# Generalized Littlewood–Paley characterizations of fractional Sobolev spaces

# Generalized Littlewood-Paley Characterizations of Fractional Sobolev Spaces

Shuichi Sato, Fan Wang, Dachun Yang $^{*}$ and Wen Yuan

**Abstract** In this article, the authors characterize the Sobolev spaces $W^{\alpha,p}(\mathbb{R}^n)$ with $\alpha \in (0,2]$ and $p \in (\max\{1, \frac{2n}{2\alpha+n}\}, \infty)$ via a generalized Lusin area function and its corresponding Littlewood-Paley $g_\lambda^*$-function. The range $p \in (\max\{1, \frac{2n}{2\alpha+n}\}, \infty)$ is also proved to be nearly sharp in the sense that these new characterizations are not true when $\frac{2n}{2\alpha+n} > 1$ and $p \in (1, \frac{2n}{2\alpha+n})$. Moreover, in the endpoint case $p = \frac{2n}{2\alpha+n}$, the authors also obtain some weak type estimates. Since these generalized Littlewood-Paley functions are of wide generality, these results provide some new choices for introducing the notions of fractional Sobolev spaces on metric measure spaces.

## 1 Introduction

The theory of Sobolev spaces is one of the central topic in modern analysis. In recent years, there is an increasing interest in developing Sobolev spaces on metric measure spaces, and the theory of Sobolev spaces with smoothness order not greater than 1 has already been intensively studied in a series of literatures (see, for example, [12, 17, 20, 15, 14]).

Recently, via establishing some new characterizations of Sobolev spaces on $\mathbb{R}^n$, Alabern et al. [1] found some ways to introduce high order Sobolev spaces on metric measure spaces. To recall their results, let $\mathcal{S}(\mathbb{R}^n)$ be the set of all *Schwartz functions* on $\mathbb{R}^n$ equipped with the well-known topology and $\mathcal{S}'(\mathbb{R}^n)$ its topological dual equipped with the weak-$*$ topology, namely, the collection of all bounded linear functionals on $\mathcal{S}(\mathbb{R}^n)$; let $\Delta := \Sigma_{i=1}^n (\frac{\partial}{\partial x_i})^2$ be the Laplace operator and, for any $\alpha \in (0,\infty)$, $f \in \mathcal{S}'(\mathbb{R}^n)$ and $\xi \in \mathbb{R}^n$, define $(-\Delta)^{\frac{\alpha}{2}}$ via $[(-\Delta)^{\frac{\alpha}{2}} f]^\wedge(\xi) := |2\pi\xi|^\alpha \widehat{f}(\xi)$. Here and hereafter, for any $f \in L^1(\mathbb{R}^n)$, we use $\widehat{f}$ to denote its *Fourier transform*, namely, for any $\xi \in \mathbb{R}^n$,

$$\widehat{f}(\xi) := \int_{\mathbb{R}^n} f(x) e^{-2\pi i x\xi}\, dx.$$

It is well known that the definition of the above Fourier transform can be extended to any $f \in \mathcal{S}'(\mathbb{R}^n)$, whose Fourier transform is still denoted by $\widehat{f}$. Recall that the *fractional*

2010 *Mathematics Subject Classification.* Primary 46E35; Secondary 42B25, 47G40, 31B15.

*Key words and phrases.* Sobolev space, $g$-function, Lusin area function, $g_\lambda^*$-function, average, Riesz potential operator.

The first named author is partly supported by Grant-in-Aid for Scientific Research (C) No. 16K05195, Japan Society for the Promotion of Science. The third and the forth named authors are partly supported by the National Natural Science Foundation of China (Grant Nos. 11571039, 11671185 and 11471042).

$^{*}$Corresponding author / August 7, 2017.

*Sobolev space* $W^{\alpha,p}(\mathbb{R}^n)$, with any $\alpha \in (0,\infty)$ and $p \in (1,\infty)$, is defined as the set of all $f \in L^p(\mathbb{R}^n)$ such that $(-\Delta)^{\frac{\alpha}{2}} f \in L^p(\mathbb{R}^n)$, equipped with the norm that, for any $f \in W^{\alpha,p}(\mathbb{R}^n)$,

$$\|f\|_{W^{\alpha,p}(\mathbb{R}^n)} := \|f\|_{L^p(\mathbb{R}^n)} + \|(-\Delta)^{\frac{\alpha}{2}} f\|_{L^p(\mathbb{R}^n)}.$$

Recently, Alabern et al. [1] proved that the Sobolev spaces $W^{\alpha,p}(\mathbb{R}^n)$ with any $\alpha \in (0,2]$ and $p \in (1,\infty)$ can be characterized, respectively, by the square functions $S_\alpha(f)$ when $\alpha \in (0,2)$ and $S(f,g)$ when $\alpha = 2$, which are respectively defined by setting, for any $f,\, g \in L^1_{\rm loc}(\mathbb{R}^n) \cap \mathcal{S}'(\mathbb{R}^n)$,

$$S_\alpha(f)(x) := \left\{ \int_0^\infty \left| \frac{B_t f(x) - f(x)}{t^\alpha} \right|^2 \frac{dt}{t} \right\}^{\frac{1}{2}}, \quad \forall\, x \in \mathbb{R}^n \tag{1.1}$$

and

$$S(f,g)(x) := \left\{ \int_0^\infty \left| \frac{B_t f(x) - f(x)}{t^2} - B_t g(x) \right|^2 \frac{dt}{t} \right\}^{\frac{1}{2}}, \quad \forall\, x \in \mathbb{R}^n. \tag{1.2}$$

Here and hereafter, $L^1_{\rm loc}(\mathbb{R}^n)$ denotes the set of all locally integrable functions on $\mathbb{R}^n$, $B(x,t)$ the open ball with center $x \in \mathbb{R}^n$ and radius $t \in (0,\infty)$, namely,

$$B(x,t) := \{y \in \mathbb{R}^n : \ |y-x| < t\},$$

and $B_t g(x)$ the integral average of $g \in L^1_{\rm loc}(\mathbb{R}^n)$ on ball $B(x,t)$, namely,

$$B_t g(x) := \fint_{B(x,t)} g(y)\, dy := \frac{1}{|B(x,t)|} \int_{B(x,t)} g(y)\, dy. \tag{1.3}$$

Indeed, Alabern et al. in [1, Theorems 1, 2 and 3] proved the following results.

**Theorem 1.A.** *Let $p \in (1,\infty)$, $\alpha \in (0,2]$, $S_\alpha$ and $S$ be as in (1.1), respectively, (1.2). Then the following statements are equivalent:*

(i) $f \in W^{\alpha,p}(\mathbb{R}^n)$*;*

(ii) $f \in L^p(\mathbb{R}^n)$ *and* $S_\alpha(f) \in L^p(\mathbb{R}^n)$ *when* $\alpha \in (0,2)$*, or there exists* $g \in L^p(\mathbb{R}^n)$ *such that* $S(f,g) \in L^p(\mathbb{R}^n)$ *when* $\alpha = 2$.

*Moreover, it holds true that* $\|(-\Delta)^{\frac{\alpha}{2}} f\|_{L^p(\mathbb{R}^n)}$ *is equivalent to* $\|S_\alpha(f)\|_{L^p(\mathbb{R}^n)}$ *when* $\alpha \in (0,2)$ *or to* $\|S(f,g)\|_{L^p(\mathbb{R}^n)}$ *when* $\alpha = 2$ *with equivalent positive constants independent of* $f$.

The characterizations of Sobolev spaces in Theorem 1.A do not depend on the differential structure of $\mathbb{R}^n$, and hence provide a way to introduce Sobolev spaces with smoothness order in $(0,2]$ on metric measure spaces. Motivated by Theorem 1.A and by noticing that $S_\alpha(f)$ and $S(f,g)$ are two kinds of the Littlewood-Paley $g$-functions, the authors of [13] and [4] considered the characterizations of $W^{\alpha,p}(\mathbb{R}^n)$ via the corresponding Lusin area function and the Littlewood-Paley $g_\lambda^*$-function. To be precise, for any $\lambda \in (1,\infty)$, $f,\, g \in L^1_{\rm loc}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, define

$$\widetilde{S}_\alpha(f)(x) := \left\{ \int_0^\infty \int_{B(x,t)} \left| \frac{B_t f(y) - f(y)}{t^\alpha} \right|^2 dy\, \frac{dt}{t^{n+1}} \right\}^{\frac{1}{2}}, \tag{1.4}$$

$$\widetilde{S}(f,g)(x) := \left\{\int_0^\infty \int_{B(x,t)} \left|\frac{B_t f(y) - f(y)}{t^\alpha} - B_t g(y)\right|^2 \,dy\,\frac{dt}{t^{n+1}}\right\}^{\frac{1}{2}}, \tag{1.5}$$

$$G^*_{\alpha,\lambda}(f)(x) := \left\{\int_0^\infty \int_{\mathbb{R}^n} \left|\frac{B_t f(y) - f(y)}{t^\alpha}\right|^2 \left(\frac{t}{t+|x-y|}\right)^{\lambda n} dy\,\frac{dt}{t^{n+1}}\right\}^{\frac{1}{2}} \tag{1.6}$$

and

$$G^*_{\lambda}(f,g)(x) := \left\{\int_0^\infty \int_{\mathbb{R}^n} \left|\frac{B_t f(y) - f(y)}{t^\alpha} - B_t g(y)\right|^2 \left(\frac{t}{t+|x-y|}\right)^{\lambda n} dy\,\frac{dt}{t^{n+1}}\right\}^{\frac{1}{2}}, \tag{1.7}$$

where $B_t$ with $t \in (0,\infty)$ is as in (1.3).

The following result was proved in [13, Theorem 1.1].

**Theorem 1.B.** *Let $n \in \mathbb{N}$, $p \in (1,\infty)$, $\widetilde{S}$ and $G^*_\lambda$ be as in (1.5), respectively, (1.7). Then the following statements are mutually equivalent:*

(i) *$f \in W^{2,p}(\mathbb{R}^n)$;*

(ii) *$f \in L^p(\mathbb{R}^n)$ and there exists $g \in L^p(\mathbb{R}^n)$ such that $\widetilde{S}(f,g) \in L^p(\mathbb{R}^n)$;*

(iii) *$f \in L^p(\mathbb{R}^n)$ and there exists $g \in L^p(\mathbb{R}^n)$ such that $G^*_\lambda(f,g) \in L^p(\mathbb{R}^n)$, provided that $p \in [2,\infty)$, $n \in \mathbb{N}$ and $\lambda \in (1,\infty)$, or $p \in (1,2)$, $n \in \{1,2,3\}$ and $\lambda \in (\frac{2}{p},\infty)$.*

*Moreover, if $f \in W^{2,p}(\mathbb{R}^n)$, then $g$ in (ii) and (iii) can be taken as $g := \Delta f/(2n+4)$; while, if either of (ii) and (iii) holds true, then $g = \Delta f/(2n+4)$ almost everywhere. In any case, $\|\widetilde{S}(f,g)\|_{L^p(\mathbb{R}^n)}$ and $\|G^*_\lambda(f,g)\|_{L^p(\mathbb{R}^n)}$ are equivalent to $\|\Delta f\|_{L^p(\mathbb{R}^n)}$ with equivalent positive constants independent of $f$ and $g$, respectively.*

This result was further completed by [4] as follows.

**Theorem 1.C.** *Let $n \in \mathbb{N}$, $n \geq 4$, $\widetilde{S}$ and $G^*_\lambda$ be as in (1.5), respectively, (1.7).*

(I) *If $p \in (\frac{2n}{4+n}, 2)$, then the following statements are mutually equivalent:*

  (i) *$f \in W^{2,p}(\mathbb{R}^n)$;*

  (ii) *$f \in L^p(\mathbb{R}^n)$ and there exists $g \in L^p(\mathbb{R}^n)$ such that $\widetilde{S}(f,g) \in L^p(\mathbb{R}^n)$;*

  (iii) *$f \in L^p(\mathbb{R}^n)$ and there exists $g \in L^p(\mathbb{R}^n)$ such that $G^*_\lambda(f,g) \in L^p(\mathbb{R}^n)$ for some $\lambda \in (\frac{2}{p},\infty)$.*

  *Moreover, if $f \in W^{2,p}(\mathbb{R}^n)$, then $g$ in (ii) and (iii) can be taken as $g := \Delta f/(2n+4)$; while, if either of (ii) and (iii) holds true, then $g = \Delta f/(2n+4)$ almost everywhere. In any case, $\|\widetilde{S}(f,g)\|_{L^p(\mathbb{R}^n)}$ and $\|G^*_\lambda(f,g)\|_{L^p(\mathbb{R}^n)}$ are equivalent to $\|\Delta f\|_{L^p(\mathbb{R}^n)}$ with equivalent positive constants independent of $f$ and $g$, respectively.*

(II) *If $p \in (1, \frac{2n}{4+2n})$, then the equivalence between (i) and either (ii) or (iii) of (I) no longer holds true.*

As a fractional version, the following result was also obtained in [4].

**Theorem 1.D.** *Let $n\in\mathbb{N}$, $\alpha\in(0,2)$, $\widetilde{S}_\alpha$ and $G^*_{\alpha,\lambda}$ be as in (1.4), respectively, (1.6).*

(I) *If $p\in(\max\{1,\frac{2n}{2\alpha+n}\},\infty)$, then the following statements are mutually equivalent:*

 (i) $f\in W^{\alpha,p}(\mathbb{R}^n)$;

 (ii) $f\in L^p(\mathbb{R}^n)$ *and* $\widetilde{S}_\alpha(f)\in L^p(\mathbb{R}^n)$;

 (iii) $f\in L^p(\mathbb{R}^n)$ *and* $G^*_{\alpha,\lambda}(f)\in L^p(\mathbb{R}^n)$ *for some* $\lambda\in(\max\{1,\frac{2}{p}\},\infty)$.

*Moreover, $\|\widetilde{S}_\alpha(f)\|_{L^p(\mathbb{R}^n)}$ and $\|G^*_{\alpha,\lambda}(f)\|_{L^p(\mathbb{R}^n)}$ are equivalent to $\|(-\Delta)^{\frac{\alpha}{2}}f\|_{L^p(\mathbb{R}^n)}$ with equivalent positive constants independent of $f$, respectively.*

(II) *If $n>2\alpha$ and $p\in(1,\frac{2n}{2\alpha+n})$, then the equivalence between (i) and either (ii) or (iii) of (I) no longer holds true.*

On the other hand, let $\vec{0}$ denote the origin of $\mathbb{R}^n$,

$$\widetilde{\chi}(x):=\frac{1}{|B(\vec{0},1)|}\chi_{B(\vec{0},1)}(x),\quad \widetilde{\chi}_t(x):=t^{-n}\widetilde{\chi}(x/t),\quad \forall\, x\in\mathbb{R}^n,\quad \forall\, t\in(0,\infty),\tag{1.8}$$

where $\chi_E$ denotes the *characteristic function* of the subset $E$ of $\mathbb{R}^n$. Then $B_tf=\widetilde{\chi}_t*f$ for any $f\in L^1_{\rm loc}(\mathbb{R}^n)$. Based on this observation and motivated by [1], Sato [16] gave a weighted generalization of Theorem 1.A when $\alpha\in(0,2)$. To state his result, we need the following assumption on $\Phi$.

**Assumption 1.E.** Let $\Phi$ be a bounded radial function on $\mathbb{R}^n$ with compact support satisfying $\int_{\mathbb{R}^n}\Phi(x)\,dx=1$.

It is easy to see that $\widetilde{\chi}$ is a special example satisfying Assumption 1.E. In what follows, for any $t\in(0,\infty)$, let $\Phi_t(\cdot):=t^{-n}\Phi(\cdot/t)$ and, for any $f\in\mathcal{S}(\mathbb{R}^n)$ and $x\in\mathbb{R}^n$,

$$T_\alpha(f)(x):=\left\{\int_0^\infty|I_\alpha(f)(x)-\Phi_t*I_\alpha(f)(x)|^2\,\frac{dt}{t^{1+2\alpha}}\right\}^{\frac12}\tag{1.9}$$

with $I_\alpha$ being the *Riesz potential operator* defined via

$$\widehat{I_\alpha f}(\xi):=(2\pi|\xi|)^{-\alpha}\widehat{f}(\xi),\quad \forall\,\xi\in\mathbb{R}^n\setminus\{\vec{0}\}.\tag{1.10}$$

In what follows, for any $p\in[1,\infty)$, we use $A_p(\mathbb{R}^n)$ to denote the class of Muckenhoupt weights on $\mathbb{R}^n$ and, for any $w\in A_p(\mathbb{R}^n)$, $L^p_w(\mathbb{R}^n)$ the weighted Lebesgue space equipped with the norm that, for any $f\in L^p_w(\mathbb{R}^n)$,

$$\|f\|_{L^p_w(\mathbb{R}^n)}:=\left\{\int_{\mathbb{R}^n}|f(x)|^pw(x)dx\right\}^{\frac1p}<\infty.$$

Sato in [16, Theorem 1.5] obtained the following result.

**Theorem 1.F.** *Let $\alpha \in (0,2)$, $p \in (1,\infty)$ and $\omega \in A_p(\mathbb{R}^n)$. Let $\Phi$ satisfy Assumption 1.E and $T_\alpha$ be as in (1.9). Then there exists a positive constant $C$ such that, for any $f \in \mathcal{S}(\mathbb{R}^n)$,*

$$C^{-1}\|f\|_{L^p_w(\mathbb{R}^n)} \le \|T_\alpha(f)\|_{L^p_w(\mathbb{R}^n)} \le C\|f\|_{L^p_w(\mathbb{R}^n)}.$$

In view of the definition of $W^{\alpha,p}(\mathbb{R}^n)$ and the density of $\mathcal{S}(\mathbb{R}^n)$ in $L^p_w(\mathbb{R}^n)$, we know that Theorem 1.F induces a generalized weighted version of Theorem 1.A, which is just [16, Corollary 1.6].

Comparing the characterizations obtained in [16] with those in [13] and [4], it is natural to ask whether or not we can also characterize Sobolev spaces as in Theorems 1.B and 1.C with the ball average function $\widetilde{\chi}_t$ used therein replaced by a general function $\Phi_t$ as in Theorem 1.F. The main purpose of this article is to answer this question.

To be precise, for any $\alpha \in (0,2)$, $\lambda \in (1,\infty)$, $f,\ g \in L^1_{\rm loc}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, define

$$U_\alpha(f)(x) := \left\{\int_0^\infty \left|\frac{\Phi_t * f(x) - f(x)}{t^\alpha}\right|^2 \frac{dt}{t}\right\}^{\frac12}, \tag{1.11}$$

$$\widetilde{U}_\alpha(f)(x) := \left\{\int_0^\infty \int_{B(x,t)} \left|\frac{\Phi_t * f(y) - f(y)}{t^\alpha}\right|^2 dy\,\frac{dt}{t^{n+1}}\right\}^{\frac12} \tag{1.12}$$

and

$$\widetilde{G}^*_{\alpha,\lambda}(f)(x) := \left\{\int_0^\infty \int_{\mathbb{R}^n} \left|\frac{\Phi_t * f(y) - f(y)}{t^\alpha}\right|^2 \left(\frac{t}{t+|x-y|}\right)^{\lambda n} dy\,\frac{dt}{t^{n+1}}\right\}^{\frac12}; \tag{1.13}$$

while for $\alpha = 2$, define

$$U(f,g)(x) := \left\{\int_0^\infty \left|\frac{\Phi_t * f(x) - f(x)}{t^2} - \Phi_t * g(x)\right|^2 \frac{dt}{t}\right\}^{\frac12},$$

$$\widetilde{U}(f,g)(x) := \left\{\int_0^\infty \int_{B(x,t)} \left|\frac{\Phi_t * f(y) - f(y)}{t^2} - \Phi_t * g(y)\right|^2 dy\,\frac{dt}{t^{n+1}}\right\}^{\frac12} \tag{1.14}$$

and

$$\widetilde{G}^*_{\lambda}(f,g)(x) := \left\{\int_0^\infty \int_{\mathbb{R}^n} \left|\frac{\Phi_t * f(y) - f(y)}{t^2} - \Phi_t * g(y)\right|^2 \times \left(\frac{t}{t+|x-y|}\right)^{\lambda n} dy\,\frac{dt}{t^{n+1}}\right\}^{\frac12}. \tag{1.15}$$

Notice that Theorem 1.F implies that, for any $\alpha \in (0,2)$ and $p \in (1,\infty)$, $f \in W^{\alpha,p}(\mathbb{R}^n)$ if and only if $f \in L^p(\mathbb{R}^n)$ and $U_\alpha(f) \in L^p(\mathbb{R}^n)$. The main results of this article read as the following several theorems.

