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A local property of absolutely convergent Laguerre polynomial series

Yuichi KANJIN[1)]

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1. Introduction

Let $L_n^\alpha(x)$ be the Laguerre polynomial of degree n and order $\alpha>-1$ defined by

$$L_n^\alpha(x) = \frac{e^x x^{-\alpha}}{n!}\left(\frac{d}{dx}\right)^n[e^{-x}x^{n+\alpha}].$$

These polynomials are orthogonal on $(0, \infty)$ with respect to the weight function $e^{-x}x^\alpha$. Denote by $R_n^\alpha(x)$ the normalized Laguerre polynomial so that $R_n^\alpha(x)=\dfrac{\Gamma(\alpha+1)n!}{\Gamma(n+\alpha+1)}L_n^\alpha(x)$. Then $R_n^\alpha(x)$ satisfies that $R_n^\alpha(0)=1$ and, for $\alpha\geq-1/2$,

(1) $$|R_n^\alpha(x)e^{-x}|\leq 1,\quad x\geq 0 \qquad [1, (5.9)].$$

Let $\{a_n\}_{n=0}^\infty$ be an absolutely convergent sequence ; $\sum_{n=0}^\infty |a_n|<\infty$. Then, by (1), the series

$$\sum_{n=0}^\infty a_nR_n^\alpha(x)$$

converges uniformly on every compact interval in $[0, \infty)$ for $\alpha\geq-1/2$, and so the series converges to a continuous function f on $[0, \infty)$; $f(x)=\sum_{n=0}^\infty a_nR_n^\alpha(x)$. We shall prove the following theorem on a local property of such functions f.

THEOREM. *Let $\alpha\geq-1/2$. Let θ be an infinitely differentiable function with compact support in* $(0, \infty)$.

(i) *If $f(x)=\sum_{n=0}^\infty a_nR_n^\alpha(x)$, $\sum_{n=0}^\infty |a_n|<\infty$, then there is a function $h\in L^1(R, (1+|t|)^{\alpha+1/2}dt)$ such that*

$$\theta(x)f(x)=\mathfrak{F}h((2x)^{1/2}),\quad x\geq 0,$$

where $\mathfrak{F}h$ is the Fourier transform of h.

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1) Department of Mathematics, College of Liberal Arts, Kanazawa University, Kanazawa 920, Japan.

(ii) *If* $h \in L^1(R, (1+|t|)^{\alpha+1/2}dt)$, *then there is a function* $f(x)=\sum_{n=0}^{\infty} a_n R_n^{\alpha}(x)$ *with* $\sum_{n=0}^{\infty} |a_n| < \infty$ *such that*

$$\theta(x)\mathcal{F}h(x)=f(x^2/2), \quad x \geq 0.$$

By using the theorem, we shall obtain some properties (spectral synthesis, operating functions) of the algebra $A^{(\alpha,\tau)}$, $\alpha \geq -1/2$, $\tau \geq 2$ of absolutely convergent Laguerre polynomial series introduced by Askey and Gasper [1, §§4, 5];

$$A^{(\alpha,\tau)} = \{f \text{ on } [0, \infty) ; f(x)=\sum_{n=0}^{\infty} a_n R_n^{\alpha}(x)e^{-\tau x}, \|f\| = \sum_{n=0}^{\infty} |a_n| < \infty\} .$$

They showed that, if $\alpha \geq -1/2$ and $\tau \geq 2$ or if $\alpha \geq \alpha_0 = (-5+(17)^{1/2})/2$ and $\tau \geq 1$, then $A^{(\alpha,\tau)}$ is an algebra with usual multiplication of functions. Kanjin [5] and [6] studied some properties of $A^{(\alpha,\tau)}$, e.g., maximal ideal space, Helson sets and spectral synthesis of the set of one point.

Some results related to the theorem will be found in Cazzaniga and Meaney [2], and Gatesoupe [3]. Cazzaniga and Meaney studied the local property of a function with absolutely convergent Jacobi polynomial series and investigated the algebra of absolutely convergent Jacobi polynomial series.

For a later convenience, we state here Gatesoupe's result. For a function F on $[0, \infty)$, we define a function $(F)_d$ on $\boldsymbol{R}^d$ by $(F)_d(t)=F(|t|)$, $t \in \boldsymbol{R}^d$. Then the mapping $F \to (F)_d$ is a one to one correspondence between functions on $[0, \infty)$ and radial functions on $\boldsymbol{R}^d$. Let $L^1_{rad}(\boldsymbol{R}^d)$ be the space of radial functions in $L^1(\boldsymbol{R}^d)$. We note that if h is a radial function on $\boldsymbol{R}^d$, then the Fourier transform $\mathcal{F}h(\xi)=(2\pi)^{-d/2}\int_{\boldsymbol{R}^d} h(t)e^{-i\xi t}\,dt$ is also radial. Let θ be the function in the theorem.

