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A COHEN TYPE INEQUALITY FOR DISK POLYNOMIAL EXPANSIONS

Yûichi KANJIN*)

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Let $F_N(\theta)=\sum_{j=1}^{N}c_j e^{in_j\theta}$, where $0<n_1<n_2<\ldots<n_N$, $N\geqq 2$ and $|c_j|\geqq 1$, and let $\|F_N\|_1=\int_0^{2\pi}|F_N(\theta)|d\theta$. In the study of rearrangements of Fourier coefficients, Littlewood conjectured that $C(\log N)$ is the lower bound for the norm $\|F_N\|_1$. P.J. Cohen [2] proved that $C(\log N/\log\log N)^{1/8}$ is a lower bound, and the terminology "Cohen type inequality" comes from his contribution to this conjecture. Recently, O.C. McGehee, L. Pigno and B.P. Smith [9] and S.V. Konjagin [6] solved independently the Littlewood conjecture. Cohen type inequalities has been established in various other contexts, *e.g.*, for Fourier expansions on compact groups or for Jacobi, Laguerre and Hermite expansions. See, S. Giulini, P.M. Soardi and G. Travaglini [5], B. Dreseler and P.M. Soardi [3], [4], and C. Markett [8].

In this note, we will establish a Cohen type inequality for disk polynomial expansions according to Dreseler and Soardi.

1. *Disk polynomials and a Cohen type inequality*

Let α be a positive real number. For nonnegative integers m, n, *disk polynomials* $R^{(\alpha)}_{m,n}(z)$ are defined by

$$R^{(\alpha)}_{m,n}(z)=R^{(\alpha,|m-n|)}_{m\wedge n}(2r^2-1)e^{i(m-n)\theta}r^{|m-n|},$$

where $z=re^{i\theta}$, $m\wedge n=\min\{m, n\}$ and $R^{(\alpha,\beta)}_n(x)$ is the Jacobi polynomial of degree n and of order (α, β) normalized so that $R^{(\alpha,\beta)}_n(1)=1$.

Disk polynomials have the following properties:

(i) $R^{(\alpha)}_{m,n}(z)$ is a polynomial of degree $m+n$ in x and y, where $z=x+iy$.

(ii) Let $\bar{D}$ be the closed unit disk in the complex plane and m_α the positive measure of total mass one on $\bar{D}$ defined by

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*)Department of Mathematics, College of Liberal Arts,
Kanazawa University, Kanazawa 920, Japan

$$dm_\alpha(z)=\frac{\alpha+1}{\pi}(1-x^2-y^2)^\alpha dxdy.$$

Then $\{R_{m,n}^{(\alpha)}\}_{m,n=0}^\infty$ is a complete orthogonal system in $L^2(\bar{D},\ m_\alpha)$, that is,

$$\int_{\bar{D}} R_{m,n}^{(\alpha)}(z)R_{k,l}^{(\alpha)}(\bar{z})dm_\alpha(z)=h_{m,n}^{(\alpha)-1}\delta_{mk}\delta_{nl},$$

and $\hat{f}(m,\ n)=0$ for all m, n implies $f=0$, where

$$h_{m,n}^{(\alpha)}=\frac{(m+n+\alpha+1)\Gamma(m+\alpha+1)\Gamma(n+\alpha+1)}{(\alpha+1)\Gamma(\alpha+1)^2\Gamma(m+1)\Gamma(n+1)},$$

$\bar{z}=x-iy$, δ_{mk} is Kronecker's symbol and

$$\hat{f}(m,\ n)=\int_{\bar{D}} f(z)R_{m,n}^{(\alpha)}(\bar{z})dm_\alpha(z).$$

(iii) $R_{m,n}^{(\alpha)}(z)=az^m\bar{z}^n+bz^{m-1}\bar{z}^{n-1}+\ \dots\ +dz^{m-(m\wedge n)}\bar{z}^{n-(m\wedge n)}$

where a, b and d are constants.

(iv) $|R_{m,n}^{(\alpha)}(z)|\leqq 1$ on $\bar{D}$.

(v) If $\alpha=1,\ 2,\ 3,\ \dots$, then disk polynomials $R_{m,n}^{(\alpha)}(z)$ are the spherical functions on the sphere $S^{2\alpha+3}$ considered as the homogeneous space $U(\alpha+2)/U(\alpha+1)$.

There are several papers concerned with disk polynomials. For example, see T.H. Koornwinder [7] that provides more references.

