

Notes on vector fields and transverse fields on foliated Riemannian manifolds

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0. Introduction

On a foliated Riemannian manifold with a bundle-like metric([9]), transverse Killing (conformal, affine) fields of the foliation have been studied by F.W. Kamber and Ph. Tondeur [2, 3, 4], P. Molino [5, 6], J.S. Pak and S. Yorozu [7] and others.

In this note, we investigate the properties of the transverse Killing field(t.K.f.) and the transverse conformal field(t.c.f.). Some of our results are as follows :

THEOREM A. *Let (M, g_M, \mathcal{F}) be a Riemannian manifold with a foliation \mathcal{F} and a bundle-like metric g_M with respect to \mathcal{F} . If $Y \in V(\mathcal{F})$ is a conformal (resp. Killing) vector field on M , then $s = \pi(Y)$ is a t.c.f. (resp. t.K.f.) of \mathcal{F} .*

THEOREM B. *Let (M, g_M, \mathcal{F}) be as in Theorem A. If there exists a nonsingular t.K.f. s of \mathcal{F} , then there exists a foliation \mathcal{F}' generated by \mathcal{F} and s such that $\mathcal{F} \subset \mathcal{F}'$, $\text{codim } \mathcal{F}' = \text{codim } \mathcal{F} - 1$ and the metric g_M is bundle-like with respect to \mathcal{F}' .*

THEOREM C. *Let $\rho : (M, g_M) \rightarrow (B, g_B)$ be a Riemannian submersion with connected fibers and \mathcal{F} be the foliation on M whose leaves are fibers of the submersion ρ . If there exists a nonsingular t.K.f. of \mathcal{F} , then $\text{Pont}^{(r)}(TB) = 0$ for $r > \dim B - 1$.*

We shall be in C^∞ -category and deal only with connected and oriented manifold without boundary. We use the following convention on the range of indices : $1 \leq i, j \leq p$, $p+1 \leq \alpha, \beta \leq p+q$. The Einstein summation convention will be used.

1. Preliminaries

Let (M, g_M, \mathcal{F}) be a $p+q$ dimensional Riemannian manifold with a foliation \mathcal{F} of codimension q and a bundle-like metric g_M with respect to \mathcal{F} ([9]). We denote by ∇ the Levi-Civita connection with respect to g_M . The foliation \mathcal{F} defines an integrable subbundle E of the tangent bundle TM over M , and let Q be the normal bundle TM/E of \mathcal{F} . Let

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σ be the splitting of the exact sequence :

$$0 \longrightarrow E \longrightarrow TM \begin{array}{c} \xrightarrow{\pi} \\ \xleftarrow{\sigma} \end{array} Q \longrightarrow 0$$

with $\sigma(Q) = E^\perp$, where E^\perp denotes the orthogonal complement bundle of E in TM with respect to g_M ([2]). The metric g_M induces the metric g_Q on Q , that is,

$$g_Q(s, t) = g_M(\sigma(s), \sigma(t))$$

for any $s, t \in \Gamma(Q)$. Here $\Gamma(Q)$ denotes the set of all sections of the bundle Q . The basic adapted (local) frame $\{X_i, X_\alpha\}$ of \mathcal{F} is given by

$$X_i = \partial / \partial x^i, \quad X_\alpha = \partial / \partial x^\alpha - A_\alpha^i \partial / \partial x^i$$

in a flat chart $U(x^i, x^\alpha)$ with respect to \mathcal{F} , where A_α^i are functions on U satisfying $g_M(X_i, X_\alpha) = 0$ ([9], [10]).

Let D be the transversal Riemannian connection of \mathcal{F} , that is, $D : \Gamma(TM) \times \Gamma(Q) \longrightarrow \Gamma(Q)$ is given by

$$\begin{aligned} D_X s &= \pi([X, Y_s]) && \text{for any } X \in \Gamma(E) \\ &&& \text{for any } s \in \Gamma(Q), \pi(Y_s) = s \\ D_X s &= \pi(\nabla_X \sigma(s)) && \text{for any } X \in \Gamma(E^\perp) \\ &&& \text{for any } s \in \Gamma(Q) \end{aligned}$$