**Theorem 1.1.** *Let $n \in \mathbb{N}$, $\alpha \in (0,2)$, $\Phi$ satisfy Assumption 1.E, and $\widetilde{U}_\alpha$ and $\widetilde{G}^*_{\alpha,\lambda}$ be as (1.12) and (1.13), respectively.*

(I) *If $p \in (\max\{1, \frac{2n}{2\alpha+n}\}, \infty)$, then the following statements are mutually equivalent:*

(i) $f \in W^{\alpha,p}(\mathbb{R}^n)$;

(ii) $f \in L^p(\mathbb{R}^n)$ *and* $\widetilde{U}_\alpha(f) \in L^p(\mathbb{R}^n)$;

(iii) $f \in L^p(\mathbb{R}^n)$ *and* $\widetilde{G}^*_{\alpha,\lambda}(f) \in L^p(\mathbb{R}^n)$ *for some* $\lambda \in (\max\{1, \frac{2}{p}\}, \infty)$.

*Moreover, $\|\widetilde{U}_\alpha(f)\|_{L^p(\mathbb{R}^n)}$ and $\|\widetilde{G}^*_{\alpha,\lambda}(f)\|_{L^p(\mathbb{R}^n)}$ are equivalent to $\|(-\Delta)^{\alpha/2} f\|_{L^p(\mathbb{R}^n)}$ with equivalent positive constants independent of $f$, respectively.*

(II) *If $n > 2\alpha$ and $p \in (1, \frac{2n}{2\alpha+n})$, then equivalence between (i) and either (ii) or (iii) of (I) no longer holds true.*

We point out that, when $\Phi = \widetilde{\chi}$ with $\widetilde{\chi}$ as in (1.8), Theorem 1.1 coincides with Theorems 1.D.

To consider the end-point case of Theorem 1.1, we need the following stronger assumption on $\Phi$.

**Assumption 1.2.** Let $\Phi$ be a bounded radial function on $\mathbb{R}^n$ with compact support satisfying $\int_{\mathbb{R}^n} \Phi(x)\,dx = 1$. Assume that there exists $t_0 \in (0,\infty)$, depending on $\Phi$, such that $\operatorname{supp}\Phi \subseteq \overline{B(\vec{0}, t_0)}$ (the closure of $B(\vec{0}, t_0)$) and, for any $t \in (0, t_0)$, there exists $x \in \mathbb{R}^n$ such that $|x| \in (t, t_0)$ and $\Phi(x) \neq 0$. Moreover, assume that there exists a positive constant $C_{(\Phi)}$, depending on $\Phi$, such that, for any $x$, $y \in B(\vec{0}, t_0)$,

$$|\Phi(x) - \Phi(y)| \le C_{(\Phi)} |x - y|. \tag{1.16}$$

Compared with Assumption 1.E, Assumption 1.2 requires that $\Phi$ satisfies the *additional local interior Lipschitz regularity (1.16).*

It is easy to see that there exist many functions $\Phi$ satisfying Assumption 1.2 and, especially, $\widetilde{\chi}$ as in (1.8) satisfies Assumption 1.2.

In what follows, for $\Phi$ as in Assumption 1.2, let

$$C_0 := \frac{1}{2} \int_{B(\vec{0}, t_0)} \Phi(x)(2\pi x_1)^2\,dx. \tag{1.17}$$

Since $\Phi$ is radial, it follows that, for any $i \in \{2, \dots, n\}$,

$$C_0 = \frac{1}{2} \int_{B(\vec{0}, t_0)} \Phi(x)(2\pi x_i)^2\,dx.$$

Also, for any locally integrable functions $f$ and $g$, let

$$\mathcal{F}(f,g)(x,t) := \left| \frac{\Phi_t * f(x) - f(x)}{t^2} + \Phi_t * g(x) \right|, \quad \forall\, x \in \mathbb{R}^n, \quad \forall\, t \in (0,\infty). \tag{1.18}$$

We have the following results.

**Theorem 1.3.** *Let $p \in (1, \infty)$ and $\Phi$ satisfy Assumption 1.2. Then the following two statements are equivalent:*

(i) $f \in W^{2,p}(\mathbb{R}^n)$;

(ii) $f \in L^p(\mathbb{R}^n)$ *and there exists* $g \in L^p(\mathbb{R}^n)$ *such that* $\mathcal{G}(\mathcal{F}(f,g)) \in L^p(\mathbb{R}^n)$, *where* $\mathcal{F}(f,g)$ *is as in (1.18) and*

$$\mathcal{G}(\mathcal{F}(f,g))(x) := \left\{ \int_0^\infty |\mathcal{F}(f,g)(x,t)|^2 \, \frac{dt}{t} \right\}^{\frac{1}{2}}, \quad \forall\, x \in \mathbb{R}^n.$$

*Moreover, if* $f \in W^{2,p}(\mathbb{R}^n)$, *then* $g$ *in (ii) can be taken as* $g := C_0 \Delta f$; *while, if (ii) holds true, then* $g = C_0 \Delta f$ *almost everywhere, where* $C_0$ *is as in (1.17). In both cases,* $\|\mathcal{G}(\mathcal{F}(f,g))\|_{L^p(\mathbb{R}^n)}$ *is equivalent to* $\|\Delta f\|_{L^p(\mathbb{R}^n)}$ *with equivalent positive constants independent of* $f$ *and* $g$.

Observe that, when $\Phi = \widetilde{\chi}$ with $\widetilde{\chi}$ as in (1.8), Theorem 1.3 coincides with Theorem 1.A in the case $\alpha = 2$.

**Theorem 1.4.** *Let* $n \in \mathbb{N}$, $\Phi$ *satisfy Assumption 1.2,* $\widetilde{U}$ *and* $\widetilde{G}^*_\lambda$ *be as in (1.14) and (1.15), respectively.*

(I) *If* $p \in (\max\{1, \frac{2n}{4+n}\}, \infty)$, *then the following statements are mutually equivalent:*

  (i) $f \in W^{2,p}(\mathbb{R}^n)$;

  (ii) $f \in L^p(\mathbb{R}^n)$ *and there exists* $g \in L^p(\mathbb{R}^n)$ *such that* $\widetilde{U}(f,g) \in L^p(\mathbb{R}^n)$;

  (iii) $f \in L^p(\mathbb{R}^n)$ *and there exists* $g \in L^p(\mathbb{R}^n)$ *such that* $\widetilde{G}^*_\lambda(f,g) \in L^p(\mathbb{R}^n)$ *for some* $\lambda \in (\max\{1, \frac{2}{p}\}, \infty)$.

  *Moreover, if* $f \in W^{2,p}(\mathbb{R}^n)$, *then* $g$ *in (ii) and (iii) can be taken as* $g := C_0 \Delta f$; *while, if either of (ii) and (iii) holds true, then* $g = C_0 \Delta f$ *almost everywhere, where* $C_0$ *is as in (1.17). In any case,* $\|\widetilde{U}(f,g)\|_{L^p(\mathbb{R}^n)}$ *and* $\|\widetilde{G}^*_\lambda(f,g)\|_{L^p(\mathbb{R}^n)}$ *are equivalent to* $\|\Delta f\|_{L^p(\mathbb{R}^n)}$ *with equivalent positive constants independent of* $f$ *and* $g$.

(II) *If* $n \geq 5$ *and* $p \in (1, \frac{2n}{4+n})$, *then the equivalence between (i) and either (ii) or (iii) of (I) no longer holds true.*

We point out that, when $\Phi = \widetilde{\chi}$ with $\widetilde{\chi}$ as in (1.8), Theorem 1.4 coincides with Theorems 1.B and 1.C.

The proof of Theorem 1.1 is given in Section 2, and the proofs of Theorems 1.3 and 1.4 are presented in Section 3. We observe that (ii) and (iii) of Theorem 1.1(I) are equivalent for all $p \in (1, \infty)$ and $n \in \mathbb{N}$ (see its proof below). Moreover, the condition $p \in (\max\{1, \frac{2n}{4+n}\}, \infty)$ is nearly sharp in the sense that, if (i) of Theorem 1.1(I) is equivalent to (ii) or to (iii) of Theorem 1.1(I), then one must have

$$p \in \left[\frac{2n}{2\alpha + n}, \infty\right) \text{ when } n > 2\alpha, \quad \text{and} \quad p \in (1, \infty) \text{ when } n \leq 2\alpha.$$

Also, the items (ii) and (iii) of Theorem 1.4(I) are equivalent for all $p \in (1,\infty)$ and $n \in \mathbb{N}$, moreover, the condition $p \in (\frac{2n}{4+n},\infty)$ is also nearly sharp in the sense that, if (i) of Theorem 1.4(I) is equivalent to (ii) or to (iii) of Theorem 1.4(I), then one must has $p \in (\frac{2n}{4+n},\infty)$ when $n \geq 5$ and $p \in (1,\infty)$ when $n \in \{1,2,3,4\}$.

Finally, in the endpoint case when $p = \frac{2n}{2\alpha+n}$, we have the following weak-type result. Recall that, for any $p \in (0,\infty)$, $f \in WL^p(\mathbb{R}^n)$ if and only if $f$ is measurable and

$$\|f\|_{WL^p(\mathbb{R}^n)} := \sup_{\gamma\in(0,\infty)} \gamma|\{x \in \mathbb{R}^n : |f(x)| > \gamma\}|^{1/p} < \infty.$$

**Theorem 1.5.** *Let $\Phi$ satisfy Assumption 1.2, $\alpha \in (0,2]$, $p = 2n/(n+2\alpha) \in (1,\infty)$ and $\lambda \in (2/p,\infty)$. Let $I_\alpha$, $\widetilde{U}_\alpha$, $\widetilde{G}^*_{\alpha,\lambda}$, $\widetilde{U}$ and $\widetilde{G}^*_\lambda$ be as in (1.10), (1.12), (1.13), (1.14) and (1.15), respectively.*

(i) *If $\alpha \in (0,2)$, then $\widetilde{U}_\alpha \circ I_\alpha$ and $\widetilde{G}^*_{\alpha,\lambda} \circ I_\alpha$ are bounded from $L^p(\mathbb{R}^n)$ to $WL^p(\mathbb{R}^n)$ and hence $\widetilde{U}_\alpha$ and $\widetilde{G}^*_{\alpha,\lambda}$ are bounded from $W^{\alpha,p}(\mathbb{R}^n)$ to $WL^p(\mathbb{R}^n)$.*

(ii) *If $\alpha = 2$, then, for any $f \in W^{2,p}(\mathbb{R}^n)$, $\widetilde{U}(f,C_0\Delta f)$ and $\widetilde{G}^*_\lambda(f,C_0\Delta f)$ belong to $WL^p(\mathbb{R}^n)$, where $C_0$ is as (1.17), and there exists a positive constant $C$, independent of $f$, such that*

$$\|\widetilde{U}(f,C_0\Delta f)\|_{WL^p(\mathbb{R}^n)} + \|\widetilde{G}^*_\lambda(f,C_0\Delta f)\|_{WL^p(\mathbb{R}^n)} \leq C\|f\|_{W^{2,p}(\mathbb{R}^n)}.$$

**Remark 1.6.** (i) Even when $\Phi = \widetilde{\chi}$ with $\widetilde{\chi}$ as in (1.8), Theorem 1.5 is also new.

(ii) On Theorem 1.5(i), if $\alpha$, $p$ and $\lambda$ are as therein and $\widetilde{U}_\alpha \in WL^p(\mathbb{R}^n)$ or $\widetilde{G}^*_{\alpha,\lambda} \in WL^p(\mathbb{R}^n)$, then it is still unclear whether or not $f \in W^{\alpha,p}(\mathbb{R}^n)$.

(iii) On Theorem 1.5(ii), it is still unclear whether or not a reverse statement still holds true. Namely, if $p$ and $\lambda$ are as in Theorem 1.5(ii) and there exists a $g \in L^p(\mathbb{R}^n)$ such that either $\widetilde{U}(f,g)$ or $\widetilde{G}^*_\lambda(f,g)$ belongs to $WL^p(\mathbb{R}^n)$, where $C_0$ is as (1.17), it is still unclear whether or not $f \in W^{2,p}(\mathbb{R}^n)$.

(iv) Observe that, if $\Phi$ satisfies either Assumption (1.E) or Assumption (1.2), then, for any $t \in (0,\infty)$, $\Phi_t * f$ is indeed an average of $f$ on a certain set, which, for some special choices of $\Phi$ (for example, when $\Phi = \widetilde{\chi}$ with $\widetilde{\chi}$ as in (1.8)), has a natural generalization in metric measure spaces. Thus, Theorems 1.1, 1.3, 1.4 and 1.5 provide some new choices for introducing fractional Sobolev spaces on metric measure spaces.

To prove Theorems 1.1 and 1.4, we borrow some ideas from [1] and [4]. The main idea is to control the Lusin area functions $\widetilde{U}_\alpha(f)$ and $\widetilde{U}(f,g)$ by a sum of a sequence of convolution operators whose kernels satisfy vector-valued Hörmander conditions. Then, applying the vector-valued Calderón-Zygmund theory (see [9, Theorem 3.4]) and the Marcinkiewicz interpolation theorem (see [10, Theorem 1.3.2]), we obtain the boundedness of all such convolution operators on $L^p(\mathbb{R}^n)$ as well as the exact decay estimates of their operator norms, which imply the desired boundedness of the Lusin area function. On the other hand, we make use of the fact that $\dot{W}^{\alpha,p}(\mathbb{R}^n) = \dot{F}^\alpha_{p,2}(\mathbb{R}^n)$ and prove that $\|f\|_{\dot{F}^\alpha_{p,2}(\mathbb{R}^n)} \lesssim$

$\|\widetilde{U}_\alpha(f)\|_{L^p(\mathbb{R}^n)}$, with $\alpha \in (0,2)$, $p \in (1,\infty)$ and the implicit positive constant independent of $f$, by means of the classical Lusin area function characterization of Triebel-Lizorkin spaces $\dot{F}^\alpha_{p,2}(\mathbb{R}^n)$ (see, for example, [19]) and the Fefferman-Stein vector-valued inequality from [7]. To prove Theorem 1.5, we apply some methods from Fefferman [8].

**Remark 1.7.** We point out that, as a generalization of the classical Sobolev spaces with integer smoothness order, the fractional Sobolev spaces $W^{\alpha,p}(\mathbb{R}^n)$ considered in this article were usually called the *Bessel-potential* (or *Lebesgue* or *Liouville*) *spaces* in other literatures, which were also denoted by $H^{\alpha,p}(\mathbb{R}^n)$ therein (see, for example, [18, p. 37]). The fractional Sobolev space $W^{\alpha,p}(\mathbb{R}^n)$ coincides with the Triebel-Lizorkin space $F^\alpha_{p,2}(\mathbb{R}^n)$ for any $\alpha \in (0,\infty)$ and $p \in (1,\infty)$ (see, for example, [18, Section 2.2.2]). It is well known that, in many literatures, there exists another approach for fractional Sobolev-type spaces, which were called *Aronszajn*, *Gagliardo* or *Slobodeckij spaces* (see, for example, [5, p. 524] or [18, p. 36]), and we denote them by the symbol $\widetilde{W}^{\alpha,p}(\mathbb{R}^n)$ here. Recall that, following [5], for any $p \in [1,\infty)$ and $\theta \in (0,1)$, the so-called Gagliardo (semi)norm of $f \in L^p(\mathbb{R}^n)$ is given by

$$[f]_{\theta,p} := \left\{\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{|x-y|^{n+\theta p}}\,dxdy\right\}^{1/p}.$$

The *Aronszajn*, *Gagliardo* or *Slobodeckij space* $\widetilde{W}^{\alpha,p}(\mathbb{R}^n)$, with any $\alpha \in (0,\infty)\setminus\mathbb{N}$ and $p \in [1,\infty)$, is then defined to be the set of all functions $f \in W^{\lfloor\alpha\rfloor,p}(\mathbb{R}^n)$ such that

$$\|f\|_{\widetilde{W}^{\alpha,p}(\mathbb{R}^n)} := \|f\|_{W^{\lfloor\alpha\rfloor,p}(\mathbb{R}^n)} + \sup_{\substack{\beta\in\mathbb{Z}^n_+\\ |\beta|=\lfloor\alpha\rfloor}} [D^\beta f]_{\alpha-\lfloor\alpha\rfloor,p}$$

is finite, where $\lfloor\alpha\rfloor$ denotes the maximal integer not greater than $\alpha$ (see, for example, [18, p. 36]). The space $\widetilde{W}^{\alpha,p}(\mathbb{R}^n)$ coincides with the Besov space $B^\alpha_{p,p}(\mathbb{R}^n)$ for any $\alpha \in (0,\infty)\setminus\mathbb{N}$ and $p \in [1,\infty)$ and hence $\widetilde{W}^{\alpha,2}(\mathbb{R}^n) = W^{\alpha,2}(\mathbb{R}^n)$ (see, for example, [18, Section 2.2.2]). However, $\widetilde{W}^{\alpha,p}(\mathbb{R}^n)$ does not coincide with $W^{\alpha,p}(\mathbb{R}^n)$ when $p \neq 2$ (see, for example, [5, Remark 3.5]). (We thank the referee for reminding us these facts.)

Finally, we make some conventions on notation. We denote by $C$ a *positive constant* which is independent of the main parameters, but may vary from line to line. The *symbol* $A \lesssim B$ means $A \le CB$. If $A \lesssim B$ and $B \lesssim A$, then we write $A \sim B$. We use $C_{(\alpha,\dots)}$ to denote a positive constant depending on the indicated parameters $\alpha,\dots$. Let $M$ denote the *Hardy-Littlewood maximal operator* defined by setting, for any $f \in L^1_{\rm loc}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$M(f)(x) := \sup_{B\ni x}\frac{1}{|B|}\int_B |f(y)|\,dy, \tag{1.19}$$

where the supremum is taken over all balls $B$ containing $x$. Also, we use $\vec{0}$ to denote the origin of $\mathbb{R}^n$ and $\mathbb{N}$ to denote the set of all positive integers. Let $\mathbb{Z}_+ := \mathbb{N}\cup\{0\}$ and $\mathbb{Z}_- := \mathbb{Z}\setminus\mathbb{Z}_+$. For any $f \in \mathcal{S}'(\mathbb{R}^n)$, we use $\widehat{f}$ and $f^\vee$ to denote its Fourier transform, respectively, its inverse Fourier transform.

## 2 Proof of Theorem 1.1

To prove Theorem 1.1, we first consider the relations among $U_\alpha$, $\widetilde{U}_\alpha$ and $\widetilde{G}^*_{\alpha,\lambda}$, respectively, from (1.11), (1.12) and (1.13). To this end, for any $\lambda \in (1,\infty)$, measurable function $\mathcal{F}$ on $\mathbb{R}^n \times (0,\infty)$ and $x \in \mathbb{R}^n$, define

$$\mathcal{G}(\mathcal{F})(x) := \left\{ \int_0^\infty |\mathcal{F}(x,t)|^2 \, \frac{dt}{t} \right\}^{\frac{1}{2}}, \tag{2.1}$$

$$\widetilde{\mathcal{G}}(\mathcal{F})(x) := \left\{ \int_0^\infty \int_{B(x,t)} |\mathcal{F}(y,t)|^2 \, dy \, \frac{dt}{t^{n+1}} \right\}^{\frac{1}{2}} \tag{2.2}$$

and

$$\mathcal{G}^*_\lambda(\mathcal{F})(x) := \left\{ \int_0^\infty \int_{\mathbb{R}^n} |\mathcal{F}(y,t)|^2 \left( \frac{t}{t+|x-y|} \right)^{\lambda n} dy \, \frac{dt}{t^{n+1}} \right\}^{\frac{1}{2}}. \tag{2.3}$$

For these operators, we have the following lemmas from [13, Lemmas 2.1(iii) and 2.2], respectively.

**Lemma 2.1.** *Let $\lambda \in (1,\infty)$ and $p \in [2,\infty)$. Then there exists a positive constant $C$ such that, for any measurable function $\mathcal{F}$ on $\mathbb{R}^n \times (0,\infty)$, $\|\mathcal{G}^*_\lambda(\mathcal{F})\|_{L^p(\mathbb{R}^n)} \le C\|\mathcal{G}(\mathcal{F})\|_{L^p(\mathbb{R}^n)}$.*

**Lemma 2.2.** *Let $p \in (1,\infty)$ and $\lambda \in (\max\{\frac{2}{p},1\},\infty)$. Then, for any measurable function $\mathcal{F}$ on $\mathbb{R}^n \times (0,\infty)$, $\widetilde{\mathcal{G}}(\mathcal{F}) \in L^p(\mathbb{R}^n)$ if and only if $\mathcal{G}^*_\lambda(\mathcal{F}) \in L^p(\mathbb{R}^n)$. Moreover, the $L^p(\mathbb{R}^n)$-norm of $\mathcal{G}^*_\lambda(\mathcal{F})$ is equivalent to that of $\widetilde{\mathcal{G}}(\mathcal{F})$ with the equivalent positive constants independent of $\mathcal{F}$.*

For any $\alpha \in (0,2)$, $\Phi$ satisfying Assumption 1.E and $x \in \mathbb{R}^n$, let

$$\Psi(x) := \Phi * I_\alpha(x) - I_\alpha(x), \tag{2.4}$$

where $I_\alpha$ is as in (1.10). Then we have

$$\widehat{\Psi}(\xi) = (2\pi|\xi|)^{-\alpha} \left[\widehat{\Phi}(\xi) - 1\right], \quad \forall\, \xi \in \mathbb{R}^n \setminus \{\vec{0}\}$$

and, by the properties of $\Phi$, we also have

$$\Psi(x) = \frac{1}{2} \int_{\mathbb{R}^n} [I_\alpha(x-y) + I_\alpha(x+y) - 2I_\alpha(x)]\Phi(y)\, dy, \quad \forall\, x \in \mathbb{R}^n \tag{2.5}$$

and hence

$$\widehat{\Psi}(\xi) = \frac{1}{2} \int_{\mathbb{R}^n} (2\pi|\xi|)^{-\alpha} \left[e^{2\pi i (y,\xi)} + e^{-2\pi i (y,\xi)} - 2\right] \Phi(y)\, dy, \quad \forall\, \xi \in \mathbb{R}^n \tag{2.6}$$

(see [16, (2.1) and (2.4)]). Then we have the following lemma.

