NON-EXISTENCE OF NON-TRIVIAL HARMONIC FORMS ON A COMPLETE RIEMANNIAN MANIFOLD WITH A BOUNDARY

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Introduction

About 25 years ago, P.E. Conner (1), G.D. Duff-D.C. Spencer (2), T. Nakae (7), T. Takahashi (9) and others studied the harmonic forms on a compact riemannial manifold with boundary.

In this note, we shall discuss L_2-harmonic forms on a complete non-compact riemannian manifold with only one compact boundary by the method owing to (5), (6), (8) and (10).

We shall be in C^∞-category. And, Greek indices run from 1 to $n+1$ and Latin ones run from 2 to $n+1$. We use the Einstein convention.

1. Preliminaries.

Let $\bar{M}=M\cup\partial M$ be an $(n+1)(n\geqq 1)$ dimensional, complete, non-compact, connected, orientable riemannian manifold with only one boundary ∂M and with metric g (or $<\ ,\ >$). We suppose that ∂M is the compact connected n dimensional manifold. We may regard $\bar{M}$ as the closure of an open submanifold of an $(n+1)$ dimensional connected, orientable riemannian manifold $\mathfrak{M}$. At each point x of ∂M there exists a coordinate patch $(U\,;(x^\lambda))$ of x in $\mathfrak{M}$ such that $U\cap\bar{M}$ is represented by $x^1\leqq 0$. In particular, $U\cap\partial M$ is represented by $x^1=0$ and (x^i) is the induced coordinate system of ∂M. We call such a $(U\,;(x^\lambda))$ a boundary coordinate patch.

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Let $(V\,;\,(y^\lambda))$ be an another boundary coordinate patch such that $U\cap V\neq\phi$. Then we have

$$\frac{\partial y^1}{\partial x^1}>0 \text{ and } \frac{\partial y^1}{\partial x^i}=0 \text{ for } i=2,\ \cdots\cdots,\ n+1.$$

Since the Jacobian of the coordinate transformation of $(U\,;\,(x^\lambda))$ and $(V\,;\,(y^\lambda))$ is positive, the Jacobian of the induced coordinate transformation of $(U\cap\partial M\,;\,(x^i))$ and $(V\cap\partial M\,;\,(y^i))$ is also positive. Therefore the boundary coordinate patch defines an orientation of ∂M. Hereafter we assume that ∂M is oriented by the orientation.

At the each point of ∂M there exist two unit vectors normal to ∂M. In each boundary coordinate patch, the first contravariant component of the one is positive and that of the other is negative. We denote the former (which is called the outer unit normal vector field) by N.

Let $\iota:\partial M\longrightarrow\bar{M}$ be the inclusion. If $(U\,;\,(x^\lambda))$ is a boundary coordinate patch of a point x of ∂M in $\mathfrak{M}$ and $(U'\,;\,(u^i))$ a coordinate patch of x in ∂M such that $U'\subset U\cap\bar{M}$, ∂M may be represented by

$$x^\lambda=x^\lambda(u^i).$$

We set $B_i^\lambda:=\dfrac{\partial x^\lambda}{\partial u^i}$ and $B_{i_1\cdots\cdots i_p}^{\lambda_1\cdots\cdots\lambda_p}=B_{i_1}^{\lambda_1}\cdots\cdots B_{i_p}^{\lambda_p}$. Then the induced metric $'g:=\iota^*g=('g_{ij})$ of ∂M is given by $'g_{ij}=g_{\lambda\mu}B_{ij}^{\lambda\mu}$, $g=(g_{\lambda\mu})$, and we have

$$\det\,('g_{ij})=\det\,(g_{\lambda\mu})(\det\,(N^\lambda,\ B_1^\lambda,\ \cdots\cdots,\ B_n^\lambda))^2.$$

Let ∇ (resp. $'\nabla$) be the riemannian connection on $\bar{M}$ (resp. ∂M) with respect to g (resp. $'g$). Then the equations of Gauss and Weingarten may be written in the form;

$$\nabla_{\iota_*X}\iota_*Y=\iota_*('\nabla_XY)+h(X,\ Y)N,$$

and

$$\nabla_{\iota_*X}N=\iota_*(-AX)$$

for any vector fields X and Y on ∂M, where A is the tensor field of type $(1,\ 1)$ such that $h(X,\ Y)='g(AX,\ Y)$.

2. L_2-harmonic forms.

Let $\Lambda^p(\partial M)$ be the space of all $C^\infty p$-forms on ∂M. $\phi\in\Lambda^p(\partial M)$ is locally represented by

$$\phi=\textstyle\sum_{i_1<\cdots\cdots<i_p}\phi_{i_1\cdots\cdots i_p}du^{i_1}\wedge\cdots\cdots\wedge du^{i_p}.$$

For any $\phi \in \Lambda^p(\partial M)$, the operators H and $\hat{H}$ are defined respectively by

$$H(\phi)(X_1, \cdots\cdots, X_p) := \phi(AX_1, X_2, \cdots\cdots, X_p) \\ -\sum_{i=2}^{p} \phi(AX_i, X_2, \cdots\cdots, \underset{i}{X_1}, \cdots\cdots, X_p)$$

and

$$\hat{H}(\phi) := (\text{trace } A)\phi - H(\phi),$$

for any vector fields $X_1, \cdots\cdots, X_p$ on ∂M.

