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A note on peak points of Reinhardt domain

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1. The study of peak points and peak functions is interesting and important for the study of function theory of several complex variables. For the pseudoconvex domains, there are much detailed discussion of these subjects, for example Basener [1], Bloom [3], Hakim-Sibony [4] and Range [5]. In these papers, the totally pseudoconvex point, that is the point where the non-singular holomorphic supporting hypersurface exists, are considered in substance. In this paper we consider the pseudoconvex Reinhardt domain. In general, there are no local C^0-peak function at the Levi-flat point. We shall show that there are points where no local C^0-peak function exists without the assumption of C^2 boundary (Lemma 3-3). Further, we shall show that except the special boundary points of pseudoconvex Reinhardt domain, global holomorphic peak functions exist (Theorem 3-7).

2. We use the following notations.

$\mathbf{C}_0^n=\{z=(z_1,z_2,\ldots,z_n)\in\mathbf{C}^n;\ z_j\neq 0,\ j=1,2,\ldots,n\}.$

$\mathbf{C}_{j_1j_2\cdots j_m}^n=\{z=(z_1,\ldots,z_n)\in\mathbf{C}^n;\ z_{j_1}=z_{j_2}=\ldots=z_{j_m}=0\}.$

$\langle z\rangle=(|z_1|,|z_2|,\ldots,|z_n|).$

$\langle\Omega\rangle=\{\langle z\rangle\in\mathbf{R}^n;\ z\in\Omega\}.$

$\log\langle z\rangle=(\log|z_1|,\ldots,\log|z_n|).$

$\log\langle\Omega\rangle=\{\log\langle z\rangle\in\mathbf{R}^n;\ z\in\Omega\cap\mathbf{C}_0^n\}.$

$\Omega_{12\cdots m}=$ all the inner points of $\bar{\Omega}\cap\mathbf{C}_{12\cdots m}^n$.

$N=\{0,1,2,\ \ldots\}$, exp = exponential.

Let Ω be a domain in $\mathbf{C}^n$.

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Definition 2-1. We say that Ω is a Reinhardt domain with center at $a=(a_1, a_2, \ldots, a_n)$ if whenever $z=(z_1, z_2, \ldots, z_n)\in\Omega$ and $\theta_1, \theta_2, \ldots, \theta_n\in\mathbf{R}$, we have $((z_1-a_1)\exp(i\theta_1), \ldots, (z_n-a_n)\exp(i\theta_n))\in\Omega$.
In the rest of this paper, we always assume that the Reinhardt domain is with center at the origin.

Definition 2-2. Let $1\leqq j_1<j_2\ldots<j_m\leqq n$. We say that Ω is complete with respect to $(j_1, j_2, \ldots, j_m)$ if whenever the point $z^0=(z_1^0, \ldots, z_n^0)\in\Omega$ then $w=(w_1, w_2, \ldots, w_n)\in\Omega$ where if $s\neq j_k$ $(k=1, 2, \ldots, m)$ then $w_s=z_s^0$ and $|w_{j_k}|\leqq|z_{j_k}^0|$.

Definition 2-3. We say that Ω is logarithmic convex if $\log\langle\Omega\rangle$ is convex in the usual sence.

Theorem 2-4. *Let Ω be a Reinhardt domain.*
(1), *if $\Omega\cap\mathbf{C}_{j_k}^n\neq\phi$ $(k=1, 2, \ldots, m)$ then Ω is a domain of holomorphy if and only if Ω is logarithmic convex and if Ω is complete with respect to $(j_1, j_2, \ldots, j_m)$,*
(1)′, *if $\Omega\cap\mathbf{C}_k^n\neq\phi$ $(k=1, 2, \ldots, n)$ then Ω is a domain of holomorphy if and only if Ω is logarithmic convex and if complete,*
(2), *if $\Omega\cap\mathbf{C}_k^n=\phi$ $(k=1, 2, \ldots, n)$ then Ω is a domain of holomorphy if and only if Ω is logarithmic convex.*

