On the structure semigroups of L-subalgebras generated by spectral measures

Yûich KANJIN

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Introduction. Let **T** be the circle group realized as **R**/**Z**. Let M (**T**) be the set of all bounded regular Borel measures on **T**. It is known that M (**T**) is a commutative Banach algebra with the convolution product and the norm of total variation and contains L^1(**T**) as a closed ideal. The object of this paper is to obtain the identification of the structure semigroup of an L-subalgebra of M(**T**) which is generated by a spectral measure.

Yu. A. Šreider [5] gave a description of the elements of the maximal ideal space Δ of M (**T**) as generalized characters. Moreover J. L. Taylor [6] showed the following result ; For every convolution measure algebra N (e.g., an L-subalgebra of M (**T**)), there exists a compact abelian jointly continuous semigroup Σ (N) (the structure semigroup of N) such that N is embedded as a weak* dense L-subalgebra of the measure algebra M (Σ(N)) and the complex homomorphisms of N are induced by the continuous semicharacters of Σ (N). G. Brown and W. Moran [1] gave the successful description of the structure semigroup of a certain single generator L-subalgebra of M (**T**). In this paper, using the work of M. Queffelec [4] , we shall obtain the description of the structure semigroup of an L-subalgebra of M (**T**) generated by a spectral measure.

1. Preliminaries and definitions. A closed subalgebra N of M (**T**) is called an L-subalgebra if $\nu \in N$ whenever $\nu \in M$ (**T**), $\mu \in N$ and $\nu \ll \mu$ (ν is absolutely continuous with respect to μ). An element $\chi = \{\chi_\mu \,;\mu \in N\}$ of the product space

$$\prod_{\mu \in N} L^\infty(\mu)$$

is called a generalized character of an L-subalgebra N of M (**T**) if

(1) $\chi_\mu = \chi_\nu$ (ν a.e.) if $\nu \ll \mu$,

(2) $\chi_{\mu*\nu}(x+y)=\chi_{\mu}(x)\chi_{\nu}(y)$ ($\mu\times\nu$ a.e. (x, y)), and

(3) $1\geqq \sup\{\|\chi_{\mu}\|_{\infty};\ \mu\in N\}>0$.

Every generalized character χ of N gives rise to a complex homomorphism of N according to the formula

$$\mu\mapsto\int\chi_{\mu}\,d\mu\ (=\hat{\mu}(\chi)=\chi(\mu))$$

for every $\mu\in N$ and in this way the maximal ideal space $\Delta(N)$ of N can be realized as the set of all generalized characters of N with the topology induced from the $\sigma(L^{\infty}(\mu), L^{1}(\mu))$-topology on each factor in the product space (cf. Yu. A. Šreider [5]). For $\chi=\{\chi_{\mu}\}$ and $\xi=\{\xi_{\mu}\}$ in $\Delta(N)$ we define $\chi\xi, \overline{\chi}$ and $|\chi|$ by $(\chi\xi)_{\mu}=\chi_{\mu}\xi_{\mu}$, $(\overline{\chi})_{\mu}=\overline{\chi}_{\mu}$ and $|\chi|_{\mu}=|\chi_{\mu}|$ respectively, where these operations are defined pointwise in $L^{\infty}(\mu)$ for each $\mu\in N$. These operations yield new elements of $\Delta(N)$ and $\Delta(N)$ forms a separately continuous semigroup.

For $\mu\in N$, we denote by $\Delta(N)_{\mu}$ the space $\{\chi_{\mu};\ \chi\in\Delta(N)\}$ with the $\sigma(L^{\infty}(\mu), L^{1}(\mu))$-topology. The space $\Delta(N)_{\mu}$ is regarded as a subsemigroup of $L^{\infty}(\mu)$. For a measure $\mu\in M(\mathbf{T})$, we denote by $N(\mu)$ the L-subalgebra of $M(\mathbf{T})$ generated by μ. It is known that $\Delta(N(\mu))$ and $\Delta(N(\mu))_{\mu}$ are homeomorphic as topological spaces and isomorphic as semigroups by the map $\mu\mapsto\chi_{\mu}$ (cf. [1]).