Let $\Lambda^p(\bar{M})$ be the space of all $C^\infty p$-forms on $\bar{M}$ and $\Lambda_0^p(\bar{M})$ the subspace of $\Lambda^p(\bar{M})$ composed of forms with compact support. For any $\phi \in \Lambda^p(\bar{M})$, $t\phi \in \Lambda^p(\partial M)$ and $n\phi \in \Lambda^{p-1}(\partial M)$ are defined respectively by

$$(t\phi)(X_1, \cdots\cdots, X_p) := (\iota^*\phi)(X_1, \cdots\cdots, X_p),$$

and

$$(n\phi)(X_1, \cdots\cdots, X_{p-1}) := \phi(N, \iota_* X_1, \cdots\cdots, \iota_* X_{p-1})$$

for any vector fields $X_1, \cdots\cdots, X_p$ on ∂M.
We call $t\phi$ (resp. $n\phi$) the tangential (resp. the normal) part of ϕ and a form ϕ with $n\phi = 0$ (resp. $t\phi = 0$) tangential (resp. normal) to ∂M. Then, considering ϕ, $t\phi$ and $n\phi$ as contravariant tensors, we have the following formula:

$$\phi = \iota_*(t\phi) + N \wedge \iota_*(n\phi).$$

The $*$-operator on $\Lambda^p(\bar{M})$ is defined by,

$$*\phi := \sum_{\substack{\lambda_1 < \cdots\cdots < \lambda_p \\ \mu_1 < \cdots\cdots < \mu_{n+1-p}}} sgn\begin{pmatrix} 1, & \cdots\cdots, & p, & p+1, & \cdots\cdots, & n+1 \\ \lambda_1, & \cdots\cdots, & \lambda_p, & \mu_1, & \cdots\cdots, & \mu_{n+1-p} \end{pmatrix} \\ \times (\det (g_{\lambda\mu}))^{1/2} g^{\lambda_1\mu_1} \cdots\cdots g^{\lambda_p\mu_p} \phi_{\mu_1 \cdots\cdots \mu_p} dx^{\mu_1} \wedge \cdots\cdots \wedge dx^{\mu_p}$$

for

$$\phi := \sum_{\lambda_1 < \cdots\cdots < \lambda_p} \phi_{\lambda_1 \cdots\cdots \lambda_p} dx^{\lambda_1} \wedge \cdots\cdots \wedge dx^{\lambda_p}.$$

Then we may define a prehilbertian metric $< \ , \ >$ on $\Lambda_0^p(\bar{M})$ by

$$<\phi, \psi> = \int_{\bar{M}} (\phi, \psi) d\sigma = \int_{\bar{M}} \phi \wedge * \psi$$

where $d\sigma$ is the volume element of $\bar{M}$. We set $\|\phi\| := <\phi, \phi>^{1/2}$ which we call the norm of ϕ.

Let $d : \Lambda^p(\bar{M}) \longrightarrow \Lambda^{p+1}(\bar{M})$ be the exterior differntial. The adjoint operator δ : $\Lambda^p(\bar{M}) \longrightarrow \Lambda^{p-1}(\bar{M})$ is defined by $\delta\phi := (-1)^{(n+1)p+n} * d * \phi$, $\phi \in \Lambda^p(\bar{M})$. Then we have $<d\phi, \psi> = <\phi, \delta\psi>$ for $\phi \in \Lambda_0^p(\bar{M})$ and $\psi \in \Lambda_0^{p+1}(\bar{M})$.

LEMMA 2.1 (cf. (9), (11)). *For any* $\phi,\ \psi \in \Lambda^p(\bar{M})$,

(1) $d(t\phi)=t(d\phi),\ n(\delta\phi)=-\delta(n\phi)$

(2) $d(n\phi)=-n(d\phi)+t(\nabla_N\phi)-H(t\phi)$

(3) $\delta(t\phi)=t(\delta\phi)+n(\nabla_N\phi)-\hat{H}(n\phi)$

(4) $(\phi,\ \psi)=(t\phi,\ t\psi)+(n\phi,\ n\psi)$.

