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Operating functions on multipliers for Jacobi and Laguerre expansions

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1. Introduction

Let $\mathbf{T}$ be the unit circle, and $L^p(\mathbf{T})$ the L^p space on $\mathbf{T}$ for $1<p<\infty$. Let $1\leq p, q\leq\infty$. Also let $m=\{m_n\}_{n=-\infty}^{\infty}$ be a bounded sequence. We define a multiplier operator T_m for Fourier series

$$T_m g\sim\sum_{-\infty}^{\infty} m_n\hat{g}(n)e^{int}.$$

We call m a (p, q)-multiplier for Fourier series if $\|T_m g\|_q\leq C\|g\|_p$ for all $g\in L^p(\mathbf{T})$. Let $M^t(p, q)$ be the set of all (p, q)-multipliers $m=\{m_n\}$. Also let A and B be the function spaces on the integer group $\mathbf{Z}$, and Φ a function on $[-1, 1]$. We say that Φ operates from A to B (if $A=B$, we simply say that Φ operates on A) if $\Phi(f)\in B$ for $f\in A$ such that $f(\mathbf{Z})\subset[-1, 1]$. Igari-Sato[IS] studied the operating functions on $M^t(p, q)$ (we also call Φ an operating function on $M^t(p, q)$), and characterized the operating functions on $M^t(p, q)$ for $1\leq p<2\leq q\leq\infty$, that is:

Theorem 1([IS]).
(I) *Let $1\leq p<\infty$ and Φ_0 be a function in $[-1, 1]$. Assume that Φ_0 is bounded near the origin if $p=1$ or $q=\infty$ and uniformly bounded in $[-1, 1]$ if $p>1$.*

(i) *Suppose $1\leq p<q<2$ or $2\leq p<q\leq\infty$. Let $\beta_0=(1/q-1/2)/(1/p-1/q)$ or $(1/2-1/p)/(1/p-1/q)$ respectively and n_0 be the smallest integer such that $n_0\geq\beta_0$. Then for any constants $\alpha_1, \alpha_2, \cdots, \alpha_{n_0}$*

$$\Phi(t)=\alpha_1 t+\alpha_2 t^2+\cdots+\alpha_{n_0}t^{n_0}+|t|^{\beta_0+1}\Phi_0(t)$$

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operates on $M^t(p, q)$.

(ii) *Suppose* $1\leq p<2\leq q\leq\infty$. *Let* $\beta_1=\min\{(1/2-1/q)/(1/p-1/2), (1/p-1/2)/(1/2-1/q)\}$. *Then for any constant* α

$$\Phi(t)=\alpha t+|t|^{\beta_1+1}\Phi_0(t)$$

operates on $M^t(p, q)$.

(II) *Let* $1\leq p<2\leq q\leq\infty$ *and* Φ *be a function in* $[-1, 1]$. *If* Φ *operates on* $M^t(p, q)$, *then* Φ *is of the form that* $\Phi(t)=\alpha t+|t|^{\beta_1}\Phi_0(t)$, *where* α *is a complex number and* β_1 *is the number given in* (I), *and* Φ_0 *is a function in* $[-1, 1]$ *bounded near the origin if* $p=1$ *or* $q=\infty$ *and uniformly bounded in* $[-1, 1]$ *if* $p>1$.

Also Igari-Sato[IS] showed that (I) in the above theorem holds for general orthogonal polynomial expansions.

In this paper, we shall investigate operating functions on (p, q)-multipliers for Jacobi and Laguerre expansions. Is §2, we shall prove an analogue of Theorem 1 for (p, q)-multipliers for Jacobi expansions by the relation between Jacobi expansions and trigonometric expansions. In §3, we shall characterize the operating functions on (p, 2)-multipliers for Laguerre expansions by Kanjin-Sato[KS] and the estimates of Laguerre polynomials (cf. [T]).