In the case of two complex variables, this theorem is proved in Behnke-Stein [2]. The proof of the general case is essentially the same and we only sketch the outline of the proof. If the Reinhardt domain is pseudoconvex, then it is a convergence domain of a Laurent series. Conversely the convergence domain of a Laurent series is a Reinhardt domain and by the transformation $\xi_j=\log|z_j|$ $(j=1, 2, \ldots, n)$ it is mapped onto the convex domain in $\mathbf{R}^n$. In regard to the completeness, we assume that $(j_1, j_2, \ldots, j_m)=(1, 2, \ldots, m)$ for simplicity. First, for any point $z^0=(z_1^0, \ldots, z_n^0)\in\Omega\cap\mathbf{C}_0^n$, it is proved that $(z_1{}',, \ldots, z_m{}', z_{m+1}^0, \ldots, z_n^0)\in\Omega$ where $|z_j{}'|\leqq|z_j^0|$ $(1\leqq j\leqq m)$. Next, for any point $z^0\in\Omega$ it is proved that $(z_1{}', z_2^0, \ldots, z_n^0)\in\Omega$ where $|z_1{}'|\leqq|z_1^0|$ and inductively we can show that the completeness holds. To prove the sufficiency, we prepare the following lemma.

Lemma. *Let K be a closed convex subset of $\mathbf{R}^n$ bounded in the direction $x_j>0$ $(1\leqq j\leqq n)$, unbounded in the direction $x_k<0$ $(1\leqq k\leqq m)$, and bounded in the direction $x_i<0$ $(m+1\leqq i\leqq n)$. Assume that for any $x^0=(x_1^0, \ldots, x_n^0)\in K$, $(x_1, \ldots, x_m, x_{m+1}^0, \ldots, x_n^0)\in K$ for all $(x_1, \ldots, x_m)$ with $x_j\leqq x_j^0$ $(1\leqq j\leqq m)$. Then for any $y\in K$ there exists an affine function $f(x)=a_1x_1+a_2x_2+\ldots+a_nx_n+c$ such that*

(i) $f(y)=0$, (ii) $f(x)<0$ *for all* $x\in K$, (iii) $a_i>0$ $(1\leqq i\leqq m)$ *and* $a_j\in\mathrm{Q}$ $(1\leqq j\leqq n)$.

The proof of sufficiency is obtained using the above lemma and the fact that holomorphically convex domain in $\mathbf{C}^n$ is a domain of holomorphy.

3. Let Ω be a Reinhardt domain and let ϕ be the mapping from $\mathbf{C}^n$ onto $\mathbf{C}^n$ defined by $\phi(z_1, \ldots, z_n)=(z_{\sigma(1)}, \ldots, z_{\sigma(n)})$, where σ is a permutation of $(1, 2, \ldots, n)$. Then $\phi(\Omega)$ is a Reinhardt domain with center at the origin and if Ω is pseudoconvex then $\phi(\Omega)$ is also pseudoconvex. Let $z^0\in\partial\Omega$, then from this reason we may assume that $z^0=(0, \ldots, 0, z^0_{m+1}, \ldots, z^0_n)$ where $z^0_j\neq 0$ $(j=m+1, \ldots, n)$. Therefore in section 3 we only consider the boundary points of this type.
Let $z^0=(0, \ldots, 0, z^0_{m+1}, \ldots, z^0_n)\in\partial\Omega$.

Definition 3-1. The point z^0 is said to belong to $\mathbf{N}^1_\Omega$ (resp. $\mathbf{N}^2_\Omega$) if the following condition (i) (resp. (ii)) is satisfied.

(i): there exists an $a=(a_1, \ldots, a_m)\in\mathbf{C}^m(a\neq 0)$, a $\nu=(\nu_1, \ldots, \nu_m)\in N^m$ and an $\varepsilon>0$ such that $a_j\nu_j\neq 0$ for some j and that the point $(a_1\zeta^{\nu_1}, \ldots, a_m\zeta^{\nu_m}, z^0_{m+1}, \ldots, z^0_n)\in\Omega$ for all $\zeta\in\mathbf{C}$ with $0<|\zeta|<\varepsilon$,

(ii): there exists a $b=(b_{m+1}, \ldots, b_n)\in\mathbf{R}^{n-m}(b\neq 0)$ and an $\varepsilon>0$ such that $(0, \ldots, 0, z^0_{m+1}\, exp\ (b_{m+1}\zeta), \ldots, z^0_n\, exp\ (b_n\zeta))\in\partial\Omega$ for all $\zeta\in\mathbf{C}$ with $|\zeta|<\varepsilon$.