Let $L^p_2(\bar{M})$ be the completion of $\Lambda^p_0(\bar{M})$ with respect to the inner product $<\ ,\ >$, and $L_2(\bar{M}) := \sum L^p_2(\bar{M})$. If $\phi \in L_2(\bar{M})$, then $t\phi$ and $n\phi$ are in $L_2(\bar{M})$ because of the compactness of ∂M. Let ∂ be the restriction of d to $\Lambda^p_0(\bar{M})$ and θ the restriction of δ to $\Lambda^p_0(\bar{M})$. We set $\bar{\partial} := (\theta)^*$ and $\bar{\theta} := (\bar{\partial})^*$ where $(\ )^*$ is the adjoint operator to $(\ )$ with respect to the inner product $<\ ,\ >$. Then $\bar{\partial}$ (resp. $\bar{\theta}$) is a closed, densely defined operator of $L^p_2(\bar{M})$ into $L^{p+1}_2(\bar{M})$ (resp. $L^{p-1}_2(\bar{M})$). Let $D^p_{\bar{\partial}}$ (resp. $D^p_{\bar{\theta}}$) be the domain of the operator $\bar{\partial}$ (resp. $\bar{\theta}$) in $L^p_2(\bar{M})$. We put

$$Z^p_{\bar{\partial}}(\bar{M}) := \{\phi \in D^p_{\bar{\partial}} \mid \bar{\partial}\phi = 0\}$$

and

$$Z^p_{\bar{\theta}}(\bar{M}) := \{\phi \in D^p_{\bar{\theta}} \mid \bar{\theta}\phi = 0\}.$$

Since $\bar{\partial}$ and $\bar{\theta}$ are closed operators, $Z^p_{\bar{\partial}}(\bar{M})$ and $Z^p_{\bar{\theta}}(\bar{M})$ are closed in $L^p_2(\bar{M})$. Let $B^p_{\bar{\partial}}(\bar{M})$ (resp. $B^p_{\bar{\theta}}(\bar{M})$) be the closure of $\bar{\partial}(D^{p-1}_{\bar{\partial}})$ (resp. $\bar{\theta}\,(^{p+1}_{\bar{\theta}})$).

DEFINITION 2.1. $H^p_2(\bar{M}) := Z^p_{\bar{\partial}}(\bar{M}) \ominus B^p_{\bar{\partial}}(\bar{M})$ *is the* L_2*-integrable cohomology space. Here* $\ominus$ *is the orthogonal complement.*

Note that $H^p_2(\bar{M}) = Z^p_{\bar{\partial}}(\bar{M}) \cap Z^p_{\bar{\theta}}(\bar{M})$. Since $Z^p_{\bar{\partial}}(\bar{M})$ and $Z^p_{\bar{\theta}}(\bar{M})$ are closed in $L^p_2(\bar{M})$, $H^p_2(\bar{M})$ has canonically the structure of Hilbert space.

THEOREM2.1. (*The orthogonal decomposition theorem*).

$$L^p_2(\bar{M}) = H^p_2(\bar{M}) \oplus B^p_{\bar{\partial}}(\bar{M}) \oplus B^p_{\bar{\theta}}(\bar{M}).$$

In fact, it is sufficient to note that $B^p_{\bar{\partial}}(\bar{M})$ and $B^p_{\bar{\theta}}(\bar{M})$ are mutually orthogonal and $Z^p_{\bar{\partial}}(\bar{M}) \cap (B^p_{\bar{\theta}}(\bar{M}))^{\perp} = H^p_2(\bar{M})$.

THEOREM 2.2. $H^p_2(\bar{M}) = H^{(n+1)-p}_2(\bar{M})$ (*isomorphic as Hilbert space*).

In fact, it is sufficient to note that the following diagram is commutative;

$$\begin{tikzcd}[row sep=large, column sep=huge]
\Lambda^p_0(\bar{M}) \arrow[r, "*"] \arrow[d, shift right, "\bar{\theta}"'] & \Lambda^{(n+1)-p}_0(\bar{M}) \arrow[d, shift left, "\bar{\partial}"] \\
\Lambda^{p-1}_0(\bar{M}) \arrow[r, "(-1)^p *"'] \arrow[u, shift right, "\bar{\partial}"'] & \Lambda^{(n+1)-p+1}_0(\bar{M}) \arrow[u, shift left, "\bar{\theta}"]
\end{tikzcd}$$

COROLLARY. *If the dimension of* $H_2^p(\bar{M})$ *is finite, dim* $H_2^p(\bar{M}) = $ *dim* $H_2^{(n+1)-p}(\bar{M})$.

We consider a function μ on R (the reals) satisfying

(1) $0 \leqq \mu \leqq 1$ on R,

(2) $\mu(t) = 1$ for $t \leqq 1$,

(3) $\mu(t) = 0$ for $t \geqq 2$.

Then we set

$$w_k(p) := \mu\left(\frac{\rho(p)}{k}\right),\ k = 1,\ 2,\ 3,\ \cdots\cdots$$

where $\rho(p)$ is the distance from $p \in \bar{M}$ to ∂M.

LEMMA 2.2. *Under the above notations, there exists a positive constant* A *depending only on* μ *such that*

(1) $\|dw_k \wedge \phi\|^2 \leqq \dfrac{(n+1)A^2}{k^2}\|\phi\|^2$ (2) $\|dw_k \wedge * \phi\|^2 \leqq \dfrac{(n+1)A^2}{k^2}\|\phi\|^2$

for all $\phi \in \Lambda_0^p(\bar{M})$.

