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Multiple Hermite-Fejĕr interpolation on infinite intervals

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Introduction.

Let S_n be the set of n distinct points

$$x_{1n}<\cdots<x_{kn}<\cdots<x_{nn}.$$

Let m be a positive integer. For a function $f(x)$, the Hermite-Fejĕr interpolation polynomial $L(m\ ;\ S_n\ ;\ f,x)$ of order m corresponding to S_n is defined to be the unique algebraic polynomial of degree at most mn-1 which satisfies

$$L(m\ ;\ S_n\ ;\ f,x_{kn})=f(x_{kn}),\ k=1,2,\cdots,n,$$

$$L^{(r)}(m\ ;\ S_n\ ;\ f,x_{kn})=0,\ k=1,2,\cdots,n,\ r=1,2,\cdots,m-1,$$

where $L^{(r)}(m\ ;\ S_n\ ;\ f,x_{kn})$ is the r-th derivative of $L_n(m\ ;\ S_n\ ;\ f,x)$ at x_{kn}. We note that $L(1\ ;\ S_n\ ;\ f,x)$ is the Lagrange interpolation polynomial. It is known that, for every $n=1,2,\cdots$ and $k=1,2,\cdots,n$, there exists a unique polynomial $h_k(m\ ;\ S_n\ ;\ x)$ of degree mn-1 satisfying

$$(0.1)\qquad \begin{aligned} &h_k(m\ ;\ S_n\ ;\ x_{pn})=\delta_{pk},\ p=1,2,\cdots,n,\\ &h_k^{(r)}(m\ ;\ S_n\ ;\ x_{pn})=0,\ p=1,2,\cdots,n,\ r=1,2,\cdots,m-1.\end{aligned}$$

We call $h_k(m\ ;\ S_n\ ;\ x)$ the fundamental polynomial of Hermite-Fejĕr interpolation of order m.

Let $l_k(S_n\ ;\ x)$, $k=1,2,\cdots$be the fundamental polynomial of Lagrange interpolation polynomial $L(1\,;\,S_n\,;\,f,x)$, that is, $l_k(S_n\,;\,x)=h_k(1\,;\,S_n\,;\,x)$. Then,

$$l_k(S_n\,;\,x)=\frac{W(x)}{(x-x_{kn})\,W'(x_{kn})},\quad k=1,2,\cdots,\,n, \tag{0.2}$$

where $W(x)=\Pi_{k=0}^{n}(x-x_{kn})$. We note that the fundamental polynomial $h_k(m\,;\,S_n\,;\,x)$ of Hermite-Fejẽr interpolation polynomial $L(m\,;\,S_n\,;\,f,x)$ of order m is divided by $l_k(S_n\,;\,x)^m$. We define $A_s(m\,;\,n,\,k)$, $s=0,1,\cdots,m\text{-}1$ to be the coefficients in the expression

$$h_k(m\,;\,S_n\,;\,x)=l_k(S_n\,;\,x)^m\sum_{t=0}^{m-1}A_t(m\,;\,n,k)\,(x-x_{kn})^t,\quad k=1,2,\cdots,n. \tag{0.3}$$

The interpolation polynomial $L(m;S_n\,;\,f,x)$ is written in the form

$$\begin{aligned} L(m\,;\,S_n\,;\,f,x)&=\sum_{k=1}^{n}f(x_{kn})\,h_k(m\,;\,S_n\,;\,x) \\ &=\sum_{t=0}^{m-1}\sum_{k=1}^{n}f(x_{kn})\,l_k(S_n\,;\,x)^m A_t(m\,;\,n,k)\,(x-x_{kn})^t,\quad k=1,2,\cdots,n. \end{aligned} \tag{0.4}$$

Many authors [1]-[22] have investigated the convergence and divergence of the sequence $\{L(m\,;S_n\,;f,\,x)\}_{n=1}^{\infty}$. Most of them dealt with the case $m=1$, 2 or 4. On the other hand, the second author studied in the series of works [13]-[17] the approximation by $\{L(m\,;S_n\,;\,f,x)\}_{n=1}^{\infty}$ for arbitrary m when S_n is the set of zeros of Chebyshev or Jacobi polynomial of degree n. In this paper, we shall consider the convergence problem of the sequence $\{L(2m\,;\,S_n\,;f,x)\}_{n=1}^{\infty}$ of even order $2m$ when S_n is the set of zeros of Hermite polynomial $H_n(x)$ of degree n and when S_n is the set of zeros of Laguerre polynomial $L_n^{(\alpha)}(x)$, $\alpha>-1$ of degree n.

We denote by $H_n(m\,;f,\,x)$ the polynomial $L(m\,;S_n\,;f,\,x)$ with the set S_n of zeros of $H_n(x)$, and by $L_n(\alpha\,;\,m\,;f,\,x)$ the polynomial $L(m\,;S_n\,;f,\,x)$ with the set S_n of zeros of $L_n^{(\alpha)}(x)$. Let $C(\boldsymbol{R})$ be the space of continuous functions on $\boldsymbol{R}$. We define the modulus of continuity of $f\in C(\boldsymbol{R})$ on an interval I by $\omega_I(f\,;h)=\sup\{|f(x)-f(y)|\,;x,\,y\in I,\,|x-y|\leq h\}$, $h>0$.