([2], [3]). We notice that D is torsion-free and metrical with respect to g_Q . The curvature R_D of D and the Ricci operator ρ_D of \mathcal{F} are defined as follows :

$$\begin{aligned} R_D(X, Y)s &= D_X D_Y s - D_Y D_X s - D_{[X, Y]}s && \text{for any } X, Y \in \Gamma(TM) \\ &&& \text{for any } s \in \Gamma(Q) \\ \rho_D(s) &= g^{\alpha\beta} R_D(\sigma(s), \pi(X_\alpha)) \pi(X_\beta) && \text{for any } s \in \Gamma(Q) \end{aligned}$$

where $(g^{\alpha\beta})$ denotes the inverse matrix of $(g_{\alpha\beta})$ with $g_{\alpha\beta} = g_M(X_\alpha, X_\beta)$ ([2], [3], [7]). Let $V(\mathcal{F})$ be the set of all infinitesimal automorphisms of \mathcal{F} , that is, $V(\mathcal{F}) = \{Y \in \Gamma(TM) \mid [Y, Z] \in \Gamma(E) \text{ for any } Z \in \Gamma(E)\}$, and we set $\bar{V}(\mathcal{F}) = \{s \in \Gamma(Q) \mid s = \pi(Y) \text{ for any } Y \in V(\mathcal{F})\}$ ([2]). Then we have some operators ([2], [3], [5], [6], [7]) :

- (i) the transverse Lie differentiation $\Theta(Y)$ with respect to $Y \in V(\mathcal{F})$, $\Theta(Y)s = \pi([Y, Y_s])$, where $\pi(Y_s) = s$,
- (ii) the transverse divergence div_D , $\text{div}_D s = g^{\alpha\beta} g_Q(D_{X_\alpha} s, \pi(X_\beta))$,
- (iii) the transverse gradient grad_D , $\text{grad}_D f = g^{\alpha\beta} X_\alpha(f) \pi(X_\beta)$,
- (iv) the Laplacian $\Delta_D = d_D d_D^* + d_D^* d_D$.

The second fundamental form α of \mathcal{F} is given by

$$\alpha(X, Y) = -(D_X \pi)(Y) \quad \text{for any } X, Y \in \Gamma(TM),$$

and the tension field τ of \mathcal{F} is given by

$$\tau = g^{ij} \alpha(X_i, X_j) (= d_D^* \pi),$$

where (g^{ij}) denotes the inverse matrix of (g_{ij}) with $g_{ij} = g_M(X_i, X_j)$ ([2], [3]). The foliation \mathcal{F} is called harmonic if $\tau=0$, and \mathcal{F} is called totally geodesic if $\alpha=0$ ([2], [3]).

2. Transverse fields

Let (M, g_M, \mathcal{F}) be as in section 1. The definitions of Killing vector field, conformal vector field, and affine vector field on M are abbreviated. In order to give the definitions of geometric transverse fields of \mathcal{F} , we have to show the following lemma :

LEMMA 2.1. *For any $X \in \Gamma(E)$, it holds that $\Theta(X)g_Q=0$ and $\Theta(X)D=0$.*

This lemma is proved by the direct calculation.

DEFINITION 2.2([2], [3], [5], [6], [7]). If $Y \in V(\mathcal{F})$ satisfies $\Theta(Y)g_Q=0$, then $s = \pi(Y)$ is called a transverse Killing field (t.K.f.) of \mathcal{F} . If $Y \in V(\mathcal{F})$ satisfies $\Theta(Y)g_Q=2f \cdot g_Q$ for some function f on M , then $s = \pi(Y)$ is called a transverse conformal field (t.c.f.) of \mathcal{F} , and $f_s=f$ is called the characteristic function of s . If $Y \in V(\mathcal{F})$ satisfies $\Theta(Y)D=0$, then $s = \pi(Y)$ is called a transverse affine field (t.a.f.) of \mathcal{F} .

Then we have

THEOREM 2.3([7]).

- (a) *If s is a t.a.f. of \mathcal{F} , then $\Delta_D s = D_{\sigma(\tau)}s + \rho_D(s)$ and $\text{div}_D s = \text{const.}$*
- (b) *If s is a t.K.f. of \mathcal{F} , then $\Delta_D s = D_{\sigma(\tau)}s + \rho_D(s)$ and $\text{div}_D s = 0$.*
- (b) *If s is a t.c.f. of \mathcal{F} , then $\Delta_D s = D_{\sigma(\tau)}s + \rho_D(s) + (1 - \frac{2}{q}) \text{grad}_D \text{div}_D s$ and $f_s = \frac{1}{q} \text{div}_D s$.*

REMARK 2.4([7]). For any $s \in \bar{V}(\mathcal{F})$, $\text{div}_D s$ is a foliated function on M .

We set

$K(\mathcal{F})$: the set of all transverse Killing fields of \mathcal{F}

$C(\mathcal{F})$: the set of all transverse conformal fields of \mathcal{F}

$A(\mathcal{F})$: the set of all transverse affine fields of \mathcal{F} .

