ON some Formulas of Integral Geometry (III)

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(Received Jan. 5, 1970)

Let K be a convex body in the 3-dimensional euclidean space E_3 and V the volume of the body K. Assuming that the straight lines G intersect with the plane E inside the body K, we have the following formula (due to Herglotz) :

$$\int dEdG = 2\pi^2 V,$$

which has been stated in "Vorlesung über Integral Geometrie" (Blashke) without proof.

In the present paper we shall give a proof for this formula and then generalize this formula to the case of n-dimensional euclidean space E_n.

Proof

Let P be the intersection point of the line G with the plane E. We shall consider an orthogonal cartesian system $(\mathbf{P} : \mathbf{A}_1\, \mathbf{A}_2\, \mathbf{A}_3)$ with a unit vector $\mathbf{A}_3$ on the line G.

Let us denote by $d\sigma_k$ the linear differential form (or pfaffe form) which is the scalar product of $d\mathbf{A}_i$ with $\mathbf{A}_j$:

$$d\sigma_k = (d\mathbf{A}_i \cdot \mathbf{A}_j) = -(d\mathbf{A}_j \cdot \mathbf{A}_i)$$

and put

$$d\varphi_1 = [\, d\sigma_1 \; d\sigma_2 \,]$$

where the square brackets standfore the exterior multiplication of the two differentials; thus $d\varphi$ means the spherical element.

Setting

$$\mathbf{A}_i = (\, a_{i1}\, a_{i2}\, a_{i3} \,) \qquad (i = 1,\ 2,\ 3,)$$

and considering that

$$\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} = a_{31}, \quad \begin{vmatrix} a_{13} & a_{11} \\ a_{23} & a_{21} \end{vmatrix} = a_{32}, \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{33}$$

we have

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$$[(d\mathbf{P}\cdot\mathbf{A}_1)\ (d\mathbf{P}\cdot\mathbf{A}_2)]$$

$$=[(dp_1\ a_{11}+dp_2\ a_{12}+dp_3\ a_{13})\ (dp_1\,a_{21}+dp_2\ a_{22}+dp_3\ a_{23})]$$

$$=\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}[dp_2\ dp_3]+\begin{vmatrix} a_{13} & a_{11} \\ a_{23} & a_{22} \end{vmatrix}[dp_3\ dp_1]+\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}[dp_1\ dp_2]$$

$$=a_{31}[dp_2\ dp_3]+a_{32}[dp_3\ dp_2]+a_{33}[dp_1\ ap_2]$$

Hence we have the following density for the lines G:

$$dG=[d\varphi_1\ (d\mathbf{P}\cdot\mathbf{A}_1)\ (d\mathbf{P}\cdot\mathbf{A}_2]$$

$$=[d\varphi_1(a_{31}[dp_2\ dp_3]+a_{32}[dp_3\ dp_1]+a_{33}[dp_1\ dp_2])]$$

Now let $\mathbf{V}$ $(V_1V_2V_3)$ be the unit vector perpendicular to the plane E, then we get

$$dE=[dp_1\ dV_2\ dV_3]+[dp_2\ dV_3\ dV_1]+[dp_3\ dV_1\ dV_2]$$

Hence

$$[dG\ dE]=[d\varphi_1(a_{31}[dp_2\ dp_3]+a_{32}[dp_1\ dp_1]+a_{33}[dp_1\ dp_2])$$

$$([dp_1\ dV_2\ dV_3]+\ [dp_2\ dV_3\ dV_1]+[dp_3\ dV_1\ dV_2])]$$

$$=[d\varphi_1(a_{31}[dpdV_2\ dV_3]+a_{32}[dpdV_3\ dV_1]+a_{32}[dpdV_1\ dV_2])]$$

because

$$[dp_1\ dp_2\ dp_3]=dp$$

Let $d\varphi_2$ be the spherical element of the end point of the unit vector $\mathbf{V}$ perpendicular to the plane E,

Then we have

$$dV_2\ dV_3=V_1\ d\varphi_2,\qquad dV_3\ dV_1=V_2\ d\varphi_2$$

$$dV_1\,dV_2=V_3\,d\varphi_2$$

Hence $[dGdE]$ can be expressed by the following formula

$$[dGdE]=(V_1\,a_{31}+V_2\ a_{32}+V_3\ a_{33})\ [dpd\varphi_1\ d\varphi_2]$$

$$=[dpd\varphi_1\ d\varphi_2\ \cos\ \theta]$$

where θ is the angle determined by the unit vector $\mathbf{A}_3$ and the unit vector $\mathbf{V}$.