Let $1\leqq p\leqq\infty$ and put $L_\alpha^p=L^p(\bar{D},\ m_\alpha)$. For f in L_α^1, define the *disk polynomial expansion* of f to be

$$f(z)\sim\sum_{m,n=0}^\infty \hat{f}(m,\ n)h_{m,n}^{(\alpha)}R_{m,n}^{(\alpha)}(z).$$

The space L_α^1 has a positive convolution structure such that for $\alpha=1,\ 2,\ 3,\ \dots$, this convolution corresponds to one for functions on $U(\alpha+2)$ which are biinvariant by $U(\alpha+1)$ (*cf*.[1]). Denote by $*$ this convolution product in L_α^1. Then it follows that $(f*g)\hat{\ }(m,\ n)=\hat{f}(m,\ n)\hat{g}(m,\ n)$ for f and g in L_α^1. For k in L_α^1, let $T:L_\alpha^p\to L_\alpha^p$ be the convolutor defined by $f\to k*f$ for f in L_α^p, and put

$$|||k|||_{p,\alpha}=\sup_{\|f\|_{p,\alpha}\leqq 1}\|Tf\|_{p,\alpha},$$

where $\|\cdot\|_{p,\alpha}$ denote the norm in L_α^p, that is,

$$\|f\|_{p,\alpha}=\left\{\int_D|f(z)|^p dm_\alpha(z)\right\}^{1/p}$$

for f in L_α^p.

Put $\boldsymbol{n}=(m,\ n)$, $\boldsymbol{1}=(1,\ 1)$ and $k\boldsymbol{n}+k'\boldsymbol{n}'=(km+k'm',\ kn+k'n')$ for $\boldsymbol{n}=(m,\ n)$, $\boldsymbol{n}'=(m',\ n')$ and $k,\ k'=0,\ 1,\ 2,\ \dots$. In the following we use the notation $R_{\boldsymbol{n}}^{(\alpha)}(z)=R_{m,n}^{(\alpha)}(z)$, $h_{\boldsymbol{n}}^{(\alpha)}=h_{m,n}^{(\alpha)}$ and $\hat{f}(\boldsymbol{n})=\hat{f}(m,\ n)$, and use the letter $C_{p,\alpha}$ for a positive constant depending only on p, α, which may be different at each occurence.

Theorem. *Let* $F(z)$ *be a finite sum of disk polynomials of the form*

$$\sum_{\boldsymbol{n}}c_{\boldsymbol{n}}h_{\boldsymbol{n}}^{(\alpha)}R_{\boldsymbol{n}}^{(\alpha)}(z),$$

where $c_{\boldsymbol{n}}$ are complex numbers with $|c_{\boldsymbol{n}}|\geqq 1$. Denote by $\boldsymbol{n}_0=(m_0, n_0)$ an $\boldsymbol{n}$ satisfying the following; $\boldsymbol{n}$ appears in $\Sigma_{\boldsymbol{n}}$ and all $\boldsymbol{n}+l\mathbf{1}=(m+l, n+l)$, $l=1, 2, 3, \ldots$ do not appear in $\Sigma_{\boldsymbol{n}}$. Then

$$|||F|||_{p,\alpha}\geqq C_{p,\alpha}\begin{cases}(m_0\vee n_0)^{-2(\alpha+1)/p+\alpha+1/2}(n_0/m_0)^{(m_0-n_0)/2} & \text{for } \dfrac{4(\alpha+1)}{2\alpha+1}<p\leqq\infty,\\[2ex] (m_0\wedge n_0)^{2(\alpha+1)/p-(\alpha+3/2)}(n_0/m_0)^{(m_0-n_0)/2} & \text{for } \dfrac{4(\alpha+1)}{2\alpha+3}>p\geqq 1,\end{cases}$$

where $m_0\vee n_0=\max\{m_0, n_0\}$.

Let $\{\Lambda_N\}_{N=1}^{\infty}$ be an increasing sequence of finite subsets Λ_N of $\{(m, n);\ m, n=0, 1, 2, \ldots\}$. If $\bigcup_{N=1}^{\infty}\Lambda_N=\{(m, n);\ m, n=0, 1, 2, \ldots\}$, then $\{\Lambda_N\}$ is called a *grouping*. The *partial sum operator* S_N with respect to a grouping $\{\Lambda_N\}$ is defined by

$$S_Nf=\sum_{\boldsymbol{n}\in\Lambda_N}\hat{f}(\boldsymbol{n})h_{\boldsymbol{n}}^{(\alpha)}R_{\boldsymbol{n}}^{(\alpha)}(z)$$

for f in L_{α}^{p}. By Theorem we have

Corollary. *Let $1\leqq p<4(\alpha+1)/(2\alpha+3)$ or $4(\alpha+1)/(2\alpha+1)<p\leqq\infty$. Then for every grouping $\{\Lambda_N\}_{N=1}^{\infty}$ there exists a function f in L_{α}^{p} such that*