In fact, ρ is a locally Lipschitz function and $|d\rho|^2 \leqq n+1$. Then $dw_k = \dfrac{1}{k}\dfrac{d\mu}{dt}d\rho$ at the point where the derivatives of ρ exist. We set $A := \sup |\dfrac{d\mu}{dt}|$.

DEFINITION 2.2. *The Laplacian acting on* $\Lambda^p(\bar{M})$ *is defined by* $\Box = -(d\delta + \delta d)$.

For any $\phi \in L_2^p(\bar{M}) \cap \Lambda^p(\bar{M})$, we have

$$\begin{aligned}(2.1)\qquad & <d\phi,\ d\psi>_{B(2k)} + <\delta\phi,\ \delta\psi>_{B(2k)} \\ & = -<\Box\phi,\ \psi>_{B(2k)} + \int_{\partial B(2k)} [(nd\phi,\ t\psi) - (t\delta\phi,\ n\psi)]d\sigma'\end{aligned}$$

for all $\psi \in \Lambda_{B(2k)}^p(\bar{M})$, where $\Lambda_{B(2k)}^p(\bar{M})$ is the space of all p-forms with compact support in $B(2k)$ and $B(2k)$ is the compact set $\{p \in \bar{M} | \rho(p) \leqq 2k\}$. Here $d\sigma'$ is the volume form of $\partial B(2k)$.

For $\psi := w_k^2\phi$, we have

$$(2.2)\qquad \begin{aligned} d\psi &= w_k^2 d\phi + 2w_k dw_k \wedge \phi \\ \delta\psi &= w_k^2 \delta\psi + (-1)^{(n+1)p+n} * (2w_k dw_k \wedge * \phi).\end{aligned}$$

Then, (2.1) may be written in the form:

$$\begin{aligned}(2.3)\qquad & <d\phi,\ w_k^2 d\phi>_{B(2k)} + <d\phi,\ 2w_k dw_k \wedge \phi>_{B(2k)} \\ & + <\delta\phi,\ w_k^2\delta\phi>_{B(2k)} + (-1)^{(n+1)p+n} <\delta\phi,\ *(2w_k dw_k \wedge * \phi)>_{B(2k)}\end{aligned}$$

$$=-<\Box\phi,\ w_k^2\phi>_{B(2k)}+\int_{\partial B(2k)}[(nd\phi,\ tw_k^2\phi)-(t\delta\phi,\ nw_k^2\phi)]d\sigma'.$$

On the other hand, we have,

$$|<d\phi,\ 2w_k dw_k\wedge\phi>_{B(2k)}|\leqq\frac{(n+1)A^2}{k^2}(\|w_k d\phi\|^2_{B(2k)}+\|\phi\|^2_{B(2k)}) \tag{2.4.1}$$

$$|<\delta\phi,\ *(2w_k dw_k\wedge *\phi>_{B(2k)}|\leqq\frac{(n+1)A^2}{k^2}(\|w_k\delta\phi\|^2_{B(2k)}+\|\phi\|^2_{B(2k)}) \tag{2.4.2}$$

$$|<-\Box\phi,\ w_k^2\phi>_{B(2k)}|\leqq\frac{1}{2}\Big(\frac{1}{\sigma}\|w_k\delta\phi\|^2_{B(2k)}+\sigma\|w_k\Box\phi\|^2_{B(2k)}) \tag{2.4.3}$$

for every $\sigma>0$.

And, we have,

$$\int_{\partial B(2k)}[(nd\phi,\ w_k t\phi)-(w_k*\delta\phi,\ w_k n\phi)]d\sigma'=\int_{\partial M}[(nd\phi,\ t\phi)-(t\delta\phi,\ n\phi)]d\sigma'. \tag{2.5}$$

In fact, it is sufficient to note that $\partial B(2k)=\partial M\cap\{p\in M|\rho(p)=2k\}$ and $w_k\phi=0$ on $\{p\in M|\rho(p)=2k\}$.

By (2.3), (2.4) and (2.5), we have the inequality

$$\begin{aligned}&\|w_k d\phi\|^2_{B(2k)}+\|w_k\delta\phi\|^2_{B(2k)}\\ &\leqq\sigma\|w_k\Box\phi\|^2_{B(2k)}+\Big(\frac{1}{\sigma}+\frac{8(n+1)A^2}{k^2}\Big)\|\phi\|^2_{B(2k)}+2\Big|\int_{\partial M}[(nd\phi,\ t\phi)-(t\delta\phi,\ n\phi)]d\sigma'\Big|.\end{aligned} \tag{2.6}$$

In particular, setting $\Box\phi=0$ and letting $\sigma\to\infty$, we have

$$\begin{aligned}&\|w_k d\phi\|^2_{B(2k)}+\|w_k\delta\phi\|^2_{B(2k)}\\ &\leqq\frac{8(n+1)A^2}{k^2}\|\phi\|^2_{B(2k)}+2\Big|\int_{\partial M}[(nd\phi,\ t\phi)-(t\delta\phi,\ n\phi)]d\sigma'\Big|\end{aligned} \tag{2.7}$$

Letting $k\to\infty$, we have

$$\|d\phi\|^2+\|\delta\phi\|^2\leqq 2\Big|\int_{\partial M}[(nd\phi,\ t\phi)-(t\delta\phi,\ n\phi)]d\sigma'\Big| \tag{2.8}$$

for any $\phi\in L_2^p(\bar{M})\cap\Lambda^p(\bar{M})$.

