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The Laplace-Beltrami operator on a Riemannian manifold with a Clairaut foliation

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0. Let $(M, g_M, \mathcal{F})$ be a $p+q$ dimensional Riemannian manifold with a Clairaut foliation $\mathcal{F}$ of codimension q and a bundle-like metric g_M with respect to $\mathcal{F}$ ([2]). Suppose that the leaf space $M/\mathcal{F}$ becomes a Riemannian manifold (B, g_B). Then the projection $\rho : (M, g_M) \longrightarrow (B, g_B)$ is a Riemannian submersion which is called a Clairaut submersion([1]). Let $\square$ (resp. $\square_B$) be the Laplace-Beltrami operator acting on $C^\infty(M)$ (resp. $C^\infty(B)$). Then the following relation holds

$$\square(h \circ \rho) = (\square_B h) \circ \rho + ((\rho_* H)h) \circ \rho \tag{\#}$$

for any $h \in C^\infty(B)$, where $\circ$ denotes the composition of mappings and H is the mean curvature vector field of $\mathcal{F}$ on M.

In this note, our aim is to show the above equality (#).

A warped product manifold is an example of Clairaut foliated manifolds as above.

1. We shall be in C^∞-category, manifolds are connected, and foliations are transversally orientable. Let $(M, g_M, \mathcal{F})$ be a $p+q$ dimensional Riemannian manifold with a foliation $\mathcal{F}$ of codimension q and a bundle-like metric g_M with respect to $\mathcal{F}$ ([4],[7]). Then the exterior derivative operator d has a decomposition : $d = d' + d'' + d'''$, and the formal adjoint operator δ of d has the induced decomposition : $\delta = \delta' + \delta'' + \delta'''$([4],[5],[6]). The Laplace-Beltrami operator $\square$ on M is an operator acting on $C^\infty(M)$ given by $\square = \delta d$, where $C^\infty(M)$ denotes the set of all functions on M. We also consider another operators acting on $C^\infty(M)$, that is, $\square' = \delta' d'$ and $\square'' = \delta'' d''$. Both $\square'$ and $\square''$ are not elliptic([6]).

In each flat chart $U(x^i, x^\alpha)$, $\{X_i, X_\alpha\} = \{\partial/\partial x^i, \partial/\partial x^\alpha - \sum A^j_\alpha\ \partial/\partial x^j\}$ is called the basic adapted frame to $\mathcal{F}$ ([4],[7]). Here functions A^j_α are chosen so as to be $g_M(X_i, X_\alpha) = 0$, and $1 \leq i, j \leq p$, $p+1 \leq \alpha, \beta \leq p+q$. Let ∇ be the Levi-Civita connection with

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respect to g_M. Then the mean curvature field H of $\mathcal{F}$ is a vector field on M whose local expression is given by

$$H=\sum g^{\alpha\beta}\, g_M(\sum g^{ij}\,\nabla_{X_i} X_j,\, X_\alpha)\cdot X_\beta \;(=(\sum g^{ij}\,\nabla_{X_i} X_j)^{\perp}),$$

where $(g^{\alpha\beta})$ and (g^{ij}) are inverse matrices of $(g_{\alpha\beta})=(g_M(X_\alpha, X_\beta))$ and $(g_{ij})=(g_M(X_i, X_j))$ respectively. The foliation $\mathcal{F}$ is called harmonic (or, minimal) if $H=0$. Since g_M is bundle-like with respect to $\mathcal{F}$, we have a decomposition of $\square$:

$$\square=\square'+\square''_0+H$$

([5]). We notice that $\square=\square'+\square''$, and a property of the operator $\square''_0$ will be shown in Lemma 2.

Let $BC^\infty(M)$ be the set of all basic (or, foliated) functions on M, that is, $BC^\infty(M)=\{f\in C^\infty(M) \mid d'f=0\}$. We notice that a foliated function is constant on each leaf of $\mathcal{F}$ ([4], [6]). Let $BX(M)$ be the set of all basic vector fields on M, that is, $BX(M)=\{X \mid X \underset{loc.}{=} \sum \xi^\alpha(x^\beta)X_\alpha\}$ ([4], [6]).

