Resolvents for One-Dimensional Diffusion Operators with Real Analytic Coefficients

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1. Introduction

In this note, we shall make improvements and supplementations to the result in my paper [3].

Given $-\infty < r_0 < r_1 < \infty$, let $I = [r_0, r_1]$ be a compact interval and let $C^2(I)$ be the space of twicely differentiable real functions on I. Denote by $C^{\omega}(I)$ the space of real analytic functions on I. Suppose that $a, b, c \in C^{\omega}(I)$, $a \geqq 0$ on I and $a(r_i) = 0 \leqq (-1)^i b(r_i)$ $(i=0, 1)$. Then we are concerned with a diffusion operator L acting on $C^2(I)$ with coefficients a, b, c:

$$L = a(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx} + c(x).$$

For functions

$$d_{kj} = \binom{k}{j-2} a^{(k-j+2)} + \binom{k}{j-1} b^{(k-j+1)} + \binom{k}{j} c^{(k-j)\;**)} \qquad (0 \leqq j \leqq k \leqq n),$$

set

$$\lambda_n = \max_{0 \leqq j \leqq n} \sum_{k=j}^{n} \|d_{kj}\|,$$

where $\| \;\|$ denotes the supremum norm.

Further assume that $a(x) > 0$ for $x \in I^0 = (r_0, r_1)$ and both r_i $(i=0, 1)$ are simple zeros of $a(x)$. Then, in [3], it is proved that the resolvent $\{G_\lambda\}$ for L has the analyticity preserving property for $\lambda > \lambda_2$ (see [3, Theorem]). Furthermore, it is shown that the condition on L is best possible. The proof of the theorem is based on the result of Ethier [1] on the differentiability preserving property of the semigroup associated with L.

In this note, it will be shown that the lower bound λ_2 can be replaced by a

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**) Set $\binom{k}{-2} = \binom{k}{-1} = 0$.

constant $\|c^+\|$, where c^+ is the positive part of the function c, that is, $c^+(x)=\max(c(x), 0)$. The lower bound $\|c^+\|$ is best possible in the general scheme of the semigroup theory. The proof is carried out by using the characterization of the infinitesimal generator of the semigroup.

2. Preliminaries

Let $C(I)$ be the space of continuous real functions on I and let

$$L_0=a(x)\frac{d^2}{dx^2}+b(x)\frac{d}{dx}.$$

Using probabilistic methods, Ethier [1] obtains the following result. An extension of L acting on $C^2(I)$ generates a unique strongly continuous non-negative semigroup $\{T_t\}$ on $C(I)$ (see [1, Proposition 1]). If it is further assumed that $a>0$ on I^0, the infinitesimal generator $\hat{L}$ of $\{T_t\}$ is precisely the restriction of L to $\mathfrak{D}_0$:

$$\mathfrak{D}_0=\{f \in C(I)\cap C^2(I^0)\cap L_0^{-1}C(I) : \exp(B_0)f' \text{ vanishes at } L_0\text{-regular boundaries of } I \text{ and } L_0 f \text{ vanishes at } L_0\text{-exit boundaries of } I\},$$

where $B_0(x)=\int_r^x \frac{b(y)}{a(y)}\,dy$ $(r \in I^0)$ (see [1, Proposition 2]).

In the paper [2], Norman gives a useful characterization of the boundary classification with respect to the operator L_0. From Theorem 2 in [2], it follows that if $a>0$ on I^0 and $a'(r_i)\neq 0$, then

$$\frac{b(r_i)}{a'(r_i)}=0 \text{ (i.e., } b(r_i)=0) \text{ iff } r_i \text{ is exit,}$$

$$0<\frac{b(r_i)}{a'(r_i)}<1 \text{ iff } r_i \text{ is regular,}$$

$$\frac{b(r_i)}{a'(r_i)}\geqq 1 \text{ iff } r_i \text{ is entrance.}$$

3. Main result

From the probabilistic representation of the semigroup $\{T_t\}$ (see [1, Proposition 1]), it follows that $\|T_t\|\leqq\exp(\|c^+\|t)$. Therefore the resolvent $\{G_\lambda\}$ for the operator $\hat{L}$, i.e., $G_\lambda=\int_0^\infty \exp(-\lambda t)T_t dt$ is defined for $\lambda>\|c^+\|$.

Before the result is stated, we shall summarize the assumption on the coefficients of L.

Assumption. (1) $a, b, c \in C^\omega(I)$,

(2) $a(x)>0$ for $x \in I^0$,

(3) $a(r_i)=0\leqq(-1)^i b(r_i)$ $\quad (i=0, 1)$,

(4) both r_i $(i=0, 1)$ are simple zeros of $a(x)$.