$$\limsup_{N\to\infty}\|S_Nf\|_{p,\alpha}=\infty.$$

2. *Proof of Theorem*

Let $4(\alpha+1)/(2\alpha+1)<p\leqq\infty$. For f in L_{α}^{p} with $\|f\|_{p,\alpha}\leqq 1$ we have

$$|||F|||_{p,\alpha}\geqq\|F*f\|_{p,\alpha}\geqq\sup_{\|g\|_{q,\alpha}\leqq 1}\left|\int_{\bar{D}}F*f(z)\overline{g(z)}\,dm_{\alpha}(z)\right|,$$

where $1/p+1/q=1$. Suppose that

(#) $\qquad \hat{f}(\boldsymbol{n})=0$ for $\boldsymbol{n}\notin\{\boldsymbol{n}_0+l\mathbf{1}: l=0, 1, 2, \ldots\}$.

Then we have

$$\int_{\bar{D}}F*f(z)\overline{g(z)}\,dm_{\alpha}(z)=c_{\boldsymbol{n}_0}h_{\boldsymbol{n}_0}^{(\alpha)}\hat{f}(\boldsymbol{n}_0)\int_{\bar{D}}R_{\boldsymbol{n}_0}^{(\alpha)}(z)\overline{g(z)}\,dm_{\alpha}(z),$$

and thus

(##) $\qquad |||F|||_{p,\alpha}\geqq|\hat{f}(\boldsymbol{n}_0)|h_{\boldsymbol{n}_0}^{(\alpha)}\|R_{\boldsymbol{n}_0}^{(\alpha)}\|_{p,\alpha}$.

We will estimate $\|R_{\boldsymbol{n}_0}^{(\alpha)}\|_{p,\alpha}$ from below. Let k be the least integer such that $2k\geqq p$. Then $(R_{\boldsymbol{n}_0}^{(\alpha)})^k$ has the following form by (iii);

$$(R_{\boldsymbol{n}_0}^{(\alpha)})^k=a_0h_{k\boldsymbol{n}_0}^{(\alpha)}R_{k\boldsymbol{n}_0}^{(\alpha)}+a_1h_{k\boldsymbol{n}_0-\mathbf{1}}^{(\alpha)}R_{k\boldsymbol{n}_0-\mathbf{1}}^{(\alpha)}+a_2h_{k\boldsymbol{n}_0-2\mathbf{1}}^{(\alpha)}R_{k\boldsymbol{n}_0-2\mathbf{1}}^{(\alpha)}+\\ \ldots+a_{k(m\wedge n)}h_{k\boldsymbol{n}_0-k(m\wedge n)\mathbf{1}}^{(\alpha)}R_{k\boldsymbol{n}_0-k(m\wedge n)\mathbf{1}}^{(\alpha)}.$$

Put

$$D^{(\alpha)}=h_{k\boldsymbol{n}_0}^{(\alpha)}R_{k\boldsymbol{n}_0}^{(\alpha)}+h_{k\boldsymbol{n}_0-\mathbf{1}}^{(\alpha)}R_{k\boldsymbol{n}_0-\mathbf{1}}^{(\alpha)}+h_{k\boldsymbol{n}_0-2\mathbf{1}}^{(\alpha)}R_{k\boldsymbol{n}_0-2\mathbf{1}}^{(\alpha)}+\\ \ldots+h_{k\boldsymbol{n}_0-k(m\wedge n)\mathbf{1}}^{(\alpha)}R_{k\boldsymbol{n}_0-k(m\wedge n)\mathbf{1}}^{(\alpha)}.$$

Then we have $(R_{n_0}^{(\alpha)})^k * D^{(\alpha)} = (R_{n_0}^{(\alpha)})^k$. By the inequality of Cauchy-Schwarz we have

$$\|(R_{n_0}^{(\alpha)})^k\|_{\infty,\alpha} \leqq \|D^{(\alpha)}\|_{2,\alpha} \|(R_{n_0}^{(\alpha)})^k\|_{2,\alpha}.$$