For any $s, t \in \bar{V}(\mathcal{F})$, we define $[s, t]$ by

$$\begin{aligned} [s, t] &= \pi([Y_s, Y_t]) & \pi(Y_s) &= s \text{ with } Y_s \in V(\mathcal{F}) \\ & & \pi(Y_t) &= t \text{ with } Y_t \in V(\mathcal{F}). \end{aligned}$$

Then we have

THEOREM 2.5. *$K(\mathcal{F})$, $C(\mathcal{F})$ and $A(\mathcal{F})$ are Lie algebras with respect to the bracket defined as above.*

Now, we have

LEMMA 2.6. *For any $Y \in V(\mathcal{F})$, $u, v \in \Gamma(Q)$, it holds that $(\Theta(Y)g_Q)(u, v) = (\mathcal{L}(Y)g_M)(\sigma(u), \sigma(v))$, where $\mathcal{L}(Y)$ denotes the Lie differentiation with respect to Y .*

If f_Y is a characteristic function of a conformal vector field $Y \in V(\mathcal{F})$, then, by Lemma 2.6, f_Y is a foliated function on M . Thus we have

THEOREM 2.7. *If $Y \in V(\mathcal{F})$ is a conformal vector field on M , then $s = \pi(Y)$ is a t.c.f. of \mathcal{F} whose characteristic function is one of Y .*

COROLLARY 2.8. *If $Y \in V(\mathcal{F})$ is a Killing vector field on M , then $s = \pi(Y)$ is a t.K.f. of \mathcal{F} .*

We denote by Y_E (resp. Y_{E^\perp}) the E (resp. E^\perp)-component of a vector field Y on M . The following results are easily proved

PROPOSITION 2.9. *Let $Y \in V(\mathcal{F})$ be a conformal vector field on M . Then Y_E is a Killing vector field on M if and only if Y_{E^\perp} is a conformal vector field on M whose characteristic function is one of Y .*

PROPOSITION 2.10. *Let $X \in \Gamma(E)$ be a Killing vector field on M and $Z \in V(\mathcal{F}) \cap \Gamma(E^\perp)$ a conformal vector field on M . Then $Y = X + Z$ is a conformal vector field on M whose characteristic function is one of Z .*

Next, by Jacobi identity, we have that $(\Theta(Y)D)_X t = 0$ for any $Y \in V(\mathcal{F})$, $X \in \Gamma(E)$, and $t \in \Gamma(Q)$. Moreover, we have

PROPOSITION 2.11. *For any $Y \in V(\mathcal{F}) \cap \Gamma(E^\perp)$, $V \in \Gamma(E^\perp)$, and $t \in \Gamma(Q)$, it holds that $(\Theta(Y)D)_V t = \pi((\mathcal{L}_Y \nabla)_V \sigma(t)) + \pi(\nabla_{\sigma(t)}(\mathcal{L}_Y V)_E + \nabla_V(\mathcal{L}_Y \sigma(t))_E)$.*

The local expression of $\pi(\nabla_{\sigma(t)}(\mathcal{L}_Y V)_E)$ is given by $-\frac{1}{2}Y^\alpha V^\beta Z^\gamma B_{\alpha\beta}^k B_{\gamma\epsilon}^h g^{\gamma\epsilon} g_{kh} \pi(X_\gamma)$, where $Y = Y^\alpha X_\alpha$, $V = V^\beta X_\beta$, $\sigma(t) = Z^\gamma X_\gamma$, and $[X_\alpha, X_\beta] = B_{\alpha\beta}^k X_k$. Thus we have

PROPOSITION 2.12. *Suppose that E^\perp is an integrable subbundle of TM . If $Y \in V(\mathcal{F}) \cap \Gamma(E^\perp)$ is an affine vector field on M , then $s = \pi(Y)$ is a t.a.f. of \mathcal{F} .*

REMARK 2.13. For a t.K.f. (resp. t.c.f., t.a.f.) s of \mathcal{F} , $\sigma(s)$ (or Y with $\pi(Y) = s$) is not always a Killing (resp. conformal, affine) vector field on M .