(G) If $(\varphi)_d \in \mathcal{F}L^1_{rad}(\boldsymbol{R}^d)$, then $\theta\varphi \in \mathcal{F}L^1(\boldsymbol{R}, (1+|t|)^{(d-1)/2}dt)$. Conversely, If $\psi \in \mathcal{F}L^1(\boldsymbol{R}, (1+|t|)^{(d-1)/2}dt)$, then $(\theta\psi)_d \in \mathcal{F}L^1_{rad}(\boldsymbol{R}^d)$.

2. Proof of the theorem

First, we shall give a simple relation between absolutely convergent Laguerre polynomial series and Hankel transforms. For a function $g(y) \in L^1((0, \infty), y^{\alpha}dy)$, the Hankel transform $\mathcal{K}^{\alpha}g(x)$, $x \geq 0$ of order $\alpha \geq -1/2$ is defined by

$$\mathcal{K}^{\alpha}g(x)=\int_0^{\infty} g(y)\frac{J_{\alpha}(2(xy)^{1/2})}{(xy)^{\alpha/2}}y^{\alpha}\,dy,$$

where J_{α} is the Bessel function of the first kind of order α.

LEMMA. *Let* $\alpha \geq -1/2$.

(i) *Let* $f(x)=\sum_{n=0}^{\infty} a_n R_n^\alpha(x)$, $\sum_{n=0}^{\infty} |a_n| \leq \infty$. Define

$$g(y)=\sum_{n=0}^{\infty} \frac{\Gamma(\alpha+1)}{\Gamma(n+\alpha+1)} a_n y^n e^{-y}.$$

Then
$$\int_0^\infty |g(y)| y^\alpha \, dy \leq \Gamma(\alpha+1) \sum_{n=0}^{\infty} |a_n|,$$

and
$$f(x)=\mathcal{K}^\alpha g(x) e^x, \quad x \geq 0.$$

(ii) *Let* $g \in L^1((0,\infty), y^\alpha dy)$. *Define* $f(x)=\sum_{n=0}^{\infty} a_n R_n^\alpha(x)$,

where
$$a_n=\frac{1}{\Gamma(\alpha+1)n!} \int_0^\infty y^n g(y) e^{-y} y^\alpha \, dy.$$

Then
$$\Gamma(\alpha+1) \sum_{n=0}^{\infty} |a_n| \leq \int_0^\infty |g(y)| y^\alpha \, dy,$$

and
$$\mathcal{K}^\alpha g(x)=f(x), \quad x \geq 0.$$

PROOF. (i): By the fact $\Gamma(n+\alpha+1)=\int_0^\infty y^{n+\alpha} e^{-y} \, dy$, we have that

$$\sum_{n=0}^{\infty} \int_0^\infty \left| \frac{\Gamma(\alpha+1)}{\Gamma(n+\alpha+1)} a_n y^n e^{-y} \right| y^\alpha \, dy \leq \Gamma(\alpha+1) \sum_{n=0}^{\infty} |a_n|, \tag{2}$$

and this imply $\int_0^\infty |g(y)| y^\alpha \, dy \leq \Gamma(\alpha+1) \sum_{n=0}^{\infty} |a_n|$. From the formula

$$L_n^\alpha(x) e^{-x} x^{\alpha/2}=\frac{1}{n!} \int_0^\infty y^{n+\alpha/2} e^{-y} J_\alpha(2(xy)^{1/2}) \, dy \qquad [10, (5.4.1)],$$

it follows that
$$\begin{aligned} f(x)&=\sum_{n=0}^{\infty} a_n R_n^\alpha(x) \\ &=\sum_{n=0}^{\infty} \left\{ \int_0^\infty \frac{\Gamma(\alpha+1)}{\Gamma(n+\alpha+1)} a_n y^{n+\alpha} e^{-y} \frac{J_\alpha(2(xy)^{1/2})}{(xy)^{\alpha/2}} y^\alpha \, dy \right\} e^x. \end{aligned}$$

For $\alpha \geq -1/2$, it holds that $J_\alpha(z)/z^\alpha = O(1)$ as $z \to \infty$ or $z \to +0$. This and (2) enable us to use the Lebesgue convergence theorem, and we have

$$\begin{aligned} f(x)&=\left\{ \int_0^\infty \left\{ \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+1)}{\Gamma(n+\alpha+1)} a_n y^n e^{-y} \right\} \frac{J_\alpha(2(xy)^{1/2})}{(xy)^{\alpha/2}} y^\alpha \, dy \right\} e^x \\ &=\mathcal{K}^\alpha g(x) \, e^x, \; x \geq 0. \end{aligned}$$