2. Operating functions on multipliers for Jacobi expansions

Let $\alpha\geq\beta\geq-\frac{1}{2}$, and $1<p<\infty$, let $d\mu(x)=(1-x)^\alpha(1+x)^\beta dx$. Also let $P_n^{(\alpha,\beta)}(x)$ be the Jacobi polynomial of degree n, that is,

$$(1-x)^\alpha(1+x)^\beta P_n^{(\alpha,\beta)}(x)=\frac{(-1)^n}{2^n n!}\frac{d^n}{dx^n}[(1-x)^{n+\alpha}(1+x)^{n+\beta}].$$

These are orthogonal polynomials with respect to $d\mu$ on $(-1, 1)$, and

$$\int_{-1}^1[P_n^{(\alpha,\beta)}(x)]^2 d\mu(x)$$

$$=\frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}(=[h_n]^2).$$

Putting $\phi_n^{(\alpha,\beta)}(\theta)=h_nP_n^{(\alpha,\beta)}(\cos\theta)(\sin\frac{\theta}{2})^{\alpha+\frac{1}{2}}(\cos\frac{\theta}{2})^{\beta+\frac{1}{2}}2^{\frac{\alpha+\beta+1}{2}}$, $\{\phi_n^{(\alpha,\beta)}(\theta)\}_{n=1}^\infty$ is a complete orthonormal system with respect to the Lebesgue measure $d\theta$ on $(0, \pi)$. Moreover.

$$\phi_n^{(-\frac{1}{2},-\frac{1}{2})}(\theta)=\sqrt{\frac{2}{\pi}}\cos n\theta\,(n\geq 1),\ \phi_0^{(-\frac{1}{2},-\frac{1}{2})}(\theta)=\sqrt{\frac{1}{\pi}}.$$

The system $\{\phi_n^{(\alpha,\beta)}\}_{n=0}^{\infty}$ leads to the formal expansion of a function $f(\theta)$ on $(0,\pi)$:

$$f(\theta)\sim\sum_{n=0}^{\infty}a_n\phi_n^{(\alpha,\beta)}(\theta),$$

where $a_n=\int_0^{\pi}f(\theta)\phi_n^{(\alpha,\beta)}(\theta)d\theta$.

Definition 1. Let $\Lambda=\{\lambda_n\}_{n=0}^{\infty}$ be a bounded sequence. We define a multiplier operator $T_\Lambda(=T_\Lambda^{(\alpha,\beta)})$ for the system $\{\phi_n^{(\alpha,\beta)}\}$ by

$$T_\Lambda f(\theta)\sim\sum_{n=0}^{\infty}\lambda_n a_n\phi_n^{(\alpha,\beta)}(\theta)$$

for $f(\theta)\sim\sum_{n=0}^{\infty}a_n\phi_n^{(\alpha,\beta)}(\theta)$. Let $1<p<q<\infty$. We call Λ a (p, q)-multiplier for the system $\{\phi_n^{(\alpha,\beta)}\}_{n=0}^{\infty}$ if $\|T_\Lambda f\|_q\leq C\|f\|_p$ for f in $L^p(0,\pi)$. We denote by $M^J(p,q)$ the set of all (p, q)-multipliers for the system $\{\phi_n^{(\alpha,\beta)}\}_{n=0}^{\infty}$. Then we obtain the following:

Theorem 2. *Let* $1<p<q<\infty$.

(I) *Let* $1<p<\infty$ *and* Φ_0 *be a function in* $[-1,1]$. *Assume that* Φ_0 *is uniformly bounded in* $[-1,1]$.

(i) *Suppose* $1<p<q<2$ *or* $2\leq p<q<\infty$. *Let* $\gamma_0=(1/q-1/2)/(1/p-1/q)$ *or* $(1/2-1/p)/(1/p-1/q)$ *respectively and* n_0 *be the smallest integer such that* $n_0\geq\gamma_0$. *Then for any constants* $\alpha_1,\alpha_2,\cdots,\alpha_{n_0}$

$$\Phi(t)=\alpha_1 t+\alpha_2 t+\cdots+\alpha_{n_0}t^{n_0}+|t|^{\gamma_0+1}\Phi_0(t)$$

operates on $M^J(p,q)$.

Suppose $1<p<2\leq q<\infty$. *Let* $\gamma_1=\min\{(1/2-1/q)/(1/p-1/2),(1/p-1/2)/(1/2-1/q)\}$. *Then for any constant* α

$$\Phi(t)=\alpha t+|t|^{\gamma_1+1}\Phi_0(t)$$

operates on $M^J(p,q)$.

