On some Formulas of Integral Geometry (II)

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(Receved October 15, 1968)

§ 1. Integral formula for arcs of the unit circle within the closed convex curve.

Let E_2 be a 2 − dimensional Euclidian space, C a unit circle and (x, y) be the orthogonal cartesian coordinates of the center of the circle C.
Then two points P (x_1, y_1), Q (x_2, y_2) which are on the circle C can be expressed as follows:

$$x_1 = x + \cos\theta_1 \qquad x_2 = x + \cos\theta_2$$
$$y_1 = y + \sin\theta_1 \qquad y_2 = y + \sin\theta_2$$

By differentiation we get

$$dx_1 = dx - \sin\theta_1\, d\theta_1 \qquad dx_2 = dx - \sin\theta_2\, d\theta_2$$
$$dy_1 = dy + \cos\theta_1\, d\theta_1 \qquad dy_2 = dy + \cos\theta_2\, d\theta_2$$

We will use square brackets in order to indicate "exterior multiplication".
By exterior multiplication

$$[dx_1\, dy_1] = [dx\, dy] + \cos\theta_1\, [dx\, d\theta_1] + \sin\theta_1\, [dy\, d\theta_1]$$
$$[dx_2\, dy_2] = [dx\, dy] + \cos\theta_2\, [dx\, d\theta_2] + \sin\theta_2\, [dy\, d\theta_2]$$

From this we get

$$\begin{aligned}[dP\, dQ] &= [dx_1\, dy_1\, dx_2\, dy_2] \\ &= \cos\theta_1 \sin\theta_2\, [dx\, d\theta_1\, dy\, d\theta_2] \\ &\qquad + \sin\theta_1 \cos\theta_2\, [dy\, d\theta_1\, dx\, d\theta_2] \\ &= \sin(\theta_1 - \theta_2)\, [dx\, dy\, d\theta_1\, d\theta_2]\end{aligned}$$

Now setting

$$\omega = \theta_1 - \theta_2 \qquad dc = [dx\, dy]$$

we obtain

$$[dP\, dQ] = \sin\omega\, [d\theta_1\, d\theta_2\, dc]$$

Let K be a convex domain in the plane E_2, and P_1, P_2 the intersection points of the circle C with the circumference of K, and α, β $(\alpha<\beta)$ are the angles between the lines CP_1, CP_2 and the x − axis respectively. When two points P and Q are in the convex domain K, we obtain

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$$\int_\alpha^\beta \int_\alpha^\beta | \sin(\theta_1 - \theta_2) | \, d\theta_1 \, d\theta_2$$
$$= \int_\alpha^\beta d\theta_2 \{ \int_\alpha^{\theta_2} \sin(\theta_2 - \theta_1) \, d\theta_1 + \int_{\theta_2}^\beta \sin(\theta_1 - \theta_2) \, d\theta_1 \}$$
$$= \int_\alpha^\beta \{ 2 - \cos(\theta_2 - \alpha) - \cos(\beta - \theta_2) \} \, d\theta_2$$
$$= 2 \{ (\beta - \alpha) - \sin(\beta - \alpha) \} = 2 (\omega - \sin \omega)$$

Since there are two unit circles through the two points P and Q, we have following ingegral formula :

$$\int_{P,Q \in K} dP \, dQ = 2 \int_\alpha^\beta \int_\alpha^\beta | \sin(\theta_1 - \theta_2) | \, d\theta_1 \, d\theta_2 \, dc$$
$$= 4 \int (\omega - \sin \omega) \, dc$$

where ω stands for the length of the arc of the unit circle C in the domain K. Consequently we have

$$F^2 = 4 \int (\omega - \sin \omega) \, dc$$

where F is the area of the domain K.

§ 2. Density of hyperplane E_{n-1}.