Our theorems are as follows:

THEOREM 1. *Let m be a positive integer. Let $I(n,\,\varepsilon)$ be an interval defined by*

$$I(n,\,\varepsilon)=[-(2n+1)^{1/2}\cos\varepsilon,\,(2n+1)^{1/2}\cos\varepsilon] \tag{0.5}$$

for $0<\varepsilon<\pi/2$ and $n=1,2,\cdots$. Then, there exist a positive constant C and a number n_0

such that, for $f\in C(\boldsymbol{R})$,

$$|H_n(2m\,;f,x)-f(x)|\leq C\{k_n(x)^{2m}e^{mx^2}\omega_{I(n,\varepsilon)}(f\,;n^{-1/2}\log n)$$
$$+\omega_{I(n,\varepsilon)}(f;|k_n(x)|n^{-1/2})\}$$

for $x\in I(n,\varepsilon)$ *and* $n\geq n_0$, *where*

$$k_n(x)=e^{-x^2/2}2^{-(2n+1)/4}(n\,!)^{-1/2}(\pi n)^{1/4}H_n(x). \tag{0.6}$$

THEOREM 2. *Let m be a positive integer and let* $\alpha\geq-\frac{1}{2}-\frac{1}{2m}$. *Let* $I(\alpha\,;n,\varepsilon)$ *be an interval defined by*

$$I(\alpha\,;n,\varepsilon)=[\varepsilon n^{-1},\ (4n+2\alpha+2)\cos^2\varepsilon] \tag{0.7}$$

for $0<\varepsilon<\pi/2$ *and* $n=1,2,\cdots$. *Then, there exist a positive constant C and a number* n_0 *such that, for* $f\in C[0,\infty)$,

$$|L_n(\alpha\,;2m\,;f,x)-f(x)|$$
$$\leq C\{k_{\alpha,n}(x)^{2m}e^{mx}(1+x^{-m})x^{-m}(\alpha+1/2)^{-1/2}\omega_{I(\alpha\,;\,n,\,\varepsilon)}(f\,;x^{-1/2}n^{-1/2}\log n)$$
$$+\omega_{I(\alpha\,;\,n,\varepsilon)}(f\,;x^{-1/2}|k_{\alpha,n}(x)|n^{-1/2})\}$$

for $x\in I(\alpha\,;n,\varepsilon)$ *and* $n\geq n_0$, *where*

$$k_{\alpha,n}(x)=e^{-x/2}x^{(2\alpha+1)/4}n^{-(2\alpha-1)/4}L_n^{(\alpha)}(x). \tag{0.8}$$

It is known that the functions $k_n(x)$ and $k_{\alpha,n}(x)$ are uniformly bounded on the interval $I(n,\varepsilon)$ and $I(\alpha\,;n,\varepsilon)$, respectively. Thus, we have the following corollary:

COROLLARY. (*i*) *If* $f\in C(\boldsymbol{R})$ *is uniformly continuous on* $\boldsymbol{R}$, *that is,* $\omega_{\boldsymbol{R}}(f\,;h)\to 0(h\to 0)$, *then for every* $b>0$,

$$\lim_{n\to\infty}\max_{-b\leq x\leq b}|H_n(2m\,;f,x)-f(x)|=0.$$

(*ii*) *If* $f\in C[0,\infty)$ *is uniformly continuous on* $[0,\infty)$, *that is,* $\omega_{[0,\infty)}(f\,;h)\to 0(h\to 0)$, *then for every a and b with* $0<a<b$,

$$\lim_{n\to\infty}\max_{a\leq x\leq b}|L_n(\alpha\,;2m\,;f,x)-f(x)|=0.$$

We shall deal with the Hermite case (Theorem 1) in §1 and the Laguerre case (Theorem 2) in §2. For the rest of the paper, if there is no confusion, we write briefly:

$$x_k = x_{kn}; \qquad h_k(m;x) = h_k(m;S_n;x); \qquad l_k(x) = l_k(S_n;x);$$

$$A_s(m;k) = A_s(m;n,k); \qquad H_n(m;x) = H_n(m;f,x);$$

$$L_n(m;x) = L_n(\alpha;m;f,x); \qquad \omega_I(h) = \omega_I(f;h).$$

§1. Hermite case.