Definition 3-1′. The point z^0 in said to belong to $\mathbf{N}^*_\Omega$

(iii): if $1\leqq m\leqq n-1$ then $\{z\in\mathbf{C}^n;\ z_j=z^0_j,\ m+1\leqq j\leqq n\}\cap\bar{\Omega}\supsetneqq\{z^0\}$,

(iv): if $m=n$, $z^0(=$the origin of $\mathbf{C}^n)\in\mathbf{N}^*_\Omega$.

It is easily seen that the conditions (i), (ii), and (iii) are equivalent to the following (i)′, (ii)′, and (iii)′ respectively.

(i)′: there exists an $a=(a_1, \ldots, a_m)\in\mathbf{R}^m(a\neq 0)$ a $\nu=(\nu_1, \ldots, \nu_m)\in N^m$ and an $\varepsilon>0$ such that $a_j\nu_j\neq 0$ for some j and that the point $(a_1|t|^{\nu_1}, \ldots, a_m|t|^{\nu_m}, |z^0_{m+1}|, \ldots, |z^0_n|)\in\langle\Omega\rangle$ for all $t\in\mathbf{R}$ with $0<|t|<\varepsilon$,

(ii)′: there exists a $b=(b_{m+1}, \ldots, b_n)\in\mathbf{R}^{n-m}(b\neq 0)$ and an $\varepsilon>0$ such that $(0, \ldots, 0, |z^0_{m+1}|exp\ (b_{m+1}t), \ldots, |z^0_n|\ exp\ (b_nt))\in\partial\langle\Omega\rangle$ for all $t\in\mathbf{R}$ with $|t|<\varepsilon$,

(iii)′: if $1\leqq m\leqq n-1$ then $\{\langle z\rangle\in\mathbf{R}^n;\ |z_j|=|z^0_j|,\ m+1\leqq j\leqq n\}\cap\langle\Omega\rangle\supsetneqq\{\langle z^0\rangle\}$.

Examples.

(1). $\Omega=\{(z_1, z_2, z_3)\in\mathbf{C}^3;\ exp\ (|z_1|)-|z_2|\ |z_3|<0\}$, then $(0, 1, 1)\in\mathbf{N}^1_\Omega\cap\mathbf{N}^2_\Omega-\mathbf{N}^*_\Omega$.

(2). $\Omega=\{(z_1, z_2)\in\mathbf{C}^2;\ |z_1|^e<|z_2|\ exp\ (-1)<|z_1|^e\}$, then $(0, 0)\in\mathbf{N}^2_\Omega\cap\mathbf{N}^*_\Omega-\mathbf{N}^1_\Omega$.

(3). $\Omega=\{(z_1, z_2)\in\mathbf{C}^2;\ |z_2|<1+|z_1|\}$, then $(0, 1)\in\mathbf{N}^1_\Omega\cap\mathbf{N}^*_\Omega-\mathbf{N}^2_\Omega$.

Now, let $A^k(\Omega)$ be the set of all functions that are holomorphic in Ω and class C^k in $\bar{\Omega}$.

Definition 3-2. A point $p \in \partial\Omega$ is a C^k-peak point if there is a function $f \in A^k(\bar{\Omega})$ such that $f(p)=1$ and $|f|<1$ in $\bar{\Omega}\backslash\{p\}$. If f is holomorphic in $\bar{\Omega}$, then p is said to be a holomorphic peak point.

Remark. Local C^k-peak point and local holomorphic peak point are defined by the same way locally.