Many fine integral formulas have been given by W. Blaschke. The following formula was stated in his "Vorlesungen über Integralgeometrie".
Let E_3 be a 3 – dimensional Euclidean space, and let E be the plane which is determined by the three points P_1, P_2 and P_3 in the space E_3.
Consider an orthogonal cartesian coordinates system
$(O: \boldsymbol{E}_1, \boldsymbol{E}_2, \boldsymbol{E}_3)$ in this space E_3
then the density for E can be expressed by the following formula

$$dE = \frac{(d\boldsymbol{P}_1 \, \boldsymbol{V}) \, (d\boldsymbol{P}_2 \, \boldsymbol{V}) \, (d\boldsymbol{P}_3 \, \boldsymbol{V})}{f}$$

where f is two times of the area of the triangle $P_1 \, P_2 \, P_3$
and $\boldsymbol{P}_1 = \vec{O}P_1$, $\boldsymbol{P}_2 = \vec{O}P_2$ and $\boldsymbol{P}_3 = \vec{O}P_3$ and $\boldsymbol{V}$ is a unit vector perpendicular to the plane E, $(d\boldsymbol{P}_i \, \boldsymbol{V})$ means scalar product of $d\boldsymbol{P}_i$ and $\boldsymbol{V}$. We try first to give a proof of this formula and next to extend it to the case of n - dimensional Euclidean space E_n.
Proof : We consider a orthogonal coordinates system $(\boldsymbol{P}; \boldsymbol{A}_1 \, \boldsymbol{A}_2)$ which is fixed on the plane E and let A_3 be a unit vector perpendicular to the plane E. i. e. $\boldsymbol{A}_3 = \boldsymbol{V}$.
Then vectors $\boldsymbol{P}_i$ can be expressed as follows.

$$\boldsymbol{P}_i = \boldsymbol{P} + r_i \cos \theta_i \, \boldsymbol{A}_1 + r_i \sin \theta_i \, \boldsymbol{A}_2 \qquad (i = 1. \, 2. \, 3) \cdots\cdots (1)$$

By differentiation we get

$$\begin{aligned} dP_i &= dP + (\cos\theta_i\, A_1 + \sin\theta_i\, A_2)\, dr_i \\ &+ (-r_i \sin\theta_i\, A_1 + r_i \cos\theta_i\, A_2)\, d\theta_i \\ &+ r_i \cos\theta_i\, dA_1 + r_i \sin\theta_i\, dA_2 \quad (i=1.\ 2.\ 3). \end{aligned}$$

Hence we have

$$\begin{aligned} (dP_i\ V) &= (dP_i\ A_3) \\ &= (dP\ A_3) + r_i \cos\theta_i\, (dA_1\ A_3) + r_i \sin\theta_i\, (dA_2\ A_3) \end{aligned}$$

Now setting $(dA_i\ A_j) = d\sigma_k$* we obtain

$$(dP_i\ A_3) = (dP\ A_3) - r_i \cos\theta_i\, d\sigma_2 + r_i \sin\theta_i\, d\sigma_1.$$

Consequently

$$[(dP_1\ A_3)\ (dP_2\ A_3)\ (dP_3\ A_3)] = -\begin{vmatrix} 1 & r_1\cos\theta_1 & r_1\sin\theta_1 \\ 1 & r_2\cos\theta_2 & r_2\sin\theta_2 \\ 1 & r_3\cos\theta_3 & r_3\sin\theta_3 \end{vmatrix} [(dP\ A_3)\ d\sigma_1\ d\sigma_2]$$

Since $dE = |\,[(dP_1\ A_3)\ d\sigma_1\ d\sigma_2]\,|$

and absolute value of $\begin{vmatrix} 1 & r_1\cos\theta_1 & r_1\sin\theta_1 \\ 1 & r_2\cos\theta_2 & r_2\sin\theta_2 \\ 1 & r_3\cos\theta_3 & r_3\sin\theta_3 \end{vmatrix}$

equals to two times of the area of triangle $P_1\ P_2\ P_3$ and dE will be taken always in absolute value, finally we have the following formula:

$$dE = \frac{[(dP_1\ V)\ (dP_2 V)\ (dP_3\ V)]}{f}$$

Next, we consider the case of n - dimensional Euclidian space E_n, and let $(O;\ E_1, E_2, \cdots\cdots, E_n)$ be an orthogonal cartesian coordinates system in the space E_n. Let E_{n-1} be the (n-1) - dimensional hyperplane which is determined by the n points $P_1, P_2, \cdots\cdots, P_n$ in the space E_n. We consider an orthogonal coordinates system $(P;\ A_1, A_2, \cdots\cdots, A_{n-1})$ which is fixed on the hyperplane E_{n-1}, and let A_n be a unit vector perpendicular to the hyperplane E_{n-1}.