Let S_n denote the set of zeros of the Hermite polynomial $H_n(x)$. We have $x_k = -x_{n+1-k}$, $k=1,2,\cdots,n$. If n is odd, then $x_{(n+1)/2}=0\in S_n$. We have for $k=1,2,\cdots,[n/2]$ (the integral part of $n/2$),

$$(1.1)\qquad \begin{aligned} &(k-1/2)(2n+1)^{-1/2} < x_{n-[n/2]+k} < (4k+1)(2n+1)^{-1/2} \qquad (n:\text{even}),\\ &\pi k(2n+1)^{-1/2} < x_{n-[n/2]+k} < (4k+3)(2n+1)^{-1/2} \qquad (n:\text{odd}). \end{aligned}$$

We see

$$(1.2)\qquad (2n+1)^{-1/2} \le x_{k+1} - x_k,\ k=1,2,\cdots,n-1 \ (\text{see [24, Ch. VI]}).$$

Let $x\in \boldsymbol{R}$ and let x_r be the nearest zero to x. If x is the midpoint of two zeros, then we define x_r to be the nearest zero to the left. If $k\neq r$, then there exists a positive integer $j_k = j(x,k)$ such that

$$(1.3)\qquad |x-x_k| \sim j_k n^{-1/2}$$

($a_n \sim b_n$ means $c_1 \le a_n/b_n \le c_2$ for $n=1,2,\cdots$ with constants c_1 and c_2). For k, we denote by J_k the set of such j_k's. Then, there exist a number N such that the number of elements of $J_{k'}\cap J_k$ is not exceed N for every k' and k with $k'\neq k$.

We have the differential equations

$$y^{(j+2)} - 2xy^{(j+1)} + 2(n-j)y^{(j)} = 0, \quad j=0,1,\cdots, \quad y=H_n(x).$$

Let $y_k^{(j)} = H_n^{(j)}(x_k)$. We can prove by induction on j that $y_k^{(j)}$ have the forms

$$y_k^{(j)}=\begin{cases}\{(-1)^{i-1}2^i n^{i-1}+P_i(n,\ k,\ x_k)\}y_k^{(1)} & (j=2i-1),\\ x_k\{(-1)^{i-1}i2^i n^{i-1}+Q_i(n,\ k,\ x_k)\}y_k^{(1)} & (j=2i),\end{cases}$$
$$j=0,1,\cdots,\quad k=1,2,\cdots,\ n,$$

where $P_i(n, k, x_k)$ and $Q_i(n, k, x_k)$ are polynomials of degree $i-2$ in n and of degree $2i-2$ in x_k. For $l_k(x)$ of (0.2), we see $l_k^{(j)}(x_k)=y_k^{(j+1)}/\{(j+1)y_k^{(1)}\}$. Thus, we see by induction on m that the j-th derivative $\{l_k(x)^m\}^{(j)}|_{x=x_k}$ at x_k has the form

$$\{l_k(x)^m\}^{(j)}|_{x=x_k}=(-1)^{[j/2]}a(m,\ j)x_k^{s(j+1)}n^{[j/2]}+P(j,\ n,\ k,\ x_k), \tag{1.4}$$

where $a(m, j)$ is a positive coefficient and $P(j, n, k, x_k)$ is a polynomaial of degree $[j/2]-1$ in n and of degree j-1 in x_k satisfying $|P(j, n, k, x_k)|=0(1)x_k^2 n^{[j/2]-1}$. Here, $s(i)=1$ (i:even) and$=0$ (i : odd).

We shall estimate the values of $H_n(x)$ and $H'_n(x)$. Let $0<\varepsilon<\pi/2$, and let $x=(2n+1)^{1/2}\cos\Phi$ for $\varepsilon\leq\Phi\leq\pi-\varepsilon$. Then, $x\in I(n,\varepsilon)$. By [24, Theorem 8.22.9], we have

$$e^{-x^2/2}H_n(x)=2^{(2n+1)/4}(n\,!)^{1/2}(\pi n)^{-1/4}(\sin\Phi)^{-1/2}\{\sin t(n,\ \Phi)+0(n^{-1})\}, \tag{1.5}$$

where $t(n, \Phi)=(n/2+1/4)(\sin 2\Phi-2\Phi)+3\pi/4$. Since $dt(n, \Phi)/d\Phi<0$, we see that $t(n, \Phi)$ is a decreasing function of Φ. For a non-negative integer j and $|\delta|<\pi/4$ such that $t(n, \pi-\varepsilon)\leq-j\pi\leq t(n, \varepsilon)$, we define $\Phi_j(\delta)$ and $x_j(\delta)$ by the equations

$$t(n,\ \Phi_j(\delta))=-j\pi+\delta,\ x_j(\delta)=(2n+1)^{1/2}\cos\Phi_j(\delta). \tag{1.6}$$

For $0<\delta<\pi/2$, we define the δ-neighborhoods

$$\begin{aligned}E_k(\delta)&=(x_k-\delta(2n+1)^{-1/2},\ x_k+\delta(2n+1)^{-1/2}),\\ E'_j(\delta)&=(x_j(0)-\delta(2n+1)^{-1/2},\ x_j(0)+\delta(2n+1)^{-1/2}).\end{aligned} \tag{1.7}$$

We have the following lemma.

