}\, p = 1$, $\mathrm{wt}\, g \geq 4$, $\mathrm{wt}\, h \geq 6$, $\mathrm{wt}\, q_1 = 1$, $\mathrm{wt}\, q_2 = 2$, $\mathrm{wt}\, q_3 = 4$, *and p, q_1, q_2, and q_3 are weighted homogeneous for the weights distributed above.*
(ii) $\gcd(q_1, q_2, q_3) = 1$.

P is of type cE_6 or cE_7.

(15) *f is the weighted blow-up with* $\mathrm{wt}(x_1, x_2, x_3, x_4) = (4, 2, 1, 1)$ *after an identification of $P \in X$ with*

$$o \in \begin{pmatrix} x_1^2 + 2x_1\{x_2 p(x_3, x_4) + q(x_3, x_4)\} \\ + x_2^3 + x_2 g(x_3, x_4) + h(x_3, x_4) = 0 \end{pmatrix} \subset \mathbb{C}^4_{x_1x_2x_3x_4}.$$

Moreover the equation defining X satisfies the following conditions:

(i) $\mathrm{wt}\, p = 1$, $\mathrm{wt}\, q \geq 2$, $\mathrm{wt}\, g \geq 4$, $\mathrm{wt}\, h \geq 6$, *and p is weighted homogeneous for the weights distributed above.*
(ii) $x_3^5 \in h$.

(iii) *Either* (a) $q_{\mathrm{wt}=2}$ *is not square, or* (b) $q_{\mathrm{wt}=2} = x_3^2$ *and there is no* $\mu \in \mathbb{C}$ *which satisfies*

$$\begin{aligned} x_2^3 + x_2 g_{\mathrm{wt}=4} + h_{\mathrm{wt}=6} &= (x_2 - \mu x_4^2)^2 (x_2 + 2\mu x_4^2), \\ \mu x_4^2 g_{\mathrm{wt}=5} + h_{\mathrm{wt}=7} &= 0, \\ \text{and } \mu x_4^2 p + q_{\mathrm{wt}=3} &= 0. \end{aligned}$$

P is of type cE_6.

(16) *f is the weighted blow-up with* $\mathrm{wt}(x_1, x_2, x_3, x_4, x_5) = (3, 2, 1, 1, 4)$ *after an identification of $P \in X$ with*

$$o \in \begin{pmatrix} x_1^2 + x_2^3 + x_5 p(x_3, x_4) + x_2 g(x_3, x_4) + h(x_3, x_4) = 0, \\ x_2 q_1(x_3, x_4) + q_2(x_3, x_4) + x_5 = 0 \end{pmatrix} \subset \mathbb{C}^5_{x_1x_2x_3x_4x_5}.$$

Moreover the equations defining X satisfy the following conditions:

(i) $\mathrm{wt}\, p = 2$, $\mathrm{wt}\, g \geq 4$, $\mathrm{wt}\, h \geq 6$, $\mathrm{wt}\, q_1 = 1$, $\mathrm{wt}\, q_2 = 3$, *and p, q_1 and q_2 are weighted homogeneous for the weights distributed above.*

(ii) *Either* (a) *p is not square, or* (b) $p = x_3^2$ *and there is no* $\mu \in \mathbb{C}$ *which satisfies*

$$\begin{aligned} x_2^3 + x_2 g_{\mathrm{wt}=4} + h_{\mathrm{wt}=6} &= (x_2 - \mu x_4^2)^2 (x_2 + 2\mu x_4^2), \\ \mu x_4^2 g_{\mathrm{wt}=5} + h_{\mathrm{wt}=7} &= 0, \\ \text{and } \mu x_4^2 q_1 + q_2 &= 0. \end{aligned}$$

P is of type cE_7.

(17) *f is the weighted blow-up with* $\mathrm{wt}(x_1, x_2, x_3, x_4) = (3, 3, 1, 1)$ *after an identification of $P \in X$ with*

$$o \in \left(x_1^2 + x_2^3 + 3x_2^2 p(x_3, x_4) + x_2 g(x_3, x_4) + h(x_3, x_4) = 0 \right) \subset \mathbb{C}^4_{x_1x_2x_3x_4}.$$

Moreover the equation defining X satisfies the following conditions:

(i) $\mathrm{wt}\, p = 2$, $\mathrm{wt}\, g \geq 3$, $\mathrm{wt}\, h \geq 6$, *and p is weighted homogeneous for the weights distributed above.*

(ii) *There is no linear form $l(x_3, x_4)$ which satisfies $l^2 \mid g_{\mathrm{wt}=3}$, $l^2 \mid h_{\mathrm{wt}=6}$, $l \mid g_{\mathrm{wt}=4}$, and $l \mid h_{\mathrm{wt}=7}$.*

P is of type cE_7.

