On Some Formulas of Integral Geometry on the Sphere (II).

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W. Blashke and L. A. Santalō have given many fine integral formulas not only on the sphere but on the general surface. In the present paper we shall give another proof one of them and show a formula of pairs of a point and a great circle on the sphere.

§ 1. Formula of pairs of great circles.

We consider a fixed great circle C_0 on a sphere of unit radius and a fixed point A on C_0.

A great circle C on the sphere can be determined for the abscissa t of one of the intersection points of C and C_0 and the angle α between the two circles.

Then the "density" for measuring sets of great circles on the sphere is given by

$$dc=\sin \alpha\ [d\alpha\ dt]\ *$$

and the "density" for measuring sets of points on the sphere is given by

$$dP=\sin \theta\ [d\theta\ d\varphi]$$

where θ and φ are the spherical coordinates of the point P. Let P_1 and P_2 are two points on the unit sphere and let C be the great circle determined by them.

If β_1 and β_2 are the abscissae of P_1 and P_2 on C in relation to the fixed origin on this circle, then

$$[dP_1\ dP_2]=|\sin (\beta_1-\beta_2)|\ [d\beta_1\ d\beta_2\ dc] \cdots\cdots (1)$$

This formula was proved by L. A Santalō using the property of dualty of great circle and its correspondence pole. We shall prove it directly by calculation. Let P be one of the intersection points of C and C_0, and N be a pole of the great circle C_0. Put $\widehat{PP_1}=r$, $\widehat{PP_2}=s$, $\widehat{PM}=\varphi_1$, and $\widehat{PK}=\varphi_2$ (see Fig. 1).

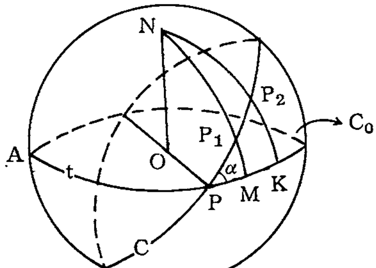

Fig. 1

then

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* We will always use square brackets in order to indicate "exterior multiplication",

$$\sin\alpha=\frac{\sin\left(\frac{\pi}{2}-\theta_1\right)}{\sin s}$$

$$\sin\alpha=\frac{\sin\left(\frac{\pi}{2}-\theta_2\right)}{\sin \gamma}$$

hence

$$\cos\theta_1=\sin s\,\sin\alpha \qquad \cdots\cdots (2)$$

$$\cos\theta_2=\sin\gamma\,\sin\alpha \qquad \cdots\cdots (2)'$$

where (θ_1, φ_1) and (θ_2, φ_2) are the spherical coordinates of the points P_1 and P_2 respectively.

Differentiating both sides of (2) and (2)′

$$-\sin\theta_1\,d\theta_1=\cos s\,\sin\alpha\,ds+\sin s\,\cos\alpha\,d\alpha$$

$$-\sin\theta_2\,d\theta_2=\cos\gamma\,\sin\alpha\,d\gamma+\sin\gamma\,\cos\alpha\,d\alpha$$

Hence we have

$$\sin\theta_1\,\sin\theta_2\,[d\theta_1\,d\theta_2]=\cos s\,\cos\gamma\,\sin^2\alpha\,[ds\,d\gamma]$$

$$+\sin\alpha\,\cos\alpha\,\{\sin\gamma\,\cos s\,[ds\,d\alpha]-\cos\gamma\,\sin s\,[d\gamma\,d\alpha]\}\cdots\cdots (3)$$

By the spherical trigonometric formulas, we have

$$\tan\varphi_1=\cos\alpha\,\tan s \qquad \cdots\cdots (4)$$

$$\tan\varphi_2=\cos\alpha\,\tan\gamma \qquad \cdots\cdots (4)'$$

Differentiating both sides of (4) and (4)′

$$\sec^2\varphi_1\,d\varphi_1=-\sin\alpha\,\tan s\,d\alpha+\cos\alpha\,\sec^2 s\,ds$$

$$\sec^2\varphi_2\,d\varphi_2=-\sin\alpha\,\tan\gamma\,d\alpha+\cos\alpha\,\sec^2\gamma\,d\gamma$$