Let $q=\{q_1, q_2,\cdots, q_n,\cdots\}$ be a sequence of integers such that $q_n\geqq 2$ $(n=1, 2, \cdots)$. Let $p_0=1$ and $p_n=q_1q_2\cdots q_n$ $(n=1, 2,\cdots)$. A complex sequence $\alpha=\{\alpha(0), \alpha(1),\cdots, \alpha(n),\cdots\}$ is called q-multiplicative if

$$\alpha(a+bp_n)=\alpha(a)\,\alpha(bp_n)$$

for all integers n, a and b such that $n\geqq 0$, $b\geqq 0$ and $0\leqq a<p_n$. For a q-multiplicative sequence α such that $|\alpha(n)|=1$ $(n=0, 1,\cdots)$, the limit

$$\gamma(k)=\lim_{N\to\infty}\frac{1}{N}\sum_{j=0}^{N-1}\alpha(j+k)\overline{\alpha(j)}$$

exists for all integers $k\geqq 0$ (cf. [2] , [4]). Set $\gamma(k)=\overline{\gamma(-k)}$ for negative integers k. By the Bochner-Herglotz theorem, there exists a probability measure λ such that

$$\hat{\lambda}(k)=\int_{\mathrm{T}}e^{2\pi ikx}d\lambda(x)=\gamma(k)\ \text{for all integers}\ k.$$

The measure λ is called the spectral measure associated with a q-multiplicative sequence α such that $|\alpha(n)|=1$ $(n=0, 1, ..)$. The class of all such measures is denoted by $\mathscr{S}$. We define some subsets of $\mathscr{S}$ in the following;

$\mathscr{S}_c=\{\lambda\in\mathscr{S};\ \lambda$ is continuous$\}$,

$\mathscr{S}_d=\{\lambda\in\mathscr{S};\ \lambda$ is discrete$\}$,

$\mathscr{S}_0=\{\lambda\in\mathscr{S};\ \lambda^n$ is in $L^1(\mathbf{T})$ for some positive integer $n\}$, and

$\mathscr{S}_1=\{\lambda\in\mathscr{S};\ \delta(x)*\lambda^n$ and λ^m are mutually singular for all $x\in\mathbf{T}$ and all positive integers n and m such that $n\neq m\}$,

where λ^n is the n times convolution product and $\delta(x)$ is the unit mass concentrated at a point x. These classes were studied by J. Coquet, T. Kamae and M. Mendès France [2] and M. Queffelec [4]. And, it was proved that $\mathscr{S}$ is the disjoint union of $\mathscr{S}_c$ and $\mathscr{S}_d$ ([2]). Let H be a countable subgroup of T. A measure $\mu\in M(\mathbf{T})$ is called H-ergodic if for all Borel sets E of $\mathbf{T}$ such that $E+H=E$, either $|\mu|(E)=0$ or $|\mu|(E)=\|\mu\|$. A measure μ is called H-quasi-invariant if $|\mu|(x+E)=0$ for every $x\in H$ and all Borel sets E such that $|\mu|(E)=0$. We denote by D the countable subgroup of $\mathbf{T}$ generated by $\{1/p_n;\ n=0, 1, \ldots\}$. M. Queffelec showed that the following results ([4]);

(A) If $\lambda\in\mathscr{S}$, then λ is D-ergodic,

(B) if $\lambda\in\mathscr{S}_c$, then λ is D-quasi-invariant, and

(C) $\mathscr{S}_c=\mathscr{S}_0\cup\mathscr{S}_1$.

2. Maximal ideal spaces. Let λ be an element of $\mathscr{S}$. We denote by $A(\lambda)$ the smallest L-subalgebra of $M(\mathbf{T})$ containing $\{\lambda, \delta(d);\ d\in D\}$. If we set $A_c(\lambda)=\{\mu\in A(\lambda);\ \mu$ is continuous$\}$ and $A_d(\lambda)=\{\mu\in A(\lambda);\ \mu$ is discrete$\}$, we have a direct sum decomposition $A(\lambda)=A_d(\lambda)\oplus A_c(\lambda)$. We regard D as a discrete group and denote by $\hat{D}$ the dual group of it.

PROPOSITION. (i) If λ is in $\mathscr{S}_c$, then $\Delta(A(\lambda))$ is identified with the disjoint union $\Delta(N(\lambda))_\lambda\cup\hat{D}$.

(ii) If λ is in $\mathscr{S}_1$, then $\Delta(M(\mathbf{T}))_\lambda\supset\{u\in\mathbf{C};\ |u|\leq 1\}$, and

(iii) $\Delta(N(\lambda))_\lambda\cup\{0\}=\Delta(M(\mathbf{T}))_\lambda$, where 0 is the null function in $L^\infty(\lambda)$.