(II) *Let* $1<p<2\leq q<\infty$ *and* Φ *be a function* $[-1,1]$. *If* Φ *operates on* $M^J(p,q)$, *then* Φ *is of the form that* $\Phi(t)=\alpha t+|t|^{\gamma_1}\Phi_0(t)$, *where* α *is a complex number and* γ_1 *is the number given in* (I), *and* Φ_0 *is uniformly bounded in* $[-1,1]$.

It is sufficient for the prooof of Theorem 2 to show Theorem 2(II). For this proof, we prepare some lemmas.

Lamma 1([A]). *Let* $1<p<\infty$, $\alpha, \beta, \gamma, \delta \geq -\frac{1}{2}$. *Let* $T_{(\gamma, \delta)}^{(\alpha, \beta)}$ *be the transplantation operator defined by*

$$T_{(\gamma, \delta)}^{(\alpha, \beta)} f(\theta) \sim \sum_{n=0}^{\infty} a_n \phi_n^{(\gamma, \delta)}(\theta)$$

for $f(\theta) \sim \Sigma_{n=0}^{\infty} a_n \phi_n^{(\alpha, \beta)}(\theta)$. *Then,*

$$C_1 \|f\|_p \leq \|T_{(\gamma, \delta)}^{(\alpha, \beta)} f\|_p \leq C_2 \|f\|_p$$

for $f \in L^p(0, \pi)$.

By this lemma, it is sufficient to prove Theorem 2(II) for $\alpha=\beta=-\frac{1}{2}$.

Now let L_{cos}^p be the set of all $f \in L^p(\mathbf{T})$ such that f is even. Also let $M^c(p, q)$ be the set of all (p, q)-multipliers $\{\hat{T}(n)\}_{n=0}^{\infty}$ associated with T which is bounded linear operator from L_{cos}^p to L_{cos}^q such that $(Tf)^{\wedge}(n) = \hat{T}(n)\hat{f}(n)$ $(n=0, 1, 2, \cdots)$ for any $f \in L_{cos}^p$. Then by Theorem 1 it is sufficient to prove that whenever Φ is an operating function on $M^c(p, q)$, Φ is an operating function on $M^t(p, q)$.

Lemma 2. *For* $f \in L_{cos}^p$, *let* $\tilde{f}$ *be the conjugate function of* f. *Then*

$$\|f\|_p \leq \|f \pm i\tilde{f}\|_p \leq C \|f\|_p$$

for some $C>0$, *where* C *is independent of* f.

Proof. Putting $F(\theta)=f(\theta)+i\tilde{f}(\theta)$, $F(-\theta)=f(\theta)-i\tilde{f}(\theta)$ since $f \in L_{cos}^p$. Then $\|f+i\tilde{f}\|_p = \|f-i\tilde{f}(\theta)\|_p$. Hence, we obtain by Riesz's Theorem ([K]) that

$$\begin{aligned} 2\|f\|_p &= \|f+i\tilde{f}+f-i\tilde{f}\|_p \\ &\leq 2\|f+i\tilde{f}\|_p \leq C\|f\|_p \end{aligned}$$

for some $C>0$. Q. E. D.

Proposition 1. *Let* Φ *be a function on* $[-1, 1]$. *If* Φ *operates on* $M^c(p, q)$, *then* Φ *operates on* $M^t(p, q)$.

Proof. Let $\{\lambda_n\}_{n=-\infty}^{\infty} \in M^t(p,q)$ such that $\{\lambda_n\}_n \subset [-1, 1]$. By Lemma 2 and Riesz's