LEMMA 1. *(i)* $|x_j(\delta)-x_j(0)|\sim|\delta|(2n+1)^{-1/2}$. *(ii) For* x *with* $x\notin\cup_{k=1}^{n}E_k(\delta)$ *and* $x\in I(n, \varepsilon)$,

$$|H_n(x)|\sim e^{x^2/2}2^{n/2}(n\,!)^{1/2}n^{-1/4}. \tag{1.8}$$

(*iii*) *For* $x \in E_k(\delta) \cap I(n, \varepsilon)$,

(1.9) $$|H'_n(x)| \sim e^{x^2/2} 2^{n/2} \{(n+1)!\}^{1/2} n^{-1/4}.$$

PROOF. (i) : We see that

$$\begin{aligned}
|\delta| &= |t(n, \Phi_j(\delta)) - t(n, \Phi_j(0))| \\
&= 4^{-1}(2n+1)|\sin 2\Phi_j(\delta) - 2\Phi_j(\delta) - \sin 2\Phi_j(0) + 2\Phi_j(0)| \\
&= 4^{-1}(2n+1)^{1/2}|x_j(\delta) - x_j(0)||\cos \Phi_j(\delta) - \cos \Phi_j(0)|^{-1} \\
&\qquad \cdot |\sin 2\Phi_j(\delta) - 2\Phi_j(\delta) - \sin 2\Phi_j(0) + 2\Phi_j(0)| \\
&= 2^{-1}(2n+1)^{1/2}|x_j(\delta) - x_j(0)| \frac{1-\cos 2\Phi}{\sin \Phi}
\end{aligned}$$

for some Φ *with* $0 < \varepsilon \leq \Phi \leq \pi - \varepsilon$. We have

$$1 \leq \frac{2 \sin \Phi}{1-\cos 2\Phi} \leq \frac{2 \sin \varepsilon}{1-\cos 2\varepsilon} = M(\varepsilon), \text{ say.}$$

Thus, we have (i).

(ii) : From (1.5),

(1.10) $$e^{-x_j(\delta')^2/2}|H_n(x_j(\delta'))|$$

$$= 2^{(2n+1)/4}(n!)^{1/2}(\pi n)^{-1/4}\{\sin x_j(\delta')\}^{-1/2}\{(-1)^j \sin \delta' + 0(n^{-1})\}.$$

Thus, if n is sufficiently large, say $n \geq n_0(\delta')$, then we see $H_n(x_j(\delta'))H_n(-\delta')) < 0$. Consequently, if we take $\delta' = (2M(\varepsilon))^{-1}\delta$, then from (i), $E'_j(\delta') \subset E_k(\delta)$ for some k. Thus, if $x \notin \cup_{k=1}^{n} E_k(\delta)$, then by (1.11) we have (1.8).

(iii) : From [24, (5.5.10)], it follws that $H'_n(x_k) = -H_{n+1}(x_k)$, $H'_{n+1}(x_k) = 0$. Thus, by (ii) we have (1.9).

Next, we shall estimate the coefficients $A_t(m\ ;\ k)$ defined by (0. 3).

LEMMA 2. (*i*) $A_0(2m\ ;\ k) = 1$ *for* $k = 1, 2, \cdots, n$.

(*ii*) $A_t(2m\ ;\ k) = b(m, t)x_k^{s(t+1)} n^{[t/2]} + Q(m, t, n, k, x_k)$,

$$k = 1, 2, \cdots, n, \quad t = 1, 2, \cdots, 2m-1,$$

where $b(m, t)$ *is a positive constant and* $Q(m. t, n, k, x_k)$ *is a polynomial of degree* $[t/$

$2]-1$ *in* n *and of degree* $t-1$ *in* x_k *satisfying* $|Q(m, t, n, k, x_k)|=0(1)x_k^2 n^{[t/2]-1}$.

PROOF. We prove by induction on t. By (0.3), (1.4) and the assumptions of induction on $t=0, 1,\cdots, j-1$, we see

$$(j\,!)A_j(m\,;)=-\sum_{t=0}^{j-1}(t\,!)A_t(m\,;k)\{l_k(x)^m\}^{(j-t)}$$

$$=-\sum_{t=0}^{j-1}(t\,!)\{b(m, t)x_k^{s(t+1)}n^{[t/2]}+Q(m, t, n, k, x_k)\}$$
$$\cdot\{(-1)^{[(j-t)/2]}a(m, j-t)x_k^{s(j-t+1)}n^{[(j-t)/2]}+P(j-t, n, k, x_k)\}.$$

Since $x^{s(t+1)+s(j-t+1)}n^{[t/2]+[(j-t)/2]}=0(1)x_k^2 n^{[j/2]-1}$($j$ even and t odd) and$=x^{s(j+1)}\ n^{[j/2]}$ (otherwise), we have the lemma.

LEMMA 3. $|x-x_k|=0(1)|k_n(x)|n^{-1/2}$ *for* $x\in E_k(\delta)$.

PROOF. Let $x\in E_k(\delta)$. We have

$$\left|\frac{k_n(x)}{x-x_k}\right|=e^{-t^2/2}n^{-(2n+1)/4}(n\,!)^{-1/2}(\pi n)^{1/4}|-tH_n(t)+H'_n(t)|$$

for some t between x and x_k. By Lemma 1, we see that the right-hand side$\sim n^{1/2}$, which completes the proof of the lemma.