THEOREM. *Under the assumption,*

$$G_\lambda: C^\omega(I) \to C^\omega(I) \text{ for } \lambda > \|c^+\|$$

and consequently

$$(L-\lambda)(C^\omega(I)) = C^\omega(I) \text{ for } \lambda > \|c^+\|.$$

PROOF. Let f be a real analytic function on I. If $\lambda > \|c^+\|$, then $u = G_\lambda f \in \mathfrak{D}_0$ (the domain of $\hat{L}$). Therefore $u \in C^2(I^0)$ and $(L-\lambda)u = -f$ on I^0. Hence $u \in C^\omega(I^0)$ (cf. the proof of Theorem in [3]). Suppose that $i=0$. Since $\lambda \neq c(r_0)$, using Lemma in [3], we see that the equation $(L-\lambda)u = -f$ has a real analytic solution u_0 in a neighborhood of r_0. Hence, for some $\delta > 0$, $u - u_0$ is a C^ω-solution of the following homogeneous equation in $(r_0, r_0+\delta)$:

$$(3.1) \qquad (x-r_0)^2 v'' + (x-r_0)P(x)v' + Q(x)v = 0,$$

where $P(x) = (x-r_0)b(x)/a(x)$ and $Q(x) = (x-r_0)^2\{c(x)-\lambda\}/a(x)$. By the assumption, $P(r_0) = b(r_0)/a'(r_0) \geqq 0$ and $Q(r_0) = 0$. Since the roots of the indicial equation $\rho(\rho-1) + P(r_0)\rho + Q(r_0) = 0$ for (3.1) relative to $x = r_0$ are 0 and $1 - P(r_0)$, a fundamental system of solutions $\{v_1(x), v_2(x)\}$ of (3.1) is given in the following form:

(i) if $\rho_1 \equiv 1 - P(r_0) = 0$ or 1, then

$$v_1(x) = (x-r_0)^{\rho_1} \sum_{n=0}^{\infty} a_n (x-r_0)^n,$$

$$v_2(x) = A v_1(x) \log(x-r_0) + \sum_{n=0}^{\infty} b_n (x-r_0)^n,$$

where a_n, A, b_n are real numbers, $a_0 \neq 0$, and $\sum a_n(x-r_0)^n$, $\sum b_n(x-r_0)^n$ converge in a neighborhood of r_0;

(ii) if $\rho_1 \neq 0, 1$, then

$$v_1(x) = (x-r_0)^{\rho_1} \sum_{n=0}^{\infty} a_n (x-r_0)^n,$$

$$v_2(x) = \sum_{n=0}^{\infty} c_n (x-r_0)^n,$$

where a_n and c_n are real numbers, $a_0 \neq 0$, $c_0 \neq 0$, and $\sum a_n(x-r_0)^n$, $\sum c_n(x-r_0)^n$ converge in a neighborhood of r_0.

Assume that δ is sufficiently small. Then there exist constants k_1 and k_2 such that

$$(3.2) \qquad u(x) - u_0(x) = k_1 v_1(x) + k_2 v_2(x) \qquad (r_0 < x < r_0 + \delta).$$

In the case (i) with $A = 0$, $u(x)$ is real analytic at r_0. If $A \neq 0$ and $\rho_1 = 0$ (i.e., r_0 is entrance), then from (3.2)

$$k_2 v_2(x) = u(x) - u_0(x) - k_1 v_1(x).$$

The right side is continuous on $[r_0, r_0+\delta)$; so that $k_2 = 0$. Therefore $u(x) = u_0(x) + k_1 v_1(x)$ is real analytic at r_0. If $A \neq 0$ and $\rho_1 = 1$ (i.e., r_0 is exit), then from (3.2)

$$k_2 A a_0(x-r_0)\log(x-r_0)+k_2 A a_1(x-r_0)^2\log(x-r_0)$$
$$=u(x)-\left\{u_0(x)+k_1 v_1(x)+k_2 A\log(x-r_0)\sum_{n=2}^{\infty} a_n(x-r_0)^{n+1}\right\}.$$

Since the function in { } of the right side is a C^2-function on $[r_0, r_0+\delta)$, it has an extension $w(x)$ of $C^2(I)$. On the other hand, $C^2(I)\subset\mathfrak{D}_0$ (see [1, Lemma 2]). Therefore $u(x)-w(x)$ belongs to $\mathfrak{D}_0$; so that the left side has an extension $g(x)$ of $\mathfrak{D}_0$. Since r_0 is exit, $g(x)$ satisfies the boundary condition:

$$\lim_{x\downarrow r_0} L_0 g(x)=k_2 A a_0 a'(r_0)=0.$$

Therefore $k_2=0$; so that $u(x)$ is real analytic at r_0.