Theorem,

$$\| \sum_{n=0}^{\infty} \lambda_n a_n \cos n\theta \|_q \leq C \| \sum_{n=0}^{\infty} \lambda_n a_n \cos n\theta + i \sum_{n=0}^{\infty} \lambda_n a_n \sin n\theta \|_q$$

$$= C \| \sum_{n=0}^{\infty} \lambda_n a_n \exp(in\theta) \|_q \leq C \| \sum_{n=0}^{\infty} a_n \exp(in\theta) \|_p$$

$$\leq C \| \sum_{n=0}^{\infty} a_n \cos n\theta \|_p,$$

where $C>0$ is independent of $\{a_n\}_n$. Hence $\{\lambda_n\}_{n=0}^{\infty} \in M^c(p, q)$. Also $\{\lambda_{-n}\}_{n=1}^{\infty} \in M^c(p, q)$. Thus $\{\Phi(\lambda_n)\}_{n=0}^{\infty} \in M^c(p, q)$ and $\{\Phi(\lambda_{-n}\}_{n=1}^{\infty} \in M^c(p, q)$. By Lemma 2,

$$\| \sum_{n=-\infty}^{\infty} \Phi(\lambda_n) a_n \exp(in\theta) \|_q$$

$$= \| \sum_{n=-\infty}^{-1} \Phi(\lambda_n) a_n \exp(in\theta) + \sum_{n=0}^{\infty} \Phi(\lambda_n) a_n \exp(in\theta) \|_q$$

$$= \| \sum_{n=-\infty}^{-1} \Phi(\lambda_n) a_n \mathrm{con} n\theta + i \sum_{n=-\infty}^{-1} \Phi(\lambda_n) a_n \sin n\theta$$

$$+ \sum_{n=0}^{\infty} \Phi(\lambda_n) a_n \cos n\theta + i \sum_{n=0}^{\infty} \Phi(\lambda_n) a_n \sin n\theta \|_q$$

$$\leq \| \sum_{n=1}^{\infty} \Phi(\lambda_{-n}) a_{-n} \cos n\theta - i \sum_{n=1}^{\infty} \Phi(\lambda_{-n}) a_{-n} \sin n\theta \|_q$$

$$+ \| \sum_{n=0}^{\infty} \Phi(\lambda_n) a_n \cos n\theta + i \sum_{n=0}^{\infty} \Phi(\lambda_n) a_n \sin n\theta \|_q$$

$$\leq C (\| \sum_{n=1}^{\infty} a_{-n} \cos n\theta \|_p + \| \sum_{n=0}^{\infty} a_n \cos n\theta \|_p)$$

$$\leq C (\| \sum_{n=1}^{\infty} a_{-n} \exp(-in\theta) \|_p + \| \sum_{n=0}^{\infty} a_n \exp(in\theta) \|_p)$$

$$\leq C \| \sum_{n=-\infty}^{\infty} a_n \exp(in\theta) \|_p.$$

Therefore

$$\{\Phi(\lambda_n)\}_{n=-\infty}^{\infty} \in M^t(p,q).$$

Q. E. D.

3. Operating functions on multipliers for Laguerre expansions

Let $L_n^{\alpha}(x)$, $\alpha > -1$, be the Laguerre polynomial of degree n and of order α defined by

$$L_n^\alpha(x)=\frac{e^x x^{-\alpha}}{n!}\left(\frac{d}{dx}\right)^n(e^{-x}x^{n+\alpha})$$

and let

$$\mathcal{L}_n^\alpha(x)=\sqrt{\frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)}}L_n^\alpha(x)e^{-\frac{x}{2}}x^{\frac{\alpha}{2}}.$$

Then the Laguerre function system $\{\mathcal{L}_n^\alpha\}_{n=0}^\infty$ is a complete orthonormal system on the interval $(0,\infty)$ with respect to the ordinary Lebesgue measure dx. This orthogonal system leads us the formal expansion of a function $f(x)$ on $(0, \infty)$:

$$f \sim \sum_{n=0}^\infty \hat{f}(n)\mathcal{L}_n^\alpha(x),$$

where $\hat{f}(n)$ is the n-th Laguerre coefficient of order α of $f(x)$ defined by

$$\hat{f}(n)=\int_0^\infty f(x)\mathcal{L}_n^\alpha(x)\,dx.$$

For $p>1$ we denote by $L^p(0, \infty)$ the Lebesgue space of all measurable functions $f(x)$ on $(0, \infty)$ such that

$$\|f\|_p=\left\{\int_0^\infty |f(x)|^p dx\right\}^{\frac{1}{p}}<\infty.$$

Let $a=\{a(n)\}_{n=0}^\infty$ be a bounded sequence. We define a multiplier operator T_a for the system $\{\mathcal{L}_n^\alpha\}$ by