**Lemma 2.3.** *Let $\alpha \in (0,\infty)$, $\Phi$ satisfy Assumption 1.E and $\Psi$ be as in (2.5). Then there exists a positive constant $C$ such that, for any $\xi \in \mathbb{R}^n \setminus \{\vec{0}\}$,*

$$|\widehat{\Psi}(\xi)| \le C \min\left\{|\xi|^{2-\alpha},\ |\xi|^{-\alpha}\right\}.$$

*Proof.* Let $M_0 := \sup_{x\in\mathbb{R}^n} |\Phi(x)|$, $E := \operatorname{supp}\Phi$ and $a := |E|$. Then, since $\Phi$ is a bounded and radial function with compact support, it follows that both $M_0$ and $a$ are finite.

On one hand, from (2.6), we deduce that, for any $\xi \in \mathbb{R}^n \setminus \{\vec{0}\}$,

$$\begin{aligned}|\widehat{\Psi}(\xi)| &= \frac{1}{2}(2\pi|\xi|)^{-\alpha}\left|\int_{\mathbb{R}^n}\left[e^{2\pi i(y,\xi)} + e^{-2\pi i(y,\xi)} - 2\right]\Phi(y)\,dy\right|\\ &\le \frac{1}{2}(2\pi|\xi|)^{-\alpha}\int_E\left[|e^{2\pi i(y,\xi)}| + |e^{-2\pi i(y,\xi)}| + 2\right]|\Phi(y)|\,dy \le 2aM_0(2\pi)^{-\alpha}|\xi|^{-\alpha}.\end{aligned}$$

On the other hand, also from (2.6), combined with the Cauchy-Schwarz inequality, we conclude that, for any $\xi \in \mathbb{R}^n \setminus \{\vec{0}\}$,

$$\begin{aligned}|\widehat{\Psi}(\xi)| &= \frac{1}{2}(2\pi|\xi|)^{-\alpha}\left|\int_{\mathbb{R}^n}[2\cos(2\pi(y,\xi)) - 2]\Phi(y)\,dy\right|\\ &\le 2(2\pi|\xi|)^{-\alpha}\int_E[\sin(\pi(y,\xi))]^2|\Phi(y)|\,dy\\ &\le 2M_0\pi^2(2\pi|\xi|)^{-\alpha}\int_E|(y,\xi)|^2\,dy \le 2M_0L\pi^2(2\pi)^{-\alpha}|\xi|^{2-\alpha},\end{aligned}$$

where $L := \int_E |y|^2\,dy$. This finishes the proof of Lemma 2.3 □

Notice that $\widehat{\Phi}$ is also a radial function. For any $\tau \in (0,\infty)$, let $F(\tau) := \widehat{\Phi}(\xi)$, where $\xi \in \mathbb{R}^n$ and $|\xi| = \tau$. Then we have the following conclusion.

**Lemma 2.4.** *Let $\alpha \in (0,2]$.*

(i) *It holds true that there exists a positive constant $C$, independent of $\alpha$, such that*

$$\sup_{\tau\in(0,\infty)}\left|\frac{F(\tau)-1}{\tau^\alpha}\right| \le C < \infty. \tag{2.7}$$

(ii) *If $j \in \mathbb{N}$ and $s \in (0,\infty)$, then there exists a positive constat $C_{(\alpha,s,j)}$, depending on $\alpha$, $s$ and $j$, such that*

$$\sup_{\tau\in(s,\infty)}\left|\frac{d^j}{d\tau^j}\left[\frac{F(\tau)-1}{\tau^\alpha}\right]\right| \le C_{(\alpha,s,j)} < \infty. \tag{2.8}$$

*Proof.* Let $e_1 := (1,0,\dots,0) \in \mathbb{R}^n$ and $x := (x_1,x_2,\dots,x_n)$. Then, by Assumption 1.E, we know that

$$\frac{F(\tau)-1}{\tau^\alpha} = \tau^{-\alpha}\left[\widehat{\Phi}(\tau e_1) - 1\right] = \tau^{-\alpha}\left[\int_{\mathbb{R}^n}\Phi(x)e^{-2\pi i x_1\tau}\,dx - 1\right]$$

$$= \int_{\mathbb{R}^n} \Phi(x)\tau^{-\alpha}[\cos(2\pi x_1\tau) - 1]\,dx. \tag{2.9}$$

We first prove (i). To this end, we consider two cases. In the case when $\tau \in [1, \infty)$, since $\Phi$ is bounded and has compact support, it follows that

$$\left|\tau^{-\alpha}\int_{\mathbb{R}^n} \Phi(x)[\cos(2\pi x_1\tau) - 1]\,dx\right| \lesssim 1. \tag{2.10}$$

When $\tau \in (0, 1)$, by the mean value theorem, we know that

$$|\cos(2\pi x_1\tau) - 1| = 2[\sin(\pi x_1\tau)]^2 = 2(\pi x_1\tau)^2[\cos(\theta\pi x_1\tau)]^2$$

for some $\theta \in (0, 1)$, which, together with $\alpha \in (0, 2)$, further implies that

$$\tau^{-\alpha}|\cos(2\pi x_1\tau) - 1| = 2(\pi x_1)^2\tau^{2-\alpha}[\cos(\theta\pi x_1\tau)]^2 \lesssim x_1^2$$

and then, by the fact that $\Phi$ is bounded and has compact support again, we conclude that

$$\left|\int_{\mathbb{R}^n} \Phi(x)\tau^{-\alpha}[\cos(2\pi x_1\tau) - 1]\,dx\right| \lesssim \int_{\mathbb{R}^n} |\Phi(x)|x_1^2\,dx \lesssim 1,$$

which, combined with (2.10), implies (2.7) and hence completes the proof of (i).

Now we prove (ii), namely, we show that (2.8) holds true. ¿From (2.9), the Leibniz formula and Assumption 1.E, we deduce that

$$\begin{aligned}
\sup_{\tau\in(s,\infty)}\left|\frac{d^j}{d\tau^j}\left[\frac{F(\tau)-1}{\tau^\alpha}\right]\right| &= \sup_{\tau\in(s,\infty)}\left|\int_{\mathbb{R}^n}\Phi(x)\frac{d^j}{d\tau^j}\{\tau^{-\alpha}[\cos(2\pi x_1\tau) - 1]\}\,dx\right| \\
&= \sup_{\tau\in(s,\infty)}\left|\int_{\mathbb{R}^n}\Phi(x)\sum_{k=0}^{j} c_k(\tau^{-\alpha})^{(k)}[\cos(2\pi x_1\tau) - 1]^{(j-k)}\,dx\right| \\
&\lesssim \sup_{\tau\in(s,\infty)}\int_{\mathbb{R}^n}|\Phi(x)|\sum_{k=0}^{j}\tau^{-\alpha-k}\,dx \lesssim s^{-\alpha-j},
\end{aligned}$$

where $c_k := \binom{j}{k}$ denotes the binomial coefficient and $f^{(n)} := (\frac{\partial}{\partial\tau})^n f$, which implies (2.8). This finishes the proof of Lemma 2.4. □

For any $\alpha \in (0, 2)$, $f \in L^1_{\rm loc}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, let

$$U^*_\alpha(f)(x) := \left\{\int_0^\infty \fint_{B(\vec{0},t)} |\Psi_t * f(x+y)|^2\,dy\,\frac{dt}{t}\right\}^{\frac12}, \tag{2.11}$$

where $\Psi$ is as in (2.4) and $\Psi_t(\cdot) = t^{-n}\Psi(\cdot/t)$ for any $t \in (0, \infty)$. Then it is easy to see that, for any $f \in L^1_{\rm loc}(\mathbb{R}^n)$.

$$\widetilde{U}_\alpha(f) \sim U^*_\alpha((-\Delta)^{\frac{\alpha}{2}} f) \tag{2.12}$$

with the equivalent positive constants independent of $f$.

Let $\varphi,\ \psi \in \mathcal{S}(\mathbb{R}^n)$ be two radial functions such that

$$\begin{cases} \operatorname{supp}\widehat{\varphi} \subset \left\{\xi \in \mathbb{R}^n :\ \frac{1}{2} \le |\xi| \le 2\right\};\ \sum_{j\in\mathbb{Z}} \widehat{\varphi}(2^{-j}\xi) = 1, \forall\, \xi \in \mathbb{R}^n \setminus \{\vec{0}\}; \\ \operatorname{supp}\widehat{\psi} \subset \left\{\xi \in \mathbb{R}^n :\ \frac{1}{4} \le |\xi| \le 4\right\};\ \widehat{\psi}(\xi) = 1 \text{ when } \frac{1}{2} \le |\xi| \le 2. \end{cases} \tag{2.13}$$

Clearly, $f*\varphi_{2^{-j}} = f*\varphi_{2^{-j}}*\psi_{2^{-j}}$ for any $j \in \mathbb{Z}$ and $f \in \mathcal{S}'(\mathbb{R}^n)$. Then, from the Minkowski inequality, we deduce that, for any $f \in L^1_{\rm loc}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$\begin{aligned} U^*_\alpha(f)(x) &\sim \left\{\int_0^\infty \int_{\mathbb{R}^n} \left|\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} [f*\varphi_{2^{-j-k}}*\Psi_t(x+y)]\,\chi_{[2^{-k},2^{-k+1})}(t)\right|^2 \right.\\ &\qquad\left. \times \chi_{B(\vec{0},t)}(y)\,dy\,\frac{dt}{t^{n+1}}\right\}^{\frac12} \\ &\lesssim \sum_{j\in\mathbb{Z}} \left\{\int_0^\infty ⨍_{B(\vec{0},t)} \left|\sum_{k\in\mathbb{Z}} [f*\varphi_{2^{-j-k}}*\Psi_t(x+y)]\,\chi_{[2^{-k},2^{-k+1})}(t)\right|^2 dy\,\frac{dt}{t}\right\}^{\frac12} \\ &\lesssim \sum_{j\in\mathbb{Z}} T_j(f)(x), \end{aligned} \tag{2.14}$$

where, for any $j \in \mathbb{Z}$,

$$T_j(f)(x) := \left[\sum_{k\in\mathbb{Z}} \int_{2^{-k}}^{2^{-k+1}} ⨍_{B(\vec{0},2^{-k+1})} |f*\varphi_{2^{-j-k}}*\Psi_t(x+y)|^2\,dy\,\frac{dt}{t}\right]^{\frac12}. \tag{2.15}$$

By Lemma 2.3, we have the following estimates.

**Lemma 2.5.** *Let $\alpha \in (0,2)$ and $\Phi$ satisfy Assumption 1.E. Then there exists a positive constant $C$ such that, for any $j \in \mathbb{Z}$ and $f \in L^2(\mathbb{R}^n)$,*

$$\|T_j(f)\|_{L^2(\mathbb{R}^n)} \le C\left[\min\left\{2^{-\alpha j}, 2^{(2-\alpha)j}\right\}\right]\|f\|_{L^2(\mathbb{R}^n)},$$

*where $T_j$ is as in (2.15).*

*Proof.* By the Plancherel theorem and Lemma 2.3, we have

$$\begin{aligned} \|T_j(f)\|^2_{L^2(\mathbb{R}^n)} &= \sum_{k\in\mathbb{Z}} \int_{2^{-k}}^{2^{-k+1}} ⨍_{B(\vec{0},2^{-k+1})} \int_{\mathbb{R}^n} |f*\varphi_{2^{-j-k}}*\Psi_t(x+y)|^2\,dx\,dy\,\frac{dt}{t} \\ &\lesssim \sum_{k\in\mathbb{Z}} \int_{2^{-k}}^{2^{-k+1}} \int_{\mathbb{R}^n} |f*\varphi_{2^{-j-k}}*\Psi_t(x)|^2\,dx\,\frac{dt}{t} \\ &\lesssim \sum_{k\in\mathbb{Z}} \int_{2^{j+k-1}\le|\xi|<2^{j+k+1}} |\widehat{f}(\xi)|^2 \int_{2^{-k}}^{2^{-k+1}} |\widehat{\Psi}(t\xi)|^2\,\frac{dt}{t}\,d\xi \end{aligned}$$

$$\lesssim \left[\min\left\{2^{-2\alpha j}, 2^{2(2-\alpha)j}\right\}\right] \|f\|_{L^2(\mathbb{R}^n)}^2,$$

which implies the desired conclusion and hence completes the proof of Lemma 2.5. □

¿From Lemmas 2.4 and 2.5, we deduce the following Lemma 2.6.

**Lemma 2.6.** *Let $\alpha \in (0,2)$, $p \in (1,\infty)$, $q := \max\{p, p'\}$ with $p'$ being the conjugate index of $p$ (namely, $\frac{1}{p} + \frac{1}{p'} = 1$), $\varepsilon \in (0, \frac{1}{q})$ and $\Phi$ satisfy Assumption 1.E. Let $\varphi$, $\psi \in \mathcal{S}(\mathbb{R}^n)$ be as in (2.13). Then there exists a positive constant $C_{(\varepsilon,\alpha,p)}$, depending on $\varepsilon$, $\alpha$, and $p$, such that, for any $j \in (-\infty, 0] \cap \mathbb{Z}$ and $f \in L^p(\mathbb{R}^n)$,*

$$\|T_j(f)\|_{L^p(\mathbb{R}^n)} \leq C_{(\varepsilon,\alpha,p)} 2^{(4-2\alpha)j(\frac{1}{q}-\varepsilon)} \|f\|_{L^p(\mathbb{R}^n)},$$

*where $T_j$ is as in (2.15).*

*Proof.* By the definition of $T_j(f)$ (see (2.15)), we know that, for any $x \in \mathbb{R}^n$,

$$|T_j(f)(x)|^2 \lesssim \sum_{k\in\mathbb{Z}} \sup_{\substack{y\in B(\vec{0}, 2^{-k+1})\\ t\in[2^{-k}, 2^{-k+1})}} |t^{-\alpha}(\Phi_t * f * I_\alpha * \varphi_{2^{-j-k}} - f * I_\alpha * \varphi_{2^{-j-k}})(x+y)|^2. \tag{2.16}$$

Notice that, for any $\xi \in \mathbb{R}^n \setminus \{\vec{0}\}$,

$$\begin{aligned} t^{-\alpha}(\Phi_t * f * I_\alpha * \varphi_{2^{-j-k}} - f * I_\alpha * \varphi_{2^{-j-k}})^\wedge(\xi) &= \widehat{\varphi}(2^{-j-k}\xi)\frac{\widehat{\Phi}(t\xi)-1}{|2\pi t\xi|^\alpha}\widehat{f_{j+k}}(\xi) \\ &=: m_{j,k}(2^{-k}\xi)\widehat{f_{j+k}}(\xi), \end{aligned}$$

where

$$m_{j,k}(\xi) := \widehat{\varphi}(2^{-j}\xi)\frac{\widehat{\Phi}(2^k t\xi)-1}{|2^{k+1}\pi t\xi|^\alpha}$$

and $f_{j+k} := f * \psi_{2^{-j-k}}$, with $\psi$ and $\varphi$ as in (2.13). Since $j \leq 0$, from Lemma 2.4, we deduce that, for any $t \in [2^{-k}, 2^{-k+1})$,

$$|\nabla^i m_{j,k}(\xi)| \lesssim 2^{-ij}\chi_{[2^{j-1}, 2^{j+1}]}(|\xi|), \quad \forall\, i \in \mathbb{Z}_+,\ \forall\, \xi \in \mathbb{R}^n$$

and hence

$$|m_{j,k}^\vee(x)| \lesssim 2^{jn}(1+2^j|x|)^{-n-1}, \quad \forall\, x \in \mathbb{R}^n. \tag{2.17}$$

Thus, by (2.17), we conclude that, for any $t \in [2^{-k}, 2^{-k+1})$ and $x \in \mathbb{R}^n$,

$$\begin{aligned} &t^{-\alpha} \sup_{|y|\leq 2^{-k+1}} |(\Phi_t * f * I_\alpha * \varphi_{2^{-j-k}} - f * I_\alpha * \varphi_{2^{-j-k}})(x+y)| \\ &= \sup_{|y|\leq 2^{-k+1}} 2^{kn} \left|\int_{\mathbb{R}^n} f_{j+k}(z) m_{j,k}^\vee(2^k(x+y-z))\,dz\right| \\ &\lesssim 2^{(k+j)n} \int_{\mathbb{R}^n} |f_{j+k}(z)| \sup_{|y|\leq 2^{-k+1}} (1+2^{k+j}|x+y-z|)^{-n-1}\,dz \end{aligned}$$

$$\lesssim 2^{(k+j)n}\int_{\mathbb{R}^n}|f_{j+k}(z)|(1+2^{k+j}|x-z|)^{-n-1}\,dz\lesssim M(f_{j+k})(x),$$

where $M$ denotes the Hardy-Littlewood maximal operator as in (1.19). ¿From this, (2.16), the Fefferman-Stein vector-valued inequality (see [7]) and the Littlewood-Paley characterization of Lebesgue spaces (see, for example, [11, Theorem 1.3.8]), we deduce that

$$\|T_j(f)\|_{L^r(\mathbb{R}^n)}\lesssim\|f\|_{L^r(\mathbb{R}^n)},\quad\forall\,r\in(1,\infty).$$

Taking $r:=\frac{1-\frac{1}{q}+\varepsilon}{\frac{1}{p}-\frac{1}{2q}+\frac{\varepsilon}{2}}$, then, by Lemma 2.5 and the Riesz-Thorin interpolation theorem (see, for example, [10, Theorem 1.3.4]), we find that

$$\|T_j(f)\|_{L^p(\mathbb{R}^n)}\lesssim 2^{(4-2\alpha)j(\frac{1}{q}-\varepsilon)}\|f\|_{L^p(\mathbb{R}^n)},$$

which completes the proof of Lemma 2.6. □

For the case $j\in[0,\infty)\cap\mathbb{Z}$, we also obtain an estimate similar to that as in the above lemma.

**Lemma 2.7.** *Let $\alpha\in(0,2),\varepsilon\in(0,1)$, $p\in(1,2]$ and $\Phi$ satisfy Assumption 1.E. Then there exists a positive constant $C_{(\varepsilon,\alpha,p)}$, depending on $\varepsilon$, $\alpha$ and $p$, such that, for any $j\in[0,\infty)\cup\mathbb{Z}$ and $f\in L^p(\mathbb{R}^n)$,*

$$\|T_j(f)\|_{L^p(\mathbb{R}^n)}\le C_{(\varepsilon,\alpha,p)}2^{[n(\frac{1}{p}-\frac{1}{2})-\alpha+\varepsilon]j}\|f\|_{L^p(\mathbb{R}^n)},$$

*where $T_j$ is as in (2.15).*

*Proof.* By Lemma 2.5 and the Marcinkiewicz interpolation theorem (see, for example, [10, Theorem 1.3.1]), to show this lemma, it suffices to show that $T_j$ is bounded from $L^1(\mathbb{R}^n)$ to $WL^1(\mathbb{R}^n)$ with operator norm not greater than $2^{j(\frac{n}{2}+\beta-\alpha)}$ modulo a positive constant, where $\beta\in(0,1)$. To this end, we assume that $f\in L^1(\mathbb{R}^n)$. Notice that, for any $t\in(0,\infty)$, $g\in L^1_{\rm loc}(\mathbb{R}^n)$ and $x\in\mathbb{R}^n$,

$$\begin{aligned}\fint_{B(\vec{0},2t)}|g*\Psi_t(x+y)|^2\,dy&=t^{-2\alpha}\fint_{B(\vec{0},2t)}|\Phi_t*g*I_\alpha(x+y)-g*I_\alpha(x+y)|^2\,dy\\&\lesssim t^{-2\alpha}\fint_{B(x,\tilde{c}t)}|g*I_\alpha(z)|^2\,dz,\end{aligned}$$

where $\tilde{c}:=2+t_0$ with $t_0$ as in Assumption 1.2. By this and (2.15), we know that, for any $x\in\mathbb{R}^n$,

$$T_j(f)(x)\lesssim\left[\sum_{k\in\mathbb{Z}}2^{2\alpha k}\fint_{B(x,\tilde{c}2^{-k})}|f*I_\alpha*\varphi_{2^{-j-k}}(z)|^2\,dz\right]^{\frac{1}{2}}=:T_{j,1}(f)(x).\qquad(2.18)$$

On the other hand, for any $\beta\in(0,1)$, from [4, (3.9)], we deduce that

$$|\{x\in\mathbb{R}^n:\ T_{j,1}(f)(x)>\lambda\}|\lesssim 2^{j(\frac{n}{2}+\beta-\alpha)}\frac{\|f\|_{L^1(\mathbb{R}^n)}}{\lambda},\quad\forall\,\lambda\in(0,\infty).$$

By this and (2.18), we find that

$$|\{x\in\mathbb{R}^n:\ T_j(f)(x)>\lambda\}|\lesssim 2^{j(\frac{n}{2}+\beta-\alpha)}\frac{\|f\|_{L^1(\mathbb{R}^n)}}{\lambda},\quad \forall\,\lambda\in(0,\infty).$$

This finishes the proof of Lemma 2.7. □

Now, we can prove the following technical lemma.