2. Let $(M, g_M, \mathcal{F})$ be as in section 1. The foliation $\mathcal{F}$ is called a Clairaut foliation if there exists a positive valued function $\mathbf{r}: M\longrightarrow \mathbf{R}$ such that, for any geodesic $\gamma(t)$ parametrized proportionally to arc-legth, $\mathbf{r}\cdot\sin\theta=$constant, where $\theta=\theta(t)$ is defined by $\cos\theta(t)=\|\dot\gamma(t)^{\perp}\|\cdot\|\dot\gamma(t)\|^{-1}$ $(0\leq\theta(t)\leq\pi/2)$ ([1],[2],[7]). The function $\mathbf{r}$ is called the girth of $\mathcal{F}$. If $\mathcal{F}$ is a Clairaut foliation with the girth $r=e^f$, then $f\in BC^\infty(M)$ (Proposition 6.1 in[7]) and $H=-p\cdot\mathrm{grad}\, f$ (Proposition 6.2 in[7]). Thus we have

PROPOSITION 1. *Let $(M, g_M, \mathcal{F})$ be as above, and let H be the mean curvature vector field of $\mathcal{F}$ on M. If $\mathcal{F}$ is a Clairaut foliation with the girth $\mathbf{r}=e^f$, then $H=-p\cdot\mathrm{grad}\, f$ is a basic vector field, that is, $H\in BX(M)$.*

We suppose that the leaf space $M/\mathcal{F}$ becomes a Riemannian manifold (B, g_B) (for example, if the bundle-like metric g_M is complete, all the leaves of $\mathcal{F}$ are closed, and the holonomy group of each leaf with respect to $\mathcal{F}$ is trivial, then $M/\mathcal{F}$ becames a Riemannian manifold). Then the natural projection $\rho:(M, g_M)\longrightarrow(B, g_B)$ is a Riemannian submersion which is a Clairaut submersion defined by Bishop[1]. Let $\square_B$ be the Laplace-Beltrami operator on B. For any $h\in C^\infty(B)$, the composition $h\circ\rho$ of h and ρ is a foliated function on M, that is, $h\circ\rho\in BC^\infty(M)$. We have

LEMMA 2. *For any $h\in C^\infty(B)$, it holds that $\square''_0(h\circ\rho)=(\square_B h)\circ\rho$.*

By Proposition 1, ρ_*H is a well-defined vector field on B and $\rho_*H=-p\cdot\mathrm{grad}_B\, \underline{f}$, where $\underline{f}\in C^\infty(B)$ with $f=\underline{f}\circ\rho$ and grad_B denotes the gradient operator with respect to g_B. Thus, by Lemma 2, we have

THEOREM 3. *Let $(M, g_M, \mathcal{F})$ and (B, g_B) be as above. The Laplace-Beltrami operators $\square$ and $\square_B$ satisfy the following relation :*

$$\square(u\circ\rho)=(\square_B u)\circ\rho-p\cdot((\mathrm{grad}_B\ \underline{f}\ )u)\circ\rho$$

for any $u\in C^\infty(B)$.

Let (B, g_B) and (F, g_F) be complete Riemannian manifolds. Then we consider a warped product manifold $(M, g_M)=(B\times F, \rho^*(g_B)+(h\circ\rho)^2\cdot\sigma^*(g_F))$, where $\rho : B\times F\longrightarrow B$ and $\sigma : B\times F\longrightarrow F$ are projections, and h is a positive valued function on B([3]). Here we notice that (M, g_M) is also written as $B\times_h F$. The fibers $\rho^{-1}(b)$ of $\rho : M\longrightarrow B$ make a foliation $\mathcal{F}$ on M, that is, leaves of $\mathcal{F}$ are fibers $\rho^{-1}(b)$ $(b\in B)$. Then g_M is bundle-like with respect to $\mathcal{F}$. By Remark 39 in [3, p. 208], $\mathcal{F}$ is a regular Clairaut foliation with the girth $\mathbf{r}=\overline{h}$, where $\overline{h}\in C^\infty(M)$ is defined by $\overline{h}(b, f)=h(b)$ for $b\in B, f\in F$. It is easily proved that $H=-p\cdot\mathrm{grad}(\log\ \overline{h})$ and $\rho_* H=-p\cdot\mathrm{grad}_B(\log\ h)$.

Thus, we have

PROPOSITION 4. *Let (M, g_M) be a complete warped product manifold $B\times_h F$. Then* $\mathrm{Ker}(\square)\cap BC^\infty(M)$ *is isomorphic to* $\mathrm{Ker}(\square_B-p\cdot\mathrm{grad}_B(\log h))$, *where* $p=\dim F$ *and* Ker( ) *denotes the kernel of an operator.*

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