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Convergence almost everywhere of Bochner-Riesz means for radial functions

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Abstract. Let $(S_R^\alpha f)\hat{}(\xi)=(1-|\xi|^2/R^2)_+^\alpha\ \hat{f}(\xi)$ and $S_*^\alpha f(t)=\sup_{0<R<\infty}|S_R^\alpha f(t)|$. It is shown that S_*^α is bounded on radial functions in $L^p(\mathbf{R}^n)$ when $2n/(n+1+2\alpha)\ (=p(\alpha))<p<2n/(n-1-2\alpha)$ and $0<\alpha<(n-1)/2$, and it implies that $S_R^\alpha f(t)$ converges to $f(t)$ almost everywhere for a radial function $f(t)\in L^p(\mathbf{R}^n)$ if $p(\alpha)<p\leqq 2$ and $0<\alpha<(n-1)/2$. It is also proved that, for $1\leqq p<p(\alpha)$ and $0<\alpha<(n-1)/2$, there exists a radial function $f(t)$ with compact support in $L^p(\mathbf{R}^n)$ such that $S_R^\alpha f(t)$ diverges almost everywhere.

Let $\hat{f}(\xi)$ be the Fourier transform of a function $f(t)$ on $\mathbf{R}^n$: $\hat{f}(\xi)=(2\pi)^{-n/2}\int_{\mathbf{R}^n} f(t)e^{-i\xi t}dt$. Let

$$S_R^\alpha f(t)=(2\pi)^{-n/2}\int_{|\xi|\leqq R}\left(1-\frac{|\xi|^2}{R^2}\right)^\alpha \hat{f}(\xi)e^{it\xi}\,d\xi, \text{ and } \qquad S_*^\alpha f(t)=\sup_{0<R<\infty}|S_R^\alpha f(t)|.$$

Throughout this paper, we suppose $n>1$, and use the notation

$$p(\alpha)=\frac{2n}{n+1+2\alpha} \quad\text{and}\quad q(\alpha)=\frac{2n}{n-1-2\alpha}.$$

For $\alpha>(n-1)/2$, the covolution kernel of S_R^α is in $L^1(\mathbf{R}^n)$, and thus S_R^α is bounded on $L^p(\mathbf{R}^n)$ for $1\leqq p\leqq\infty$. Herz [H] proved that, if S_R^0 is restricted to $L^p(\mathbf{R}^n)$ radial functions, S_R^0 is bounded for $p(0)<p<q(0)$. By applying the complex interpolation theorem of E. Stein to these results, Welland [W] obtained that $\|S_R^\alpha f\|_p\leqq C\|f\|_p$ for radial $f(t)$ if $p(\alpha)<p<q(\alpha)$ and $0<\alpha<(n-1)/2$. For the maximal operator S_*^α, Stein proved that

(A) [S, §5, (D)] If $\dfrac{2(n-1)}{n-1+2\alpha}<p<\dfrac{2(n-1)}{n-1-2\alpha}$ *and* $0<\alpha\leqq(n-1)/2$, then $\|S_*^\alpha f\|_p\leqq C\|f\|_p$ with a constant C not depending on $f(t)$.

Carbery [C] showed that, if $n=2$, then S_*^α is bounded on $L^p(\mathbf{R}^n)$ for $p(\alpha)<p<q(\alpha)$ and $0<\alpha<(n-1)/2$. Restricting to $L^p(\mathbf{R}^n)$ radial functions, we shall show the following:

THEOREM. *Let* $0<\alpha<(n-1)/2$.

(i) *Let* $f(t)$ *be radial. Then, for* $p(\alpha)<p<q(\alpha)$, $\| S_*^\alpha f \|_p \leqq C \| f \|_p$ *with a constant* C *not depending on* $f(t)$.

(ii) $S_R^\alpha f(t)$ *converges to* $f(t)$ *almost everywhere for a radial function* $f(t)$ *in* $L^p(\mathbf{R}^n)$ *if* $p(\alpha)<p\leqq 2$.