(18) *f is the weighted blow-up with* $\mathrm{wt}(x_1, x_2, x_3, x_4, x_5) = (5, 3, 2, 1, 7)$ *after an identification of $P \in X$ with*

$$o \in \begin{pmatrix} x_1^2 + x_2 x_5 + x_2 g(x_3, x_4) + h(x_3, x_4) = 0, \\ x_2^2 + 3x_2 p(x_3, x_4) + q(x_3, x_4) + x_5 = 0 \end{pmatrix} \subset \mathbb{C}^5_{x_1x_2x_3x_4x_5}.$$

Moreover the equations defining X satisfy the following conditions:

(i) $\mathrm{wt}\, g \geq 7$, $\mathrm{wt}\, h \geq 10$, $\mathrm{wt}\, p = 3$, $\mathrm{wt}\, q = 6$, *and p and q are weighted homogeneous for the weights distributed above.*

(ii) $x_2^3 + 3x_2 p + q$ *is irreducible.*

(iii) *There is no* $\mu \in \mathbb{C}$ *which satisfies*

$$(x_3 - \mu x_4^2) \mid g_{\mathrm{wt}=7}, (x_3 - \mu x_4^2)^2 \mid h_{\mathrm{wt}=10}, (x_3 - \mu x_4^2) \mid h_{\mathrm{wt}=11},$$
$$(x_3 - \mu x_4^2) \mid p, \text{ and } (x_3 - \mu x_4^2)^2 \mid q.$$

P is of type cE_7 or cE_8.

•*Exceptional case*

(19) *f is the weighted blow-up with* $\mathrm{wt}(x_1, x_2, x_3, x_4) = (3, 3, 2, 1)$ *after an identification of $P \in X$ with*

$$o \in \Big(x_1^2 + \{x_2 - p(x_3, x_4)\}^3 + x_2 g(x_3, x_4) + h(x_3, x_4) = 0\Big) \subset \mathbb{C}^4_{x_1x_2x_3x_4}.$$

Moreover the equation defining X satisfies the following conditions:

(i) $\mathrm{wt}\, g \geq 3$, $\mathrm{wt}\, h \geq 6$, *and p is weighted homogeneous of weight* 2 *for the weights distributed above.*

(ii) $\deg g \geq 3$ *and* $\deg h \geq 4$.

(iii) $x_3 \in p$ *and* $x_4^3 \in g$.

P is of type cE_6.

(20) *f is the weighted blow-up with* $\mathrm{wt}(x_1, x_2, x_3, x_4, x_5) = (5, 3, 2, 2, 7)$ *after an identification*

$$P \in X \simeq o \in \begin{pmatrix} x_1^2 + x_2x_5 + p(x_3, x_4) = 0, \\ x_2^2 + q(x_3, x_4) + x_5 = 0 \end{pmatrix} \subset \mathbb{C}^5_{x_1x_2x_3x_4x_5}.$$

Moreover the equations defining X satisfy the following conditions:

(i) $\mathrm{wt}\, p \geq 10$, $\mathrm{wt}\, q \geq 6$ *for the weights distributed above.*

(ii) $\gcd(p_5, q_3) = 1$.

P is of type cE_7.

(21) *f is the weighted blow-up with* $\mathrm{wt}(x_1, x_2, x_3, x_4) = (7, 5, 3, 2)$ *after an identification of $P \in X$ with*

$$o \in \Big(x_1^2 + x_2^3 + \lambda x_2^2 x_4^2 + x_2 g(x_3, x_4) + h(x_3, x_4) = 0\Big) \subset \mathbb{C}^4_{x_1x_2x_3x_4}.$$

Moreover the equation defining X satisfies the following conditions:

(i) $\lambda \in \mathbb{C}$ *and* $\mathrm{wt}\, g \geq 9$, $\mathrm{wt}\, h \geq 14$ *for the weights distributed above.*

(ii) *If P is of type cE_7 (resp. cE_8), then $x_3^3 \in g$ (resp. x_3^5 or $x_3^4 x_4 \in h$).*

(iii) $x_4^7 \in h$.

P is of type cE_7 or cE_8.

Conversely, weighted blow-ups of cDV points as above are divisorial contractions.