In the case (ii) with $0<\rho_1<1$, r_0 is regular. From (3.2)

$$k_1 a_0(x-r_0)^{\rho_1}+k_1 a_1(x-r_0)^{\rho_1+1}$$
$$=u(x)-\left\{u_0(x)+k_2 v_2(x)+k_1\sum_{n=2}^{\infty} a_n(x-r_0)^{\rho_1+n}\right\}.$$

The function in { } of the right side is a C^2-function on $[r_0, r_0+\delta)$. Therefore, in the same way as in the case (i) with $A\neq 0$ and $\rho_1=1$, the left side has an extension $h(x)$ of $\mathfrak{D}_0$. Since r_0 is regular, $h(x)$ satisfies the boundary condition:

$$\lim_{x\downarrow r_0}\exp(B_0(x))h'(x)=k_1 a_0\rho_1\exp(B(r_0))=0,$$

where $B(x)$ is a continuous function on I. Therefore $k_1=0$; so that $u(x)=u_0(x)+k_2 v_2(x)$ is real analytic at r_0.

Finally, if $\rho_1<0$, r_0 is entrance. Then the right side of

$$k_1 v_1(x)=u(x)-u_0(x)-k_2 v_2(x)$$

is continunus on $[r_0, r_0+\delta)$. Hence $k_1=0$; so that $u(x)$ is real analytic at r_0.

The function $u(x)$ is also real analytic at r_1. Consequently, $u=G_\lambda f \in C^\omega(I)$ for $\lambda>\|c^+\|$. Q.E.D.

In [3], some examples are given to show that the conditions (2) and (4) are necessary to the conclusion of Theorem. Concerning such examples, we give a remark.

Remark. Let $L=x^2(1-x)\dfrac{d^2}{dx^2}$, $I=[0, 1]$ and $f(x)=x^2$. Then $G_\lambda f \notin C^\omega(I)$ for every $\lambda>0$ (cf. [3, Example 1]). Moreover, from the proof of Theorem, it follows that $G_\lambda f$ is real analytic on $(0, 1]$ for every $\lambda>0$ and for $\lambda\neq m(m-1)$ $(m=2, 3, \cdots\cdots)$ the main singular part of $G_\lambda f$ is $x^{1/2(1+\sqrt{1+4\lambda})}$.

4. Appendix

The following proposition is used in [3]. We give some addition to the proof.

Proposition. *Suppose that a linear ordinary differential equation with analytic data has a* C^∞*-solution in an interval. Then there exists a formal power-series soluiton* *of the equation at each point of the interval. If it is further assumed that all the singular points of the equation are regular, the* C^∞*-solution is a* C^ω*-solution in the interval.*

The proof of the first part is easy. The second part can be shown in the same way as in the proof of Theorem in [3]. However we need the following lemma for higher order equations.

LEMMA. *Let* λ_k $(k=1, 2, \cdots\cdots, n)$ *be mutually distinct real numbers and let* c_k $(k=1, 2, \cdots\cdots, n)$ *be complex numbers. If* $\lim_{t\to\infty} \sum_{k=1}^{n} c_k \exp(i\lambda_k t)=0$ $(i=\sqrt{-1})$, *then* $c_k=0$ *for* $k=1, 2, \cdots\cdots, n$.

PROOF. The function $\sum_{k=1}^{n} c_k \exp(i\lambda_k t)$ of the real variable t is a uniformly almost periodic function. Therefore this lemma is proved by using the following property of uniformly almost periodic functions: Let $f(t)$ be a uniformly almost periodic function with $\lim_{t\to\infty} f(t)=c$. Then $f(t)\equiv c$.

References

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2. M.F. Norman, *A "psychological" proof that certain Markov semigroups preserve differentiability,* SIAM-AMS Proc. **13** (1981), 197-211.
3. M. Tsuchiya, *Analyticity preserving properties of resolvents for degenerate diffusion operators in one dimension,* Proc. Amer. Math. Soc. **90** (1984) (to appear).

Note added in proof. Let $I=[0, 1]$, and for a negative number b, set $L=x\,(1-x)\,(d^2/dx^2)+b\,(d/dx)$. Then $C^2(I) \subsetneqq \mathfrak{D}_0$, and we can show that $G_\lambda: C^\omega(I) \not\to C^\omega(I)$ for every $\lambda>0$ and hence $G_\lambda: C^\infty(I) \not\to C^\infty(I)$ for every $\lambda>0$. This show that the condition (3) of Assumption is also necessary to the conclusion of Theorem.