(iii) *For* $1\leqq p<p(\alpha)$, *there exists a radial function* $f(t)$ *with compact support in* $L^p(\mathbf{R}^n)$ *such that* $S_R^\alpha f(t)$ *diverges almost everywhere.*

Hereafter, we let $\lambda = n/2-1$. For a function $g(x)$ on $(0, \infty)$, we define the Hankel transform $\tilde{g}(y)$ of order λ by $\tilde{g}(y)=\int_0^\infty g(x) J_\lambda(yx)(yx)^{-\lambda}x^{2\lambda+1}dx$, where $J_\lambda(x)$ is the Bessel function of the first kind of order λ. Let

$$T_R^\alpha g(x)=\int_0^R \left(1-\frac{y^2}{R^2}\right)^\alpha \tilde{g}(y) \frac{J_\lambda(xy)}{(xy)^\lambda} y^{2\lambda+1}\, dy,$$

$$T_*^\alpha g(x)= \sup_{0<R<\infty} | T_R^\alpha g(x) |, \text{ and}$$

$$L_\lambda^p = \{ g(x) \text{ on } (0, \infty)\, ;\ \| g \|_{\lambda,p}=(\textstyle\int_0^\infty | g(x) |^p x^{2\lambda+1}dx)^{1/p}<\infty \}.$$

For a radial function $f(t)$ on $\mathbf{R}^n$, we define a function $g(x)$ on $(0, \infty)$ by $g(|t|)=f(t)$, $t\in\mathbf{R}^n$. Then, by Bochner's formula, we have $\hat{f}(\xi)=\tilde{g}(|\xi|)$, $\xi\in\mathbf{R}^n$, and thus $S_R^\alpha f(t)=T_R^\alpha g(|t|)$ and $S_*^\alpha f(t)=T_*^\alpha g(|t|)$, $t\in\mathbf{R}^n$. The relation $g(|t|)=f(t)$, $t\in\mathbf{R}^n$ gives a norm equivalent isomorphism of L_λ^p onto $L^p(\mathbf{R}^n)$ radial functions. Instead of proving the theorem, we shall prove the corresponding statements for the operators T_R^α and T_*^α on L_λ^p.

PROOF OF (i). We obtain (i) by interpolating between (A) and the following result:

(B) [K, COROLLARY 3] If $p(0)<p<q(0)$, then $\| T_*^0 g \|_{\lambda,p}\leqq C \| g \|_{\lambda,p}$ for $g(x)\in L_\lambda^p$ with a constant C not depending on $g(x)$.

The relation stated above between S_*^α and T_*^α enables us to prove (i) by translating the lines of [SW, p. 279, *l*. 13~p. 281, *l*. 28] into our situation, and we omit it.

PROOF OF (ii). By using routine method, we obtain (ii) from (i).

PROOF OF (iii). The argument is similar to that used in Stanton and Tomas [ST] to prove divergence of central Fourier series on compact Lie groups, or in Meaney [M] to prove divergence of Jacobi polynomial series. See also [K]. We state two lemmas.

LEMMA 1. *Let* $1\leq p<p(\alpha)$. *Define a sequence* $\{\varphi_k\}_{k=1}^{\infty}$ *of bounded linear functionals of the space* $L_\lambda^p(0, 1)=\{g\in L_\lambda^p\,;\, supp\ g\subset(0, 1)\}$ *by* $\varphi_k(g)=\int_k^{k+1}\tilde{g}(y)\, y^{\lambda-\alpha+1/2}dy$. *Then the norms* $\|\varphi_k\|$ *of the functionals* φ_k *satisfy that* $\lim_{k\to\infty}\|\varphi_k\|=\infty$.

LEMMA 2. *Let* $\frac{4\lambda+2}{2\lambda+3+2\alpha}\leq p\leq 2$. *If a function* $g(x)$ *in* L_λ^p *satisfies the condition that* $T_R^\alpha g(x)$ *converges on a set of positive measure, then* $\lim_{R\to\infty}\int_R^{R+h}\tilde{g}(y)\, y^{\lambda-\alpha+1/2}dy=0$ *uniformly in* $0\leq h\leq 1$.