LEMMA 4. *For every* $i=1,2,\cdots,2n-1$,

$$\sum_{k=1}^{n}|x_k^i|H'_n(x_k)^{-2}=0(1)2^{-n}(n\,!)^{-1}.$$

PROOF. The lemma follows from [24, Theorem 3.4.1, (15. 3. 6)].

PROOF of THEOREM 1. We denote $I(n, \varepsilon)$ by I in the proof. By the identity $\sum_{k=1}^{n} h_k(2m\,;x)=1$, we have

$$|f(x)-H_n(2m\,;x)|=0(1)\sum_{k=1}^{n}|f(x)-f(x_k)||h_k(2m\,;x)|$$
$$=0(1)\sum_{k=1}^{n}\omega_I(|x-x_k|)|h_k(2m\,;x)|$$
$$=0(1)\{\omega_I(|x-x_r|)|h_r(2m\,;x)|+\sum_{k:k\neq r}\omega_I(|x-x_k|)|h_k(2m\,;x)|\}.$$

It follows from Lemma 3 that $\omega_I(|x-x_r|)|h_r(2m\,;x)|\ =0(1)\omega_I(|k_n(x)|n^{-1/2})$. To estimate $\sum_{k:k\neq r}$, we first divide it into two parts

$$\sum_{k:k\neq r}=\sum_{k:k\neq r,x_k\in I}+\sum_{k:k\neq r,x_k\notin I}=\sum{}^{(1)}+\sum{}^{(2)}, \text{ say.}$$

By (1.3) and Lemma 2, we have

$$\begin{aligned}\sum{}^{(1)}&=0(1)\sum_{k:k\neq r,x_k\in I}\omega_I(j_kn^{-1/2})\left|\frac{H_n(x)}{H'_n(x_k)}\right|^{2m}\\&\quad\cdot(1+j_k^{-1}n^{1/2}|x-x_k|)\sum_{t=0}^{2m-1}|x-x_k|^{-2m+t}A_t(2m\,;k)\\&=0(1)\sum_{k:k\neq r,x_k\in I}\omega_I(j_kn^{-1/2})\left|\frac{H_n(x)}{H'_n(x_x)}\right|^{2m}(1+j_k^{-1}n^{1/2}|x-x_k|)\,n^m\\&\quad\cdot\{|x_k|j_k^{-1}n^{-1/2}+\sum_{t=0}^{2m-2}|x_k^{s(t+1)}|j_k^{-2m+t}n^{[t/2]-t/2}\\&\qquad+\sum_{t=0}^{2m-1}j_k^{-2m+t}n^{[t/2]-t-1}\sum_{0\leq i\leq t-1}|x_k^i|\}\\&=\sum{}_1^{(1)}+\sum{}_2^{(1)}+\sum{}_3^{(1)}, \text{ say.}\end{aligned}$$

By (1.8) and (1.9), we have

$$\begin{aligned}\sum{}_1^{(1)}&=O(1)k_n(x)^{2m}e^{mx^2}\omega_I(n^{-1/2})\,n^{-1/2}\sum_{k=1}^{n}|x_k|\,e^{-mx_k^2}\\&=O(1)k_n(x)^{2m}e^{mx^2}\omega_I(n^{-1/2})\int_0^\infty xe^{-mx^2}\,dx\\&=O(1)k_n(x)^{2m}e^{mx^2}\omega_I(n^{-1/2}).\end{aligned}$$

Similarly, we get $\sum_3^{(1)}=O(1)k_n(x)^{2m}e^{mx^2}\omega_I(n^{-1/2})$. Simple calculation leads us to

$$\begin{aligned}\sum{}_2^{(1)}&=O(1)k_n(x)^{2m}e^{mx^2}\sum_{k=1}^{n}j_k^{-2}\omega_I(j_kn^{-1/2})\,|x_k|\,e^{-mx_k^2}\\&=O(1)k_n(x)^{2m}e^{mx^2}\omega_I(n^{-1/2}\log n).\end{aligned}$$

To estimate $\sum^{(2)}$, we note that $n^{m/2}|x_k{}^{-m}|=O(1)$ for $x_k\notin I$. By Lemma 4, we have

$$\begin{aligned}\sum{}^{(2)}&=O(1)H_n(x)^{2m}n^m\omega_I(n^{-1/2})\,n^{-m/2}\sum_{k:k\neq r,x_k\notin I}|H'_n(x_k)|^{-2m}\textstyle\sum_{0\leq i\leq t-1}|x_k^{i+m}|\\&=O(1)k_n(x)^{2m}e^{mx^2}\omega_I(n^{-1/2})2^{mn}(n\,!)^m\;\{\sum_{k=1}^{n}|H'_n(x_k)|^{-2}\sum_{0\leq i\leq t-1}|x_k^{(i+m)/m}|\}^m\\&=O(1)k_n(x)^{2m}e^{mx^2}\omega_I(n^{-1/2}).\end{aligned}$$

Consequently, Theorem 1 is now established.