(ii): We have

$$\begin{aligned} \Gamma(\alpha+1) \sum_{n=0}^{\infty} |a_n| &\leq \sum_{n=0}^{\infty} \int_0^\infty \frac{y^n}{n!} |g(y)| e^{-y} y^\alpha \, dy \\ &=\int_0^\infty \left(\sum_{n=0}^{\infty} \frac{y^n}{n!} \right) e^{-y} |g(y)| y^\alpha \, dy = \int_0^\infty |g(y)| y^\alpha \, dy. \end{aligned}$$

Now, by using the formula

$$e^{y}\frac{J_{\alpha}(2(xy)^{1/2})}{(xy)^{\alpha/2}}=\sum_{n=0}^{\infty}\frac{y^{n}}{\Gamma(n+\alpha+1)}L_{n}^{\alpha}(x) \qquad [10, (5.1.16)]$$

$$=\frac{1}{\Gamma(\alpha+1)}\sum_{n=0}^{\infty}\frac{y^{n}}{n!}R_{n}^{\alpha}(x),$$

we have

$$\mathcal{K}^{\alpha}g(x)=\int_{0}^{\infty}g(y)e^{y}\frac{J_{\alpha}(2(xy)^{1/2})}{(xy)^{\alpha/2}}e^{-y}y^{\alpha}\,dy$$

$$=\frac{1}{\Gamma(\alpha+1)}\int_{0}^{\infty}g(y)\sum_{n=0}^{\infty}\frac{y^{n}}{n!}R_{n}^{\alpha}(x)\,e^{-y}y^{\alpha}\,dy.$$

Because of (1) and the fact $\sum_{n=0}^{\infty}\int_{0}^{\infty}\frac{y^{n}}{n!}|g(y)|e^{-y}y^{\alpha}\,dy<\infty$, we can apply the Lebesgue convergence theorem to the last integral, and we have $\mathcal{K}^{\alpha}g(x)=\sum_{n=0}^{\infty}a_{n}R_{n}^{\alpha}(x)=f(x)$, $x\geq 0$.

Next, we shall prove the theorem by combining the lemma, (G) and Bochner's formula. We define another Hankel transform $\mathcal{J}^{\alpha}g$, $\alpha\geq -1/2$ of a function $g\in L^{1}((0,\infty), y^{2\alpha+1}dy)$ by

$$\mathcal{J}^{\alpha}g(x)=\int_{0}^{\infty}g(y)\frac{J_{\alpha}(xy)}{(xy)^{\alpha}}y^{2\alpha+1}\,dy,\ x\geq 0.$$

Then Bochner's formula states that

$$\mathcal{F}((g)_{d})=(\mathcal{J}^{(d-2)/2}g)_{d}, \qquad g\in L^{1}((0,\infty), y^{d-1}dy).$$

We notice that the mapping $g\to(g)_{d}$ gives a norm equivalent isomorphism of $L^{1}((0,\infty), y^{d-1}dy)$ onto $L^{1}_{rad}(\boldsymbol{R}^{d})$. We shall use the notations

$$g_{\sigma}(x)=g((2x)^{1/2}),\quad g^{\sigma}(x)=g(x^{2}/2),\quad x\geq 0.$$

Then the mappings $g\to g_{\sigma}$ and $g\to g^{\sigma}$ are each other's inverses. Also, the mapping $g\to g_{\sigma}$ is a norm equivalent isomorphism of $L^{1}((0,\infty), y^{2\alpha+1}dy)$ onto $L^{1}((0,\infty), y^{\alpha}dy)$, and

$$\mathcal{K}^{\alpha}(g_{\sigma})=(\mathcal{J}^{\alpha}g)_{\sigma}, \qquad g\in L^{1}((0,\infty), y^{2\alpha+1}dy).$$

Let $T_{\theta}^{(\alpha)}$ and $S_{\theta}^{(\alpha)}$ be operators defined by

$$\mathcal{F}(T_{\theta}^{(\alpha)}g)=\theta\cdot\mathcal{J}^{\alpha}g, \qquad g\in L^{1}((0,\infty), y^{2\alpha+1}dy)$$

and $\quad \mathcal{J}^{\alpha}(S_{\theta}^{(\alpha)}h)=\theta\cdot\mathcal{F}h, \qquad h\in L^{1}(\boldsymbol{R}, (1+|t|)^{\alpha+1/2}dt)$, respectively.

Then it follows from (G) that the following (G_{α}) holds for $\alpha=(d-2)/2$, $d=1, 2, 3, \ldots$.

(G_{α}) $T_{\theta}^{(\alpha)}$ is a bounded operator of $L^{1}((0,\infty), y^{2\alpha+1}dy)$ to $L^{1}(\boldsymbol{R}, (1+|t|)^{\alpha+1/2}dt)$.