(iii) is a easy consequence of the lemmas. Let $1\leq p<p(\alpha)$. We choose p' so that $\frac{4\lambda+2}{2\lambda+3+2\alpha}$, $p\leq p'<p(\alpha)$. By LEMMA 1, there exists a function $g(x)$ in $L_\lambda^{p'}(0, 1)$ such that

$$\limsup_{k\to\infty}\left|\int_k^{k+1}\tilde{g}(y)\, y^{\lambda-\alpha+1/2}dy\right|=\infty.$$

It follows from LEMMA 2 that $T_*^\alpha g(x)$ diverges almost everywhere in $(0, \infty)$ as $R\to\infty$. Since the function $g(x)$ belongs to L_λ^p, we have (iii).

Now we prove LEMMA 1. By Fubini's theorem, we have

$$\varphi_k(g)=\int_0^1 g(x)\ \left\{\int_k^{k+1}\frac{J_\lambda(yx)}{(yx)^\lambda}\, y^{\lambda-\alpha+1/2}\, dy\right\}\, x^{2\lambda+1}dx,$$

and thus

$$\|\varphi_k\|^q=\int_0^1\left|\int_k^{k+1}\frac{J_\lambda(yx)}{(yx)^\lambda}\, y^{\lambda-\alpha+1/2}\, dy\right|^q x^{2\lambda+1}dx,\quad 1/p+1/q=1.$$

Let r be an arbitrary positive integer. By the asymptotic formula

$$(*)\ \ J_\lambda(z)=\left(\frac{2}{\pi z}\right)^{1/2}\cos(z-\gamma)+O(z^{-3/2})\ \ (z\to\infty),\qquad \gamma=(2\lambda+1)\pi/4,$$

we have

$$\|\varphi_k\|^q\geq\int_{r/k}^1\left|\int_k^{k+1}\frac{J_\lambda(yx)}{(yx)^\lambda}\, y^{\lambda-\alpha+1/2}dy\right|^q x^{2\lambda+1}\, dx$$

$$\geq C\int_{r/k}^1\left|\int_k^{k+1} y^{-\alpha}\cos(yx-\gamma)\, dy\right|^q x^{2\lambda+1-(\lambda+1/2)q}\, dx$$

$$-C\int_{r/k}^1\left|\int_k^{k+1} y^{-1-\alpha}dy\right|^q x^{2\lambda+1-(\lambda+3/2)q}\, dx$$

$$=CA_k-CB_k,\ \text{say}.$$

In this proof, the letter C means positive costants not depending or r and k, and it may be different in each occurrence. The terms B_k satisfy that $B_k \leqq C\ r^{-\varepsilon-\alpha q-q}k^{\varepsilon}$, where $\varepsilon=(\lambda-\alpha+1/2)\ q-2\ (\lambda+1)$. We shall estimate the terms A_k. For k satisfying $2r\pi+\gamma+\pi/6 \leqq (k+1)\pi/6$, we put $x_r=(2r\pi+\gamma-\pi/6)/k$ and $x'_r=(2r\pi+\gamma+\pi/6)/(k+1)$. Then, we notice that $x'_r-x_r \geqq \pi/(6k)$ and $r/k \leqq x_r \leqq x'_r \leqq 1$. Since $\cos(xy-\gamma) \geqq \sqrt{3}/2$ for $x_r \leqq x \leqq x'_r$ and $k \leqq y \leqq k+1$, it follows that

$$A_k^q \geqq C\ (k+1)^{-\alpha q} \int_{x_r}^{x'_r} x^{2\lambda+1-(\lambda+1/2)q}\ dx \geqq C\ k^{\varepsilon}\ r^{-\varepsilon-\alpha q-1}.$$

Therefore, we have

$$\| \varphi_k \|^q \geqq C r^{-\varepsilon-\alpha q-1}(1-r^{1-q})\ k^{\varepsilon}$$

for k satisfying $2r\pi+\gamma+\pi/6 \leqq (k+1)\pi/6$. Since $\varepsilon>0$ and $(1-r^{1-q})>0$ for large r, it follows that $\lim_{k\to\infty} \| \varphi_k \| = \infty$, which completes the proof of LEMMA 1.