PROOF. Since $A_c(\lambda)$ is a closed ideal in $A(\lambda)$ and $A_d(\lambda)$ is a subalgebra of $A(\lambda)$, $\Delta(A(\lambda))$ is identified with the disjoint union $\Delta(A_c(\lambda))\cup\Delta(A_d(\lambda))$. The algebra $A_d(\lambda)$ is isomorphic with $L^1(D)$ and $A_c(\lambda)=N(\lambda)$ since λ is D-quasi-invariant by (B). Note that $\Delta(N(\lambda))=\Delta(N(\lambda))_\lambda$ and $\Delta(L^1(D))=\hat{D}$. Then we have (i). (A), (B) and C. C. Graham and O. C. McGehee [3, Theorem 6. 1. 5. (iii)] imply that, for every u with $|u| \leqq 1$, there exists $\chi \in \Delta(M(\mathbf{T}))$ such that $\chi_\lambda = u$ (λ a.e.). This means (ii). The inclusion $\Delta(N(\lambda))_\lambda \cup \{0\} \supset \Delta(M(\mathbf{T}))_\lambda$ is obvious. For $\chi \in \Delta(N(\lambda))$, we decompose $\chi = \chi_1\chi_2$ where $|(\chi_1)_\lambda|^2 = |(\chi_1)_\lambda|$ (λ a.e.) and $(\chi_2)_\lambda \geqq 0$ (λ a.e.) (cf. [6]). (A), (B) and [3, Theorem 6. 1. 8. (i)] imply that $|\chi_\lambda| = a$ (λ a.e.) for some $0 < a \leqq 1$. Therefore $|(\chi_1)_\lambda| = 1$ (λ a. e.) and $(\chi_2)_\lambda = a$ (λ a.e.). By the extension theorem (cf. [1, (2.2)]), there exists $\chi_1' \in \Delta M(\mathbf{T}))$ such that $(\chi_1')_\lambda = (\chi_1)_\lambda$($\lambda$ a.e.). By (ii), there exists $\chi_2' \in \Delta(M(\mathbf{T}))$ such that $(\chi_2')_\lambda = a$ (λ a.e.). Since $\chi_1'\chi_2' \in \Delta(M(\mathbf{T}))$ and $(\chi_1'\chi_2')_\lambda = \chi_\lambda$, we have (iii).

3. Structure semigroups. We recall the following definitions ([6], cf. [1, § 6]).

Let N and N' be L-subalgebras of $M(\mathbf{T})$. Let θ be an algebra homomorphism of N into N'. Then θ is called a CM-morphism if the following conditions are satisfied;

(1) If $0 \leqq \mu \in N$, then $\|\theta\mu\| = \|\mu\|$,
(2) if $0 \leqq \mu \in N$, then $\theta\mu \geqq 0$, and
(3) if $\mu \in N$, $\nu \in N'$, $\mu \geqq 0$ and $0 \leqq \nu \leqq \theta\mu$, then

there exists $\omega \in N$ such that $0 \leqq \omega \leqq \mu$ and $\theta\omega = \nu$. It is known that for every L-subalgebra N of $M(\mathbf{T})$ there uniquely exists a compact abelian topological semigroup $\Sigma(N)$ which satisfies the following condition; There exists a CM-morphism $\theta: N \to M(\Sigma(N))$ such that

(1) $\theta(N)$ is dense in $M(\Sigma(N))$ by the $\sigma(M(\Sigma(N)), C(\Sigma(N)))$-topology,
(2) $\hat{\Sigma}(N)$ separates points of $\Sigma(N)$, and
(3) the complex homomorphisms of N are given by

$\mu \to \int f\, d\theta\mu$ for $f \in \hat{\Sigma}(N)$. Here $M(\Sigma(N))$ is the Banach algebra of all bounded regular Borel measures on $\Sigma(N)$ and $\hat{\Sigma}(N)$ is the set of all nontrivial continuous semicharacters on $\Sigma(N)$. We call $\Sigma(N)$ the structure semigroup of N.