$S_{\theta}^{(\alpha)}$ is a bounded operator of $L^{1}(\boldsymbol{R}, (1+|t|)^{\alpha+1/2}dt)$ to $L^{1}((0,\infty), y^{2\alpha+1}dy)$.

We define $\quad V_{\alpha}(g)(t)=T_{\theta}^{(\alpha)}(g(y)y^{2\alpha+1})(t)\cdot(1+|t|)^{\alpha+1/2}$,

$$U_\alpha(h)(y)=S_\theta^{(\alpha)}(h(t)(1+|t|)^{\alpha+1/2})(y)\cdot y^{2\alpha+1},$$

and apply the complex interpolation theorem for the analytic families of operators $\{V_\alpha\}$ and $\{U_\alpha\}$ between $\alpha=(d-2)/2$ and $\alpha=(d-1)/2$. Then (G_α) holds for every $\alpha\geq-1/2$.

Let $f(x)=\sum_{n=0}^\infty a_n R_n^\alpha(x)$, $\sum_{n=0}^\infty |a_n|\leq\infty$. By Lemma (i), we have

$$\theta(x)f(x)=\theta(x)\mathcal{K}^\alpha g(x)e^x,\quad x\geq 0$$

for some $g\in L^1((0,\infty), y^\alpha dy)$. We put $k(x)=\theta(x)e^x$, $x\geq 0$ and $\tau=k^\sigma$. Then τ is infinitely differentiable and compact support in $(0,\infty)$. By (G_α), we have that

$$\theta f=\{\tau\cdot\mathcal{J}^\alpha g^\sigma\}_\sigma=\{\mathcal{F}(T_\tau^{(\alpha)}g^\sigma)\}_\sigma,\ T_\tau^{(\alpha)}g^\sigma\in L^1(\boldsymbol{R}, (1+|t|)^{\alpha+1/2}dt).$$

This complete the proof of Theorem (i).

By (G_α), we have that, for $h\in L^1(\boldsymbol{R}, (1+|t|)^{\alpha+1/2}dt)$,

$$\theta\cdot\mathcal{F}h=\mathcal{J}^\alpha(S_\theta^{(\alpha)}h)=[\mathcal{K}^\alpha((S_\theta^{(\alpha)}h)_\sigma)]^\sigma,\quad (S_\theta^{(\alpha)}h)_\sigma\in L^1((0,\infty), y^\alpha dy).$$

It follows from Lemma (ii) that $\mathcal{K}^\alpha((S_\theta^{(\alpha)}h)_\sigma)=f$ for some $f(x)=\sum_{n=0}^\infty a_nR_n^\alpha(x)$ with $\sum_{n=0}^\infty |a_n|\leq\infty$, and this completes the proof of Theorem (ii).

3. Some properties of the algebra $A^{(\alpha,\tau)}$

Let $\mathcal{F}L^1(\boldsymbol{R}, (1+|t|)^{\alpha+1/2}\,dt)$ be the algebra of Fourier transforms of functions $L^1(\boldsymbol{R}, (1+|t|)^{\alpha+1/2}\,dt)$. Let

$$A_\alpha=\ \{f\ ;f(\theta)=\sum_{n=0}^\infty a_ne^{in\theta},\ \|f\|=\sum_{n=0}^\infty|a_n|\ (1+|n|)^{\alpha+1/2}<\infty\}\ .$$

Then A_α is locally coincident with $\mathcal{F}L^1(\boldsymbol{R}, (1+|t|)^{\alpha+1/2}\,dt)$ and with $(A^{(\alpha,\tau)})^\sigma$ from the theorem, where $(A^{(\alpha,\tau)})^\sigma=\ \{f^\sigma\ ;f\in A^{(\alpha,\tau)}\}$. This means that local properties of A_α and those of $A^{(\alpha,\tau)}$ are same. We state here two local properties of $A^{(\alpha,\tau)}$ transferred from A_α.

(I) (cf. [5, Theorem 2], [6]) *If $\alpha\geq 1/2$ and $\tau\geq 2$, then the singleton $\{x_0\}$, $x_0>0$ is not a set of spectral synthesis for $A^{(\alpha,\tau)}$.*

This is induced from the fact that the sets $\{f\in A_\alpha\ ;f(\theta_0)=0\}$ and $\{f\in A_\alpha\ ;f(\theta_0)=f'(\theta_0)=0\}$ are distinct closed dieals in A_α for $\alpha\geq 1/2$.

(II) *If $\alpha>-1/2$ and $\tau\geq 2$, then a closed interval in $(0,\infty)$ is not a set of analyticity for $A^{(\alpha,\tau)}$.*

For this, see [4, p. 80], [7], [8]. Also, refer to [2, §3] and [9].

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