Since $d\varphi_1$ and $d\varphi_2$ can be expressed in the form of linear combination of $d\alpha$, ds and $d\alpha$, $d\gamma$ respectively, we have

$$-[dP_1, dP_2]=\sin\theta_1\,\sin\theta_2\,[d\theta_1\,d\theta_2\,(dt+d\varphi_1)\,(dt+d\varphi_2)]$$

$$=\sin\theta_1\,\text{siu}\,\theta_2\,[d\theta_1\,d\theta_2\,(d\varphi_1-d\varphi_2)\,dt] \qquad \cdots\cdots (5)$$

By (3) and (5) we have

$$[dP_1\,dP_2]=[\{-\cos s\,\cos\gamma\,\sin^2\alpha\,[ds\,d\gamma]-\sin\alpha\,\cos\alpha$$
$$(\sin\gamma\,\cos s\,[ds\,d\alpha]-\cos\gamma\,\sin s\,[ds\,d\alpha]\}$$
$$\{\cos^2\varphi_1\,(-\sin\alpha\,\tan s\,d\alpha+\cos\alpha\,\sec^2 s\,ds)-\cos^2\varphi_2$$
$$(-\sin\alpha\,\tan\gamma\,d\alpha+\cos\alpha\,\sec^2\gamma\,d\gamma)\}\,dt]$$
$$=(\sin s\,\cos\gamma\,\sin^2\alpha\,\cos^2\varphi_1+\sin s\,\cos\gamma\,\cos^2\alpha\,\sec^2 s\,\cos^2\varphi_1$$
$$-\cos s\,\sin\gamma\,\sin^2\alpha\,\cos^2\varphi_2-\cos s\,\sin\gamma\,\cos^2\alpha\,\sec^2\gamma\,\cos^2\varphi_2)\,\sin a\,[d\alpha\,dt\,ds\,d\gamma]$$
$$=[\sin s\,\cos\gamma\,\{\sin^2\alpha+\cos^2\alpha\,(1+\tan^2 s)\}\,\cos^2\varphi_1$$
$$-\cos s\,\sin\gamma\,\{\sin^2\alpha+\cos^2\alpha\,(1+\tan^2\gamma)\}\,\cos^2\varphi_2]\,[dC\,d\gamma\,ds]$$

$=(\sin s \cos \gamma-\cos s \sin \gamma)\ [dCds\ d\gamma]$

$=\sin (s-\gamma)\ [dC\ ds\ d\gamma]$

$=|\sin (\beta_1-\beta_2)|[d\beta_1\ d\beta_2\ dC]$ ·

hence we can proved the following formula.

$$[dP_1\ dP_2]=|\sin (\beta_1-\beta_2)|[d\beta_1\ d\beta_2\ dC]$$

§ 2. Formula of pairs of a point and a great circle.

Let us consider a point P and a great circle C on the unit sphere, the great circle C intersect the fixed great circle C_0 on the sphere with angle α and abscissa is t of one of the intersecting points of C and C_0.
Let the spherical coordinates of the point P be (θ, φ), and the great circle which is determined by the points N and P intersect the great circles C and C_0 at the points Q and R respectively. (see Fig. 2),

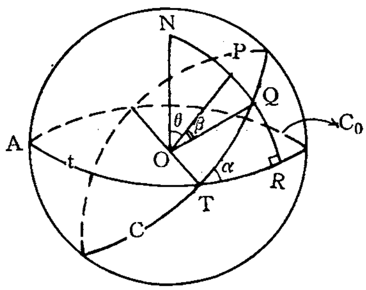

Fig. 2

Put $\widehat{PQ}=\beta,\ \widehat{TR}=\gamma$

then

$$\tan \alpha \sin (\varphi-t)=\tan (\frac{\pi}{2}-\theta-\beta) \qquad \cdots\cdots (1)$$

Differentiating both sides of the equation (1), we have

$$\sec^2\alpha \sin (\varphi-t)\ d\alpha+\tan \alpha \cos (\varphi-t)\ (d\varphi-dt)$$
$$=-\text{cosec}^2\ (\theta+\beta)\ (d\theta+d\beta) \qquad \cdots\cdots (2)$$

Multipling both sides of the (2) by the product $[dt\ d\gamma\ d\varphi]$

$\sec^2\alpha \sin (\varphi-t)\ [d\alpha\ dt\ d\theta\ d\varphi]=\text{cosec}^2\ (\theta+\beta)\ [d\beta\ dt\ d\theta\ d\varphi]$