LEMMA 2.3. *For any $\phi\in L_2^p(\bar{M})\cap\Lambda^p(\bar{M})$ such that $\Box\phi=0$,*

(1) *if $t\phi=0$ and $t\delta\phi=0$ on ∂M, then $d\phi=0$ and $\delta\phi=0$*

(2) *if $n\phi=0$ and $nd\phi=0$ on ∂M, then $d\phi=0$ and $\delta\phi=0$.*

Let N_d^p (resp. N_δ^p) be the subspace $\{\phi\in\Lambda^p(\bar{M})|d\phi=0\}$ (resp. $\{\phi\in\Lambda^p(\bar{M})|\delta\phi=0\}$) of $\Lambda^p(\bar{M})$. For any $\phi\in\Lambda^p(\bar{M})$, $w_k\phi\in D_{\bar\partial}^p\cap D_{\bar\theta}^p$ and,

$$\bar{\partial}(w_k\phi)=d(w_k\phi),\ \ \bar{\theta}(w_k\phi)=\delta(w_k\phi).$$

Then, noting that for any $\phi \in N^p_d \cap L^p_2(\bar{M})$,

$$\begin{aligned}\bar{\partial}(w_k\phi) &= d(w_k\phi)\\ &= dw_k \wedge \phi + w_k d\phi\\ &= dw_k \wedge \phi,\end{aligned}$$

we have

$$\|\bar{\partial}(w_k\phi)\|^2 \leqq \frac{(n+1)A^2}{k^2}\|\phi\|^2.$$

Setting $\phi_k = w_k\phi$, we have

$$\bar{\partial}\phi_k \to 0 \text{ (as } k \to \infty) \text{ (strong)},$$

and,

$$\phi_k \to \phi \text{ (as } k \to \infty) \text{ (strong)}.$$

Since $\bar{\partial}$ is a closed operator, $\phi \in D^p_{\bar{\partial}}$ and $\bar{\partial}\phi = 0$, which implies that $\phi \in Z^p_{\bar{\partial}}(\bar{M})$. In the same way, we have $N^p_{\bar{\partial}} \cap L^p_2(\bar{M}) \subset Z^p_{\vartheta}(\bar{M})$.

LEMMA 2.4. $N^p_d \subset L^p_2(\bar{M}) \cap Z^p_{\bar{\partial}}(\bar{M})$, $N^p_\delta \subset L^p_2(\bar{M}) \cap Z^p_{\vartheta}(\bar{M})$.

Lemma 2.3 and 2.4 imply the following,

PROPOSITION 2.1. *For any* $\phi \in L^p_2(\bar{M}) \cap \Lambda^p(\bar{M})$ *such as* $\Box\phi = 0$,

(1) *if* $t\phi = 0$ *and* $t\delta\phi = 0$ *on* ∂M, *then* $\phi \in H^p_2(\bar{M})$,

(2) *if* $n\phi = 0$ *and* $nd\phi = 0$ *on* ∂M, *then* $\phi \in H^p_2(\bar{M})$.

3. Non-existence of L_2-harmonic fields.

The following formula is well-known:

$$-\Box\phi = \sum_{\lambda_1 < \cdots\cdots < \lambda_p}\Big(-\nabla^\lambda \nabla_\lambda \phi_{\lambda_1 \cdots\cdots \lambda_p} + \sum_{t=1}^p R_{\lambda_t}{}^{\alpha} \phi_{\lambda_1 \cdots \underset{\hat{t}}{\alpha} \cdots \lambda_p} \tag{3.1}$$
$$+ \sum_{1 \leqq t < s \leqq p} R_{\lambda_t \lambda_s}{}^{\alpha\beta} \phi_{\lambda_1 \cdots \underset{\hat{t}}{\alpha} \cdots \underset{\hat{s}}{\beta} \cdots \lambda_p}\Big) dx^{\lambda_1} \wedge \cdots\cdots \wedge dx^{\lambda_p}$$

for $\phi = \sum_{\lambda_1 < \cdots\cdots < \lambda_p} \phi_{\lambda_1 \cdots\cdots \lambda_p} dx^{\lambda_1} \wedge \cdots\cdots \wedge dx^{\lambda_p} \in \Lambda^p(\bar{M})$. Here and hereafter, we set