Since $\|(R_{n_0}^{(\alpha)})^k\|_{2,\alpha} \leqq \|R_{n_0}^{(\alpha)}\|_{\infty,\alpha}^{(2k-p)/2} \|R_{n_0}^{(\alpha)}\|_{p,\alpha}^{p/2}$ and $R_{n_0}^{(\alpha)}(1)=1$, we have

$$\|R_{n_0}^{(\alpha)}\|_{p,\alpha} \geqq \|D^{(\alpha)}\|_{2,\alpha}^{-2/p}.$$

On the other hand,

$$\|D^{(\alpha)}\|_{2,\alpha}^2 = h_{kn_0}^{(\alpha)} + h_{kn_0-1}^{(\alpha)} + h_{kn_0-21}^{(\alpha)} + \ \dots\ + h_{kn_0-k(m\wedge n)1}^{(\alpha)}$$
$$\leqq C_{p,\alpha} m_0^\alpha n_0^\alpha (m_0+n_0)(m_0 \wedge n_0)$$

since $h_n^{(\alpha)} \sim m^\alpha n^\alpha (m+n)$ as $m,\ n \to \infty$, and thus

(###) $$\|R_{n_0}^{(\alpha)}\|_{p,\alpha} \geqq C_{p,\alpha} \{ m_0^\alpha n_0^\alpha (m_0+n_0)(m_0 \wedge n_0)\}^{-1/p}.$$

Put $\beta = |m_0 - n_0|$ and denote by s the smallest odd integer such that $s \geqq 2\alpha+1$ and $s \geqq \beta+1$. Define $f_{n_0}^{(\alpha)}$ in L_α^∞ by

$$f_{n_0}^{(\alpha)}(z) = e^{i(m_0-n_0)\theta} \cos^{s-(\beta+1)} \frac{\phi}{2} \sin^{s-(2\alpha+1)} \frac{\phi}{2} \sin l\phi,$$

where $l = s + m_0 \wedge n_0$ and $z = \cos \frac{\phi}{2}\, e^{i\theta}$, $0 < \phi \leqq \pi$, $0 \leqq \theta < 2\pi$.

Then we have $\|f_{n_0}^{(\alpha)}\|_{p,\alpha} \leqq \|f_{n_0}^{(\alpha)}\|_{\infty,\alpha} \leqq 1$ and

$$\hat{f}_{n_0}^{(\alpha)}(\boldsymbol{n}) = \frac{2(\alpha+1)}{\pi} \int_0^{2\pi} \int_0^\pi f_{n_0}^{(\alpha)}\left(\cos \frac{\phi}{2}\, e^{i\theta}\right) R_n^{(\alpha)}\left(\cos \frac{\phi}{2}\, e^{i\theta}\right) \sin^{2\alpha+1} \frac{\phi}{2}\, d\phi d\theta$$
$$= \frac{(\alpha+1)}{2^s} \int_0^\pi \sin l\phi\, \sin^s \phi\, R_{m\wedge n}^{(\alpha,\beta)}(\cos \phi) d\phi$$

if $m - n = m_0 - n_0$, and $=0$ if $m - n \neq m_0 - n_0$. The functions $f_{n_0}^{(\alpha)}$ are essentially introduced by Dreseler and Soardi, who showed that

$$\int_0^\pi \sin l\phi\, \sin^s \phi\, R_{m\wedge n}^{(\alpha,\beta)}(\cos \phi) d\phi$$
$$= \frac{(\alpha+1)(-1)^{m_0\wedge n_0}(m_0 \wedge n_0 + \alpha + |m_0 - n_0| + 1)_{m_0\wedge n_0}}{2^{s+1}(2i)^{s-1+2(m_0\wedge n_0)}(\alpha+1)_{m_0\wedge n_0}}$$

for $\boldsymbol{n} = \boldsymbol{n}_0$, and $=0$ for $\boldsymbol{n} = \boldsymbol{n}_0 - k\mathbf{1}$, $k = 1, 2, 3, \dots, m_0 \wedge n_0$ (see the proof of Lemma 3 in [4]), where $(a)_n = a(a+1) \dots (a+n-1)$. Thus we have

(####) $$|\hat{f}_{n_0}^{(\alpha)}(\boldsymbol{n}_0)| \geqq C_\alpha \frac{\Gamma(m_0+n_0+\alpha+1)}{2^{m_0+n_0}\Gamma(m_0+\alpha+1)\Gamma(n_0+\alpha+1)}.$$

Since $f_{n_0}^{(\alpha)}$ satisfies (#) and $\|f_{n_0}^{(\alpha)}\|_{p,\alpha} \leqq 1$, it follows from (##), (###) and (####) that

$$|||F|||_{p,\alpha} \geqq (m_0 n_0)^{-\alpha/p} (m_0+n_0)^{1-1/p+\alpha} (m_0 \wedge n_0)^{-1/p} \cdot \frac{\Gamma(m_0+n_0+1)}{2^{m_0+n_0}\Gamma(m_0+1)\Gamma(n_0+1)}.$$

By Stirling's formula and duality we have the theorem.

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