Since $[d\beta\ dt\ d\theta\ d\varphi]=-[d\beta\ d(\varphi-t)\ d\theta\ d\varphi]$

we have

$\sec^2\alpha \sin (\varphi-t)\ [d\alpha\ dt\ d\theta\ d\varphi]=-\text{cosec}^2\ (\theta+\beta)\ [d\beta\ d\gamma\ d\theta\ d\varphi]$

Multipling both sides of the last equation by $\sin \alpha \sin \theta$

$$\sin \alpha \sin \theta \sec^2\alpha \sin (\varphi-t)\ [da\ dt\ d\theta\ d\varphi]$$

$=\sin \alpha \sin \theta\ \text{cosec}^2\ (\theta+\beta)\ [d\gamma\ d\beta\ d\theta\ d\varphi]$

Hence we have the following formaula

$$[dC\ dP]=\frac{\sin \alpha}{\sin \gamma}\cdot\frac{\cos^2\alpha}{\cos^2(\frac{\pi}{2}-\theta-\beta)}\ [d\gamma\ d\beta\ dP]$$

Next we want to show an another formula of pairs of a point and a great circle, Denote the pole of the great circle C by N_1. Let the great cicle C_0 intersects the great circle through the points N and P, the great circle through the points N_1 and P, and the great circle through the points N and N_1 at the points Q, R, S respectively. (see Fig. 3)

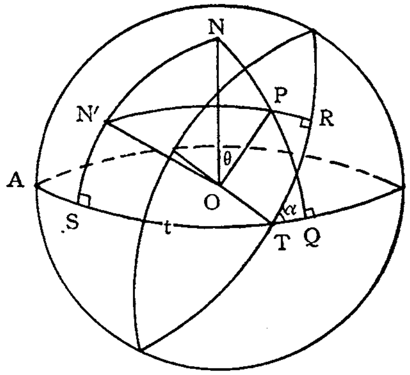

Fig. 3

Put $\widehat{PR}=a$, $\widehat{RT}=b$

then by the spherical trigonometric formulas, we have

$$\cos a \cos b=\cos(\varphi-t)\sin\theta \qquad \cdots\cdots (3)$$

$$\sin a=\cos\alpha\cos\theta+\sin\alpha\sin\theta\sin(t-\varphi) \qquad \cdots\cdots (4)$$

Differentiating both sides of (3) and (4)

$$-\sin a\cos b\, da-\cos a\sin b\, db$$
$$=-\sin(\varphi-t)\sin\theta\,(d\varphi-dt)$$
$$+\cos(\varphi-t)\cos\theta\, d\theta$$

$$\cos a\, da=\{-\sin\alpha\cos\theta+\cos\alpha\sin\theta\sin(t-\varphi)\}\, d\alpha$$
$$+\{-\cos\theta\alpha\sin\theta+\sin\alpha\cos\theta\sin(t-\varphi)\}\, d\theta$$
$$+\sin\alpha\sin\theta\cos(t-\varphi)\,(dt-d\varphi)$$

By the last two equations, we have

$$-\cos^2 a\sin b\,[da\ db]=A\ D\,[d\alpha\ d\theta]$$
$$+(CD+BE)\,[dt\ d\theta]-(CD+BE)\,[d\varphi\ d\theta]$$
$$+AE[d\alpha\ d\varphi]-AE\,[d\alpha\ dt] \qquad \cdots\cdots (5)$$

where $A=-\sin\alpha\cos\theta+\cos\alpha\sin\theta\sin(t-\varphi)$

$B=-\cos\alpha\sin\theta+\sin\alpha\cos\theta\sin(t-\varphi)$

$C=\sin\alpha\sin\theta\cos(t-\varphi)$

$D=\cos(\varphi-t)\cos\theta$

$E=-\sin(\varphi-t)\sin\theta$

Next multipling both sides of (5) by the product $[d\alpha\ dt]$, then we have

$$(CD+BE)\,[d\theta\ d\varphi\ d\alpha\ dt]=-\cos^2 a\sin b\,[da\ db\ d\alpha\ dt]$$

$$\sin\theta\,\{\sin\alpha\cos\theta+\cos\alpha\sin\theta\sin(\varphi-t)\}\,[d\theta\ d\varphi\ d\alpha\ dt]$$
$$=-\cos^2 a\sin b\,[da\ db\ d\alpha\ dt]$$

hence we have the following formula

$$\{\sin\alpha\cos\theta+\cos\alpha\sin\theta\sin(\varphi-t)\}\,[dP\ dC]$$
$$=-\cos^2 a\sin b[dC\ da\ db]$$

Bibligraphy

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