Since

$$\int d\varphi\ =\ 2\pi,\qquad \int dp=V\ \text{ and }\ \int\cos\ \theta\ d\varphi=\pi,$$

we have

$$\int dGdE=\int dpd\varphi_1\ d\varphi_2=2\pi^2V$$

Next, we consider a convex body K in the n-dimensional euclidean space E_n, and let the straight lines G intersect with the $(n-1)$-dimensional hyperplane E_{n-1} inside the body K. We shall calculate the following integral

$$\int dGdE_{n-1}$$

Let P be the intersection point of the line G with the plane E_{n-1}, consider an orthogonal cartesian coordinate system ($\mathbf{P}$; $\mathbf{A}_1$, $\mathbf{A}_2, \ldots\ldots$, $\mathbf{A}_n$) with a unit vector $\mathbf{A}_n$ on the line G.

Now setting

$$d\mathbf{A}_n=\omega_{1n}\mathbf{A}_1+\omega_{2n}\mathbf{A}_2+\omega_{3n}\mathbf{A}_3+\text{-----}+\omega_{n-1n}\mathbf{A}_{n-1}$$

we get

$$(\mathbf{A}_i\cdot d\mathbf{A}_n)=\omega_{in}\qquad (i=1,\ 2,\ 3,\ \text{----},\ n\text{-}1)$$

Further, setting

$$\mathbf{A}_i=\ (a_{i1}\ a_{i2}\ a_{i2}.\ \text{-----}\ a_{in})\qquad (i=1,\ 2,\ \text{----},\ n-1)$$

we get

$$[(d\mathbf{P}\cdot\mathbf{A}_1)\ (d\mathbf{P}\cdot\mathbf{A}_2)\ \text{-------}(d\mathbf{P}\cdot\mathbf{A}_{n-1})]$$

$$=[(dp_1\,a_{11}+dp_2\,a_{12}+\text{-----}+dp_n\,a_{1n})\ (dp_1\,a_{21}+dp_2\,a_{22}+\text{----}$$
$$+dp_n\,a_{2n})$$
$$\text{--------}(dp_1\,a_{n-1n-1}+dp_2\,a_{n-12}+\text{------}+dp_n\,a_{n-1n})]$$

$$=\left|\begin{array}{cccc|c|} a_{11} & a_{12} & \text{-------} & a_{1.n-1} & \overset{\checkmark}{a_{1n}} \\ a_{21} & a_{22} & \text{-------} & a_{2.n-1} & a_{2n} \\ \text{----} & \text{----} & \text{----} & \text{----} & \text{----} \\ \text{---------} & & & & \\ a_{-1} & a_{n-2} & \text{-----} & a_{n-1.n-1} & a_{n-1n} \end{array}\right|[dp_1dp_2\,\text{-------}\,dp_{n-1}]$$

$$+\left|\begin{array}{ccc|c|c} a_{11} & a_{12} & \text{-------} & \overset{\checkmark}{a_{1.n-1}} & a_{1n} \\ a_{21} & a_{22} & \text{-------} & a_{2.n-1} & a_{2n} \\ \text{----} & \text{----} & \text{----} & \text{----} & \text{----} \\ a_{n-1.1} & a_{n-1.2} & \text{----} & a_{n-1.n-1} & a_{n-1n} \end{array}\right|[dp_1\ dp_2\,\text{-----}\,dp_{n-2}\ dp_n]$$

$$+\left|\begin{array}{ccc|c|cc} a_{11} & a_{12} & \text{----} & \overset{\checkmark}{a_{1i}} & \text{---} & a_{1n} \\ a_{21} & a_{22} & \text{----} & a_{2i} & \text{---} & a_{2n} \\ \text{----} & \text{----} & \text{----} & \text{----} & \text{----} & \text{----} \\ a_{n-11} & a_{n-12} & \text{--} & a_{n-1i} & \text{--} & a_{n-1n} \end{array}\right|[dp_1\ dp_2\text{---}\,dp_{i-1}\ dp_{i+1}\text{---}dp_n]$$

$$+ - - - - - - - - - - - - -$$

$$+\begin{vmatrix} \overset{\checkmark}{a_{11}} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ - - - & - - - & - - - & - - - & - - - \\ a_{n-1 1} & a_{n-1 2} & a_{n-1 3} & \cdots & a_{n-1 n} \end{vmatrix} [dp_2 \ dp_3 - - - - - dp_n] - - - (1, \ 1)$$

Where $\surd$ signed columns of the determinants are lacking.