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

for any vector fields X, Y, Z on $\bar{M}$,

$$R(\partial/\partial x^\lambda, \partial/\partial x^\mu)\partial/\partial x^\kappa = R_{\lambda\mu\kappa}{}^\nu \partial/\partial x^\nu$$

and

$$\mathrm{Ricci}(\bar{M})=(R_{\lambda\mu})=(R_{\alpha\lambda\mu}{}^{\alpha}).$$

Setting

$$(F\phi)_{\lambda_1\cdots\cdots\lambda_p}:=\sum_{t=1}^{p}R_{\lambda_t}{}^{\alpha}\phi_{\lambda_1\cdots\cdots\underset{\hat{t}}{\alpha}\cdots\cdots\lambda_p}+\sum_{1\le t<s\le p}R_{\lambda_t\lambda_s}{}^{\alpha\beta}\phi_{\lambda_1\cdots\cdots\underset{\hat{t}}{\alpha}\cdots\cdots\underset{\hat{s}}{\beta}\cdots\cdots\lambda_p},$$

(3.1) is written in the form;

$$-\square\phi=\sum_{\lambda_1<\cdots\cdots<\lambda_p}(-\nabla^{\lambda}\nabla_{\lambda}\phi_{\lambda_1\cdots\cdots\lambda_p}+(F\phi)_{\lambda_1\cdots\cdots\lambda_p})dx^{\lambda_1}\wedge\cdots\cdots\wedge dx^{\lambda_p}.$$

And, for $\phi\in\Lambda^p(\bar{M})$, we set

$$\Phi:=\Phi_{\lambda}dx^{\lambda},\ \text{where}\ \Phi_{\lambda}=(\nabla_{\lambda}\phi_{\lambda_1\cdots\cdots\lambda_p})\phi^{\lambda_1\cdots\cdots\lambda_p}.$$

Noting that $*(w_k^2\Phi)$ is the n-form with compact support in $B(2k)$, by means of Stokes' formula, we have

$$\int_{B(2k)}d(*(w_k^2\Phi))=\int_{\partial B(2k)}*(w_k^2\Phi). \tag{3.2}$$

Then we have

$$\int_{B(2k)}d*(w_k^2\Phi)=\int_{\partial M}*\Phi \tag{3.3}$$

In fact, since $\partial B(2k)=\partial M\cup\{p\in M|\rho(p)=2k\}$, $w_k^2\Phi=0$ on $\{p\in M|\rho(p)=2k\}$ and $w_k^2\Phi=\Phi$ on ∂M.

The right hand side of (3.3) is

$$\int_{\partial M}(\nabla_{\lambda}\phi_{\lambda_1\cdots\cdots\lambda_p})\phi^{\lambda_1\cdots\cdots\lambda_p}N^{\lambda}d\sigma'=\int_{\partial M}(\nabla_N\phi,\ \phi)d\sigma'.$$

On the other hand, the left hand side of (3.3) is

$$\begin{aligned}\int_{B(2k)}*(*d*(w_k^2\Phi))&=-\int_{B(2k)}*\delta(w_k^2\Phi)\\&=-\int_{B(2k)}*\{*(\mathrm{w}_k^2\delta\Phi)-(2w_kdw_k\wedge*\Phi)\}\\&=\int_{B(2k)}w_k^2\nabla^{\lambda}((\nabla_{\lambda}\phi_{\lambda_1\cdots\cdots\lambda_p})\phi^{\lambda_1\cdots\cdots\lambda_p})d\sigma+\int_{B(2k)}(2w_kdw_k\wedge*\Phi)\\&=<w_k\square\phi,\ w_k\phi>_{B(2k)}+<w_kF(\phi),\ w_k\phi>_{B(2k)}\\&\qquad+<w_k\nabla\phi,\ w_k\nabla\phi>_{B(2k)}+\int_{B(2k)}2w_kdw_k\wedge*\Phi.\end{aligned}$$

In fact, it is sufficient to note that

$$\begin{aligned}&\nabla^{\lambda}((\nabla_{\lambda}\phi_{\lambda_1\cdots\cdots\lambda_p})\phi^{\lambda_1\cdots\cdots\lambda_p})\\&=(\nabla^{\lambda}\nabla_{\lambda}\phi_{\lambda_1\cdots\cdots\lambda_p})\phi^{\lambda_1\cdots\cdots\lambda_p}+(\nabla_{\lambda}\phi_{\lambda_1\cdots\cdots\lambda_p})(\nabla^{\lambda}\phi^{\lambda_1\cdots\cdots\lambda_p})\\&=(\square\phi)_{\lambda_1\cdots\cdots\lambda_p}\phi^{\lambda_1\cdots\cdots\lambda_p}+(F(\phi))_{\lambda_1\cdots\cdots\lambda_p}\phi^{\lambda_1\cdots\cdots\lambda_p}+(\nabla_{\lambda}\phi_{\lambda_1\cdots\cdots\lambda_p})(\nabla^{\lambda}\phi^{\lambda_1\cdots\cdots\lambda_p}).\end{aligned}$$

Then, we have the equality

$$(3.4)\qquad -<w_k\Box\phi,\ w_k\phi>_{B(2k)}=<w_kF(\phi),\ w_k\phi>_{B(2k)}+<w_k\nabla\phi,\ w_k\nabla\phi>_{B(2k)} + \int_{B(2k)} 2w_k dw_k\wedge *\Phi-\int_{\partial M}(\nabla_N\phi,\ \phi)d\sigma'.$$