Lemma 3-3. *Let Ω be a pseudoconvex Reinhardt domain in* $\mathbf{C}^n$. *If* $z^0=(0, \ldots, 0, z^0_{m+1}, \ldots, z_n) \in \mathrm{N}^1_\Omega(m \geqq 1)$ *or if* $z^0=(z^0_1, \ldots, z^0_n) \in \mathrm{N}^2_\Omega \cap \mathbf{C}^n_0$, *then* z^0 *is not a local* C^0*-peak point.*

Proof. Suppose that z^0 is a local C^0-peak point. Then there exists a neighbourhood U of z^0 and a function f holomorphic in $U \cap \Omega$ and continuous in $U \cap \bar{\Omega}$ such that $f(z^0)=1$ and $|f|<1$ in $U \cap \bar{\Omega}\backslash\{z^0\}$. If $z^0=(0, \ldots, 0, z^0_{m+1}, \ldots, z^0_n) \in \mathrm{N}^1_\Omega(m \geqq 1)$, then by the definition, $\{(a_1\zeta^{\nu_1}, \ldots, a_m\zeta^{\nu_m}, z^0_{m+1}, \ldots, z^0_n);\ 0<|\zeta|<\varepsilon\} \subset \Omega$. Put

$$g(\zeta)=f(a_1\zeta^{\nu_1}, \ldots, a_m\zeta^{\nu_m}, z^0_{m+1}, \ldots, z^0_n)$$

then g is holomorphic in $0<|\zeta|<\varepsilon$ and continuous in $|\zeta|<\varepsilon$, so that g can be continued analytically to $\zeta=0$. Therefore g must be 1 by the maximum principle. But this is a contradiction since we assumed that f is a local C^0-peak function at z^0. Next, if $z^0 \in \mathrm{N}^2_\Omega \cap \mathbf{C}^n_0$, then $\{(z^0_1\ exp\ (a_1\zeta), \ldots, z^0_n\ exp\ (a_n\zeta));\ |\zeta|<\varepsilon\} \subset \partial\Omega$. Take a point $z^{(1)} \in U \cap \Omega \cap \mathbf{C}^n_0$ and put

$$l(t)=\{(|z^0_1|\ exp\ (c_1t), \ldots, |z^0_n|\ exp\ (c_nt));\ t \in \mathbf{R}\}$$

where $c_j=\log|z^{(1)}_j|-\log|z^0_j|(j=1,2, \ldots, n)$.
Since $\log\langle\bar{\Omega}\rangle$ is convex, there exists a positive numbers $\varepsilon'<\varepsilon$ such that

$$\{(z^{(k)}_1\ exp\ (a_1\zeta), \ldots, z^{(k)}_n\ exp\ (a_n\zeta));\ |\zeta|<\varepsilon'\} \subset U \cap \Omega,$$

where $z^{(k)}=l(1/k)$ and k is a positive integer. Put

$$g_k(\zeta)=f(z^{(k)}_1\ exp\ (a_1\zeta), \ldots, z^{(k)}_n\ exp\ (a_n\zeta))$$

then g_k is holomorphic in $|\zeta|<\varepsilon'$ and since f is continuous in $U \cap \bar{\Omega}$, the sequence $\{g_k\}$ converges uniformly to

$$g_0(\zeta)=f(z^0_1\ exp\ (a_1\zeta), \ldots, z^0_n\ exp\ (a_n\zeta)).$$

Therefore g_0 is holomorphic in $|\zeta|<\varepsilon'$ and $g_0(0)=1$, $|g|<1$ in $0<|\zeta|<\varepsilon'$. Then by the maximum principle g is constant, but this is a contradiction.

Lemma 3-4. *Let Ω be a pseudoconvex Reinhardt domain and let $z^0=(0, \ldots, 0, z^0_{m+1}, \ldots, z^0_n)\varepsilon\bar{\Omega}$. If $z=(z_1, \ldots, z_n)\in\bar{\Omega}\cap \mathbf{C}^n_0$, then $z'=(0, \ldots, 0, z_{m+1}, \ldots, z_n)\in\bar{\Omega}\cap\mathbf{C}^n$.*

Proof. Given a positive number ε, we can choose a x_0 with $0<x_0<1$ such that

$$|z_j|\left\{1-\left(\frac{|z^0_j|}{|z_j|}\right)^{x_0}\right\}<\frac{\varepsilon}{\sqrt{n}}\ (j=1, 2, \ldots, n)$$