**Lemma 2.8.** *Let $\alpha\in(0,2)$, $p\in(\max\{1,\frac{2n}{2\alpha+n}\},\infty)$, $\Phi$ satisfy Assumption 1.E and $U_\alpha^*$ be as in (2.11). Then there exists a positive constant $C$ such that, for any $f\in L^p(\mathbb{R}^n)$,*

$$\|U_\alpha^*(f)\|_{L^p(\mathbb{R}^n)}\le C\|f\|_{L^p(\mathbb{R}^n)}. \tag{2.19}$$

*In particular, if $n\le 2\alpha$, then (2.19) holds true for any $p\in(1,\infty)$.*

*Conversely, if (2.19) holds true for some $\alpha\in(0,2)$, $p\in(1,\infty)$ and any $f\in L^p(\mathbb{R}^n)$, then $p\ge\frac{2n}{2\alpha+n}$.*

*Proof.* We first show (2.19). For any $\alpha\in(0,2)$, $p\in[2,\infty)$ and $f\in L^p(\mathbb{R}^n)$, let

$$\widetilde{\mathcal{F}}_\alpha(x,t):=\left|\frac{\Phi_t*I_\alpha*f(x)-I_\alpha*f(x)}{t^\alpha}\right|,\quad \forall\,(x,t)\in\mathbb{R}^n\times(0,\infty),$$

where $I_\alpha$ is as in (1.10), $\Phi$ satisfies Assumption 1.E, and $\Phi_t(\cdot)=t^{-n}\Phi(\cdot/t)$ for any $t\in(0,\infty)$. ¿From Lemmas 2.1 and 2.2, and [16, Theorem 1.5], we deduce that

$$\begin{aligned}\|U_\alpha^*(f)\|_{L^p(\mathbb{R}^n)}&=\left\|\widetilde{\mathcal{G}}(\widetilde{\mathcal{F}})\right\|_{L^p(\mathbb{R}^n)}\lesssim\left\|\mathcal{G}_\lambda^*(\widetilde{\mathcal{F}})\right\|_{L^p(\mathbb{R}^n)}\\&\lesssim\left\|\mathcal{G}(\widetilde{\mathcal{F}})\right\|_{L^p(\mathbb{R}^n)}\sim\|f\|_{L^p(\mathbb{R}^n)},\end{aligned}$$

where $U_\alpha^*$, $\widetilde{\mathcal{G}}$, $\mathcal{G}_\lambda^*$ and $\mathcal{G}$ are as in (2.11), (2.2), (2.3) and (2.1), respectively.

For any given $p\in(1,2)$, let $\varepsilon\in(0,\frac12)$ be small enough such that $\theta:=\frac{2}{p}-1+\frac{\varepsilon}{2}\in(0,1)$, and $q_\varepsilon$ be such that $\frac1p=\frac{\theta}{q_\varepsilon}+\frac{1-\theta}{2}$. Then $\frac1p-\frac12=\theta(\frac{1}{q_\varepsilon}-\frac12)$, which implies that $q_\varepsilon\in(1,2)$ because $\theta>2(\frac1p-\frac12)$. If $j\in\mathbb{Z}_+$, then, by Lemmas 2.5 and 2.7, and the Marcinkiewicz interpolation theorem (see, for instance, [10, Theorem 1.3.2]), we have

$$\|T_j(f)\|_{L^p(\mathbb{R}^n)}\lesssim 2^{|j|\theta[n(\frac{1}{q_\varepsilon}-\frac12)-\alpha+\frac{\varepsilon}{\theta}]}2^{-\alpha(1-\theta)|j|}\|f\|_{L^p(\mathbb{R}^n)}\sim 2^{-[\alpha-n(\frac1p-\frac12)-\varepsilon]|j|}\|f\|_{L^p(\mathbb{R}^n)}.$$

If $j\in\mathbb{Z}_-$, then, by Lemmas 2.5 and 2.6, and the Marcinkiewicz interpolation theorem again, we obtain

$$\begin{aligned}\|T_j(f)\|_{L^p(\mathbb{R}^n)}&\lesssim 2^{|j|\theta[-2(2-\alpha)(\frac{1}{q_\varepsilon}-\varepsilon)]}2^{-(2-\alpha)(1-\theta)|j|}\|f\|_{L^p(\mathbb{R}^n)}\\&\sim 2^{-(2-\alpha)[(\frac{2}{q_\varepsilon}-2\varepsilon-1)\theta+1]|j|}\|f\|_{L^p(\mathbb{R}^n)}.\end{aligned}$$

Notice that $p\in(\frac{2n}{2\alpha+n},2)$ implies $\alpha\in(n(\frac1p-\frac12),2)$. We then choose an appropriate $\varepsilon\in(0,1)$ such that

$$\delta:=\min\left\{\alpha-n\left(\frac1p-\frac12\right)-\varepsilon,\ (2-\alpha)\left[\left(\frac{2}{q_\varepsilon}-2\varepsilon-1\right)\theta+1\right]\right\}>0.$$

Then we find that

$$\|T_j(f)\|_{L^p(\mathbb{R}^n)} \lesssim 2^{-|j|\delta}\|f\|_{L^p(\mathbb{R}^n)}, \quad \forall\, j \in \mathbb{Z},$$

which, together with (2.14), implies (2.19). Thus, (2.19) holds true.

Next, we show that, if (2.19) holds true for some $p \in (1,\infty)$ and any $f \in L^p(\mathbb{R}^n)$, then $p \geq \frac{2n}{2\alpha+n}$. Since we always assume $p \in (1,\infty)$, it only needs to consider the case $n > 2\alpha$. Let $\phi$ be a radial Schwartz function on $\mathbb{R}^n$ such that

$$\chi_{[\frac{1}{2},2]}(|\xi|) \leq \widehat{\phi}(\xi) \leq \chi_{[\frac{1}{4},4]}(|\xi|), \quad \forall\, \xi \in \mathbb{R}^n$$

and $\phi_j(\cdot) := 2^{jn}\phi(2^j\cdot)$ for any $j \in \mathbb{Z}$. Then, for any $x \in \mathbb{R}^n$, we have

$$U_\alpha^*(\phi_j)(x) \gtrsim \left\{\int_1^2 \int_{B(\vec{0},1)} \left|\frac{\Phi_t * \phi_j * I_\alpha(x+y) - \phi_j * I_\alpha(x+y)}{t^\alpha}\right|^2 dy\, dt\right\}^{\frac{1}{2}} \gtrsim \mathrm{J}_{j,1} - \mathrm{J}_{j,2},$$

where

$$\mathrm{J}_{j,1}(x) := 2^{-\alpha}\left[\int_{B(\vec{0},1)} |\phi_j * I_\alpha(x+y)|^2\, dy\right]^{\frac{1}{2}}$$

and

$$\mathrm{J}_{j,2}(x) := \left[\int_1^2 \int_{B(\vec{0},1)} |\Phi_t * \phi_j * I_\alpha(x+y)|^2\, dy\, dt\right]^{\frac{1}{2}}.$$

For $\mathrm{J}_{j,1}$, from [4, (3.11)], we deduce that

$$\|\mathrm{J}_{j,1}\|_{L^p(\mathbb{R}^n)} \gtrsim 2^{-\alpha j + \frac{jn}{2}}.$$

For $\mathrm{J}_{j,2}$, we rewrite $\phi_j * I_\alpha(x) = 2^{-\alpha j}P_j(x)$, where

$$P_j(x) := 2^{jn}\left(\frac{\widehat{\phi}(\cdot)}{|2\pi\cdot|^\alpha}\right)^\vee (2^j x), \quad \forall\, x \in \mathbb{R}^n.$$

Since $\Phi$ is bounded, it follows that, for any $x \in \mathbb{R}^n$,

$$\begin{aligned}
\mathrm{J}_{j,2}(x) &= \left[\int_1^2 \int_{B(\vec{0},1)} \left| t^{-n} \int_{\frac{x+y-z}{t} \in B(\vec{0},t_0)} \Phi\left(\frac{x+y-z}{t}\right)(\phi_j * I_\alpha)(z)\, dz\right|^2 dy\, dt\right]^{\frac{1}{2}} \\
&\lesssim \left\{\int_1^2 \int_{B(\vec{0},1)} \left[t^{-n}\int_{x-z \in B(\vec{0},2tt_0)} |\phi_j * I_\alpha(z)|\, dz\right]^2 dy\, dt\right\}^{\frac{1}{2}} \\
&\lesssim M(\phi_j * I_\alpha)(x) \sim 2^{-j\alpha}M(P_j)(x),
\end{aligned}$$

which, combined with the boundedness of $M$ on $L^p(\mathbb{R}^n)$ with any $p \in (1,\infty]$, further implies that

$$\|\mathrm{J}_{j,2}\|_{L^p(\mathbb{R}^n)} \lesssim 2^{-\alpha j}\|P_j\|_{L^p(\mathbb{R}^n)} \lesssim 2^{-\alpha j}2^{jn(1-\frac{1}{p})}. \tag{2.20}$$

Therefore, by the estimates of $\mathrm{J}_{j,1}$ and $\mathrm{J}_{j,2}$, and (2.19), we conclude that, for any sufficiently large $j \in \mathbb{N}$,

$$\begin{aligned} 2^{j(-\alpha+\frac{n}{2})} &\lesssim \|\mathrm{J}_{j,1}\|_{L^p(\mathbb{R}^n)} \lesssim \|U^*_\alpha(\phi_j)\|_{L^p(\mathbb{R}^n)} + \|\mathrm{J}_{j,2}\|_{L^p(\mathbb{R}^n)} \\ &\lesssim \|\phi_j\|_{L^p(\mathbb{R}^n)} + 2^{-\alpha j} 2^{jn(1-\frac{1}{p})} \lesssim 2^{jn(1-\frac{1}{p})}. \end{aligned}$$

This implies that $-\alpha + \frac{n}{2} \le n(1-\frac{1}{p})$, namely, $p \ge \frac{2n}{2\alpha+n}$ and hence finishes the proof of Lemma 2.8. □

Next we turn to establish the inverse inequality of (2.19), which follows from the following lemma. We prove this lemma by borrowing some ideas from [2, 3].

**Lemma 2.9.** *Let $p \in (1,\infty)$, $\Phi$ satisfy Assumption 1.E, $\alpha \in (0,2)$ and $\widetilde{U}_\alpha$ be as in (1.12). Then there exists a positive constant $C$ such that, for any $f \in L^p(\mathbb{R}^n)$,*

$$\left\|(-\Delta)^{\alpha/2} f\right\|_{L^p(\mathbb{R}^n)} \le C \left\|\widetilde{U}_\alpha(f)\right\|_{L^p(\mathbb{R}^n)}.$$

*Proof.* Let $\rho \in \mathcal{S}(\mathbb{R}^n)$ be such that

$$\mathrm{supp}\ \widehat{\rho} \subset \left\{\xi \in \mathbb{R}^n:\ 2^{k_0-1} \le |\xi| \le 2^{k_0+1}\right\}$$

and $|\widehat{\rho}(\xi)| \ge$ constant $> 0$ when $\frac{3}{5}2^{k_0} \le |\xi| \le \frac{5}{3}2^{k_0}$ for some $k_0 \in \mathbb{Z}$ which is determined later. Notice that

$$\begin{aligned} \left\|(-\Delta)^{\alpha/2} f\right\|_{L^p(\mathbb{R}^n)} &\sim \left\|(-\Delta)^{\alpha/2} f\right\|_{\dot{F}^0_{p,2}(\mathbb{R}^n)} \sim \|f\|_{\dot{F}^\alpha_{p,2}(\mathbb{R}^n)} \\ &\sim \left\|\left\{\int_0^\infty \left[\fint_{B(\cdot,t)} |\rho_t * f(y)|\,dy\right]^2 \frac{dt}{t^{2\alpha+1}}\right\}^{\frac{1}{2}}\right\|_{L^p(\mathbb{R}^n)}, \end{aligned} \tag{2.21}$$

where $\dot{F}^0_{p,2}(\mathbb{R}^n)$ and $\dot{F}^\alpha_{p,2}(\mathbb{R}^n)$ denote the homogeneous Triebel-Lizorkin spaces, and the second equivalence is due to the lifting property of Triebel-Lizorkin spaces, and the third one follows from the Lusin area function characterization of Triebel-Lizorkin spaces (see, for example, [19, Theorem 2.8] and its proof).

On the other hand, for any $\xi \in \mathbb{R}^n$, we have

$$(\Phi_t * f - f)^\wedge(\xi) = \left[\widehat{\Phi}(t\xi) - 1\right] \widehat{f}(\xi) =: A(t|\xi|)\widehat{f}(\xi)$$

and

$$\rho_t * f = \left[\widehat{\rho}(t\cdot)\widehat{f}(\cdot)\right]^\vee = \left[\frac{\widehat{\rho}(t\cdot)}{A(t|\cdot|)} A(t|\cdot|)\widehat{f}(\cdot)\right]^\vee =: \left[\eta(t\cdot)A(t|\cdot|)\widehat{f}(\cdot)\right]^\vee,$$

where

$$A(s) := A(s|e_1|) = \widehat{\Phi}(se_1) - 1, \quad \forall\, s \in (0,\infty),$$

and

$$\eta(\xi) := \frac{\widehat{\rho}(\xi)}{A(|\xi|)}, \quad \forall\, \xi \in \mathrm{supp}\ \widehat{\rho}$$

and otherwise $\eta(\xi) := 0$. Since $\Phi \in L^1(\mathbb{R}^n)$ with compact support, it follows that $A(s)$ is smooth when $2^{k_0-1} \le s \le 2^{k_0+1}$, and $A(s) \neq 0$ therein provided that $k_0$ is sufficiently small. Thus, we choose $k_0$ small enough such that $\eta \in C_c^\infty(\mathbb{R}^n)$, and hence $\eta^\vee$ is a Schwartz function. Then, for any $N \in \mathbb{N}$ and $x \in \mathbb{R}^n$, $|\eta^\vee(x)| \lesssim (1+|x|)^{-N}$ with the implicit positive constant depending on $N$, and we also find that, for any $t \in (0,\infty)$ and $x \in \mathbb{R}^n$,

$$\begin{aligned}
\fint_{B(x,t)} |\rho_t * f(y)|\,dy &= \fint_{B(x,t)} |\eta_t^\vee * (\Phi_t * f - f)(y)|\,dy \\
&\le \int_{\mathbb{R}^n} |\eta_t^\vee(\nu)| \fint_{B(\nu-x,t)} |(\Phi_t * f - f)(y)|\,dy\,d\nu \\
&\lesssim \int_{\mathbb{R}^n} \frac{t^{-n}}{(1+\frac{|\nu+x|}{t})^N} \fint_{B(\nu,t)} |(\Phi_t * f - f)(y)|\,dy\,d\nu \\
&\sim \int_{|\nu+x|\le t} \frac{t^{-n}}{(1+\frac{|\nu+x|}{t})^N} \fint_{B(\nu,t)} |(\Phi_t * f - f)(y)|\,dy\,d\nu \\
&\quad + \sum_{k=1}^\infty \int_{2^{k-1}t<|\nu+x|\le 2^k t} \frac{t^{-n}}{(1+\frac{|\nu+x|}{t})^N} \fint_{B(\nu,t)} |(\Phi_t * f - f)(y)|\,dy\,d\nu \\
&\lesssim \fint_{|\nu+x|\le t} \fint_{B(\nu,t)} |(\Phi_t * f - f)(y)|\,dy\,d\nu \\
&\quad + \sum_{k=1}^\infty 2^{nk} 2^{-N(k-1)} \fint_{|\nu+x|\le 2^k t} \fint_{B(\nu,t)} |(\Phi_t * f - f)(y)|\,dy\,d\nu \\
&\lesssim M\left(\fint_{B(\cdot,t)} |(\Phi_t * f - f)(y)|\,dy\right)(-x)\left[1+\sum_{k=1}^\infty 2^{-(N-n)k}\right] \\
&\sim M\left(\fint_{B(\cdot,t)} |(\Phi_t * f - f)(y)|\,dy\right)(-x),
\end{aligned}$$

where we took $N > n$. Therefore, by (2.21), the Fefferman-Stein vector-valued inequality (see [7]) and the Hölder inequality, we have

$$\begin{aligned}
\left\|(-\Delta)^{\alpha/2} f\right\|_{L^p(\mathbb{R}^n)} &\lesssim \left\|\left\{\int_0^\infty \left[M\left(\fint_{B(\cdot,t)} |(\Phi_t * f - f)(y)|\,dy\right)\right]^2 \frac{dt}{t^{2\alpha+1}}\right\}^{\frac12}\right\|_{L^p(\mathbb{R}^n)} \\
&\lesssim \left\|\left\{\int_0^\infty \left[\fint_{B(\cdot,t)} |(\Phi_t * f - f)(y)|\,dy\right]^2 \frac{dt}{t^{2\alpha+1}}\right\}^{\frac12}\right\|_{L^p(\mathbb{R}^n)} \\
&\lesssim \left\|\left\{\int_0^\infty \int_{B(\cdot,t)} |(\Phi_t * f - f)(y)|^2\,dy\,\frac{dt}{t^{2\alpha+1+n}}\right\}^{\frac12}\right\|_{L^p(\mathbb{R}^n)} \\
&\sim \left\|\widetilde{U}_\alpha(f)\right\|_{L^p(\mathbb{R}^n)},
\end{aligned}$$

which completes the proof of Lemma 2.9. □

Based on these lemmas discussed above, we are now ready to prove Theorem 1.1.

*Proof of Theorem 1.1.* For any $\Phi$ satisfying Assumption 1.E and $f \in W^{\alpha,p}(\mathbb{R}^n)$ with $\alpha \in (0,2)$ and $p \in (1,\infty)$, let

$$\mathcal{F}_\alpha(x,t) := \left|\frac{\Phi_t * f(x) - f(x)}{t^\alpha}\right|, \quad \forall\,(x,t) \in \mathbb{R}^n \times (0,\infty).$$

Then, applying Lemma 2.2 with $\mathcal{F}$ replaced by $\mathcal{F}_\alpha$, we know that the equivalence between (ii) and (iii) of Theorem 1.1(I) when $p \in (1,\infty)$ holds true. The items (i) $\Rightarrow$ (ii) of Theorem 1.1(I) follows from Lemma 2.8 and (2.12), while the items (ii) $\Rightarrow$ (i) of Theorem 1.1(I) comes from Lemma 2.9 and (2.12). Moreover, Theorem 1.1(II) is an immediate consequence of Lemma 2.8. This finishes the proof of Theorem 1.1. □

# 3 Proofs of Theorems 1.3 and 1.4

To prove Theorem 1.3, for $\Phi$ satisfying Assumption 1.2 and $x \in \mathbb{R}^n$, let

$$K(x) := \Phi * I_2(x) - I_2(x) + C_0\Phi(x), \tag{3.1}$$

where $C_0$ is as in (1.17) and $I_2$ is as in (1.10) with $\alpha = 2$. Then, we have

$$\widehat{K}(\xi) = (2\pi|\xi|)^{-2}\left[\widehat{\Phi}(\xi) - 1\right] + C_0\widehat{\Phi}(\xi), \quad \forall\,\xi \in \mathbb{R}^n \setminus \{\vec{0}\}. \tag{3.2}$$

In what follows, $t \to 0^+$ means that $t \in (0,\infty)$ and $t \to 0$.