$$T_af(x) \sim \sum_{n=0}^\infty a(n)\hat{f}(n)\mathcal{L}_n^\alpha(x)$$

for a function $f(x)$ on $(0,\infty)$. Let $1<p, q<\infty$. We call a (p, q)-multiplier for the system $\{\mathcal{L}_n^\alpha\}$ if $\|T_af\|_q\leq C\|f\|_p$ for f in L^p $(0, \infty)$. We denote by $M_\alpha(p, q)$ the set of all (p, q)-multiplier for system $\{\mathcal{L}_n^\alpha\}$. We define $\|a\|_{M_\alpha(p,q)}=\|T_a\|_{p,q}$, where $\|T_a\|_{p,q}$ is the (p, q)-multiplier operator norm of T_a.

Definition 2. Let $1<p, q, r, s<\infty$. A function Φ on $[-1, 1]$ is said to operate from $M_\alpha(p, q)$ to $M_\alpha(r, s)$ (when $p=r$, $q=s$, we simply say that Φ operates on $M_\alpha(p, q)$), if $\{\Phi(a(n))\}\in M_\alpha(r, s)$ for every $a=\{a(n)\}\in M_\alpha(p, q)$ such that $\{a(n)\}\subset[-1, 1]$.

Theorem 3. *Let $\alpha\geq 0$. Also Let $\frac{4}{3}<p<2\leq q\leq p'$ and Φ be a function on $[-1, 1]$, where $\frac{1}{p}+\frac{1}{p'}=1$. Then Φ operates from $M_\alpha(p, 2)$ to $M_\alpha(p, q)$, if and only if,*

$$\Phi(t) = |t|^{\beta}\Phi_0(t),$$

where $\beta = (1/p - 1/q)/(1/p - 1/2)$ *and* Φ_0 *is a bounded function.*

Corollary. *Let* $\frac{4}{3} < p < 2$, *and* Φ *a function on* $[-1, 1]$. *Then* Φ *operates on* $M_\alpha(p, 2)$, *if and only if,*

$$|\Phi(t)| \le C|t|$$

for all $t \in [-1, 1]$ *with some* $C > 0$.

Since Kanjin[Kj] shows $M_\alpha(p, q) = M_0(p, q)$, it is sufficient to prove Theorem 3 for the case $M_0(p, q)$.

The proof of Theorem 3. Let $\Phi(t) = |t|^{\beta}\Phi_0(t)$, where β, Φ_0 are in Theorem 3. Then by the same method to the proof of [IS ; Theorem 1] we shall show that Φ operates from $M_\alpha(p,2)$ to $M_\alpha(p, q)$. We remark that if $\{a(n)\} \in M_\alpha(p, 2)$, and $\{a(n)\} \subset [-1, 1]$, then $\{a(n)\} \in M_\alpha(2, p')$ by duality. We note that a bounded sequence $a = \{a(n)\}$ is in $M_\alpha(2, 2)$. Then by Parseval's equality,

$$\| TS_a \|_{p,2} \le \| a(.) \|_{l^\infty} \| T \|_{p,2}, \tag{1}$$

where T is a multiplier operator for Laguerre expansion associated with a (p, 2)-multiplier $\{\hat{T}(n)\}$ and S_a a multiplier operator for Laguerre expansion associated with a (2, 2)-multiplier $a = \{a(n)\}$. Moreover, we obtain that

$$\| T^2 S_a \|_{p,p'} \le \| a(.) \|_{l^\infty} \| T \|_{p,2}^2. \tag{2}$$

Next for $0 \le \mathbf{Re}z \le 1$, we define $\boldsymbol{R}_z$ by

$$\boldsymbol{R}^z(n) = (\mathrm{sign}\,\hat{T}(n))\, |\hat{T}(n)|^z \Phi_0(\hat{T}(n))\, \hat{T}(n).$$

Then by (1),

$$\| \boldsymbol{R}^{iy} \|_{p,2} \le \| \Phi_0(.) \|_{l^\infty} \| T \|_{p,2}. \tag{3}$$