Further, take a $\tilde{z}\in\Omega\cap\mathbf{C}^n_0$ such that

$$|z_j|\left\{1-\left(\frac{|\tilde{z}_j|}{|z_j|}\right)^{x_0}\right\}<\frac{\varepsilon}{\sqrt{n}}\ (j=m+1, \ldots, n)$$

$$|z_i|^{1-x_0}|\tilde{z}_i|^{x_0}<\frac{\varepsilon}{\sqrt{n}}\ (i=1, 2, \ldots, m).$$

Since $\log\langle\Omega\rangle$ is convex, $(1-x_0)\log\langle z\rangle+x_0\log\langle\tilde{z}\rangle\in\log\langle\bar{\Omega}\rangle$. Then $\langle w\rangle=(|z_1|^{1-x_0}|z_1|^{x_0}, \ldots, |z_n|^{1-x_0}|z_n|^{x_0})\in\bar{\Omega}$ and by a simple calculation, we have $|\langle z'\rangle-\langle w\rangle|<\varepsilon$. This means that $z'\in\bar{\Omega}$.

Proposition 3-5. *Let Ω be a pseudoconvex Reinhardt domain and let $z^0\in\partial\Omega$. If $z^0\in\mathbf{N}^2_\Omega\cap\mathbf{N}^*_\Omega$, then z^0 is a local holomorphic peak point.*

Proof. If $z^0\in\mathbf{C}^n_0$, then we may assume that $z^0=(1, 1, \ldots, 1)$ without loss of generality. Since $\log\langle\Omega\rangle$ is convex, there exists a linear function $L_1(x)=a_1x_1+\ldots+a_nx_n$ such that for any $x\in\log\langle\Omega\rangle$ $L_1(x)<0$. Put

$$D_1=\overline{\log\langle\Omega\rangle}\cap\{x\in\mathbf{R}^n;\ L_1(x)=0\}.$$

Since our proposition is local, we have only to consider in a small neighbourhood U of z^0. Then we may assume that $L_1(x)=x_1$ without loss of generality. Firstly, we consider the case that $D_1=\{(0, \ldots, 0)\}$. For $0\leqq x_1\leqq 1$ put

$$\rho_j(x_1)=\sup\{|z_j|;\ (x_1, |z_2|, \ldots, |z_n|)\in\langle U\cap\bar{\Omega}\rangle\}.$$

It is easily seen that $\rho_j(1)=1$ and for small positive number δ, $\rho_j(x_1)$ is bounded continuous decreasing in $1-\delta\leqq x_1\leqq 1$. Here we need the following lemma ([6]pp. 205-206).

Lemma. *Let f be an incresing, continuous zero free function on $[0, 1]$ such that $f(1)=1$. Then for any positive series $\sum_j a_j=1$, there is a sequence of positive integers n_j such that $g(t)=\sum_j a_jt^{n_j}$ converges on $[0, 1]$, $g(t)<f(t)$ when $0\leqq t<1$ and $g(1)=1$.*

From this lemma, and from the fact that $\sum_k 1/k!=e$, we can choose positive integers m^j_k large enough such that they are all prime numbers and monotone increasing with respect to k and that

$$\rho_j(x_1)\cdot\sum\left(\frac{x_1{}^{m_k^j}}{e\cdot k!}\right)<1 \text{ if } 1-\delta<x_1<1.$$

Put

$$h_j(z_1)=\frac{1}{e}\cdot\sum_k\frac{z_1{}^{m_k^j}}{k!}$$

Then $|h_j(z_1)|\leqq 1$ in $|z_1|\leqq 1$ and $h_j(z_1)=1$ if and only if $z_1=1$. In fact if $h_j(z_1)=1$, then all the $z_1{}^{m_k^j}$ must be 1. Put $z_1=\exp(i\theta)$ then $m_k^j\theta=2n_k^j\pi$ for some positive integers n_k^j. Therefore $\theta/2\pi$ is a rational number. Put $\theta/2\pi=\alpha/\beta$ where $(\alpha, \beta)=1$. Then since m_k^j are monotone increasing prime numbers and since $(\alpha, \beta)=1$, we have $\beta=1$, that is $z_1=1$. Put