**Lemma 3.1.** *Let $K$ be as in (3.1). Then there exists a positive constant $C$ such that, for any $\xi \in \mathbb{R}^n \setminus \{\vec{0}\}$,*

$$\left|\widehat{K}(\xi)\right| \le \begin{cases} C\min\left\{|\xi|^2, |\xi|^{-\frac{1}{2}}\right\} & \text{when } n = 2, \\ C\min\left\{|\xi|^2, |\xi|^{-1}\right\} & \text{when } n \in \{1,3,4\}, \\ C\min\left\{|\xi|^2, |\xi|^{-2}\right\} & \text{when } n \in [5,\infty)\cap\mathbb{N}. \end{cases} \tag{3.3}$$

*Proof.* Obviously, for any $\xi \in \mathbb{R}^n \setminus \{\vec{0}\}$, we have

$$\begin{aligned}\left|\widehat{K}(\xi)\right| &= \left|(2\pi|\xi|)^{-2}\left[\widehat{\Phi}(\xi) - 1\right] + C_0\widehat{\Phi}(\xi)\right| \\ &\le \left|(2\pi|\xi|)^{-2}\left[\widehat{\Phi}(\xi) - 1\right] + C_0\right| + C_0\left|\widehat{\Phi}(\xi) - 1\right|,\end{aligned}$$

where $C_0$ is as in (1.17). By the Taylor expansion and the definition of $C_0$ in (1.17), we conclude that, when $0 < |\xi| \le 1$,

$$\left|(2\pi|\xi|)^{-2}\left[\widehat{\Phi}(\xi) - 1\right] + C_0\right|$$

$$
\begin{aligned}
&= \left|\frac{1}{2}\int_{\mathbb{R}^n}\Phi(x)(2\pi i x_1)^2\,dx + \frac{|\xi|^2}{4!}\int_{\mathbb{R}^n}\Phi(x)(2\pi i x_1)^4\ dx\right.\\
&\quad\left. + \frac{o(|\xi|^2)}{6!}\int_{\mathbb{R}^n}\Phi(x)(2\pi i x_1)^6\,dx + C_0\right|\\
&= \left|\frac{|\xi|^2}{4!}\int_{\mathbb{R}^n}\Phi(x)(2\pi i x_1)^4\,dx + \frac{o(|\xi|^2)}{6!}\int_{\mathbb{R}^n}\Phi(x)(2\pi i x_1)^6\,dx\right|,
\end{aligned}
$$

where $o(|\xi|^2)$ means $o(|\xi|^2) \lesssim |\xi|^2$ and $\frac{|o(|\xi|^2)|}{|\xi|^2} \to 0$ as $|\xi| \to 0^+$. ¿From this, it further follows that

$$
\left|(2\pi|\xi|)^{-2}\left[\widehat{\Phi}(\xi) - 1\right] + C_0\right| \lesssim |\xi|^2.
$$

By Lemma 2.4 with $\alpha = 2$ and $j = 0$, we have $C_0|\widehat{\Phi}(\xi) - 1| \lesssim |\xi|^2$. This proves (3.3) when $|\xi| \le 1$.

If $|\xi| > 1$, by [10, Appendix B.5, pp. 577-578], letting $\Phi_0(r) := \Phi(x)$ with $r := |x|$, we have

$$
\widehat{\Phi}(\xi) = \frac{2\pi}{|\xi|^{\frac{n-2}{2}}}\int_0^\infty \Phi_0(r)\mathrm{J}_{\frac{n}{2}-1}(2\pi r|\xi|)r^{\frac{n}{2}}\,dr,
$$

where $\mathrm{J}_\nu$ denotes the *Bessel function of order* $\nu$. Then, by the well-known fact that $|\mathrm{J}_\nu(t)| \sim t^{\frac{n}{2}-1}$ as $t \to 0^+$ and $|\mathrm{J}_\nu(t)| \sim t^{-1/2}$ as $t \to \infty$, together with the facts that $\Phi_0$ is bounded and $|\xi| > 1$, we find that, when $n \ge 2$,

$$
\begin{aligned}
\left|\widehat{\Phi}(\xi)\right| &\lesssim \frac{1}{|\xi|^{\frac{n-2}{2}}}\left|\int_0^{t_0}\Phi_0(r)\mathrm{J}_{\frac{n}{2}-1}(2\pi r|\xi|)r^{\frac{n}{2}}\,dr\right| \lesssim \frac{1}{|\xi|^{\frac{n-2}{2}}}\int_0^{t_0}|\mathrm{J}_{\frac{n}{2}-1}(2\pi r|\xi|)|r^{\frac{n}{2}}\,dr\\
&\lesssim \frac{1}{|\xi|^{\frac{n-1}{2}}}\left[\int_0^{\frac{1}{2\pi|\xi|}}|\xi|^{\frac{1}{2}}(2\pi r|\xi|)^{\frac{n}{2}-1}r^{\frac{n}{2}}\,dr + \int_{\frac{1}{2\pi|\xi|}}^{t_0}|\xi|^{\frac{1}{2}}(2\pi r|\xi|)^{-\frac{1}{2}}r^{\frac{n}{2}}\,dr\right] \lesssim \frac{1}{|\xi|^{\frac{n-1}{2}}},
\end{aligned}
$$

where $t_0$ is as in Assumption 1.2 and, when $\frac{1}{2\pi|\xi|} \ge t_0$, the integral

$$
\int_{\frac{1}{2\pi|\xi|}}^{t_0}|\xi|^{\frac{1}{2}}(2\pi r|\xi|)^{-\frac{1}{2}}r^{\frac{n}{2}}\,dr
$$

in the above argument is void. On the other hand, when $n = 1$, since $\Phi$ satisfies Assumption 1.2, we know that $\Phi$ is a Lipschitz function on $(-t_0, t_0)$, and hence a function of bounded variation on $(-t_0, t_0)$. Then, by integrating by parts and the Riemannian-Stieltjes integral theory, we know that, for any given $\epsilon \in (0, t_0/2)$ and any $\xi \in \mathbb{R}$ with $|\xi| > 1$,

$$
\begin{aligned}
\left|\int_{-t_0+\epsilon}^{t_0-\epsilon}\Phi(x)e^{-2\pi i x\xi}\,dx\right| &\sim \frac{1}{|\xi|}\left|\int_{-t_0+\epsilon}^{t_0-\epsilon}\Phi(x)\,d\left(e^{-2\pi i x\xi}\right)\right|\\
&\lesssim \frac{1}{|\xi|}\left|\Phi(x)e^{-2\pi i x\xi}|_{x=-t_0+\epsilon}^{t_0-\epsilon}\right| + \frac{1}{|\xi|}\left|\int_{-t_0+\epsilon}^{t_0-\epsilon}e^{-2\pi i x\xi}\,d\Phi(x)\right|\\
&\lesssim \frac{1}{|\xi|} + \frac{1}{|\xi|}\left|\int_{-t_0+\epsilon}^{t_0-\epsilon}e^{-2\pi i x\xi}\,d\Phi(x)\right|.
\end{aligned}
$$

Since $\Phi$ satisfies Assumption 1.2, it follows that

$$\begin{aligned}\frac{1}{|\xi|}\left|\int_{-t_0+\epsilon}^{t_0-\epsilon} e^{-2\pi i x\xi}\,d\Phi(x)\right| &= \frac{1}{|\xi|}\lim_{N\to\infty}\left|\sum_{j=1}^{N} e^{-2\pi i\eta_j}[\Phi(x_j)-\Phi(x_{j-1})]\right|\\ &\lesssim \frac{1}{|\xi|}\lim_{N\to\infty}\sum_{j=1}^{N}|\Phi(x_j)-\Phi(x_{j-1})|\\ &\lesssim \frac{1}{|\xi|}\lim_{N\to\infty}\sum_{j=1}^{N}|x_j-x_{j-1}| \lesssim 2(t_0-\epsilon)\frac{1}{|\xi|},\end{aligned}$$

where $\{x_0,\dots,x_N\}$ is a partition of $[-t_0+\epsilon,t_0-\epsilon]$, $\eta_j\in[x_{j-1},x_j]$ and the implicit positive constants are independent of $\epsilon$ and $\xi$. Therefore, we conclude that, when $n=1$,

$$\left|\widehat{\Phi}(\xi)\right| = \lim_{\epsilon\to 0^+}\left|\int_{-t_0+\epsilon}^{t_0-\epsilon}\Phi(x)e^{-2\pi i x\xi}\,dx\right| \lesssim \frac{1}{|\xi|}.$$

On the other hand, since $\Phi$ is a bounded function with compact support, it follows that $|\widehat{\Phi}(\xi)|\lesssim 1$ for any $\xi\in\mathbb{R}^n$. Thus, combining the previous estimates, we know that, for any $|\xi|>1$,

$$\left|\widehat{K}(\xi)\right| \lesssim |\xi|^{-2}\left|\widehat{\Phi}(\xi)-1\right| + C_0\left|\widehat{\Phi}(\xi)\right| \lesssim \begin{cases}|\xi|^{-\frac12} & \text{when } n=2,\\ |\xi|^{-1} & \text{when } n\in\{1,3,4\},\\ |\xi|^{-2} & \text{when } n\ge 5.\end{cases}$$

This finishes the proof of Lemma 3.1. □

Now, we are ready to prove Theorem 1.3.

*Proof of Theorem 1.3.* We prove this theorem by borrowing some ideas from the proof of [1, Theorem 2]. To prove (i) ⇒ (ii), we only need to show that, for any $f\in W^{2,p}(\mathbb{R}^n)$,

$$\|\mathcal{G}(\mathcal{F}(f,C_0\Delta f))\|_{L^p(\mathbb{R}^n)} \sim \|\Delta f\|_{L^p(\mathbb{R}^n)}, \tag{3.4}$$

where $C_0$ is as in (1.17), $\mathcal{F}(f,C_0\Delta f))$ as in (1.18) with $g:=C_0\Delta f$ and $\mathcal{G}$ as in Theorem 1.3(ii).

To prove (3.4), we first show that, for any $f\in W^{2,2}(\mathbb{R}^n)$,

$$\|\mathcal{G}(\mathcal{F}(f,C_0\Delta f))\|_{L^2(\mathbb{R}^n)} \sim \|\Delta f\|_{L^2(\mathbb{R}^n)}. \tag{3.5}$$

Applying the Plancherel theorem, we find that, for any $f\in W^{2,2}(\mathbb{R}^n)$,

$$\begin{aligned}\|\mathcal{G}(\mathcal{F}(f,C_0\Delta f))\|_{L^2(\mathbb{R}^n)}^2 &= \int_{\mathbb{R}^n}\int_0^\infty\left|\frac{\Phi_t*f(x)-f(x)}{t^2}+C_0\Phi_t*\Delta f(x)\right|^2\frac{dt}{t}\,dx\\ &= \int_0^\infty\int_{\mathbb{R}^n}\left|\frac{\widehat{\Phi}(t\xi)\widehat{f}(\xi)-\widehat{f}(\xi)}{t^2}+C_0\widehat{\Phi}(t\xi)|\xi|^2\widehat{f}(\xi)\right|^2 d\xi\,\frac{dt}{t}\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^n} \left[ \int_0^\infty \left| \frac{\widehat{\Phi}(t\xi) - 1}{|t\xi|^2} + C_0 \widehat{\Phi}(t\xi) \right|^2 \frac{dt}{t} \right] |\xi|^4 |\widehat{f}(\xi)|^2 \, d\xi \\
&= \int_{\mathbb{R}^n} \left[ \int_0^\infty \left| \widehat{K}(t\xi) \right|^2 \frac{dt}{t} \right] |\xi|^4 |\widehat{f}(\xi)|^2 \, d\xi. \qquad (3.6)
\end{aligned}
$$

By Lemma 3.1, we know that, for any $\xi \in \mathbb{R}^n \setminus \{\vec{0}\}$,

$$
\int_0^\infty \left| \widehat{K}(t\xi) \right|^2 \frac{dt}{t} \le \int_0^{\frac{1}{|\xi|}} |t\xi|^4 \frac{dt}{t} + \int_{\frac{1}{|\xi|}}^\infty \frac{1}{|t\xi|} \frac{dt}{t} \lesssim 1. \qquad (3.7)
$$

Since $\Phi$ is a radial function, from [10, p. 577, §B.5], it follows that $\widehat{\Phi}$ is also a radial function, which, together with (3.2), implies that $\widehat{K}$ is a radial function and hence, for any $\xi \in \mathbb{R}^n \setminus \{\vec{0}\}$, $\widehat{K}(\xi) =: k(|\xi|)$. Therefore, for any $\xi \in \mathbb{R}^n \setminus \{\vec{0}\}$,

$$
\int_0^\infty \left| \widehat{K}(t\xi) \right|^2 \frac{dt}{t} = \int_0^\infty |k(t|\xi|)|^2 \frac{dt}{t} = \int_0^\infty |k(s)|^2 \frac{ds}{s},
$$

which, combined with (3.7), further implies that $\int_0^\infty \left| \widehat{K}(t\xi) \right|^2 \frac{dt}{t}$ is a positive constant independent of $\xi \in \mathbb{R}^n \setminus \{\vec{0}\}$. By this and (3.6), we know that (3.5) holds true.

Now we turn our attention to the case $p \in (1, \infty)$. Let $h := \Delta f$. Then we can translate (3.5) into

$$
\int_{\mathbb{R}^n} \|K_t * h\|_{L^2(dt/t)}^2 dx \sim \|h\|_{L^2(\mathbb{R}^n)}^2,
$$

where $K$ is as in (3.1) and $K_t(x) := t^{-n} K(x/t)$ for any $x \in \mathbb{R}^n$ and $t \in (0, \infty)$. If the kernel $K_t$ satisfies the following Hörmander condition:

$$
\int_{|x| \ge 2|y|} \|K_t(x - y) - K_t(x)\|_{L^2(dt/t)} \, dx \lesssim 1, \quad \forall \, y \in \mathbb{R}^n \setminus \{\vec{0}\}, \qquad (3.8)
$$

then, by [9, p. 492, Theorem 3.4], we conclude that, when $p \in (1, \infty)$, for any $f \in L^p(\mathbb{R}^n)$,

$$
\int_{\mathbb{R}^n} \|K_t * f\|_{L^2(dt/t)}^p dx \lesssim \|f\|_{L^p(\mathbb{R}^n)}^p,
$$

which means that, for any $f \in W^{2,p}(\mathbb{R}^n)$,

$$
\|\mathcal{G}(\mathcal{F}(f, C_0 \Delta f))\|_{L^p(\mathbb{R}^n)} \lesssim \|\Delta f\|_{L^p(\mathbb{R}^n)}. \qquad (3.9)
$$

We now prove the following stronger version of the Hörmander condition (3.8):

$$
\|K_t(x - y) - K_t(x)\|_{L^2(dt/t)} \lesssim \frac{|y|^\gamma}{|x|^{n+\gamma}}, \quad \forall \, |x| \ge 2|y| > 0, \qquad (3.10)
$$

with certain $\gamma \in (0, \infty)$.

To show (3.10), we deal with the kernels $H_t := \Phi_t * I_2 - I_2$ and $t^2 \Phi_t$ separately.

For $t^2\Phi_t$, we first observe that the quantity $|\Phi_t(x-y)-\Phi_t(x)|$ is non-zero only when $t>\min\{\frac{|x-y|}{t_0},\frac{|x|}{t_0}\}$, due to the compact support of $\Phi$, where $t_0$ is as in Assumption 1.2. We only consider the case $|x-y|\le|x|$, because the case $|x|<|x-y|$ is similar, the details being omitted. Then, by (1.16) and the mean value theorem, we know that, when $|x-y|\le|x|$ and $|x|\ge 2|y|>0$,

$$\begin{aligned}\left\{\int_0^\infty |t^2[\Phi_t(x-y)-\Phi_t(x)]|^2\frac{dt}{t^5}\right\}^{\frac12} &\lesssim \left[\int_{\frac{|x-y|}{t_0}}^{\frac{|x|}{t_0}}\frac{dt}{t^{2n+1}}\right]^{\frac12}+\left[\int_{\frac{|x|}{t_0}}^\infty |y|^2\frac{dt}{t^{2n+3}}\right]^{\frac12}\\ &\lesssim\left[\frac{1}{|x-y|^{2n}}-\frac{1}{|x|^{2n}}\right]^{\frac12}+\frac{|y|}{|x|^{n+1}}\lesssim\frac{|y|^{\frac12}}{|x|^{n+\frac12}},\end{aligned}\tag{3.11}$$

where, when $|x-y|=|x|$, the term $\int_{\frac{|x-y|}{t_0}}^{\frac{|x|}{t_0}}\frac{dt}{t^{2n+1}}$ automatically disappears. This is the desired estimate for $t^2\Phi_t$.

We now consider $H_t$. By similarity, we also assume that $|x-y|\le|x|$. If $t<\frac{|x|}{3t_0}$, then the origin $\vec{0}$ does not belong to the balls $B(x,tt_0)$ and $B(x-y,tt_0)$. By the mean value theorem, we know that, when $|x-y|\le|x|$ and $|x|\ge 2|y|>0$,

$$|H_t(x-y)-H_t(x)|\le|y|\sup_{z\in[x-y,x]}|\nabla H_t(z)|,\tag{3.12}$$

where, when $x-y\neq x$, $z\in[x-y,x]$ means that $z$ lies in the segment connecting $x-y$ and $x$, otherwise, (3.12) automatically holds true.

In the remainder of the proof of this case, we always assume that $x-y\neq x$, otherwise, all wanted conclusions automatically hold true. By the Taylor expansion, we find that, for any $z\in[x-y,x]$,

$$\begin{aligned}\nabla H_t(z)&=\Phi_t*\nabla I_2(z)-\nabla I_2(z)=\int_{\mathbb{R}^n}\Phi_t(w)[\nabla I_2(z-w)-\nabla I_2(z)]\,dw\\ &=t^{-n}\int_{B(\vec{0},tt_0)}\Phi\left(\frac{w}{t}\right)\left[\sum_{|\beta|=1}D^\beta\nabla I_2(z)(-w)^\beta+\sum_{|\beta|=2}\frac{D^\beta\nabla I_2(z)(-w)^\beta}{\beta!}\right.\\ &\quad\left.+\sum_{|\beta|=3}\frac{D^\beta\nabla I_2(z-\theta w)(-w)^\beta}{\beta!}\right]dw,\end{aligned}\tag{3.13}$$

where $\theta\in(0,1)$. Since $\Phi$ is radial, it follows that $\int_{B(\vec{0},tt_0)}\Phi(\frac{w}{t})w^\beta dw=0$ if one of $\beta_i$ in $\beta:=(\beta_1,\beta_2,\dots,\beta_n)$ is odd, which implies that

$$\begin{aligned}&\int_{B(\vec{0},tt_0)}\Phi\left(\frac{w}{t}\right)\left[\sum_{|\beta|=1}^{2}D^\beta\nabla I_2(z)w^\beta\right]dw\\ &\quad=[\Delta\nabla I_2(z)]\int_{B(\vec{0},tt_0)}\Phi\left(\frac{w}{t}\right)\left(\frac{\sum_{i=1}^n w_i^2}{n}\right)dw=0,\end{aligned}\tag{3.14}$$

where the last equality follows from the fact that $I_2$ in (1.10) with $\alpha = 2$ is harmonic.

Notice that, if $z \in [x-y, x]$, $\theta \in (0,1)$, $w \in B(\vec{0}, tt_0)$ and $t < \frac{|x|}{3t_0}$, then $|z-\theta w| \ge |x|/6$. From (3.13) and (3.14), we deduce that

$$|\nabla H_t(z)| \lesssim t^{-n}|x|^{-n-2}\int_{B(\vec{0},tt_0)} |w|^3\,dw \sim t^3|x|^{-n-2}.$$

By this and (3.12), we obtain

$$\left\{\int_0^{\frac{|x|}{3t_0}} |H_t(x-y)-H_t(x)|^2\,\frac{dt}{t^5}\right\}^{\frac12} \lesssim \left\{\int_0^{\frac{|x|}{3t_0}} \frac{|y|^2}{|x|^{2n+4}} t\,dt\right\}^{\frac12} \lesssim \frac{|y|}{|x|^{n+1}}, \tag{3.15}$$

which is the desired estimate. If $t \ge \frac{|x|}{3t_0}$, then

$$|H_t(x-y)-H_t(x)| \le |y| \sup_{z\in[x-y,x]} |\nabla H_t(z)| \lesssim \frac{|y|}{|x|^{n-1}},$$

where the last inequality follows from the facts that

$$\nabla H_t(z) = \Phi_t * \nabla I_2(z) - \nabla I_2(z),$$

$$|\nabla I_2(z)| \lesssim \frac{1}{|z|^{n-1}} \lesssim \frac{1}{|x|^{n-1}}$$

and

$$|\Phi_t * \nabla I_2(z)| \lesssim \fint_{B(z,tt_0)} |\nabla I_2(\omega)|\,d\omega \lesssim \fint_{B(z,tt_0)} \frac{d\omega}{|\omega|^{n-1}} \lesssim \frac{1}{|x|^{n-1}}.$$

Therefore,

$$\int_{\frac{|x|}{3t_0}}^\infty |H_t(x-y)-H_t(x)|^2\,\frac{dt}{t^5} \lesssim \int_{\frac{|x|}{3t_0}}^\infty \frac{|y|^2}{|x|^{2n-2}}\,\frac{dt}{t^5} \lesssim \frac{|y|^2}{|x|^{2n+2}},$$

which, combined with (3.15), further implies that

$$\|H_t(x-y)-H_t(x)\|_{L^2(dt/t^5)} \lesssim \frac{|y|^{\frac12}}{|x|^{n+\frac12}}, \quad \forall\, |x| \ge 2|y| > 0.$$

This, together with (3.11), implies (3.10) and hence (3.9) holds true.

On the other hand, the reverse inequality of (3.9) follows from a polarization from (3.5) via a well-known duality argument (see, for example, [9, p. 507]). Thus, (3.4) holds true, which completes the proof that (i) $\Rightarrow$ (ii).