Now let A_{n1}, A_{n2} -----, A_{nn} respectively be the cofactors of the elements a_{n1}, a_{n2}, ---, a_{nn}, in the determinant

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ - - - & - - - & - - - & - - - \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

then we have

$$(1,1) = A_n[dp_1 \ dp_2 - - - - dp_{n-1}] + (-1)A_{n-1}[dp_1 \ dp_2 - - - dp_{n-2} \ dp_n]$$

$$+ - - - + (-1)^{n-i} A_i [dp_1 \ dp_2 - - - - dp_{i-1} \ dp_{i+1} - - - - dp_n]$$

$$+ - - - + (-1)^{n-1} A_1 [dp_2 \ dp_3 - - - - dp_n I$$

$$= a_{nn} [dp_1 \, dp_2 - - - dp_{n-1}] + (-1) \ a_{n \ n-1} [dp_1 \, dp_2 - - - dp_{n-2} \, dp_n]$$

$$+ - - - - + (-1)^{n-i} a_{ni} [dp_1 \ dp_2 - - - dp_{i-1} \ dp_{i+1} - - - - dp_n]$$

$$+ - - - - + (-1)^{n-1} a_{n1} [dp_2 \ dp_3 - - - dp_n]$$

because $A_i = a_{ni} \ (i = 1, \ 2, \ - - - - - n)$

Hence we have

$$dG = [d\Omega_1 (a_{nn}[dp_1 \ dp_2 - - - dp_{n-1}] + (-1)^{n-1} a_{n \ n1}[dp_1 \ dp_2 - - - -$$

$$dp_{n-2} \ dp_n]$$

$$+ - - - - + (-1)^{n-i} a_{ni} [dp_1 \ dp_2 - - - dp_{i-1} \ dp_{i+1} - - - dp_n] + - - -$$

$$+ (-1)^{n-i} a_{n1} [dp_2 \ dp_3 - - - - dp_n]))]$$

Let $\mathbf{V}$ $(V_1 V_2 V_3 —— V_n)$ be a unit normal vector to the plane E_{n-1}, then

$$dE_{n-1} = [d\Omega_2 (d\mathbf{P} \cdot \mathbf{V})]$$

$$= [d\Omega_2 \ (dp^1 \ V_2 + dp_2 \ V_2 + - - - dp_n \ V_n)]$$

Consequently we get the following formula :

$$\int dG dE_{n-1} = \int [[d\Omega_1 (a_{nn}[dp_1 \ dp_2 - - - - dp_{n-1}]$$

$$+ (-1)^{n-1} a_{n \ n-1} [dp_1 \, dp_2 - - - - dp_{n-2} \, dp_n] + - - - -$$

$$+(-1)^{n-i}a_{ni}[dp_1\, dp_2 ---- dp_{i-1}\, dp_{i+1} ---- dp_n]$$

$$++(-1)^{n-1}a_{n1}[dp_2 ---- dp_n])]$$

$$[d\Omega_2(dp_1\, V_1+dp_2\, V_2+ ---- +dp_n\, V_n)]]$$

$$=\int d\Omega_1\, d\Omega_2\, dp(a_{n1}V_1+a_{n2}V_2+ ----+ a_{nn}V_n)$$

$$=\int dp\ d\Omega_1\, d\Omega_2 \cos\theta,$$

where θ is the angle formed by the vector $\mathbf{A}_n$ with the vector $\mathbf{V}$. Finally, considering that

$$\int dp = V \text{ (volume of the } K)$$

we get

$$\int dGdE_{n-1} = V\ S_n\ S_{n-1}/4$$

with

$$S_x = \frac{2(\sqrt{\pi})^n}{\Gamma\left(\frac{n}{2}\right)}$$

where S_n is the area of the surface of the n-dim sphere.

Bibliography

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