3. Nonsingular transverse Killing fields

Let (M, g_M, \mathcal{F}) be as in section 2. We consider a nonsingular t.K.f. s of \mathcal{F} with $s = \pi(Y) \in \overline{V}(\mathcal{F})$. Let E' be a subbundle of TM generated by E and Y . Then E' is an integrable subbundle. Thus we have a new foliation \mathcal{F}' defined by E' , and $\text{codim } \mathcal{F}' = q-1$. For any $u, v \in \Gamma(Q)$, the equality $(\Theta(Y)g_Q)(u, v) = 0$ implies $g_M(\nabla_{\sigma(u)} Y, \sigma(v)) + g_M(\sigma(u), \nabla_{\sigma(v)} Y) = 0$. And we easily have that $g_M(\nabla_{\sigma(u)} Y_E, \sigma(v)) + g_M(\sigma(u), \nabla_{\sigma(v)} Y_E) = 0$, where Y_E denotes the E -component of Y . Thus we have that $g_M(\nabla_{\sigma(u)} \sigma(s), \sigma(v)) + g_M(\sigma(u), \nabla_{\sigma(v)} \sigma(s)) = 0$. Here $\sigma(s)$ is the E^\perp -component of Y . Therefore, by Theorem 3.1

in[10], we have

THEOREM 3.1([1]). *Let (M, g_M, \mathcal{F}) be as above. If there exists a nonsingular t.K.f. of \mathcal{F} , then there exists a foliation \mathcal{F}' such that $\mathcal{F} \subset \mathcal{F}'$ and $\text{codim } \mathcal{F}' = \text{codim } \mathcal{F} - 1$, and the metric g_M is bundle-like with respect to \mathcal{F}'*

Let $\rho : (M, g_M) \rightarrow (B, g_B)$ be a Riemannian submersion with connected fibers. Then we have a foliation \mathcal{F} on M whose leaves are fibers $\rho^{-1}(b)$ with $b \in B$. If $s = \pi(Y) \in \overline{V}(\mathcal{F})$ is a nonsingular t.K.f. of \mathcal{F} , then $\rho_*(Y)$ is a nonsingular Killing vector field on B and defines an integrable subbundle $\{\rho_*(Y)\}$ of the tangent bundle TB over B . Then, by Pasternack's theorem[8], we have

THEOREM 3.2([1]). *Let $\rho : (M, g_M) \rightarrow (B, g_B)$ be a Riemannian submersion with connected fibers and \mathcal{F} be the foliation on M whose leaves are fibers of the submersion ρ . If there exists a nonsingular t.K.f. of \mathcal{F} , then $\text{Pont}^{(r)}(TB) = 0$ for $r > \dim B - 1$, where $\text{Pont}^{(*)}(TB)$ denotes the characteristic ring generated by the real Pontryagin classes of TB .*

PROOF. Let $s = \pi(Y) \in \overline{V}(\mathcal{F})$ be a nonsingular t.K.f. of \mathcal{F} . Then $\rho_*(Y)$ defines an integrable subbundle $\{\rho_*(Y)\}$ of TB and a foliation \mathcal{F}_B on B . The codimension of \mathcal{F}_B is equal to $\dim B - 1$. Since $\rho_*(Y)$ is a Killing vector field on B , g_B is a bundle-like metric with respect to \mathcal{F}_B . Thus, by Pasternack's vanishing theorem, we have that $\text{Pont}^{(r)}(TB) = 0$ for $r > \dim B - 1$.

4. Harmonic foliations and transverse Killing fields

Let (M, g_M, \mathcal{F}) be as in section 1. We have the following theorems :

THEOREM 4.1([7]). *Suppose that M is compact and \mathcal{F} is harmonic. Then it holds that $\int_M \text{div}_D s \, dM = 0$ for any $s \in \Gamma(Q)$.*

THEOREM 4.2([7]). *Let (M, g_M, \mathcal{F}) be as in Theorem 4.1. If $s \in \overline{V}(\mathcal{F})$ satisfies $\Delta_D s = \rho_D(s)$ and $\text{div}_D s = 0$, then s is a t.K.f. of \mathcal{F} .*

Then we have

THEOREM 4.3([7]). *Let (M, g_M, \mathcal{F}) be as in Theorem 4.1. Then every t.a.f. of \mathcal{F} is a t.K.f. of \mathcal{F} .*

THEOREM 4.4 *Let (M, g_M, \mathcal{F}) be as in Theorem 4.1. Suppose that \mathcal{F} has dense leaves. Then every t.c.f. of \mathcal{F} is a t.K.f. of \mathcal{F} .*

PROOF. Let s be a t.c.f. of \mathcal{F} . By Theorem 2.3 and Remark 2.4, $\text{div}_D s = q \cdot f$, is a foliated function on M . Since \mathcal{F} has dense leaves, $\text{div}_D s$ is a constant function on M . Thus, by Theorem 4.1, $\text{div}_D s = 0$. Therefore, Theorem 4.2 implies that s is a t.K.f. of \mathcal{F} .

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