Now, we prove (ii) $\Rightarrow$ (i). Assume that $f$, $g \in L^p(\mathbb{R}^n)$ such that $\mathcal{G}(\mathcal{F}(f,g)) \in L^p(\mathbb{R}^n)$. We shall prove that $g$ coincides with $C_0\Delta f$ almost everywhere, where $C_0$ is as in (1.17). To this end, take a non-negative radial smooth function $\zeta$ which is supported in the closure of $B(\vec{0},1)$ such that $\|\zeta\|_{L^1(\mathbb{R}^n)} = 1$ and, for any $\varepsilon \in (0,\infty)$ and $x \in \mathbb{R}^n$, let $\zeta_\varepsilon(x) := \varepsilon^{-n}\zeta(x/\varepsilon)$, $f_\varepsilon := f * \zeta_\varepsilon$ and $g_\varepsilon := g * \zeta_\varepsilon$. Then, by [1, Lemma 2(i)], we know that $f_\varepsilon \in W^{2,p}(\mathbb{R}^n)$. Therefore, by the conclusion that (i) $\Rightarrow$ (ii), we further find that

$\mathcal{G}(\mathcal{F}(f_\varepsilon, C_0\Delta f_\varepsilon)) \in L^p(\mathbb{R}^n)$. Moreover, from the Minkowski inequality, we deduce that, for any $\varepsilon \in (0,\infty)$ and $x \in \mathbb{R}^n$,

$$\begin{aligned}
\mathcal{G}(\mathcal{F}(f_\varepsilon, g_\varepsilon))(x) &= \left\{\int_0^\infty \left|\left(\frac{\Phi_t * f - f}{t^2} - C_0\Phi_t * g\right) * \zeta_\varepsilon(x)\right|^2 \frac{dt}{t^{n+1}}\right\}^{\frac12} \\
&= \left\{\int_0^\infty \left|\int_{\mathbb{R}^n} \left[\frac{\Phi_t * f(x-z) - f(x-z)}{t^2}\right.\right.\right. \\
&\qquad \left.\left.\left. - C_0\Phi_t * g(x-z)\right] \zeta_\varepsilon(z)\, dz\right|^2 \frac{dt}{t^{n+1}}\right\}^{\frac12} \\
&\lesssim \int_{\mathbb{R}^n} \left\{\int_0^\infty \left|\frac{\Phi_t * f(x) - f(x)}{t^2} - C_0\Phi_t * g(x)\right|^2 \frac{dt}{t^{n+1}}\right\}^{\frac12} \zeta_\varepsilon(z)\, dz \\
&\sim \int_{\mathbb{R}^n} \mathcal{G}(\mathcal{F}(f,g))(x-z)\zeta_\varepsilon(z)dz = \mathcal{G}(\mathcal{F}(f,g)) * \zeta_\varepsilon(x).
\end{aligned}$$

For any $\varepsilon \in (0,\infty)$ and $x \in \mathbb{R}^n$, define

$$D_\varepsilon(x) := \left\{\int_0^\infty |\Phi_t * (g_\varepsilon - C_0\Delta f_\varepsilon)(x)|^2 \frac{dt}{t^{n+1}}\right\}^{\frac12}.$$

Then we find that

$$\begin{aligned}
D_\varepsilon(x) &= \left\{\int_0^\infty \left|-\left(\frac{\Phi_t * f_\varepsilon - f_\varepsilon}{t^2} - \Phi_t * g_\varepsilon\right)(x)\right.\right. \\
&\qquad \left.\left. + \left(\frac{\Phi_t * f_\varepsilon - f_\varepsilon}{t^2} - C_0\Phi_t * \Delta f_\varepsilon\right)(x)\right|^2 \frac{dt}{t^{n+1}}\right\}^{\frac12} \\
&\lesssim \mathcal{G}(\mathcal{F}(f_\varepsilon, g_\varepsilon))(x) + \mathcal{G}(\mathcal{F}(f_\varepsilon, C_0\Delta f_\varepsilon))(x) \\
&\lesssim \mathcal{G}(\mathcal{F}(f,g)) * \zeta_\varepsilon(x) + \mathcal{G}(\mathcal{F}(f_\varepsilon, C_0\Delta f_\varepsilon))(x),
\end{aligned}$$

which implies that $D_\varepsilon \in L^p(\mathbb{R}^n)$, in particular, $D_\varepsilon(x) < \infty$ for almost every $x \in \mathbb{R}^n$. Combining this with [6, Corollary 2.9], we find that, for almost every $x \in \mathbb{R}^n$,

$$|g_\varepsilon(x) - C_0\Delta f_\varepsilon(x)| = \lim_{t\to 0} |g_\varepsilon * \Phi_t(x) - C_0\Delta f_\varepsilon * \Phi_t(x)| = 0$$

and hence $\Delta f_\varepsilon \to g$ in $L^p(\mathbb{R}^n)$ as $\varepsilon \to 0^+$. Since $f_\varepsilon \to f$ in $L^p(\mathbb{R}^n)$ as $\varepsilon \to 0^+$, then it follows that $\Delta f_\varepsilon \to \Delta f$ in $\mathcal{S}'(\mathbb{R}^n)$ as $\varepsilon \to 0^+$. Therefore, $C_0\Delta f = g$ almost everywhere, which completes the proof of Theorem 1.3. □

Let $K$ be as in (3.1) and, for any suitable $f$ and $x \in \mathbb{R}^n$, let

$$V^*(f)(x) := \left\{\int_0^\infty \fint_{B(\vec{0},t)} |K_t * f(x+y)|^2\, dy\, \frac{dt}{t}\right\}^{\frac12}. \tag{3.16}$$

It is clear that, for any $f \in W^{2,p}(\mathbb{R}^n)$ with $p \in (1,\infty)$, $\widetilde{U}(f, C_0\Delta f) \sim V^*(g)$, where $\widetilde{U}$ is as in (1.14), $C_0$ as in (1.17), $g := \Delta f$ and the implicit equivalent positive constants are independent of $f$. Similarly to (2.14), we know that, for any $f \in L^p(\mathbb{R}^n)$ with $p \in (1,\infty)$ and $x \in \mathbb{R}^n$,

$$V^*(f)(x) \lesssim \sum_{j\in\mathbb{Z}} \widetilde{T}_j(f)(x),$$

where the implicit equivalent positive constant is independent of $f$ and, for any $j \in \mathbb{Z}$,

$$\widetilde{T}_j(f)(x) := \left[\sum_{k\in\mathbb{Z}} \int_{2^{-k}}^{2^{-k+1}} \fint_{B(\vec{0},2^{-k+1})} |f * \varphi_{2^{-j-k}} * K_t(x+y)|^2 \, dy \, \frac{dt}{t}\right]^{\frac{1}{2}} \tag{3.17}$$

with $\varphi$ as in (2.14). Then, similarly to Lemmas 2.5 and 2.6, we have the following technical lemmas.

**Lemma 3.2.** *Let $n \in \mathbb{N}\cap[3,\infty)$. Let $\Phi$ satisfy Assumption 1.2 and $\widetilde{T}_j$ for any $j \in \mathbb{Z}$ be as in (3.2). Then there exists a positive constant $C$ such that, for any $j \in \mathbb{Z}$ and $f \in L^2(\mathbb{R}^n)$,*

$$\left\|\widetilde{T}_j(f)\right\|_{L^2(\mathbb{R}^n)} \le C\|f\|_{L^2(\mathbb{R}^n)} \begin{cases} \min\{2^{2j},\, 2^{-j}\} & \text{when } n \in \{3,4\}, \\ \min\{2^{2j},\, 2^{-2j}\} & \text{when } n \in [5,\infty)\cap\mathbb{N}. \end{cases}$$

*Proof.* When $n \in \{3,4\}$, by the Plancherel theorem and Lemma 3.1, we find that

$$\begin{aligned} \left\|\widetilde{T}_j(f)\right\|_{L^2(\mathbb{R}^n)}^2 &= \sum_{k\in\mathbb{Z}} \int_{2^{-k}}^{2^{-k+1}} \fint_{B(\vec{0},2^{-k+1})} \int_{\mathbb{R}^n} |f * \varphi_{2^{-j-k}} * K_t(x+y)|^2 \, dx \, dy \, \frac{dt}{t} \\ &\lesssim \sum_{k\in\mathbb{Z}} \int_{2^{-k}}^{2^{-k+1}} \int_{\mathbb{R}^n} |f * \varphi_{2^{-j-k}} * K_t(x)|^2 \, dx \, \frac{dt}{t} \\ &\lesssim \sum_{k\in\mathbb{Z}} \int_{2^{j+k-1}\le|\xi|\le 2^{j+k+1}} |\widehat{f}(\xi)|^2 \int_{2^{-k}}^{2^{-k+1}} |\widehat{K}(t\xi)|^2 \, \frac{dt}{t} \, d\xi \\ &\lesssim \min\left\{2^{4j}, 2^{-2j}\right\} \|f\|_{L^2(\mathbb{R}^n)}^2. \end{aligned}$$

The proof for the case $n \in [5,\infty)\cap\mathbb{N}$ is similar, the details being omitted. This finishes the proof of Lemma 3.2. □

**Lemma 3.3.** *Let $n \in [3,\infty)\cap\mathbb{N}$. Let $p \in (1,\infty)$, $p'$ be its conjugate index as in Lemma 2.6, $q := \max\{p, p'\}$, and $\varepsilon \in (0, \frac{1}{q})$. Let $\Phi$ satisfy Assumption 1.2 and $\widetilde{T}_j$ for any $j \in \mathbb{Z}$ be as in (3.17). Let $\varphi$, $\psi \in \mathcal{S}(\mathbb{R}^n)$ be as in (2.13). Then there exists a positive constant $C_{(\varepsilon,p)}$, depending on $\varepsilon$ and $p$, such that, for any $j \in \mathbb{Z}_-$ and $f \in L^p(\mathbb{R}^n)$,*

$$\left\|\widetilde{T}_j(f)\right\|_{L^p(\mathbb{R}^n)} \le C_{(\varepsilon,p)} 2^{4j(\frac{1}{q}-\varepsilon)} \|f\|_{L^p(\mathbb{R}^n)}.$$

*Proof.* Let $r \in (1,\infty)$. Similarly to (2.16), by (3.17), we conclude that, for any $j \in \mathbb{Z}_-$, $f \in L^r(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$\left|\widetilde{T}_j(f)(x)\right|^2 \lesssim \sum_{k\in\mathbb{Z}} \sup_{\substack{y\in B(\vec{0},2^{-k+1}) \\ t\in[2^{-k},2^{-k+1})}} |t^{-2}(\Phi_t * f * I_2 * \varphi_{2^{-j-k}} - f * I_2 * \varphi_{2^{-j-k}})(x+y)|^2$$

$$+\sum_{k\in\mathbb{Z}}[M(f*\varphi_{2^{-k}})(x)]^2,$$

where $M$ is as in (1.19). For the first part of the right-hand side of the above inequality, similarly to the proof of Lemma 2.6, letting $f_{j+k} := f*\psi_{2^{-j-k}}$, we then have the following analogous estimate: for any $k\in\mathbb{Z}$, $t\in[2^{-k},2^{-k+1})$, $f\in L^r(\mathbb{R}^n)$ and $x\in\mathbb{R}^n$,

$$\begin{aligned}
&t^{-2}\sup_{|y|\le 2^{-k+1}}|(\Phi_t*f*I_2*\varphi_{2^{-j-k}}-f*I_2*\varphi_{2^{-j-k}})(x+y)|\\
&\quad\lesssim 2^{(k+j)n}\int_{\mathbb{R}^n}|f_{j+k}(z)|(1+2^{k+j}|x-z|)^{-n-1}\,dz\lesssim M(f_{j+k})(x).
\end{aligned}$$

Thus, we conclude that, for any $f\in L^r(\mathbb{R}^n)$ and $x\in\mathbb{R}^n$,

$$\left|\widetilde{T}_j(f)(x)\right|^2\lesssim\sum_{k\in\mathbb{Z}}\left\{[M(f_k)(x)]^2+[M(f*\varphi_{2^{-k}})(x)]^2\right\},$$

which, together with the Fefferman-Stein vector-valued inequality (see [7]) as well as the Littlewood-Paley characterization of $L^r(\mathbb{R}^n)$ (see, for example, [11, Theorem 1.3.8]), further implies that, for any $f\in L^r(\mathbb{R}^n)$,

$$\left\|\widetilde{T}_j(f)\right\|_{L^r(\mathbb{R}^n)}\lesssim\|f\|_{L^r(\mathbb{R}^n)}.$$

Taking $r:=\frac{1-\frac{2}{q}+2\varepsilon}{\frac{1}{p}-\frac{1}{q}+\varepsilon}$, then, by Lemma 3.2 and applying the Riesz-Thorin interpolation theorem (see, for example, [10, Theorem 1.3.4]) to $T_j$ (namely, taking interpolation between $L^r(\mathbb{R}^n)$ and $L^2(\mathbb{R}^n)$), we find that, for any $j\in\mathbb{Z}_-$ and $f\in L^p(\mathbb{R}^n)$,

$$\|T_j(f)\|_{L^p(\mathbb{R}^n)}\lesssim 2^{4j(\frac{1}{q}-\varepsilon)}\|f\|_{L^p(\mathbb{R}^n)},$$

which completes the proof of Lemma 3.3. □

Using the same notation as in (3.17), similarly to (2.18), we conclude that, for any $x\in\mathbb{R}^n$,

$$\begin{aligned}
\widetilde{T}_j(f)(x)&\lesssim\left[\sum_{k\in\mathbb{Z}}2^{4k}⨍_{B(x,2^{-k+4})}|f*I_2*\varphi_{2^{-j-k}}(z)|^2\,dz\right]^{\frac{1}{2}}\\
&\quad+\left\{\sum_{k\in\mathbb{Z}}[M(f*\varphi_{2^{-j-k}})(x)]^2\right\}^{\frac{1}{2}}\sim\widetilde{T}_{j,1}(f)(x)+\widetilde{T}_{j,2}(f)(x),
\end{aligned}\tag{3.18}$$

where the implicit equivalent positive constants are independent of $j$, $f$ and $x$,

$$\widetilde{T}_{j,1}(f)(x):=\left[\sum_{k\in\mathbb{Z}}2^{4k}⨍_{B(x,2^{-k+4})}|f*I_2*\varphi_{2^{-j-k}}(z)|^2\,dz\right]^{\frac{1}{2}}\tag{3.19}$$

and

$$\widetilde{T}_{j,2}(f)(x) := \left\{\sum_{k\in\mathbb{Z}}[M(f*\varphi_{2^{-j-k}})(x)]^2\right\}^{\frac{1}{2}}.$$

Dai et al. [4, Lemma 2.9] established the following lemma.

**Lemma 3.4.** *Let $\varepsilon \in (0,1)$ and $p \in (1,2]$. Then there exists a positive constant $C_{(\varepsilon,p)}$, depending on $\varepsilon$ and $p$, such that, for any $j \in \mathbb{Z}_+$ and $f \in L^p(\mathbb{R}^n)$,*

$$\left\|\widetilde{T}_{j,1}(f)\right\|_{L^p(\mathbb{R}^n)} \le C_{(\varepsilon,p)} 2^{[n(\frac{1}{p}-\frac{1}{2})-2+\varepsilon]j}\|f\|_{L^p(\mathbb{R}^n)},$$

*where $\widetilde{T}_{j,1}$ is as in (3.19).*

After these preparations, now we can prove the following conclusion.

**Lemma 3.5.** *Let $n \in \mathbb{N}\cap[4,\infty)$ and $p \in (\frac{2n}{4+n},\infty)$. Then there exists a positive constant $C$ such that, for any $f \in L^p(\mathbb{R}^n)$,*

$$\|V^*(f)\|_{L^p(\mathbb{R}^n)} \le C\|f\|_{L^p(\mathbb{R}^n)}, \tag{3.20}$$

*where $V^*$ is as in (3.16).*

*Conversely, if (3.20) holds true for some $p \in (1,\infty)$ and any $f \in L^p(\mathbb{R}^n)$, then $p \in [\frac{2n}{4+n},\infty)$.*

*Proof.* We first prove (3.20). When $p \in [2,\infty)$, from Lemmas 2.1 and 2.2, and Theorem 1.3, we deduce that (3.20) holds true in this case.

For any given $p \in (1,2)$, let $\varepsilon \in (0,\frac{1}{2})$ be small enough such that $\theta := \frac{1}{p}-\frac{1}{2}+\frac{\varepsilon}{4} \in (0,\frac{1}{2})$. Let $q_\varepsilon$ be such that $\frac{1}{p} = \frac{2\theta}{q_\varepsilon}+\frac{1-2\theta}{2}$. Then $\frac{1}{2}(\frac{1}{p}-\frac{1}{2}) = \theta(\frac{1}{q_\varepsilon}-\frac{1}{2})$, which implies that $q_\varepsilon \in (1,2)$ because $\theta > \frac{1}{p}-\frac{1}{2}$. If $j \in \mathbb{Z}_+$, then, by (3.18), Lemmas 3.2 and 3.4, and the Marcinkiewicz interpolation theorem (see, for instance, [10, Theorem 1.3.2]) to $\widetilde{T}_j$ (namely, taking interpolation between $L^2(\mathbb{R}^n)$ and $L^{q_\varepsilon}(\mathbb{R}^n)$), we conclude that, when $n \ge 5$,

$$\left\|\widetilde{T}_j(f)\right\|_{L^p(\mathbb{R}^n)} \lesssim 2^{|j|2\theta[n(\frac{1}{q_\varepsilon}-\frac{1}{2})-2+\frac{\varepsilon}{4\theta}]}2^{-(1-2\theta)2|j|}\|f\|_{L^p(\mathbb{R}^n)} \sim 2^{-[2-n(\frac{1}{p}-\frac{1}{2})-\frac{\varepsilon}{2}]|j|}\|f\|_{L^p(\mathbb{R}^n)},$$

while when $n = 4$, by the definition of $\theta$, we have

$$\begin{aligned}\left\|\widetilde{T}_j(f)\right\|_{L^p(\mathbb{R}^n)} &\lesssim 2^{|j|2\theta[n(\frac{1}{q_\varepsilon}-\frac{1}{2})-2+\frac{\varepsilon}{4\theta}]}2^{-(1-2\theta)|j|}\|f\|_{L^p(\mathbb{R}^n)}\\ &\sim 2^{|j|[n(\frac{1}{p}-\frac{1}{2})-2\theta+\frac{\varepsilon}{2}-1]}\|f\|_{L^p(\mathbb{R}^n)} \sim 2^{-|j|[1-2(\frac{1}{p}-\frac{1}{2})]}\|f\|_{L^p(\mathbb{R}^n)}.\end{aligned}$$

Here we point out that $1-2(\frac{1}{p}-\frac{1}{2})$ is positive because $p \in (1,2)$. Similarly, if $j \in \mathbb{Z}_-$, then, by Lemmas 3.2 and 3.3, and the Marcinkiewicz interpolation theorem again, we obtain

$$\left\|\widetilde{T}_j(f)\right\|_{L^p(\mathbb{R}^n)} \lesssim 2^{8j\theta(\frac{1}{q_\varepsilon-\varepsilon})}2^{-2|j|(1-2\theta)}\|f\|_{L^p(\mathbb{R}^n)} \sim 2^{-2[2n(\frac{1}{p}-\frac{1}{2})-4\theta\varepsilon+1]|j|}\|f\|_{L^p(\mathbb{R}^n)}.$$

Now, let $p \in (\frac{2n}{4+n}, 2)$, which implies $n(\frac{1}{p} - \frac{1}{2}) < 2$. We then choose an appropriate $\varepsilon \in (0,1)$ such that

$$\delta := \min\left\{\frac{1}{2}\left[2 - n\left(\frac{1}{p} - \frac{1}{2}\right) - \frac{\varepsilon}{2}\right],\ 2n\left(\frac{1}{p} - \frac{1}{2}\right) - 4\theta\varepsilon + 1, 1 - 2\left(\frac{1}{p} - \frac{1}{2}\right)\right\} > 0.$$

Then we find that

$$\|T_j(f)\|_{L^p(\mathbb{R}^n)} \lesssim 2^{-|j|\delta}\|f\|_{L^p(\mathbb{R}^n)}, \quad \forall\, j \in \mathbb{Z},$$

which, together with (3.16), implies (3.20). Thus, (3.20) holds true.