Also by (2),

$$\| \boldsymbol{R}^{1+iy} \|_{p,p'} \leq \| \Phi_0(.) \|_{l^\infty} \| T \|_{p,2}^2. \tag{4}$$

Thus by (3) and (4),

$$\| \boldsymbol{R}^{\theta} \|_{p,q} \leq \| \Phi_0(.) \|_{l^\infty} \| T \|_{p,2}^{1-\theta} \| T \|_{p,2}^{2\theta}, \tag{5}$$

where $\frac{1}{q}=\frac{1-\theta}{2}+\frac{\theta}{p'}$, that is, $1+\theta=\beta$. Also we obtain that

$$\begin{aligned} \boldsymbol{R}^{\theta}(n) &= \operatorname{sign}(\hat{T}(n)) \mid \hat{T}(n) \mid^{\theta} \Phi_0(\hat{T}(n)) \hat{T}(n) \\ &= \mid \hat{T}(n) \mid^{\theta+1} \Phi_0(\hat{T}(n)), \end{aligned}$$

that is, $\boldsymbol{R}^{\theta}(n)=\Phi(\hat{T}(n))$.

Hence, Φ operates from $M_\alpha(p, 2)$ to $M_\alpha(p, q)$.

Next we show the converse of the above statement. First we use the following:

Proposition 2([KS]). *Let* $\Gamma=\{\gamma_n\}_{n=-\infty}^{\infty}$ *be a bounded sequence and define* $\Gamma^+=\{\gamma_n\}_{n=0}^{\infty}$.

(1) *Let* $\alpha \geq 0$. *Suppose* $1<p\leq 2\leq q<\infty$. *If* Γ *is a* (p, q)*-multiplier for Fourier series, then* Γ^+ *is a* (p, q)*-multiplier for Laguerre expansions of order* α.

(2) *Let* $-1<\alpha<0$. *If* $(1+\frac{\alpha}{2})^{-1}<p\leq 2\leq q<-\frac{2}{\alpha}$, *then the assertion of* (1) *remains true.*

In Proposition 2, when $\Lambda=\{\lambda_n\}_{n=-\infty}^{\infty}$ is a (p, q)-multiplier for Fourier series, we denote $\Lambda^+=\{\lambda_n\}_{n=0}^{\infty}$ the (p, q)-multiplier for Laguerre expansions corresponding to Λ.

Lemma 3. *There exist* C, $\eta>0$, *such that whenever* $\| \Lambda \|_{M^t(p,2)}<\eta$, *then* $\| \Phi(\Lambda^+) \|_{M_0(p,q)} \leq C$.

Proof. We assume that the conclusion is negative. For any integer $m\geq 1$, there exists $\Lambda_m \in M^t(p, 2)$ such that $\| \Lambda_m \|_{M^t(p,2)}<\frac{1}{m^2}$ and $\| \Phi(\Lambda_m^+) \|_{M_0(p,q)}>m$. Then we may assume that the multiplier operator $T_m=T_{\Lambda_m}$ associated with Λ_m is a trigonometric polynomial. In fact, let D_N be the Dirichlet kernel of order N for Fourier series. Then by using D_N, we see that there exists an absolute constant C_p such that for any N

$$\| \Lambda_m^N \|_{M^t(p,2)} \leq \frac{C_p}{m^2},$$

where $\Lambda_m^N=\{\lambda_n\}_{n=-N}^{N}$. Also let D_N^α be the Dirichlet kernel for Laguerre expansions of order α (cf. [AW]). By using D_N^0 and $\| \Phi(\Lambda_m^+) \|_{M_0(p,q)}>m$, we see that there exists sufficiently large N such that

$$\| \{\Phi(\lambda_n)\}_{n=0}^{N} \|_{M_0(p,q)} > \frac{m}{2}$$

for $\frac{4}{3}<p<q<4$ (cf. [AS]). Hence, we may assume that the multiplier operator T_m associated with Λ_m is a trigonometric polynomial.