$$f(z)=z_2\cdot z_3 \ \dots\ z_n\cdot h_2(z_1)\ \dots\ h_n(z_1)$$

then it is easily seen that f is a local holomorphic peak function at z^0. If $D_1\supsetneqq\{(0, \dots, 0)\}$, then since D_1 is convex in the real hyperplane $L_1(x)=0$ and since $z^0\notin \mathbf{N}_{\Omega}^2$, we have $(0, \dots, 0)\in\partial D_1$. Therefore there exists a linear function $L_2(x)$ such that D_1 lies in the side $L_2(x)<0$. Now we consider the case where

$$D_1\cap\{x\in\mathbf{R}^n;\ L_2(x)=0\}=\{(0, \dots, 0)\}.$$

Then as the first case we may assume that $L_1(x)=x_1$, $L_2(x)=x_2$. For $0\leqq x_1\leqq 1$, $0\leqq x_2\leqq 1$, we put

$$\rho_j(x_1, x_2)=\sup\ \{|z_j|;\ (x_1, x_2, |z_3|, \dots, |z_n|)\in\langle U\cap\bar{\Omega}\rangle\}.$$

Then as before we can choose positive integers $m_{k_1}^j$, $m_{k_2}^j$ large enough such that

$$\frac{1}{e^2}\rho_j(x_1, x_2)\cdot\sum\left(\frac{x_1{}^{m_{k_1}^j}\ x_2{}^{m_{k_2}^j}}{k_1!\ k_2!}\right)<1 \text{ if } 1-\delta<x_1<1,\ 1-\delta<x_2<1.$$

Put

$$h_j(z_1, z_2)=\frac{1}{e^2}\cdot\sum\left(\frac{z_1{}^{m_{k_1}^j}\ z_2{}^{m_{k_2}^j}}{k_1!\ k_2!}\right)$$

then $f(z)=z_3\ \dots\ z_n\cdot h_3(z_1, z_2)\ \dots\ h_n(z_1, z_2)$ is a local holomorphic peak function at z^0. We can repeat this process. Finally if $\overline{\log\langle\Omega\rangle}\cap\{x\in\mathbf{R}^n;\ L_1(x)=\ \dots\ =L_n(x)=0\}=\{(0, \dots, 0)\}$, then

$$f(z)=\frac{1}{2-z_1}\ \cdots\cdots\ \frac{1}{2-z_n}$$

is a local holomorphic peak function at z^0. Next if $z^0=(0, \dots, 0, z_{m+1}^0, \dots, z_n^0)$ $(m\geqq 1)$, then by Lemma 3-4, $\Omega_{12\cdots m}$ is a pseudoconvex Reinhardt domain in $\mathbf{C}^{n-m}$. Put $(z^0)'=(z_{m+1}^0, \dots, z_n^0)$, then it is easily seen that $(z^0)'\in\partial\Omega_{12\cdots m}$ and that $(z^0)'$ satisfies the condition (ii) of Definition 3-1. Therefore there is a local holomorphic peak function

$g(z_{m+1}, \ldots, z_n)$ at $(z^0)'$. Since $z^0 \notin \mathbf{N}^*_\Omega$, $f(z)=g(z_{m+1}, \ldots, z_n)$ is a local holomorphic peak function at z^0.

By Lemma 3-3 and Proposition 3-5, we have

Corollary 3-6. $\mathbf{N}^1_\Omega \subset \mathbf{N}^2_\Omega \cup \mathbf{N}^*_\Omega$.