Let λ be an element of $\mathscr{S}_1$. For every $\chi \in \Delta(A(\lambda))$, set

$$\phi(\chi)(d) = \hat{\delta}(d)(\chi) \quad (d \in D).$$

Then $\phi(\chi)$ is in $\hat{D}$ for every χ in $\Delta(A(\lambda))$ and ϕ is a continuous semigroup homomorphism of $\Delta(A(\lambda))$ onto $\hat{D}$ such that $\phi(\overline{\chi}) = \overline{\phi(\chi)}$ for every $\chi \in \Delta(A(\lambda))$. We use the following result;

(D) ([4, Lemma 5]) Let λ be an element of $\mathscr{S}_c$ and χ an element of $\Delta(M(\mathbf{T}))$

such that χ_λ is not a null function. Then χ_λ is a constant function if and only if $\phi(\chi)$ is equal to the constant one.

We denote by G the set of all elements of $\Delta(M(\mathbf{T}))_\lambda$ such that the absolute values are equal to the constant function 1. We note that G becomes a group under the multiplication induced from $\Delta(M(\mathbf{T}))_\lambda$, and also by Proposition we can regard G as a subgroup of $\Delta(A(\lambda))$. Set $H=\phi(G)$. Then H is a subgroup of $\hat{D}$ and we regard H as a discrete group. We denote by π the dual homomorphism of the embedding of H into $\hat{D}$ with the discrete topology. Note that π is a surjection of $\bar{D}$ onto $\hat{H}$, where $\bar{D}$ is the Bohr compactification of D.

Let $\mathbf{N}$ be the semigroup of positive integers with the discrete topology. We denote by $\bar{\mathbf{N}}$ the almost periodic compactification of $\mathbf{N}$. And, recall that $\mathbf{N}$ is naturally contained in $\bar{\mathbf{N}}$ and continuous semicharacters of $\bar{\mathbf{N}}$ separate points (cf. [1, § 6]).

Under the above notations, we have the following theorem.

THEOREM. If λ is in $\mathscr{S}_1$, then
(i) $\Sigma(N(\lambda))=\bar{\mathbf{N}}\times\hat{H}$, and
(ii) $\Sigma(A(\lambda))=\bar{D}\cup(\bar{\mathbf{N}}\times\hat{H})$,

where the topology is that of the disjoint union and the multiplication is that of disjoint union together with the linking formula $x+(y, z)=(y, \pi(x)+z)$ $(z\in\bar{D}, y\in\bar{\mathbf{N}}, z\in\hat{H})$.

PROOF. By Proposition, we note that G contains the unit circle $\{u\in\mathbf{C}; |u|=1\}$. Consider the sepuence

$$0\rightarrow\mathbf{T}\xrightarrow{\iota}G\xrightarrow{\phi}H\rightarrow 0,$$

where ι is the map taking $t\in\mathbf{T}$ to the constant function with value $e^{2\pi i t}$. By (D), we have Ker $\phi=\mathrm{Im}\,\iota$, and so the sequence is exact. Since $\mathbf{T}$ is divisible, the exact sequence splits, i. e., there exist homomorphisms $\kappa: G\rightarrow\mathbf{T}$ and $\psi: H\rightarrow G$ such that $\kappa\circ\iota$ and $\phi\circ\psi$ are the identity maps on $\mathbf{T}$ and H respectively. Define a homomorphism $\tau: G\rightarrow\mathbf{T}\times H$ by $\tau(\chi)=(\kappa(\chi), \phi(\chi))$ for each χ in G. Then τ is an isomorphism. Topologize G so that τ gives rise to a homeomorphism. Then the topology of G is stronger than topology induced on G as a subset of $\Delta(A(\lambda))$. Thus for each positive measure $\nu\in A(\lambda)$, $\hat{\nu}|_G$ is a continuous positive definite function on G, where $\hat{\nu}|_G$ is the restriction of the Gelfand transform $\hat{\nu}$ to G. Using Bochner's theorem and the fact that the value $\hat{\nu}|_G$

at the identity element of G is equal to $\|\nu\|$ we hav a positive isometric algebra homomorphism $\theta: A(\lambda)\rightarrow M(\hat{G})$ such that for $\nu\in A(\lambda)$ the Fourier-Stieltjes transform of $\theta(\nu)$ coincides with $\hat{\nu}$ on G. By the same argument as [1, (6.3)], it follows that θ is a CM-morphism. The dual homomorphism τ^{*} induces a CM-isomorphism of $M(\hat{G})$ onto $M(\mathbf{Z}\oplus\hat{H})$. Denote by Θ this map composition Θ. Then Θ is a CM-morphism $A(\lambda)$ into $M(\mathbf{Z}\oplus\hat{H})$.

