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On some Formulas of Integral Geometry.

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M. W. Crofton has given first some remarkable integral formulas for plane convex figures. Recently meny fine integral formulas have been given by W. Blashke and L. A. Santaló.

In the present we shall give another proof for some of these formulas by an elementary method used by W. Blashke and L. A. Santaló. This method is very simple and less complicated so that it becomes easier to understand the integral geometry.

§ 1. Integral formulas for the chords of convex body.

Let E_3 be a 3-dimensional Euclidian space, and let K be a convex body in the space E_3. Let G be a straight line in E_3, L the chord determined as the intersection $G \cdot K$ and let s be the length of L. We want to evaluate the integral

$$I=\int_{G\cdot K\neq 0} s\,dG$$

where the integral is extended over all the sstraight lines G which intersect the body K. Now let P be a point on the chord L and consider the integral

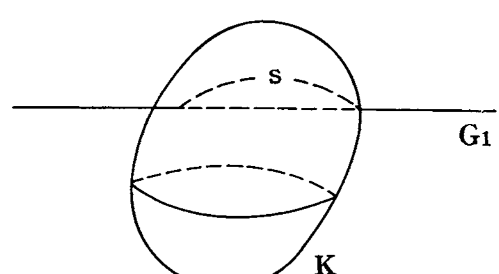
$$I = \int_{G\cdot K=L\neq 0.\ P\in L} dP\ dG.$$

If we leave G fixed, then the integral gives

$$I=\int dG\,dP=\int s\,dG \tag{1. 1}$$

We may also leave the point P fixed and we get

$$I=\int dP\,dG=2\pi\int dP = 2\pi V \tag{1. 2}$$



第 1 図

where V is the volume of K.

Combining (1. 1) and (1. 2) we get the formula

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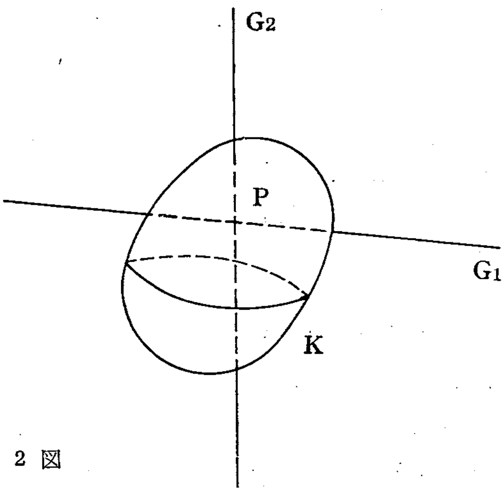
$$\int s dG = 2\pi V \qquad * \qquad [1]$$

In the case of E_2, a similar formula to [1] has been proved by W. Blashke by the method using two straight lines G_1, G_2 which intersect in K.

We want to show that the formula [1], in the case of E_3, can also be proved by the method similar to that of W. Blashke.



第 2 図

Let G_1 and G_2 be two straight lines in E_3 and let s be the length of the intersection $G_1 \cdot K$.
We assume that the point P of intersection of G_1 with G_2 is in K. Now we consider the integral

$$I = \int_{G_1 \cdot G_2 = P \in K} dG_1\, dG_2 \qquad (1.\ 3)$$

Where the integral is extended over G_1 and G_2 which intersect in K. Let the plane E be determined by two lines G_1 and G_2, then above integral becomes

$$I = \int dG_1\, dG_2 = \int dE\, dG_1{}^{(E)}\, dG_2{}^{(E)}$$

where $dG_1{}^{(E)}$ and $dG_2{}^{(E)}$ indicate respectivily the density of the lines G_1 and G_2 in E.
First leave E and G_1 fixed, then the integral gives

$$I = 2\int s dG_1{}^{(E)}\, dE = 2\pi \int s dG_1 \qquad ** \qquad (1.\ 4)$$

On the other hand the integral (1. 1) can be expressed as follows:

$$I = \int_{P \in K} dP\, d\varphi_1\, d\varphi_2$$

where $d\varphi_1$ and $d\varphi_2$ are elements of sphere, and the integral is extended over all points P in K. Hence we have

$$I = (2\pi)^2 \int dP = (2\pi)^2\, V \qquad (1.\ 5)$$

Since (1. 5) must be equal to (1.4), we obtain the formula

* W. Blashke has proved more general integral formula for the powers of the chords of a convex body.

** A. Santaló

$$\int sdG = 2\pi V.$$

In the above proof take a plane E, in place of G_2, and let P be the intersection $G_1 \cdot E$.
Now consider the integral

$$I = \int dG\ dE. \tag{1. 6}$$

First if we leave G fixed, then the integral gives

$$I = \pi \int sdG \tag{1. 7}$$

according to W. Blashke's formula

$$\int_{E \cdot L \neq 0} ndE = \pi l$$

where L denotes a curve whose length is l and n means the number of intersection points of E with L.
Next if we leave E fixed, the integral of (1. 6) gives,

$$I = \int dE\ dG = \pi \int fdE$$

where f is the area of the intersection $E \cdot K$.
But we have the Caushy's formula

* $$\int fdE = 2\pi V.$$

Hence we obtain the following formula;

** $$I = \int dE\ dG = 2\pi^2\ V. \tag{1. 8}$$

By (1. 7) and (1. 8) we have the formula

$$\int sdG = 2\pi V.$$

§ 2. Integral formulas for the surface area of a convex body.