§2. Laguerre case.

In this section, we let S_n be the set of zeros of the Laguerre polynomial $L_n^{(\alpha)}(x)$. From [24, Theorem 6.31.3, and Problems and Exercises 35], it follows that

$$c_1k^2/n\le x_k\le c_2k^2/n,\ k=1,2,\cdots,n, \tag{2.1}$$
$$\min_{1\le k\le n-1}(x_{k+1}^{1/2}-x_k^{1/2})\sim n^{-1/2},$$

where c_1 and c_2 are positive constants independent of n and k. Let $x>0$ and let x_r be the nearest zero to x. If $x_k\neq x_m$, then there exists a positive integer $j_k=j(x,\ k)$ such that

$$|x^{1/2}-x_k^{1/2}|\sim j_kn^{-1/2}. \tag{2.2}$$

For k, we denote by J_k the set of such j_k's. Then, there exist a number N such that the number of elements of $J_{k'}\cap J_k$ is not exceed N for every k' and k with $k'\neq k$. We note

$$|x-x_k|^{-1}\sim\frac{j_k^{-1}n^{1/2}}{x^{1/2}+x_k^{1/2}}=O(1)j_k^{-1}n^{1/2}(x^{-1/2}+x_k^{-1/2}). \tag{2.3}$$

The polynomial $y=L_n^{(\alpha)}(x)$ satisfies the differential equations

$$xy^{(j+2)}+(\alpha+j+1-x)y^{(j+1)}+(n-j)y^{(j)}=0,\qquad j=0,1,\cdots.$$

The j-th derivative $y_k^{(j)}=L_n^{(\alpha)}(x_k)$ at x_k has the form

$$y_k^{(j)}=\{\sum_{\substack{s,t:t\le s,\ s+t\le j\\ t\le[j/2]}}c_{st}(j)n^tx_k^{-s}\}y_k^{(1)},$$

where $c_{st}(j)$ are some coefficients. Thus, we see

$$\{l_k(x)^m\}^{(j)}|_{x=x_k}=\sum_{\substack{s,t:t\le s,\ s-t\le j\\ t\le[j/2]}}b_{st}(j,\ m)n^tx_k^{-s}, \tag{2.4}$$

where $b_{st}(j,\ m)$ are some coefficients. By (2.4), we have

LEMMA 5. (*i*) $A_0(2m\ ;\ k)=1,\ k=1,2,\cdots,n.$

(*ii*) $A_j(2m\ ;\ k)=\sum_{\substack{s,t:t\le s,\ s+t\le j\\ t\le[j/2]}}a_{st}(j,\ m)n^tx_k^{-s},\ j=1,2,\cdots,2m-1,\ k=1,2,\cdots,n,$

where $a_{st}(j, m)$ *are some coefficients.*

We set

$$I'(\alpha\,;n,\varepsilon)=[(4n+2\alpha+2)\sin^2(\varepsilon n^{-1/2}),\ (4n+2\alpha+2)\cos^2\varepsilon]. \tag{2.5}$$

Then, we note $I'(\alpha\,;n,\varepsilon)\subset I(\alpha\,;n,\varepsilon)$. Let $0<\varepsilon<\pi/2$, and let $x=(4n+2\alpha+2)\cos^2\Phi$ for $\varepsilon\leq\Phi\leq\pi/2-\varepsilon n^{-1/2}$. Then, $x\in I'(\alpha\,;n,\varepsilon)$. From [24, Theorem 8.22.8], it follows that

$$e^{-x/2}L_n^{(\alpha)}(x)=(-1)^n(\pi\sin\Phi)^{-1/2}x^{-(2\alpha+1)/4}n^{(2\alpha-1)/4}\cdot\{\sin t(\alpha\,;n,\Phi)+O(1)(nx)^{-1/2}\}, \tag{2.6}$$

where $t(\alpha\,;n,\Phi)=\{n+(\alpha+1)/2\}(\sin 2\Phi-2\Phi)+3\pi/4$. We set

$$E_k(\delta)=\{x\,;|x^{1/2}-x_k^{1/2}|<\delta n^{-1/2}\}. \tag{2.7}$$

If we define $x_k(\delta)$ as (1.6), we see

$$|x_k^{1/2}(\delta)-x_k^{1/2}(0)|\sim|\delta|n^{-1/2}. \tag{2.8}$$

Thus, by (2.6) and (2.8) we have the following lemma.

LEMMA 6. *Let* $x\in I'(\alpha\,;n,\varepsilon)$.

(*i*) *For* x *with* $x\notin\cup_{k=1}^{n}E_k(\delta)$,

$$|L_n^{(\alpha)}(x)|\sim e^{x/2}x^{-(2\alpha+1)/4}n^{(2\alpha-1)/4}.$$

(*ii*) *For* $x\in E_k(\delta)$,

$$|(L_n^{(\alpha)}(x))'|\sim e^{x/2}x^{-(2\alpha+3)/4}n^{(2\alpha+1)/4}.$$

Moreover, we need the following lemmas.