We show that supp $\Theta(\lambda^{n})=\{n\}\times\hat{H}$ for each positive integer n. Set $f(h)=\hat{\lambda}(\psi(h))$ for every $h\in H$. Then f is a positive definite function on the discrete group H. By Bochner's theorem there exists a positive measure ρ on $\hat{H}$ such that $\hat{\rho}(h)=\hat{\lambda}(\psi(h))$ for every $h\in H$. And, $\Theta(\lambda)$ is equal to the product measure $\delta_{1}\times\rho$ on $\mathbf{Z}\oplus\hat{H}$, where δ_{1} is the unit mass concentrated at 1 in $\mathbf{Z}$. In fact,

$$\begin{aligned}\widehat{(\delta_{1}\times\rho)}(\tau(\chi))&=\hat{\delta}_{1}(\kappa(\chi))\,\hat{\rho}(\phi(\chi))\\&=\iota(\kappa(\chi))\hat{\lambda}(\psi(\phi(\chi)))\\&=\hat{\lambda}(\iota(\kappa(\chi))\,\psi(\phi(\chi)))\\&=\hat{\lambda}(\chi)\\&=\widehat{\Theta(\lambda)}(\tau(\chi))\quad(\chi\in G).\end{aligned}$$

Thus it follows that the support of $\Theta(\lambda)$ is contained in $\{1\}\times\hat{H}$. By (B), we have $\Theta(\lambda)\approx\Theta(\lambda)*\Theta(\delta(d))$ for all $d\in D$. It is not difficult that $\Theta(\delta(d))=\delta(0,\pi(d))$ for every $d\in D$ and $\pi(D)$ is dense in $\hat{H}$. Thus we have supp$\Theta(\lambda)=\{1\}\times\hat{H}$. Since supp $\Theta(\lambda^{n})$ is contained in supp $\Theta(\lambda^{n-1})+$supp $\Theta(\lambda)$ and $\Theta(\lambda^{n})\approx\Theta(\lambda^{n})*\delta(0,\pi(d))$, it follows that supp $\Theta(\lambda^{n})=\{n\}\times\hat{H}$. Note that $\Theta(\nu)$ is supported on $\bar{\mathbf{N}}\times\hat{H}$ for every $\nu\in N(\lambda)$.

We prove assertion (i). The canonical injection $\mathbf{N}\rightarrow\bar{\mathbf{N}}$ induces a CM-morphism from $M(\mathbf{N}\times\hat{H})$ to $M(\bar{\mathbf{N}}\times\hat{H})$. Composing this map with Θ, we obtain a CM-morphism $\Lambda: N(\lambda)\rightarrow M(\bar{\mathbf{N}}\times\hat{H})$. Since semicharacters of $\bar{\mathbf{N}}$ separate points, the same is true of $\bar{\mathbf{N}}\times\hat{H}$. The union of the supports of the measures $\Theta(\lambda^{n})$ $(n=1, 2, \cdots)$ is dense in $\bar{\mathbf{N}}\times\hat{H}$, and hence $\Lambda(N(\lambda))$ is weak* dense in $M(\bar{\mathbf{N}}\times\hat{H})$.

Next we show that the complex homomorphisms of $N(\lambda)$ are given by $\nu\rightarrow\int f\,d\Lambda(\nu)$ for $f\in(\bar{\mathbf{N}}\times\hat{H})$. Let χ be an element of $\Delta(N(\lambda))$. We decompose $\chi=\chi_{1}\chi_{2}$, where $|(\chi_{1})_{\lambda}|=1$ (λ a. e.), $0<a\leqq 1$ (cf. Proof of Proposition). By (ii) and (iii) of Proposition, χ_{1} belongs to G. Thus $\tau(\chi_{1})$ can be regarded as a character of $\mathbf{Z}\oplus\hat{H}$. We have a semicharacter ζ_{1} of $\bar{\mathbf{N}}\times\hat{H}$ which is naturally induced by $\tau(\chi_{1})$. For every $\nu\in N(\lambda)$,

$$\hat{\nu}(\chi_1)=\widehat{\Theta(\nu)}(\tau(\chi_1))$$

$$=\int \tau(\chi_1)\, d\Theta(\nu)$$

$$=\int \xi_1\, d\Lambda(\nu).$$

We define ξ' by $\xi'(n, x)=a^n(n=1, 2, \cdots)$ and denote by ξ_2 the semicharacter on $\bar{\mathrm{N}}\times\hat{H}$ induced by the semicharacter ξ' of $\mathrm{N}\times\hat{H}$, and set $\xi=\xi_1\xi_2$. Then we show that