Next, we prove LEMMA 2. Suppose that $T_R^{\alpha} g(x)$ converges on a set $F \subset (0, \infty)$ of positive measure. Then, by a theorem concerning typical means, we have that

$$\int_0^R \tilde{g}(y) \frac{J_\lambda(xy)}{(xy)^\lambda} y^{2\lambda+1} dy = o(R^{\alpha}) \quad (R \to \infty)$$

for every $x \in F$ (cf. [HR, Ch. 5, THEOREM 22]). By Egorov's theorem, we have a closed set E of positive measure such that the above convergence is uniform in $x \in E$. Without loss of generality, we may assume that $E \subset (a, b)$ with $0<a<b<\infty$. It follows from the second mean value theorem that the integral

$$U_R = \int_R^{R+h} \tilde{g}(y) \frac{J_\lambda(xy)}{(xy)^\lambda}\ y^{2\lambda+1} y^{-\alpha}\ dy$$

converges to 0 as $R \to \infty$, uniformly in $x \in E$ and $0 \leqq h \leqq 1$. By asymptotic formula (*), we have that, as $R \to \infty$,

$$U_R = \frac{(2/\pi)^{1/2}}{x^{\lambda+1/2}} \int_R^{R+h} \tilde{g}(y) \cos(xy-\gamma)\ y^{\lambda-\alpha+1/2}\ dy$$

$$+ \int_R^{R+h} \tilde{g}(x)\ O((xy)^{-3/2})\ (xy)^{-\lambda} y^{2\lambda+1} y^{-\alpha}\ dy$$

$$= \frac{(2/\pi)^{1/2}}{x^{\lambda+1/2}} V_R + W_R, \text{ say,}$$

and thus $|V_R| \leqq C\ (|U_R| + |W_R|)$. In the proof, the letter C means positive constants depending only on a, b and λ, and it may be different in each occurrence. We have

$$|W_R| \leqq C \left| \int_R^{R+h} \tilde{g}(y)\ y^{-(\lambda+\alpha+3/2)} y^{2\lambda+1} dy \right|$$

$$\leqq C\left(\int_R^{R+h} |\tilde{g}(y)|^q y^{2\lambda+1}\,dy\right)^{1/q}\cdot\left(\int_R^{R+h} y^{-p(\lambda+\alpha+3/2)}y^{2\lambda+1}\,dy\right)^{1/p}$$

$$=o(1)\quad (R\to\infty),$$

uniformly in $0\leqq h\leqq 1$, since $\|\tilde{g}\|_{\lambda,q}<\infty$ and $(4\lambda+2)/(2\lambda+3+2\alpha)\leqq p\leqq 2$. Thus we have that V_R converges uniformyly in $x\in E$ and $0\leqq h\leqq 1$ as $R\to\infty$. We write V_R in the form

$$\int_R^{R+h}\cos xy\; d\chi_1(y)+\sin xy\; d\chi_2(y),$$

where $\qquad d\chi_1(y)=(\cos\gamma)\,\tilde{g}(y)\,y^{\lambda-\alpha+1/2}\,dy,$ and

$$d\chi_2(y)=(\sin\gamma)\tilde{g}(y)\,y^{\lambda-\alpha+1/2}\,dy.$$

By the proof of [Z, Ch. XVI THEOREM (8.4)], which is a trigonometric integral analogue of the Cantor-Lebesgue theorem, we have that $\int_R^{R+h} d\chi_j(y)=o(1)$ $(R\to\infty)$ uniformly in $0\leqq h\leqq 1$ for $j=1$, 2, and thus

$$\int_R^{R+h}\tilde{g}(y)\,y^{\lambda-\alpha+1/2}\,dy=\int_R^{R+h}\cos\gamma\; d\chi_1(y)\;+\;\sin\gamma\; d\chi_2(y)=o(1)$$

$(R\to\infty)$, uniformly in $0\leqq h\leqq 1$. This completes the proof of LEMMA 2.

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