LEMMA 7. ([24, Theorem 8. 22. 5, Theorem 8. 9. 2]).

(i)
$$|L_n^{(\alpha)}(x)| = \begin{cases} e^{x/2}x^{-(2\alpha+1)/4}n^{(2\alpha-1)/4} & (\varepsilon n^{-1} \le x \le \varepsilon), \\ n^{\alpha} & (o \le x \le \varepsilon n^{-1}). \end{cases}$$

(ii)
$$|(L_n^{(\alpha)}(x))'|_{x=x_k}| \sim e^{x_k/2}x_k^{-(2\alpha+3)/4}n^{(2\alpha+1)/4} \qquad (x_k \le \varepsilon).$$

LEMMA 8. $|x-x_k| = O(1)x^{1/2}|k_{\alpha,n}(x)|n^{-1/2}$ *for* $x \in E_k(\delta)$.

PROOF. Let $x \in E_k(\delta)$. We have

$$\left|\frac{x^{1/2}k_{\alpha,n}(x)}{x-x_k}\right| = e^{-t/2}t^{(2\alpha+1)/4}n^{-(2\alpha-1)/4} \cdot \left|\left\{-\frac{t^{1/2}}{2}+\frac{2\alpha+1}{4}t^{-1/2}\right\}L_n^{(\alpha)}(x) + t^{1/2}\{(L_n^{(\alpha)}(x)\}'|_{x=t}\}\right|$$

for some t between x and x_k. By Lemma 6, we see that the right-hand side$\sim n^{1/2}$, which completes the proof of the lemma.

LEMMA 9. ([24, Theorem 3. 4. 1,(15. 3. 5)]). *For every* $i=1,2,\cdots,2n-1$,

$$\sum_{k=1}^{n} x_k^{i-1}\{(L_n^{(\alpha)}(x))'|_{x=x_k}\}^{-2} = O(1)n^{-\alpha}.$$

PROOF of THEOREM 2. We first note that Lemma 6 with $I(\alpha\ ;\ n,\ \varepsilon)$ instead of $I'(\alpha\ ;\ n,\ \varepsilon)$ holds. We denote $I=I(\alpha\ ;\ n,\ \varepsilon)$ in the proof. By the identity $\sum_{k=1}^{n} h_k(2m\ ;\ x) = 1$, we have

$$\begin{aligned} |f(x)-L_n(\alpha\ ;2m\ ;x)| &= O(1)\sum_{k=1}^{n}|f(x)-f(x_k)||h_k(2m\ ;x)| \\ &= O(1)\sum_{k=1}^{n}\omega_I(|x-x_k|)|h_k(2m\ ;x)| \\ &= O(1)\{\omega_I(|x-x_r|)|h_r(2m\ ;x)| + \sum_{K:k\neq r}\omega_I(|x-x_k|)|h_k(2m\ ;x)|\}. \end{aligned}$$

It follows from Lemma 8 that $\omega_I(|x-x_r|)|h_r(2m\ ;\ x)| = O(1)\omega_I(x^{1/2}|k_{\alpha,n}(x)|n^{-1/2})$. To estimate $\sum_{k:k\neq r}$, we first divide it into two parts

$$\sum_{k:k\neq r} = \sum_{k:k\neq r, x_k\in I} + \sum_{k:k\neq r, x_k\notin I} = \sum{}^{(1)} + \sum{}^{(2)}, \text{ say.}$$

By (2.3), we have

$$\sum^{(1)}=O(1)\sum_{k:k\neq r,x_k\in I}\omega_I(x^{1/2}j_k n^{-1/2})\left|\frac{L_n^{(\alpha)}(x)}{(L_n^{(\alpha)})'(x_k)}\right|^{2m}(1+x^{-1/2}j_k^{-1}n^{1/2}|x-x_k|)$$
$$\cdot\{|x-x_k|^{-1}A_{2m-1}(2m;k)+\sum_{t=0}^{2m-2}|x-x_k|^{-2m+t}A_t(2m;k)\}$$

$$=\sum_1^{(1)}+\sum_2^{(1)},\text{ say.}$$

Let $\alpha\geq-1/2-1/(2m)$. By Lemma 6, Lemma 7, (2.3) and the fact $x_k^{-1}=O(1)n$, we have