Hence, we have the inequality

$$(3.5)\qquad |<w_k\Box\phi,\ w_k\phi>_{B(2k)}|\ \geqq <w_kF(\phi),\ w_k\phi>_{B(2k)}+\|w_k\nabla\phi\|^2_{B(2k)} -|\int_{B(2k)} 2w_k dw_k\wedge *\Phi|-\int_{\partial M}(\nabla_N\phi,\ \phi)d\sigma'.$$

By (2.4.3) and the inequality

$$|\int_{B(2k)} 2w_k dw_k\wedge *\Phi|\leqq\frac{(n+1)^{1/2}A}{k}(\|w_k\nabla\phi\|^2_{B(2k)}+\|\phi\|^2_{B(2k)}),$$

we have

$$(3.6)\qquad \sigma\|w_k\Box\phi\|^2_{B(2k)}+\frac{1}{\sigma}\|w_k\phi\|^2_{B(2k)} \geqq 2<w_kF(\phi),\ w_k\phi>_{B(2k)}+2\left(1-\frac{(n+1)^{1/2}A}{k}\right)\|w_k\nabla\phi\|^2_{B(2k)} -\frac{2(n+1)^{1/2}A}{k}\|\phi\|^2_{B(2k)}-2\int_{\partial M}(\nabla_N\phi,\ \phi)d\sigma'.$$

In particular, setting $\Box\phi=0$ and letting $\sigma\to\infty$, we have

$$0\geqq\lim_{k\to\infty}\ \sup<w_kF(\phi),\ w_k\phi>_{B(2k)}+\|\nabla\phi\|^2-\int_{\partial M}(\nabla_N\phi,\ \phi)d\sigma.$$

And, by Lemma 2.1, we have,

$$\int_{\partial M}(\nabla_N\phi,\ \phi)d\sigma'=\int_{\partial M}\{(Ht\phi,\ t\phi)+(\hat{H}n\phi,\ n\phi) +(nd\phi,\ t\phi)-(t\delta\phi,\ n\phi)+2(dn\phi,\ t\phi)\}d\sigma'.$$

Therefore, we have,

$$0\geqq\lim_{k\to\infty}\ \sup<w_kF(\phi),\ w_k\phi>_{B(2k)}+\|\nabla\phi\|^2 -\int_{\partial M}\{(Ht\phi,\ t\phi)+(\hat{H}n\phi,\ n\phi)+(nd\phi,\ t\phi)-(t\delta\phi,\ n\phi)+2(dn\phi,\ t\phi)\}d\sigma'.$$

DEFINITION 3.1. *A p-form ϕ on $\bar{M}$ is called a harmonic field if $d\phi=0$ and $\delta\phi=0$.*

Note that harmonic field ϕ satisfies the equation $\Box\phi=0$, but the converse does not hold because of the non-compactness of $\bar{M}$.

THEOREM I. *Suppose that there exists a non-negative function* λ *on* $\bar{M}$ *such that* $(F(\phi), \phi) \geqq \lambda(\phi, \phi)$ *for all* $\phi \in \Lambda^p(\bar{M})$. *If* $\lambda > 0$ *and bounded away from zero and* H (*resp.* $\hat{H}$) *is negative semi-definite on* ∂M, *there exists no non-trivial harmonic* p*-field in* $L_2^p(\bar{M})$ *tangential (resp. normal) to* ∂M. *Moreover, if* $\lambda = 0$, *a harmonic* p*-field* ϕ *in* $L_2^p(\bar{M})$ *tangential (resp. normal) to* ∂M *is parallel and* $H(t\phi) = 0$ (*resp.* $\hat{H}(n\phi) = 0$).

In fact, it is sufficient to note that there exists a non-negative constant $K(\lambda \geqq K \geqq 0)$ such that

$$\lim_{k\to\infty} \sup < w_k F(\phi),\ w_k \phi >_{B(2k)} \geqq K\|\phi\|^2.$$

EXAMPLE. We consider the product manifold $\bar{M} = S^1 \times [0, \infty)$ with a riemannian metric $ds^2 = f^2(t)d\theta^2 + dt^2$, $f(t) > 0$. Then a 1-form $\phi = \phi_1 d\theta + \phi_2 dt$ is harmonic field if and only if

$$\frac{\partial\phi_2}{\partial\theta} = \frac{\partial\phi_1}{\partial t} \quad \text{and} \quad \frac{\partial\phi_1}{\partial\theta} + f\frac{df}{dt}\phi_2 + f^2\frac{\partial\phi_2}{\partial t} = 0.$$

The Ricci curvature Ricci $(\bar{M})$ of $\bar{M}$ is given by $-\dfrac{1}{f}\dfrac{d^2 f}{dt^2}$ and A by $A(\partial/\partial\theta) = -\left(\dfrac{1}{f}\dfrac{df}{dt}\right)_{t=0}\partial/\partial\theta$. Setting $f(t) = e^{t^2}$, we have Ricci $(M) = -(4t^2+2) < 0$ and $A(\partial/\partial\theta) = 0$. If $\phi = \alpha d\theta$ (α : a non-zero constant), then $\phi \in L_2(\bar{M})$. Further, it is easily seen that ϕ is harmonic field. But $(\nabla_{\partial/\partial\theta}\phi)(\partial/\partial t) = -\dfrac{1}{f}\dfrac{df}{dt} = -2\alpha t$ is not zero unless $t = 0$. Therefore ϕ is not parallel.