Then n vectors $P_1, P_2, \cdots\cdots, P_n$ can be expressed as follows:

$$\begin{aligned} P_1 &= P + r_1 (l_{11} A_1 + l_{12} A_2 + \cdots\cdots + l_{1n-1} A_{n-1}) \\ P_2 &= P + r_2 (l_{21} A_1 + l_{22} A_2 + \cdots\cdots + l_{2n-1} A_{n-1}) \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ P_n &= P + r_n (l_{n1} A_1 + l_{n2} A_2 + \cdots\cdots + l_{n\ n-1} A_{n-1}) \end{aligned}$$

* See W. Blasche "Vorlesung über Integraı geometrie".
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where r_i is $\overline{P\ P_i}$ and $(l_{i1}, l_{i2} \dots\dots, l_{in})$
are direction cosines of the lines $P\ P_i$ $(i = 1.\ 2.\ 3.\ \dots.\ n)$.
By differentiation we get

$$\begin{aligned} d\boldsymbol{P}_i &= d\boldsymbol{P} + (l_{i1}\ \boldsymbol{A}_1 + l_{i2}\ \boldsymbol{A}_2 + \dots\dots + l_{i\ n-1}\ \boldsymbol{A}_{n-1})\ dr_i \\ &+ r_i\ (dl_{i1}\ \boldsymbol{A}_1 + dl_{i2}\ \boldsymbol{A}_2 + \dots\dots + dl_{i\ n-1}\ \boldsymbol{A}_{n-1}) \\ &+ r_i\ (l_{i1}\ d\boldsymbol{A}_1 + l_{i2}\ d\boldsymbol{A}_2 + \dots\dots + l_{i\ n-1}\ d\boldsymbol{A}_{n-1}) \end{aligned}$$

Since $(\boldsymbol{A}_i\ \boldsymbol{A}_j) = \delta_{ij}$
we get

$$\begin{aligned} (d\boldsymbol{P}_i\ \boldsymbol{A}_n) &= (d\boldsymbol{P}\ \boldsymbol{A}_n) + r_i\ ((l_{i1}\ d\boldsymbol{A}_1 + l_{i2}\ d\boldsymbol{A}_2 + \dots\dots + l_{i\ n-1} \\ &\quad d\boldsymbol{A}_{n-1})\ \boldsymbol{A}_n) \\ &= (d\boldsymbol{P}\ \boldsymbol{A}_n) + r^1\ (l_{i1}\ d\sigma_1 + l_{i2}\ d\sigma_2 + \dots\dots + l_{i\ n-1}\ d\sigma_{n-1}) \end{aligned}$$

where $(d\boldsymbol{A}_i\ \boldsymbol{A}_n) = d\sigma_i$ $(i = 1.\ 2.\ \dots\dots\ n)$

Finally we have the following formula

$$[(d\boldsymbol{P}\ \boldsymbol{A}_n)\ (d\boldsymbol{P}\ \boldsymbol{A}_n)\ \dots\dots\ (d\boldsymbol{P}_n\ \boldsymbol{A}_n)]$$

$$= \begin{vmatrix} 1 & r_1\ l_{11} & r_1\ l_{12} \dots\dots r_1\ l_{1\ n-1} \\ 1 & r_2\ l_{21} & r_2\ l_{22} \dots\dots r_2\ l_{2\ n-1} \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ 1 & r_n\ l_{n1} & r_n\ l_{n2} \dots\dots r_n\ l_{n\ n-1} \end{vmatrix} [(d\boldsymbol{P}\ \boldsymbol{A}_n)\ d\sigma_1\ d\sigma_2 \dots d\sigma_n]$$

$$= n\ !\ V\ dE_{n-1}$$

where V is the volume of the simplex $P_1\ P_2\ \dots\dots P_n$.

Hence we have

$$dE_{n-1} = \frac{[(d\boldsymbol{P}_1\ \boldsymbol{A}_n)\ (d\boldsymbol{P}_2\ \boldsymbol{A}_n)\ \dots\dots\ (d\boldsymbol{P}_n\ \boldsymbol{A}_n)]}{n\ !\ V}$$

Bibliography

(1) W. Blaschke. Vorlesung über Integral Geometrie

(2) L. A. Santalo. Introduction to Integral Geometry