We tabulate the divisorial contractions $f\colon (Y \supset E) \to (X \ni P)$ in Theorem 5.1, its dicrepancy, and the non-Gorenstein singularities on Y.

type	terminal	discrepancy	non-Gorenstein terminal on Y
(I-1)	smooth	$a + b$	$\frac{1}{a}(1, -1, a - b)$, $\frac{1}{b}(1, -1, b - a)$
(II-1)	cA	a	$\frac{1}{r_1}(1, -1, a)$, $\frac{1}{r_2}(1, -1, a)$

type	terminal	discrepancy	non-Gorenstein terminal on Y
(II-2)	cA_1	4	$\frac{1}{5}(1,4,3)$
(II-3)	cA_2	3	$cAx/4$ deforming to $\frac{1}{2}(1,1,1),\ \frac{1}{4}(1,3,3)$
(III-1)	cD	1	$\frac{1}{r}(1,-1,1),\ \begin{cases} cAx/2, \\ cA/2,\ \text{or} \\ 2\times\frac{1}{2}(1,1,1) \end{cases}$
(III-2)			cA/r deforming to $2\times\frac{1}{r}(1,-1,1)$
(III-3)			$\frac{1}{r}(1,-1,2)$
(III-4)			$\frac{1}{r}(1,-1,1),\ \frac{1}{r+1}(1,-1,1)$
(III-5)			$\frac{1}{r}(1,-1,1),\ \frac{1}{r+2}(1,-1,1)$
(III-6)		a	$\frac{1}{r}(1,-1,a),\ \frac{1}{r+1}(1,-1,a)$
(III-7)			$\frac{1}{r}(1,-1,a),\ \frac{1}{r+2}(1,-1,a)$
(III-8)		4	$\frac{1}{r}(1,-1,8)$
(III-9)		2	$\frac{1}{r}(1,-1,4)$
(III-10)			cA/r deforming to $2\times\frac{1}{r}(1,-1,2)$
(III-11)	cD_4		$cD/3$ deforming to $2\times\frac{1}{3}(1,2,2)$
(III-12)		3	$cAx/4$ deforming to $\frac{1}{2}(1,1,1),\ \frac{1}{4}(1,3,3)$
(IV-1)	cE_6	1	$cAx/2$ deforming to $2\times\frac{1}{2}(1,1,1)$
(IV-2)	$cE_{6,7}$		$\frac{1}{2}(1,1,1),\ \frac{1}{3}(1,2,2)$
(IV-3)	cE_6 cE_7 cE_8		$cA/2$ $cD/2$ deforming to $3\times\frac{1}{2}(1,1,1)$ $cE/2$
(IV-4)	cE_6 cE_7 cE_8		$\frac{1}{3}(1,2,2), 2\times\frac{1}{2}(1,1,1)$ $\frac{1}{3}(1,2,2), cA/2$ $\frac{1}{3}(1,2,2)$, "$cA/2$ or $cAx/2$"
(IV-5)	cE		$\frac{1}{2}(1,1,1),\ \frac{1}{5}(1,4,2)$
(IV-6)	cE_7 cE_8		$\frac{1}{2}(1,1,1), cAx/4$ $cAx/4$ deforming to $2\times\frac{1}{2}(1,1,1),\ \frac{1}{4}(1,3,3)$
(IV-7)	cE_6 cE_7 cE_8		$\frac{1}{2}(1,1,1), 2\times\frac{1}{3}(1,2,2)$ $\frac{1}{2}(1,1,1), cA/3$ $\frac{1}{2}(1,1,1)$, "$cA/3$ or $cD/3$"
(IV-8)	$cE_{7,8}$		$\frac{1}{3}(1,2,2),\ \frac{1}{5}(1,4,2)$
(IV-9)	cE_8		$\frac{1}{8}(1,7,3)$
(IV-10)	cE_7 cE_8		$\frac{1}{2}(1,1,1), \frac{1}{3}(1,2,2), \frac{1}{4}(1,3,3)$ $\frac{1}{3}(1,2,2), cAx/4$
(IV-11)	cE_8		$\frac{1}{2}(1,1,1),\ \frac{1}{7}(1,6,2)$

type	terminal	discrepancy	non-Gorenstein terminal on Y
(IV-12)	cE_8		$\frac{1}{4}(1,3,3)$, $\frac{1}{5}(1,4,2)$
(IV-13)	cE_8		$\frac{1}{2}(1,1,1)$, $\frac{1}{3}(1,2,2)$, $\frac{1}{5}(1,4,1)$
(IV-14)	$cE_{6,7}$		$\frac{1}{5}(1,4,2)$
(IV-15)	cE_6	1	$cAx/4$ deforming to $\frac{1}{2}(1,1,1)$, $\frac{1}{4}(1,3,3)$
(IV-16)	cE_7		$cAx/4$ deforming to $\frac{1}{2}(1,1,1)$, $\frac{1}{4}(1,3,3)$
(IV-17)	cE_7		$cD/3$ deforming to $2 \times \frac{1}{3}(1,2,2)$
(IV-18)	$cE_{7,8}$		$\frac{1}{7}(1,6,3)$
(IV-19)	cE_6		$cD/3$ deforming to $2 \times \frac{1}{3}(1,2,2)$
(IV-20)	cE_7	2	$\frac{1}{7}(1,6,6)$
(IV-21)	$cE_{7,8}$		$\frac{1}{3}(1,2,2)$, $\frac{1}{5}(1,4,4)$

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