$$\sum_1^{(1)}=O(1)\ k_{\alpha,n}(x)^{2m}e^{mx}x^{-m(\alpha+1/2)-1/2}\omega_I(x^{1/2}n^{-1/2})\,n^{-m+1/2}$$
$$\cdot\sum_{k:k\neq r,x_k\in I}j_k^{-1}x_k^{m(\alpha+3/2)}e^{-mx_k}\sum_{\substack{s,t:t\leq s,\ s+t\leq 2m-1\\ t\leq m-1}}n^t x_k^{-s}$$
$$=O(1)\ k_{\alpha,n}(x)^{2m}e^{mx}x^{-m(\alpha+1/2)-1/2}\omega_I(x^{1/2}n^{-1/2})\,n^{-m+1/2}$$
$$\cdot\sum_{k:k\neq r,x_k\in I}j_k^{-1}\{\sum{}'\ n^t x_k^{-\sigma}e^{-mx_k}+\sum{}''\ k^{2\sigma}n^{s+t-m(\alpha+3/2)}\},$$
$$=O(1)\ k_{\alpha,n}(x)^{2m}e^{mx}x^{-m(\alpha+1/2)-1/2}\omega_I(x^{1/2}n^{-1/2})$$
$$\cdot\{n^{-1/2}\sum_{s:0\leq\sigma}\sum_{k=1}^{n}x_k^{\sigma}e^{-mx_k}+\sum_{s:0>\sigma}\sum_{k=1}^{n}j_k^{-1}k^{2\sigma}\},$$

where $\sigma=m(\alpha+3/2)-s$ and $\sum'$ is the sum over s, t satisfying $t\leq s$, $s+t\leq 2m-1$, $t\leq m-1$, $\sigma\geq 0$, and $\sum''$ is the sum over s, t satisfying $t\leq s$, $s+t\leq 2m-1$, $t\leq m-1$, $\sigma<0$. Simple observation implies that both terms in the last $\{\cdots\}$ are bounded as $n\to\infty$. Thus, we have $\sum_1^{(1)}=O(1)\ k_{\alpha,n}(x)^{2m}e^{mx}x^{-m(\alpha+1/2)-1/2}\omega_I(x^{1/2}n^{-1/2})$.

we estimate $\sum_2^{(1)}$. We have

$$\sum_2^{(1)}=O(1)\ k_{\alpha,n}(x)^{2m}(1+x^{-m})\,e^{mx}x^{-m(\alpha+1/2)-1/2}$$
$$\cdot\sum_{k:k\neq r,x_k\in I}j_k^{-2}\omega_I(x^{1/2}j_k n^{-1/2})\,x_k^{m(\alpha+3/2)}e^{-mx_k}\sum_{p=0}^{2m-2}n^{-p/2}\{\sum{}'+\sum{}''\}n^t x_k^{-s},$$

where $\sum'$ is the sum over s, t satisfying $t\leq s$, $s+t\leq p$, $t\leq[p/2]$, $\sigma\geq 0$ and $\sum''$ is the sum over s, t satisfying $t\leq s$, $s+t\leq p$, $t\leq[p/2]$, $\sigma<0$. We use $x_k^{\sigma}e^{-mx_k}=O(1)$ for $\sigma>0$ and $x_k^{-1}=O(n)$. The sum on the right-hand side of the above identity is estimated as follows.

$$\sum_{k:k\neq r,x_k\in I}=O(1)\sum_{k\,:\,k\neq r,x_k\in I}j_k^{-2}\omega_I(x^{1/2}j_k n^{-1/2})\sum_{p=0}^{2m-2}\{\sum{}'\ n^{-p/2+t}x_k^{\sigma}e^{-mx_k}+\sum{}''\ n^{-p/2+t-\sigma}\}$$
$$=O(1)\sum_{j=1}^{n}j^{-2}\omega_I(x^{1/2}j\ n^{-1/2})=O(1)\,\omega_I(x^{1/2}n^{-1/2}\log n).$$

Lastly, we estimate $\sum^{(2)}$. We denote by $\sum^*$ the sum over k satisfying $k\neq r$, $x_k\notin I$ and by $\sum^{\#}$ the sum over s, t satisfying $t \leq s$, $s+t \leq p$, $t \leq [p/2]$. By Lemma 9 and the fact $n^m \cdot x_k^{-m} = O(1)$ for $x_k \notin I(\alpha; n, \varepsilon)$, we have

$$\sum^{(2)} = O(1)\ k_{\alpha,n}(x)^{2m}(1+x^{-m})e^{mx}x^{-m(\alpha+1/2)-1/2}\omega_I(x^{1/2}n^{-1/2}) \cdot n^{m(\alpha-1/2)}\sum{}^{*}\{L_n^{(\alpha)\prime}(x_k)\}^{-2m}\sum_{p=0}^{2m-2}(j_k^{-1}n^{1/2})^{2m-p}\sum{}^{\#}n^t x_k^{-s}.$$

We estimate the sum $\sum^{*}$. We have

$$\begin{aligned}\sum{}^{*} &= O(1)\sum{}^{*}\{L_n^{(\alpha)}(x_k)\}^{-2m}\sum_{p=0}^{2m-2}\sum{}^{\#}n^{t-p/2}x_k^{-s+m}\\ &= O(1)\sum_{k=1}^{n}x_k^{-s+m}\{L_n^{(\alpha)}(x_k)\}^{-2m}\\ &= O(1)\left(\sum_{k=1}^{n}x_k^{(-s+m)/m}\{L_n^{(\alpha)}(x_k)\}^{-2}\right)^m = O(1)\,n^{-m\alpha},\end{aligned}$$

which completes the proof of Theorem 2.

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