Now we show that, if (3.20) holds true, then it must hold true that $p \geq \frac{2n}{4+n}$. Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ be as in the proof of Lemma 2.8. Then, for any $x \in \mathbb{R}^n$, we have

$$\begin{aligned} V^*(\phi_j)(x) &\gtrsim \left\{\int_1^2 \int_{B(\vec{0},1)} \left|\frac{\Phi_t * \phi_j * I_2(x+y) - \phi_j * I_2(x+y)}{t^2} + \Phi_t * \phi_j(x+y)\right|^2 dy\, dt\right\}^{\frac{1}{2}} \\ &\gtrsim \widetilde{\mathrm{J}}_{j,1} - \widetilde{\mathrm{J}}_{j,2} - \widetilde{\mathrm{J}}_{j,2}, \end{aligned}$$

where

$$\widetilde{\mathrm{J}}_{j,1}(x) := 2^{-2}\left[\int_{B(\vec{0},1)} |\phi_j * I_2(x+y)|^2\, dy\right]^{\frac{1}{2}},$$

$$\widetilde{\mathrm{J}}_{j,2}(x) := \left[\int_1^2 \int_{B(\vec{0},1)} |\Phi_t * \phi_j * I_2(x+y)|^2\, dy\, dt\right]^{\frac{1}{2}}$$

and

$$\widetilde{\mathrm{J}}_{j,3}(x) := \left[\int_1^2 \int_{B(\vec{0},1)} |\Phi_t * \phi_j(x+y)|^2\, dy\, dt\right]^{\frac{1}{2}}.$$

For $\widetilde{\mathrm{J}}_{j,1}$, Dai et al. ([4, (2.22)]) obtained

$$\left\|\widetilde{\mathrm{J}}_{j,1}\right\|_{L^p(\mathbb{R}^n)} \gtrsim 2^{-2j+\frac{jn}{2}}. \tag{3.21}$$

For $\widetilde{\mathrm{J}}_{j,2}$, similarly to (2.20), we have

$$\left\|\widetilde{\mathrm{J}}_{j,2}\right\|_{L^p(\mathbb{R}^n)} \lesssim 2^{-2j} 2^{jn(1-\frac{1}{p})}. \tag{3.22}$$

For $\widetilde{\mathrm{J}}_{j,3}$, we find that, for any $x \in \mathbb{R}^n$,

$$\left|\widetilde{\mathrm{J}}_{j,3}(x)\right|^2 \lesssim \int_1^2 \int_{B(\vec{0},1)} \left[⨍_{B(x,t_0)} |\phi_j(z)|\, dz\right]^2 dy\, dt \lesssim [M(\phi_j)(x)]^2,$$

where $M$ denotes the Hardy-Littlewood maximal operator as in (1.19). ¿From this and the boundedness of $M$ on $L^p(\mathbb{R}^n)$ with any $p \in (1,\infty]$, it follows that

$$\left\|\widetilde{\mathrm{J}}_{j,3}\right\|_{L^p(\mathbb{R}^n)} \lesssim \|\phi_j\|_{L^p(\mathbb{R}^n)} \lesssim 2^{jn(1-\frac{1}{p})}. \tag{3.23}$$

Therefore, by (3.21), (3.22), (3.23) and (3.20), we conclude that, for any sufficiently large $j \in \mathbb{N}$,

$$\begin{aligned} 2^{j(-2+\frac{n}{2})} &\lesssim \left\|\widetilde{\mathrm{J}}_{j,1}\right\|_{L^p(\mathbb{R}^n)} \lesssim \|V^*(\phi_j)\|_{L^p(\mathbb{R}^n)} + \left\|\widetilde{\mathrm{J}}_{j,2}\right\|_{L^p(\mathbb{R}^n)} + \left\|\widetilde{\mathrm{J}}_{j,3}\right\|_{L^p(\mathbb{R}^n)} \\ &\lesssim \|\phi_j\|_{L^p(\mathbb{R}^n)} + 2^{jn(1-\frac{1}{p})} \lesssim 2^{jn(1-\frac{1}{p})}. \end{aligned}$$

This implies that $-2+\frac{n}{2} \le n(1-\frac{1}{p})$, namely, $p \in [\frac{2n}{4+n}, \infty)$, which completes the proof of Lemma 3.5. □

For the case $n \in \{1,2,3\}$, we have the following conclusion.

**Lemma 3.6.** *Let $n \in \{1,2,3\}$ and $p \in (1,\infty)$. Then there exists a positive constant $C_{(n)}$, depending only on $n$, such that, for any $f \in L^p(\mathbb{R}^n)$,*

$$\|V^*(f)\|_{L^p(\mathbb{R}^n)} \le C_{(n)}\|f\|_{L^p(\mathbb{R}^n)},$$

*where $V^*$ is as in (3.16).*

*Proof.* Recall that, for any $x \in \mathbb{R}^n \setminus \{\vec{0}\}$,

$$I_2(x) := \begin{cases} -\dfrac{1}{2}|x| & \text{when } n=1, \\ -\dfrac{1}{2n}\log|x| & \text{when } n=2, \\ c_{(n)}|x|^{2-n} & \text{when } n=3, \end{cases}$$

where $c_{(n)}$ is a constant, depending on $n$, such that, for any $f \in W^{2,p}(\mathbb{R}^n)$, $I_2*(-\Delta f) = f$. Recall that, for any $f \in W^{2,p}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, $K(x) := \Phi * I_2(x) - I_2(x) + C_0\Phi(x)$ and

$$\begin{aligned} V^*(f)(x) &:= \left\{\int_0^\infty \fint_{B(\vec{0},t)} |K_t * f(x+y)|^2\,dy\,\frac{dt}{t}\right\}^{\frac{1}{2}} \\ &\sim \left\{\int_0^\infty \int_{B(\vec{0},t)} \left|\int_{\mathbb{R}^n} K_t(x+y-z)f(z)\,dz\right|^2 dy\,\frac{dt}{t^{n+5}}\right\}^{\frac{1}{2}} =: \|Tf(x)\|_{L^2(\Sigma)}, \end{aligned}$$

where $C_0$ is as in (1.17),

$$\Sigma := \left(\mathbb{R}^n \times (0,\infty),\ \frac{\chi_{B(\vec{0},t)}}{t^{n+5}}\,dy\,dt\right)$$

and

$$Tf(x)(y,t) := \int_{\mathbb{R}^n} K_t(x+y-z)f(z)\,dz, \quad \forall\,(y,t) \in \mathbb{R}^n \times (0,\infty).$$

When $p=2$, for any $f \in W^{2,2}(\mathbb{R}^n)$, by the Fubini theorem and (3.5), we know that

$$\int_{\mathbb{R}^n} \|Tf(x)\|_{L^2(\Sigma)}^2\,dx \sim \int_{\mathbb{R}^n}\int_0^\infty \int_{B(x,t)} |K_t * f(y)|^2\,dy\,\frac{dt}{t^{n+1}}\,dx$$

$$\begin{aligned}&\sim \int_{\mathbb{R}^n}\int_0^\infty \int_{B(y,t)} |K_t * f(y)|^2\,dx\,\frac{dt}{t^{n+1}}\,dy\\ &\sim \|\mathcal{G}(\mathcal{F})\|_{L^2(\mathbb{R}^n)} \sim \|\Delta f\|_{L^2(\mathbb{R}^n)}.\end{aligned}$$

If the kernel $K_t$ satisfies the following Hörmander condition:

$$\int_{|x-z|\ge 2|w-z|} \|K_t(x+\cdot-z)-K_t(x+\cdot-w)\|_{L^2(\Sigma)}\,dx \le C, \quad \forall\, w,\, z\in\mathbb{R}^n \text{ and } w\ne z, \tag{3.24}$$

then, by [9, p. 492, Theorem 3.4], we conclude that, for any $f\in L^p(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \|K_t * f\|_{L^2(\Sigma)}^p\,dx \lesssim \|f\|_{L^p(\mathbb{R}^n)}^p,$$

which means that, for any $f\in L^p(\mathbb{R}^n)$,

$$\|V^*(f)\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}.$$

We now prove the following stronger version of the Hörmander condition (3.24):

$$\|K_t(x+\cdot-z)-K_t(x+\cdot-w)\|_{L^2(\Sigma)}^2 \lesssim \frac{|w-z|^\gamma}{|x-z|^{n+\gamma}}, \quad \forall\,|x-z|\ge 2|w-z|>0, \tag{3.25}$$

where $\gamma\in(0,\infty)$ is a positive constant independent of $x$, $w$ and $z$.

Let $\widetilde{x}:=x-z$ and $\widetilde{z}:=w-z$. Then (3.25) becomes

$$\|K_t(\widetilde{x}+\cdot)-K_t(\widetilde{x}+\cdot-\widetilde{z})\|_{L^2(\Sigma)}^2 \lesssim \frac{|\widetilde{z}|^\gamma}{|\widetilde{x}|^{n+\gamma}}, \quad \forall\,|\widetilde{x}|\ge 2|\widetilde{z}|>0. \tag{3.26}$$

The proof of (3.26) is similar to that of (3.10), and we estimate $K_{t,1}:=\Phi_t * I_2 - I_2$ and $t^2\Phi_t$, separately.

For $t^2\Phi_t$, we first observe that the quantity $|\Phi_t(\widetilde{x}+y)-\Phi_t(\widetilde{x}+y-\widetilde{z})|$ is non-zero only when $t>\min\{\frac{|\widetilde{x}+y-\widetilde{z}|}{t_0}, \frac{|\widetilde{x}+y|}{t_0}\}$ with $t_0$ as in Assumption 1.2. Assume that $|\widetilde{x}+y-\widetilde{z}|\le|\widetilde{x}+y|$, because the proof for another case $|\widetilde{x}+y-\widetilde{z}|>|\widetilde{x}+y|$ is similar, the details being omitted. Notice that, when $t>\frac{|\widetilde{x}+y-\widetilde{z}|}{t_0}$ and $y\in B(\vec{0},t)$, we have $t>\frac{|\widetilde{x}-\widetilde{z}|}{1+t_0}>\frac{|\widetilde{x}|}{2(1+t_0)}$. On the other hand, we also notice that, when $t>\frac{|\widetilde{x}+y|}{t_0}$ and $y\in B(\vec{0},t)$, then $tt_0>|\widetilde{x}|-|y|\ge|\widetilde{x}|-t$, that is, $t>\frac{|\widetilde{x}|}{2(1+t_0)}$. Thus, there exists a positive constant $c_0$, independent of $t$ and $\widetilde{x}$, such that $t>c_0|\widetilde{x}|$. Let

$$E_1:=\{[B(\widetilde{x},tt_0)\setminus B(\widetilde{x}-\widetilde{z},tt_0)]\cup[B(\widetilde{x}-\widetilde{z},tt_0)\setminus B(\widetilde{x},tt_0)]\}\cap B(\vec{0},t)$$

and

$$E_2:=B(\widetilde{x},tt_0)\cap B(\widetilde{x}-\widetilde{z},tt_0)\cap B(\vec{0},t).$$

¿From the definitions of $E_1$ and $E_2$, we deduce that, if $y\in E_2$, then $t>\frac{|\widetilde{x}+y-\widetilde{z}|}{t_0}$ and $t>\frac{|\widetilde{x}+y|}{t_0}$. We also have

$$|E_1|\lesssim t^{n-1}|\widetilde{z}| \quad\text{and}\quad |E_2|\lesssim t^n,$$

where $|E_1|$ and $|E_2|$ denote their Lebesgue measures, respectively. Then, combining (1.16) and the fact that $\Phi$ is bounded, we find that

$$\begin{aligned}&\left[\int_0^\infty \int_{B(\vec{0},t)} |t^2[\Phi_t(\widetilde{x}+y)-\Phi_t(\widetilde{x}+y-\widetilde{z})]|^2\,dy\,\frac{dt}{t^{n+5}}\right]^{\frac12}\\ &\quad\lesssim \left[\int_{c_0|\widetilde{x}|}^\infty \int_{E_1} dy\,\frac{dt}{t^{3n+2}}\right]^{\frac12}+\left[\int_{c_0|\widetilde{x}|}^\infty \int_{E_2} |\widetilde{z}|^2\,dy\,\frac{dt}{t^{3n+3}}\right]^{\frac12}\\ &\quad\lesssim \left[\int_{c_0|\widetilde{x}|}^\infty |\widetilde{z}|\,\frac{dt}{t^{2n+2}}\right]^{\frac12}+\left[\int_{c_0|\widetilde{x}|}^\infty |\widetilde{z}|^2\,\frac{dt}{t^{2n+3}}\right]^{\frac12}\lesssim \frac{|\widetilde{z}|^{\frac12}}{|\widetilde{x}|^{n+\frac12}},\end{aligned}$$

which is the desired estimate.

We now consider $K_{t,1}$. Similarly, without loss of generality, we may assume that $|\widetilde{x}+y-\widetilde{z}|\le|\widetilde{x}+y|$. If $t<\frac{|\widetilde{x}|}{5(t_0+1)}$, then the origin $\vec{0}$ does not belong to the balls $B(\widetilde{x}+y-\widetilde{z},tt_0)$ and $B(\widetilde{x}+y,tt_0)$. Since $y\in B(\vec{0},t)$ and $t<\frac{|\widetilde{x}|}{5(t_0+1)}$, it follows that $|\widetilde{x}+y|\gtrsim|\widetilde{x}|$. Similarly to (3.13) and (3.15), we have

$$\begin{aligned}&\int_0^{\frac{|\widetilde{x}|}{5(t_0+1)}} \int_{B(\vec{0},t)} |K_{t,1}(\widetilde{x}+y)-K_{t,1}(\widetilde{x}+y-\widetilde{z})|^2\,dy\,\frac{dt}{t^{n+5}}\\ &\quad\lesssim \int_0^{\frac{|\widetilde{x}|}{5(t_0+1)}} \int_{B(\vec{0},t)} \frac{|\widetilde{z}|^2}{|\widetilde{x}+y|^{2n+4}}\,dy\,\frac{dt}{t^{n-1}}\lesssim\frac{|\widetilde{z}|^2}{|\widetilde{x}|^{2n+2}}.\end{aligned}$$

If $t\ge\frac{|\widetilde{x}|}{5(t_0+1)}$, then we have

$$\begin{aligned}&\int_{\frac{|\widetilde{x}|}{5(t_0+1)}}^\infty \int_{B(\vec{0},t)} |K_{t,1}(\widetilde{x}+y)-K_{t,1}(\widetilde{x}+y-\widetilde{z})|^2\,dy\,\frac{dt}{t^{n+5}}\\ &\quad\le \int_{\frac{|\widetilde{x}|}{5(t_0+1)}}^\infty \int_{B(\vec{0},t)} |I_2(\widetilde{x}+y)-I_2(\widetilde{x}+y-\widetilde{z})|^2\,dy\,\frac{dt}{t^{n+5}}\\ &\qquad+\int_{\frac{|\widetilde{x}|}{5(t_0+1)}}^\infty \int_{B(\vec{0},t)} |\Phi_t*I_2(\widetilde{x}+y)-\Phi_t*I_2(\widetilde{x}+y-\widetilde{z})|^2\,dy\,\frac{dt}{t^{n+5}}=:\mathrm{J}_1+\mathrm{J}_2.\end{aligned}$$

By the mean value theorem, we know that, for any $|\widetilde{x}+y-\widetilde{z}|\le|\widetilde{x}+y|$,

$$|\Phi_t*I_2(\widetilde{x}+y)-\Phi_t*I_2(\widetilde{x}+y-\widetilde{z})|\le|\widetilde{z}|\sup_{\theta\in[0,1]}|\Phi_t*\nabla I_2(\widetilde{x}+y-\theta\widetilde{z})|,$$

which, together with the fact $|\widetilde{z}|<\frac{|\widetilde{x}|}{2}\le\frac52 t(t_0+1)$ whenever $|\widetilde{x}|\le 5t(t_0+1)$, further implies that, for any $y\in B(\vec{0},t)$ and $\theta\in[0,1]$,

$$|\Phi_t*\nabla I_2(\widetilde{x}+y-\theta\widetilde{z})|\lesssim\frac{1}{t^n}\int_{B(\vec{0},tt_0)}\frac{1}{|\widetilde{x}+y-\theta\widetilde{z}-h|^{n-1}}\,dh$$

$$\lesssim \frac{1}{t^n}\int_{B(\vec{0},10t(t_0+1))}\frac{1}{|h|^{n-1}}\,dh\lesssim\frac{1}{t^{n-1}},$$

because $|\nabla I_2(x)|\lesssim|x|^{1-n}$ for any $x\in\mathbb{R}^n\setminus\{\vec{0}\}$. Therefore, we obtain

$$\mathrm{J}_2\lesssim\int_{\frac{|\widetilde{x}|}{5(t_0+1)}}^{\infty}\int_{B(\vec{0},t)}\frac{|\widetilde{z}|^2}{t^{2n-2}}\,dy\,\frac{dt}{t^{n+5}}\sim|\widetilde{z}|^2\int_{\frac{|\widetilde{x}|}{5(t_0+1)}}^{\infty}\int_{B(\vec{0},t)}\frac{1}{t^{2n+3}}dt\lesssim\frac{|\widetilde{z}|^2}{|\widetilde{x}|^{2n+2}},$$

which is the desired estimate.

For $\mathrm{J}_1$, similarly to the estimation for $\mathrm{J}_1$ as in the proof of [13, Lemma 2.5], we also have

$$\mathrm{J}_1\lesssim\frac{|\widetilde{z}|^2}{|\widetilde{x}|^{2n+2}},$$

which is also the desired estimate and hence completes the proof of Lemma 3.6. □

Now, we are ready to show Theorem 1.4.

*Proof of Theorem 1.4.* To prove Theorem 1.4, for any $f\in W^{2,p}(\mathbb{R}^n)$ with $p\in(1,\infty)$ and $\Phi$ satisfying Assumption 1.2, we define

$$\mathcal{F}(x,t):=\left|\frac{\Phi_t*f(x)-f(x)}{t^2}-\Phi_t*g(x)\right|,\quad\forall\,(x,t)\in\mathbb{R}^n\times(0,\infty).$$

Then, applying Lemma 2.2 to this $\mathcal{F}$, we conclude the equivalence between (ii) and (iii) of Theorem 1.4(I).

To prove (i) $\Rightarrow$ (ii) of Theorem 1.4(I), let $g:=C_0\Delta f$ in (ii) of Theorem 1.4(I). Then, when $n\geq 4$ and $p\in(\frac{2n}{4+n},\infty)$, by Lemma 3.5, we know that (i) $\Rightarrow$ (ii) holds true. When $n\in\{1,2,3\}$ and $p\in(1,\infty)$, by Lemma 3.6, we know that (i) $\Rightarrow$ (ii) holds true. Therefore, to complete the proof of Theorem 1.4(I), we only need to prove (ii) $\Rightarrow$ (i) of Theorem 1.4(I).

Assume that $f,\ g\in L^p(\mathbb{R}^n)$ such that $\widetilde{U}(f,g)\in L^p(\mathbb{R}^n)$. We prove that $g$ coincides with $\Delta f$ modulo a positive constant. To this end, take a non-negative radial smooth function $\zeta$ which is supported in $B(\vec{0},1)$ such that $\|\zeta\|_{L^1(\mathbb{R}^n)}=1$ and, for any $\varepsilon\in(0,\infty)$ and $x\in\mathbb{R}^n$, let $\zeta_\varepsilon(x):=\varepsilon^{-n}\zeta(x/\varepsilon)$, $f_\varepsilon:=f*\zeta_\varepsilon$ and $g_\varepsilon:=g*\zeta_\varepsilon$. Then, by [1, Lemma 2(i)], we know $f_\varepsilon\in W^{2,p}(\mathbb{R}^n)$. Therefore, by the conclusion that (i) $\Rightarrow$ (ii) of Theorem 1.4(I), we conclude that $\widetilde{U}(f_\varepsilon,C_0\Delta f_\varepsilon)\in L^p(\mathbb{R}^n)$. ¿From the Minkowski inequality, we deduce that, for any $\varepsilon\in(0,\infty)$ and $x\in\mathbb{R}^n$,

$$\begin{aligned}\widetilde{U}(f_\varepsilon,g_\varepsilon)(x)&=\left\{\int_0^\infty\int_{B(x,t)}\left|\left(\frac{\Phi_t*f-f}{t^2}-\Phi_t*g\right)*\zeta_\varepsilon(y)\right|^2dy\,\frac{dt}{t^{n+1}}\right\}^{\frac12}\\&=\left\{\int_0^\infty\int_{B(x,t)}\left|\int_{\mathbb{R}^n}\left[\frac{\Phi_t*f(y-z)-f(y-z)}{t^2}\right.\right.\right.\\&\qquad\left.\left.\left.-\Phi_t*g(y-z)\right]\zeta_\varepsilon(z)\,dz\right|^2dy\,\frac{dt}{t^{n+1}}\right\}^{\frac12}\end{aligned}$$

$$\lesssim \int_{\mathbb{R}^n} \left\{ \int_0^\infty \int_{B(x-z,t)} \left| \frac{\Phi_t * f(y) - f(y)}{t^2} - \Phi_t * g(y) \right|^2 dy \, \frac{dt}{t^{n+1}} \right\}^{\frac{1}{2}} \zeta_\varepsilon(z)\, dz$$
$$\sim \int_{\mathbb{R}^n} \widetilde{U}(f,g)(x-z)\zeta_\varepsilon(z)dz \sim \widetilde{U}(f,g) * \zeta_\varepsilon(x).$$

For any $\varepsilon \in (0,\infty)$ and $x \in \mathbb{R}^n$, define

$$D_\varepsilon(x) := \left\{ \int_0^\infty \int_{B(x,t)} |\Phi_t * (g_\varepsilon - C_0 \Delta f_\varepsilon)(y)|^2 \, dy \, \frac{dt}{t^{n+1}} \right\}^{\frac{1}{2}},$$

where $C_0$ is as in (1.17). Then we know that

$$D_\varepsilon(x) = \left\{ \int_0^\infty \int_{B(x,t)} \left| - \left( \frac{\Phi_t * f_\varepsilon - f_\varepsilon}{t^2} - \Phi_t * g_\varepsilon \right)(y) \right.\right.$$
$$\left.\left. + \left( \frac{\Phi_t * f_\varepsilon - f_\varepsilon}{t^2} - C_0 \Phi_t * \Delta f_\varepsilon \right)(y) \right|^2 dy \, \frac{dt}{t^{n+1}} \right\}^{\frac{1}{2}}$$
$$\lesssim \widetilde{U}(f_\varepsilon, g_\varepsilon)(x) + \widetilde{U}(f_\varepsilon, C_0 \Delta f_\varepsilon)(x)$$
$$\lesssim \widetilde{U}(f,g) * \zeta_\varepsilon(x) + \widetilde{U}(f_\varepsilon, C_0 \Delta f_\varepsilon)(x),$$

which implies that $D_\varepsilon \in L^p(\mathbb{R}^n)$, in particular, $D_\varepsilon(x) < \infty$ for almost every $x \in \mathbb{R}^n$. This, combined with [6, Corollary 2.9], implies that, for almost every $x \in \mathbb{R}^n$,

$$|g_\varepsilon(x) - C_0 \Delta f_\varepsilon(x)| = \lim_{t\to 0} |g_\varepsilon * \Phi_t(x) - C_0 \Delta f_\varepsilon * \Phi_t(x)| = 0$$

and hence $C_0 \Delta f_\varepsilon \to g$ in $L^p(\mathbb{R}^n)$ as $\varepsilon \to 0^+$. Since $f_\varepsilon \to f$ in $L^p(\mathbb{R}^n)$ as $\varepsilon \to 0^+$, then $\Delta f_\varepsilon \to \Delta f$ in $\mathcal{S}'(\mathbb{R}^n)$ as $\varepsilon \to 0^+$. Therefore, $C_0 \Delta f = g$ almost everywhere in $\mathbb{R}^n$, which completes the proof of Theorem 1.4(I).