$$\hat{\nu}(\chi)=\int \xi\, d\Lambda(\nu)$$

for every $\nu\in N(\lambda)$. In fact, for a measure $\nu\in N(\lambda)$, we have a norm convergent decomposition

$$\nu=\sum_{n=1}^{\infty} \nu_n,$$

where each ν_n is a measure which is absolutely continuous with respect to λ^n. Since $(\chi_2)_{\lambda^n}=a^n(\lambda^n \text{a.e.})$ and $\Lambda(\nu_n)$ is supported on $\{n\}\times\hat{H}$, we have

$$\int \xi\, d\Lambda(\nu_n)=a^n\int \xi_1\, d\Lambda(\nu_n)$$

$$=a^n\hat{\nu}_n(\chi_1).$$

Since Λ is bounded, it follows that

$$\int \xi\, d\Lambda(\nu)=\sum_{n=1}^{\infty}\int \xi\, d\Lambda(\nu_n)$$

$$=\sum_{n=1}^{\infty} a^n\hat{\nu}_n(\chi_1)$$

$$=\hat{\nu}(\chi).$$

It is clear that every semicharacter of $\bar{\mathrm{N}}\times\hat{H}$ gives rise to an element of $\Delta(N(\lambda))$. This completes the proof of (i).

We denote by Σ the semigroup $\bar{D}\cup(\bar{\mathrm{N}}\times\hat{H})$ described in (ii). Every $\nu\in A(\lambda)$ can be decomposed in the form $\nu=\nu'+\nu''$, where $\nu'\in N(\lambda)$ and ν'' is the discrete part of ν. We define $\Lambda' : A(\lambda) \to M(\Sigma)$ by

$$\Lambda'(\nu) = \Lambda(\nu') + \Phi(\nu'') \quad (\nu \in A(\lambda)),$$

where Φ is the canonical map from $A_d(\lambda)$ to $M_d(\bar{D})$ regarded as a subalgebra of $M(\Sigma)$. It is not difficult that Λ' is a CM-morphism with weak* dense image. It is clear that the elements of $\hat{\Sigma}$ separate points of Σ. Using the fact that the non zero complex homomorphisms of $N(\lambda)$ correspond to evaluation at a semicharacter of $\bar{\mathbf{N}} \times \hat{H}$, we have that for χ in $\Delta(A(\lambda))$ there exists an semicharacter of Σ which gives rise to χ.

Let η be an element of $\hat{\Sigma}$ which is not identically zero on $\bar{\mathbf{N}} \times \hat{H}$. Since $\eta|_{\bar{\mathbf{N}} \times \hat{H}}$ is a non-trivial semicharacter of $\bar{\mathbf{N}} \times \hat{H}$, there exist $h \in H$ and $a \in$ Cwith $0 < |a| \leq 1$ such that $\eta((n, z)) = a^n h(z)$ for all $n \in \mathbf{N}$ and $z \in \hat{H}$. For all n in $\mathbf{N}$, x in $\overline{D}$ and z in $\hat{H}$, we have

$$\begin{aligned} a^n h(\pi(x)+z) &= \eta((n, \pi(x)+z)) \\ &= \eta(x+(n, z)) \\ &= \eta(x)\, \eta((n, z)) \\ &= \eta(x)\, a^n h(z), \end{aligned}$$

and hence $\eta(x) = h(\pi(x))$ for all $x \in \overline{D}$. Let χ be the element of $\Delta(A(\lambda))$ defined by $\chi_\lambda(s) = a\, \psi(h)(s)$ (λ a.e.). Then we have

$$\int \eta \, d\Phi(\nu'') = \int h \, d\nu''$$

for all $\nu \in A(\lambda)$. Let η be an element of $\hat{\Sigma}$ which is identically zero on $\bar{\mathbf{N}} \times \hat{H}$. Then η is induced by some $\gamma \in \hat{D}$ on $\bar{D}$. Let χ be the generalized character which is zero on $N(\lambda)$ and is induced by γ on $A_d(\lambda)$. We have

$$\int \chi_\nu \, d\nu = \int \gamma \, d\nu''$$

$$= \int \eta \, d\Lambda'(\nu)$$

for all $\nu \in A(\lambda)$. This completes the proof of the theorem.

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