Now we choose natural numbers $\{n_m\}$ such that for $\Gamma_m=\{\lambda_{n+n_m}\}$ are pairwise disjoint. Putting $\Lambda=\Sigma_{m=1}^{\infty}\Lambda_m$, $\Lambda\in M^t(p, 2)$. Here, by Proposition 2, we obtain $\Phi(\Lambda^+)\in M_0(p, q)$. Then there exist $\{N_m\}$ such that

$$\Phi(S_m^+)=\Phi(T^+)(D_{n_m+N_m}^0-D_{n_m-1}^0),$$

where $\Phi(S_m^+)$ is a multiplier operator associated with $\Phi(\Gamma_m^+)$, and $\Phi(T^+)$ a multiplier operator associated with $\Phi(\Lambda^+)$. Thus we obtain that

$$\infty > \sup_N \| D_N^0 \|_{q,q} \| \Phi(\Lambda^+) \|_{M_0(p,q)}$$

$$\geq \| \Phi(\Gamma_m^+) \|_{M_0(p,q)} > m.$$

This is a contradiction. Q. E. D.

We continue the proof of Theorem 3. Now by Proposition 2, Lemma 3, and $\| D_N \|_{M^t(p,2)} \leq N^{\frac{1}{p}-\frac{1}{2}}$ (cf. [E]), we see that there exists $C>0$ such that

$$\| \{\Phi(1/CN^{(1/p-1/2)})\}_{n=0}^{N} \|_{M_0(p,q)} \leq C.$$

Then

$$|\Phi(1/(CN^{(1/p-1/2)})| \ \| D_N^0 \|_q \leq C \| D_N^0 \|_p$$

where $D_N^0=\Sigma_{n=0}^{N}\mathcal{L}_n^0$. Here, by

$$\sum_{n=1}^{N} \mathcal{L}_n^0(x)=\sqrt{N+1}\,\mathcal{L}_N^1(x)x^{-\frac{1}{2}}$$

and [T; Lemma 1. 5. 4],

$$\| D_N^0 \|_p \sim N^{\frac{1}{p'}} \qquad (\tfrac{4}{3}<p<4).$$

Thus we obtain

$$|\Phi(t)| \leq C|t|^{(1/p-1/q)/(1/p-1/2)}$$

for all $t \in [-1, 1]$. Then it is sufficient to prove that Φ is bounded. In fact, we assume that Φ is unbounded. Then there exist t_0 and $\{t_n\}$ such that $|t_n - t_0| < 4^{-n}$ and $|\Phi(t_n)| > n$ for all $n \geq 1$. Defining $E = \{2^k\}_{k=1}^{\infty}$, we have $\chi_E \in M_0(p, 2)$ by Proposition 2. Let $\Lambda = t_0\chi_E + \Sigma_{n\in E}(t_n - t_0)\chi_{\{n\}}$. Since

$$\begin{aligned} |t_n - t_0| \, \|\chi_{\{n\}}\|_{M_0(p,2)} &= |t_n - t_0| \sup\{|\hat{f}(n)|; \|f\|_p \leq 1\} \\ &\leq C4^{-n} n^{(1/p'-1/2)} \leq C4^{-n} \end{aligned}$$

(cf. [T]), we obtain $\Lambda \in M_0(p, 2)$. Hence, we have $\Phi(\Lambda) \in M_0(p, q)$. Here, by

$$|\Phi(\lambda_n)| \, \|\mathcal{L}_n^0\|_{p,q} = \|\{\Phi(\lambda_k)\}_{k=1}^{n} - \{\Phi(\lambda_k)\}_{k=1}^{n-1}\|_{M_0(p,q)},$$

where $\Lambda = \{\lambda_k\}$, we have

$$|\Phi(\lambda_n)| \, \|\mathcal{L}_n^0\|_{p,q} \leq C \|\Phi(\Lambda)\|_{M_0(p,q)}$$

with some $C > 0$. Moreover,

$$\|\mathcal{L}_n^0\|_{p,q} \geq \|\mathcal{L}_n^0\|_q / \|\mathcal{L}_n^0\|_p \sim n^{(1/q-1/p)}$$

and $\lambda_n = t_n$ $(n \in E)$. Then

$$C\|\Phi(\Lambda)\|_{M_0(p,q)} \geq n^{1-(1/p-1/q)} \quad (n \in E).$$

By $1 - (1/p - 1/q) > 0$, this is a contradiction. Q. E. D.

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