Let O be the surface area of a convex body K and let u be the length of the circumference of the intersection of a plane E with the convex body K. Then we have the formula

$$\int_{E \cdot K \neq 0} udE = \frac{\pi^2}{2} O$$

which was first proved by W. Blashke. Now we shall give another proof for

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* W. Blashke proved this formula by the method of integral geometry.
** G. Hergloty's formula.

this formula. Let E and E' be two planes whose intersection is a straight line G. Consider the integral

$$I=\int_{E\cdot E'=G,\,G\cdot K\neq 0} dE\, dE'.$$

First if we leave the plane E fixed the integrel gives

$$* \qquad I=\int dE\; dE'=\frac{\pi}{2}\int udE. \tag{2. 1}$$

On the other hand the integral can be expressed as follows :

$$I=\int dE\; dE'=\int dG\; dE^{(G)}\; dE'^{(G)}$$

Let ω be the angle between two supporting planes for convex body K which contains the line G. Then we get

$$I=\frac{1}{2}\int(\omega^2-\sin^2\omega)dG=\frac{1}{4}\pi^3 O. \tag{2. 2}$$

By (2. 1) and (2. 2) we have the formula

$$\int udE=\frac{1}{2}\pi^2 O.$$

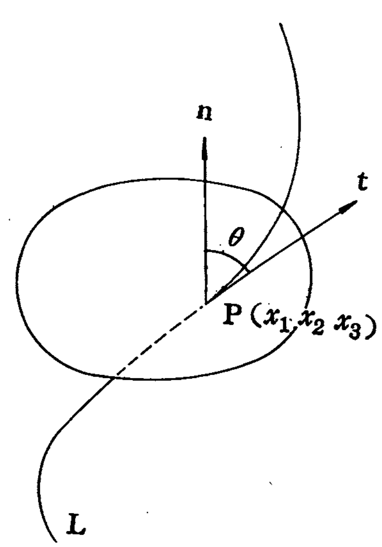
§ 3. Integral formula for curves and part of plane.

Let F be a part of plane and f be the area of F and let L be a rectifiable curve of length S. Now we want to evaluate the integral

$$I=\int_{L\cdot F\neq 0} ndL$$

where n means the number of intersection points of F with L. Since the kinematic measure is invariant under inversion of the motion, we may consider that L is fixed and F is moved. Hence we get

$$I=\int_{L\cdot F\neq 0} ndL=\int_{L\cdot F\neq 0} ndF.$$



第3図

* Blashke's formula

Expressing the density dF in terms of the coordinates of P (x_1, x_2, x_3) and the angle θ determined by the tangent to L at the point P and the normal to F at the same point we get

$$* \qquad dF = dx_1\ dx_2\ dx_3\ d\sigma_1\ d\sigma_2\ d\sigma_3 = dx_1\ dx_2\ ds \cos\theta\ d\varphi\ d\sigma_3$$

Hence we have

$$\int n dF = \pi \int dx_1\ dx_2\ ds d\sigma_3 = \pi f \int ds d\sigma_3 = 2\pi^2 f S \qquad (3.\ 1)$$

Application. Let L be a rectifiable curve of length S and let l be the length of the intersection $L \cdot K$.

We consider the integral

$$I = \int_{L \cdot K \neq 0} l dL.$$

In order to evaluate this integral, we take a plane E which intersects with the curve L in K and consider the integral

$$I = \int_{L \cdot E \in K} n dL dE. \qquad (3.\ 2)$$

where n means the number of intersection points of E with L in K. If we leave E fixed, then the integral (3.2) gives

$$I = 2\pi^2 S \int f dE$$

where f denotes the area of the intersection $E \cdot K$. By the Cauchy's formula

$$\int f dE = 2\pi V,$$

the integral (3. 2) gives

$$I = 4\pi^3 S V \qquad (3.\ 3)$$

On the other hand if we leave the curve L fixed. the integral (3. 2) gives

$$I = \int n dL dE = \pi \int l dL. \qquad (3.\ 4)$$

By (3. 3) and (3. 4) we get the formula

$$\int l dL = 4\pi^2 S V.$$

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* See, W. Blashke In Die kinematishe Dichten in Raum.

Bibliography

(1) W. Blashke, Vorlesungen über Integral Geometrie.
(2) L. A. Santaló, Introduction to integral geometry.
(3) M. Kurita, Integral Geometry (written in Japanese).