Theorem 3-7. *Let Ω be a bounded preudoconvex Reinhardt domain and let $z^0 \in \partial\Omega$. If $z^0 \notin \mathbf{N}^2_\Omega \cup \mathbf{N}^*_\Omega$, then there exists a holomorphic peak function at z^0.*

Proof. For simplicity, we assume that Ω is $(1, 2, \ldots, k)$-complete. Firstly we consider the case $z^0 \in \mathbf{C}^n_0$. Put

$$\Omega'_\varepsilon = \{z \in \mathbf{C}^n_0;\ d(\log \langle z\rangle, \log \langle \Omega\rangle) < \varepsilon\},$$

$$\Omega_\varepsilon = \{(z'_1, \ldots, z'_k, z_{k+1}, \ldots, z_n) \in \mathbf{C}^n;\ (z_1, \ldots, z_n) \in \Omega'_\varepsilon, |z'_j| \leqq |z_j| j=1, 2, \ldots, k\}$$

where d is a euclidean distance. Then Ω_ε is a pseudoconvex Reinhardt domain by Theorem 2-4. Let U be a small neighbourhood of z^0 which does not contain the origin and let f be a local holomorphic peak function at z^0 defined in U. Put $L(z)=\log g(z)$ where we take the principal branch. Choose positive numbers λ and ε such that

(i): $U(\lambda)=\{z \in \mathbf{C}^n;\ |z-z^0|<\lambda\} \subset U$

(ii): $\{z \in U;\ L(z)=0\} \cap \Omega_\varepsilon \backslash U(\lambda/3)=\phi$.

Take a C^∞ function $\eta(t)$ with $0 \leqq \eta(t) \leqq 1$ such that

$$\eta(t)=\begin{cases} 1 & \text{if } t<\lambda/3 \\ 0 & \text{if } t \geqq 2\lambda/3 \end{cases}$$

and put

$$g(z)=\begin{cases} \bar\partial\eta(|z-z^0|)/L(z) & \text{if } z \in U(\lambda) \\ 0 & \text{if } z \in \Omega_\varepsilon - U(\lambda). \end{cases}$$

Then $g(z)$ is a $\bar\partial$-closed $C^\infty(0, 1)$ form in Ω_ε. Since Ω_ε is pseudoconvex there exists a C^∞ function $u(z)$ in Ω_ε such that $\bar\partial u=g$. Since $\bar\Omega \subset \Omega_\varepsilon$, $|u(z)|<\mathrm{M}$ in $\bar\Omega$ for some positive constant M. Put

$$\Phi(z)=\frac{\eta(|z-z^0|)}{L(z)} - u(z) - \mathrm{M}$$

then $\bar\partial\Phi=0$ in $\bar\Omega\backslash\{z^0\}$, so that Φ is holomorphic in $\bar\Omega\backslash\{z^0\}$. It is easily seen that Re $(1/L(z))<0$ in $\bar\Omega \cap U\backslash\{z^0\}$. Then by a simple calculation we have Re $\Phi(z)<0$ in $\bar\Omega\backslash\{z^0\}$, that is

$$\Phi(\bar\Omega\backslash\{z^0\}) \subset \{w \in \mathbf{C};\ \mathrm{Re}\ w<0\}.$$

Put $h(w)=(w+1)/(w-1)$, then h maps the left half plane into the unit disc. Put

$$\Psi(z)=h(\Phi(z))=\frac{\eta(|z-z^0|)-(u(z)+M-1)L(z)}{\eta(|z-z^0|)-(u(z)+M+1)L(z)}$$

Then $\Psi(z)$ is holomorphic and $|\Psi(z)|<1$ in $\bar{\Omega}\backslash\{z^0\}$. For $z\in U(\lambda/3)$,

$$\Psi(z)=\frac{1-(u(z)+M-1)L(z)}{1-(u(z)+M+1)L(z)}$$

so that Ψ can be continued analytically to z^0. Evidently this function is the global holomorphic peak function at z^0. Next, we consider the case $z^0=(0, \ldots, 0, z^0_{m+1}, \ldots, z^0_n)$. Since $\Omega_{12\cdots m}$ is a pseudoconvex Reinhardt domain by the same method we can find a holomorphic peak function $f(z_{m+1}, \ldots, z_n)$ at $(z^0)'$. Put $\Psi(z)=f(z_{m+1}, \ldots, z_n)$ then since $z^0\notin N^*_\Omega$, Ψ is a global holomorphic peak function at z^0.

Note. By the detailed observation of the boundary, we can show that if Ω is a bounded pseudoconvex Reinhardt domain with smooth real analytic boundary, then every points of $\partial\Omega$ are holomorphic peak points. The proof will be appeared elsewhere.

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