4. Non-existence of L_2-harmonic 1-forms.

The following theorem is proved by R. Ichida (3) and A. Kasue (4) independently;

THEOREM 4.1. *Let* $\bar{M} = M \cup \partial M$ *be an* $(n+1)(n \geqq 1)$ *dimensional, connected, complete riemannian manifold with only one compact doundary* ∂M. *Suppose that* $\bar{M}$ *is of non-negative Ricci curvature and the second fundamental form of* ∂M *with respect to the outer unit normal vector field is non-positive everywhere. Then* $\bar{M}$ *is isometric to a riemannian product* $[0, \infty) \times \partial M$.

Let $\bar{M} = M \cup \partial M$ be the above manifold in theorem 4.1, where the riemannian metric is $ds^2 = dx_1^2 + \sum_{i,j=2}^n g_{ij}(x^k)dx^i dx^j$ in each coordinate patch $(U; (x^\lambda))$. Noting that ∂M is totally geodesic, we may construct a complete riemannian manifold $\bar{\bar{M}}$ which is the simple double of $\bar{M}$ (cf. (1)). If we replace ρ as the distance $\bar{\rho}$ from a fixed point $o \in \partial M$ to $p \in \bar{\bar{M}}$, we may prove Lemma 4.1 by the similar argument in §2.

LEMMA 4.1. *Under the above assumptions, if* $\phi \in L_2^p(\bar{M})$ *satisfies* $\square\phi=0$ *then* $d\phi=0$ *and* $\delta\phi=0$, i.e. $\phi \in H_2^p(\bar{M})$.

THEOREM II. *Let* $\bar{M}=M\cup\partial M$ *be an* $(n+1)(n\geqq 1)$ *dimensional, connected, complete riemannian manifold with only one compact doundary* ∂M. *Suppose that* $\bar{M}$ *is of non-negative Ricci curvature and the second fundamental form of* ∂M *is non-positive everywhere with respect to the outer unit normal vector field. Moreover, suppose that there exists a non-negative function* λ *on* $\bar{M}$ *such that* $(F(\phi), \phi)\geqq\lambda(\phi, \phi)$ *for all* $\phi\in\Lambda^p(\bar{M})$. *If* $\lambda>0$ *and bounded away from zero, there exists no non-zero harmonic p-form in* $L_2^p(\bar{M})$ *tangential (or normal) to* ∂M. *Moreover, if* $\lambda=0$, *a harmonic p-form in* $L_2^p(\bar{M})$ *tangential (or normal) to* ∂M *is parallel.*

Now, we consider only 1-forms. For an 1-form

$$\phi=\phi_1(x^\lambda)dx^1+\textstyle\sum_{j=2}^{n}\phi_j(x^\lambda)dx^j,$$

we have

$$\nabla_i\phi_j={}'\nabla_i\phi_j,\ \ \nabla_1\phi_j=\frac{\partial\phi_j}{\partial x^1},\ \ \nabla_i\phi_1=\frac{\partial\phi_1}{\partial x^i}\ \text{and}\ \nabla_1\phi_1=\frac{\partial\phi_1}{\partial x^1}. \tag{4.1}$$

LEMMA 4.2. *If* $\phi=\phi_1dx^1+\sum_{j=2}^{n+1}\phi_jdx^j$ *in* $L_2^1(\bar{M})\cap\Lambda^1(\bar{M})$ *is parallel, then* ϕ *is zero.*

In fact, if ϕ is parallel, then $\phi_1=C$ (constant) and $\phi_j=\phi_j(x^i)$ by (4.1). Then $\int_{\bar{M}}<\phi, \phi>d\sigma=\int_0^\infty\left(\int_{\partial M}g^{ij}\phi_i\phi_jd\sigma'\right)dt+$ vol $(\partial M)\int_0^\infty C\ dt$. Therefore, if $\phi\in L_2(\bar{M})$ then $\phi=0$.

THEOREM III. *Let* $\bar{M}=M\cup\partial M$ *be an* $(n+1)(n\geqq 1)$ *dimensional, connected, complete riemannian manifold with only one compact boundary* ∂M. *Suppose that* $\bar{M}$ *is of non-negative Ricci curvature and the second fundamental form of* ∂M *with respect to the outer normal vector field is non-positive everywhere. Then there exists no nontrivial harmonic 1-form in* $L_2(\bar{M})$ *tangential (or normal) to* ∂M.

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