Finally, Theorem 1.4(II) is just deduced from Lemma 3.5. This finishes the proof of Theorem 1.4. □

# 4 Proof of Theorem 1.5

Before we prove Theorem 1.5, we need several technical lemmas as follows. The first one is from [8, p. 15, Decomposition Theorem].

**Lemma 4.1** ([8]). *Let $\beta \in (0,\infty)$, $p \in (1,\infty)$ and $f \in L^p(\mathbb{R}^n)$. Then there exist a family $\{Q_j\}_j$ of disjoint cubes, functions $g$ and $b$, and a positive constant $C$, independent of $f$, such that $f = g + b$ and*

(i) $\sum_j |Q_j| \le C\beta^{-p} \|f\|^p_{L^p(\mathbb{R}^n)}$;

(ii) *if $f_j := b\chi_{Q_j}$, then $\int_{\mathbb{R}^n} f_j(x)dx = 0$ and*

$$\int_{\mathbb{R}^n} |f_j(x)|^p \, dx \le C\beta^p |Q_j|;$$

(iii) $\|g\|_{L^\infty(\mathbb{R}^n)} \le C\beta$, $\|g\|_{L^p(\mathbb{R}^n)} \le C\|f\|_{L^p(\mathbb{R}^n)}$;

(iv) $b = \sum_j f_j$;

(v) $10\,\mathrm{diam}(Q_j) \le \mathrm{dist}\,(Q_j, \mathbb{R}^n \setminus \Omega) \le 20\,\mathrm{diam}(Q_j)$, *where* $\Omega := \cup_j Q_j$.

The following lemma can be found in [1, p. 615, Lemma].

**Lemma 4.2.** *Let* $E$ *be a measurable subset of* $\mathbb{R}^n$ *and* $\beta \in (0, n)$. *Then there exists a positive constant* $C$, *independent of* $E$, *such that*

$$\int_E \frac{1}{|z|^{n-\beta}}\,dz \le C|E|^{\frac{\beta}{n}},$$

*where* $|E|$ *is the Lebesgue measure of* $E$.

The following lemma is similar to [1, p. 598, Lemma 1], we give some details for the completeness.

**Lemma 4.3.** *Let* $t \in (0,\infty)$, $x \in \{x \in \mathbb{R}^n : \frac{t_0 t}{3} < |x| < 3t_0 t\}$ *and* $\Phi$ *satisfy Assumption 1.2, where* $t_0$ *is as in Assumption 1.2. Then there exists a positive constant* $C$, *independent of* $x$ *and* $t$, *such that*

$$\left|\mathrm{p.\,v.}\int_{B(x,t_0t)} \Phi\left(\frac{x-w}{t}\right)\frac{w}{|w|^{n+1}}\,dw\right| \le C\log\frac{|x|+t_0t}{||x|-t_0t|}. \tag{4.1}$$

*Proof.* If $\frac{t_0t}{3} < |x| < t_0t$, then $B(\vec{0}, t_0t-|x|) \subset B(x,t_0t)$ and hence we can write

$$\begin{aligned}
&\left|\mathrm{p.\,v.}\int_{B(x,t_0t)} \Phi\left(\frac{x-w}{t}\right)\frac{w}{|w|^{n+1}}\,dw\right| \\
&= \left|\int_{B(x,t_0t)\setminus B(\vec{0},t_0t-|x|)} \Phi\left(\frac{x-w}{t}\right)\frac{w}{|w|^{n+1}}\,dw + \mathrm{p.\,v.}\int_{B(\vec{0},t_0t-|x|)} \Phi\left(\frac{x-w}{t}\right)\frac{w}{|w|^{n+1}}\,dw\right|.
\end{aligned}$$

Since $\Phi$ is bounded, it follows that

$$\left|\int_{B(x,t_0t)\setminus B(\vec{0},t_0t-|x|)} \Phi\left(\frac{x-w}{t}\right)\frac{w}{|w|^{n+1}}\,dw\right| \lesssim \int_{t_0t-|x|}^{|x|+t_0t}\frac{dr}{r} \sim \log\frac{|x|+t_0t}{||x|-t_0t|}.$$

On the other hand, by the fact that $\Phi$ is radial and (1.16), we know that

$$\begin{aligned}
&\left|\mathrm{p.\,v.}\int_{B(\vec{0},t_0t-|x|)} \Phi\left(\frac{x-w}{t}\right)\frac{w}{|w|^{n+1}}\,dw\right| \\
&\quad= \lim_{\substack{\varepsilon\to 0\\ \varepsilon\in(0,t_0t-|x|)}}\left|\int_{B(\vec{0},t_0t-|x|)\setminus B(\vec{0},\varepsilon)} \Phi\left(\frac{x-w}{t}\right)\frac{w}{|w|^{n+1}}\,dw\right| \\
&\quad= \lim_{\substack{\varepsilon\to 0\\ \varepsilon\in(0,t_0t-|x|)}}\frac{1}{2}\left|\int_{B(\vec{0},t_0t-|x|)\setminus B(\vec{0},\varepsilon)} \left[\Phi\left(\frac{x-w}{t}\right)-\Phi\left(\frac{x+w}{t}\right)\right]\frac{w}{|w|^{n+1}}\,dw\right|
\end{aligned}$$

$$\lesssim t^{-1}\lim_{\varepsilon\to 0^+}\int_\varepsilon^{t_0t-|x|}\,dr\lesssim \frac{t_0t-|x|}{t}\lesssim \log\frac{|x|+t_0t}{||x|-t_0t|}.$$

This proves (4.1) when $|x|<tt_0$. If $t_0t\le |x|<3t_0t$, then, by the fact that $\Phi$ is bounded, we conclude that

$$\left|\mathrm{p.\,v.}\int_{B(x,t_0t)}\Phi\left(\frac{x-w}{t}\right)\frac{w}{|w|^{n+1}}\,dw\right|=\left|\int_{B(x,t_0t)}\Phi\left(\frac{x-w}{t}\right)\frac{w}{|w|^{n+1}}\,dw\right|$$
$$\lesssim\int_{|x|-t_0t}^{t_0t+|x|}\frac{dr}{r}\sim\log\frac{|x|+t_0t}{||x|-t_0t|}.$$

Combining all above estimates, we then complete the proof of Lemma 4.3. □

Applying Lemmas 4.2 and 4.3, we can obtain the following conclusion, which can be proved by an argument similar to that used in [1, pp. 597-601]. Here we give some details for the completeness.

**Lemma 4.4.** *There exists a positive constant $C$ such that, for any $z\in\mathbb{R}^n$,*

$$\int_{|x|>2|z|}\int_0^\infty|\Psi_u(x-z)-\Psi_u(x)|\,\frac{du}{u}\,dx\le C$$

*and*

$$\int_{|x|>2|z|}\int_0^\infty|K_u(x-z)-K_u(x)|\,\frac{du}{u}\,dx\le C,$$

*where $\Psi$ is as in (2.4) with $\alpha\in(0,2)$, $K$ is as in (3.1) and $\Psi_u(\cdot):=u^{-n}\Psi(\cdot/u)$ and $K_u(\cdot):=u^{-n}K(\cdot/u)$ for any $u\in(0,\infty)$.*

*Proof.* For the first inequality, we are going to prove the following stronger version of the Hörmander condition:

$$\|\Psi_u(x-y)-\Psi_u(x)\|_{L^1(du/u)}\lesssim\frac{|y|^\gamma}{|x|^{n+\gamma}},\qquad\forall\ |x|\ge 2|y|>0,$$

with some $\gamma\in(0,\infty)$.

By the mean value theorem, we know that

$$|\Psi_u(x-y)-\Psi_u(x)|\le|y|\sup_{z\in[x-y,x]}|\nabla\Psi_u(z)|,$$

where $z\in[x-y,x]$ means $z$ lies in the segment with endpoints $x-y$ and $x$.

If $u\ge\frac{|x|}{3t_0}$ with $t_0$ as in Assumption 1.2, we first consider $\alpha\in(1,2)$. Since, for any $z\in[x-y,x]$,

$$\nabla\Psi_u(z)=u^{-\alpha}[\Phi_u*\nabla I_\alpha(z)-\nabla I_\alpha(z)],$$

$$|\nabla I_\alpha(z)|\lesssim\frac{1}{|z|^{n-\alpha+1}}\lesssim\frac{1}{|x|^{n-\alpha+1}},$$

and

$$|\Phi_u * \nabla I_\alpha(z)| \lesssim \fint_{B(z,ut_0)} |\nabla I_\alpha(\omega)|\, d\omega \lesssim \fint_{B(z,ut_0)} \frac{d\omega}{|\omega|^{n-\alpha+1}} \lesssim \frac{1}{|x|^{n-\alpha+1}},$$

it follows that

$$|\Psi_u(x-y) - \Psi_u(x)| \lesssim \frac{|y|}{u^\alpha |x|^{n-\alpha+1}}.$$

Therefore, we have

$$\int_{\frac{|x|}{3t_0}}^\infty |\Psi_u(x-y) - \Psi_u(x)| \frac{du}{u} \lesssim \int_{\frac{|x|}{3t_0}}^\infty \frac{|y|}{|x|^{n+1-\alpha}} \frac{du}{u^{1+\alpha}} \lesssim \frac{|y|}{|x|^{n+1}}. \tag{4.2}$$

When $\alpha = 1$, we consider two cases. When $|x| < \frac{t_0 u}{3}$, for each $z$ in the segment $[x-y, x]$, we have $B(\vec{0}, t_0 u/6) \subset B(z, t_0 u)$ and hence

$$\begin{aligned}
\nabla\Psi_u(z) &= (-n+1)u^{-1}\left[\text{p. v.} \frac{1}{|B(z,t_0u)|} \int_{B(z,t_0u)} \Phi\left(\frac{z-w}{u}\right) \frac{w}{|w|^{n+1}}\, dw - \frac{z}{|z|^{n+1}}\right] \\
&= (-n+1)u^{-1}\left[\frac{1}{|B(z,t_0u)|} \int_{B(z,t_0u)\setminus B(\vec{0},t_0u/6)} \Phi\left(\frac{z-w}{u}\right) \frac{w}{|w|^{n+1}}\, dw\right. \\
&\qquad \left. + \text{p. v.} \frac{1}{|B(z,t_0u)|} \int_{B(\vec{0},t_0u/6)} \Phi\left(\frac{z-w}{u}\right) \frac{w}{|w|^{n+1}}\, dw - \frac{z}{|z|^{n+1}}\right].
\end{aligned}$$

Since $\Phi$ is bounded and $|x| \sim u \sim |z|$, it is easy to see that the first and the third items of the last quantity are controlled by $\frac{1}{u|x|^n}$ modulo a positive constant. For the second term, since $\Phi$ is radial and satisfies (1.16), it follows that

$$\begin{aligned}
&\left|\text{p. v.} \frac{1}{|B(z,t_0u)|} \int_{B(\vec{0},t_0u/6)} \Phi\left(\frac{z-w}{u}\right) \frac{w}{|w|^{n+1}}\, dw\right| \\
&\quad = \lim_{\substack{\varepsilon\to 0 \\ \varepsilon \in (0, \frac{t_0u}{6})}} \frac{1}{2|B(z,t_0u)|} \left|\int_{B(\vec{0},t_0u/6)\setminus B(\vec{0},\varepsilon)} \left[\Phi\left(\frac{z-w}{u}\right) - \Phi\left(\frac{z+w}{u}\right)\right] \frac{w}{|w|^{n+1}}\, dw\right| \\
&\quad \lesssim \frac{1}{|B(z,t_0u)|} u^{-1} \lim_{\substack{\varepsilon\to 0 \\ \varepsilon \in (0, \frac{t_0u}{6})}} \int_\varepsilon^{t_0u/6} dr \lesssim \frac{1}{u|x|^n}.
\end{aligned}$$

Hence, for any $|x| < \frac{t_0u}{3}$, we have

$$|\nabla\Psi_u(z)| \lesssim \frac{1}{u|x|^n}, \quad \forall\, z \in [x-y, x],$$

which, combined with the mean value theorem, further implies that

$$\int_{\frac{3|x|}{t_0}}^\infty |\Psi_u(x-y) - \Psi_u(x)| \frac{du}{u} \lesssim \int_{\frac{3|x|}{t_0}}^\infty \frac{|y|}{|x|^n} \frac{du}{u^2} \lesssim \frac{|y|}{|x|^{n+1}}.$$

When $\frac{t_0 u}{3} \le |x| < 3t_0 u$, by Lemma 4.3, similarly to the proof presented in [1, pp. 599-600], we conclude that

$$\begin{aligned}
\int_{\frac{|x|}{3t_0}}^{\frac{3|x|}{t_0}} |\Psi_u(x-y) - \Psi_u(x)| \frac{du}{u} &\lesssim \frac{|y|}{|x|^{n+1}} \int_{\frac{|x|}{3t_0}}^{\frac{3|x|}{t_0}} \int_0^1 \left| 1 + \log \frac{|x-\tau y| + t_0 u}{||x-\tau y| - t_0 u|} \right| d\tau \frac{du}{u} \\
&\lesssim \frac{|y|}{|x|^{n+1}} \int_0^1 \int_{\frac{|x|}{3t_0}}^{\frac{3|x|}{t_0}} \left| 1 + \log \frac{|x-\tau y| + t_0 u}{||x-\tau y| - t_0 u|} \right| \frac{du}{u} d\tau \\
&\lesssim \frac{|y|}{|x|^{n+1}} \int_{\frac{2}{9}}^{6} \left| 1 + \log \frac{1+s}{|1-s|} \right| ds \lesssim \frac{|y|}{|x|^{n+1}},
\end{aligned}$$

where the third inequality follows from letting $s := \log \frac{t_0 u}{|x-\tau y|}$.

Now we consider the case $\alpha \in (0,1)$. Let $D_1$ be the symmetric difference between $B(x, ut_0)$ and $B(x-y, ut_0)$ and $D_2 := B(x, ut_0) \cap B(x-y, ut_0)$. Then $|D_1| \lesssim u^{n-1}|y|$ and $|D_2| \lesssim u^n \lesssim |y|^{1-\frac{n}{\alpha}} u^{n-1+\frac{n}{\alpha}}$. By (1.16), Lemma 4.2 and the fact that $\Phi$ is bounded, we find that

$$\begin{aligned}
&|\Phi_u * I_\alpha(x-y) - \Phi_u * I_\alpha(x)| \\
&\quad = u^{-n} \left| \int_{B(x-y,ut_0)} \Phi\left(\frac{x-y-z}{u}\right) I_\alpha(z)\, dz - \int_{B(x,ut_0)} \Phi\left(\frac{x-z}{u}\right) I_\alpha(z)\, dz \right| \\
&\quad \lesssim u^{-n} \int_{D_1} \frac{1}{|z|^{n-\alpha}}\, dz + |y| u^{-n-1} \int_{D_2} \frac{1}{|z|^{n-\alpha}}\, dz \\
&\quad \lesssim u^{-n} (u^{n-1}|y|)^{\frac{\alpha}{n}} + |y| u^{-n-1} (|y|^{1-\frac{n}{\alpha}} u^{n-1+\frac{n}{\alpha}})^{\frac{\alpha}{n}} \lesssim u^{\alpha-n-\frac{\alpha}{n}} |y|^{\frac{\alpha}{n}},
\end{aligned}$$

which, together with the fact that

$$\int_{\frac{|x|}{3t_0}}^{\infty} u^{\alpha-n-\frac{\alpha}{n}} \frac{1}{u^{1+\alpha}}\, du \sim |x|^{-n-\frac{\alpha}{n}},$$

further implies that $\int_{\frac{|x|}{3t_0}}^{\infty} |\Psi_u(x-y) - \Psi_u(x)| \frac{du}{u} \lesssim \frac{|y|^{\frac{\alpha}{n}}}{|x|^{n+\frac{\alpha}{n}}}$ whenever $\alpha \in (0,1)$. This finishes the proof of the desired estimate when $u \ge \frac{|x|}{3t_0}$.

If $u < \frac{|x|}{3t_0}$, by the Taylor formula and the fact that $\Phi$ is radial, we know that, for any $z \in [x-y, x]$,

$$\begin{aligned}
|\nabla \Psi_u(z)| &= |\Phi_u * \nabla I_\alpha(z) - \nabla I_\alpha(z)| \\
&= \left| \int_{\mathbb{R}^n} \Phi_u(s) \left[\nabla I_\alpha(z-s) - \nabla I_\alpha(z)\right] ds \right| \\
&= u^{-n} \left| \int_{B(\vec{0},ut_0)} \Phi_u(s) \left[ \nabla^2 I_\alpha(z) \cdot (-s) + \sum_{|\beta|=2} \frac{D^\beta \nabla I_\alpha(z-\theta s)(-s)^\beta}{\beta!} \right] ds \right| \\
&\lesssim u^2 \sup_{w \in B(z,ut_0)} |\nabla^3 I_\alpha(w)|,
\end{aligned}$$

where $\theta \in (0,1)$.

Notice that, if $z \in [x-y,x]$, $w \in B(z,ut_0)$ and $u \le \frac{|x|}{3t_0}$, then $|w| \ge \frac{|x|}{6t_0}$ and

$$|\nabla^3 I_\alpha(w)| \lesssim |w|^{\alpha-n-3}.$$

Then we find that

$$|\Psi_u(x-y)-\Psi_u(x)| \le |y| \sup_{z\in[x-y,x]} |\nabla\Psi_u(z)| \lesssim |y|u^2|x|^{\alpha-n-3},$$

which further implies that

$$\int_0^{\frac{|x|}{3t_0}} |\Psi_u(x-y)-\Psi_u(x)|\,\frac{du}{u} \lesssim \int_0^{\frac{|x|}{3t_0}} |y|u^2|x|^{n-\alpha+3}\frac{du}{u^{1+\alpha}} \lesssim \frac{|y|}{|x|^{n+1}}. \tag{4.3}$$

Combining (4.2) and (4.3), we obtain

$$\|\Psi_u(x-y)-\Psi_u(x)\|_{L^1(du/u)} \lesssim \frac{|y|^{\frac{\alpha}{n}}}{|x|^{n+\frac{\alpha}{n}}}, \quad \forall\, |x| \ge 2|y| > 0.$$

This proves the first inequality of Lemma 4.4.

The proof of the second inequality of Lemma 4.4 is similar to that of (3.10), the details being omitted. This finishes the proof of Lemma 4.4. □

Now, we are ready to prove Theorem 1.5.

*Proof of Theorem 1.5.* We first prove Theorem 1.5(i). As in (2.11), we write $U_\alpha^* := \widetilde{U}_\alpha \circ I_\alpha$ for any $\alpha \in (0,2)$. We also notice that the assumption on $p$ in Theorem 1.5 implies $\alpha < n$. Let $f \in L^p(\mathbb{R}^n)$ with $p = \frac{2n}{n+2\alpha}$. Then, for any $\beta \in (0,\infty)$, by Lemma 4.1, we can decompose $f$ as $f = g + b$ with two functions $g$ and $b$ as in Lemma 4.1. Hence, to estimate $U_\alpha^*(f)$, it suffices to consider $U_\alpha^*(g)$ and $U_\alpha^*(b)$, separately.

By Lemma 2.8, we know that $U_\alpha^*$ is bounded on $L^2(\mathbb{R}^n)$, which, together with the Chebyshev inequality, $p<2$ and Lemma 4.1(iii), implies that, for any $\beta \in (0,\infty)$,

$$|\{x \in \mathbb{R}^n:\ U_\alpha^*(g)(x) > \beta\}| \lesssim \beta^{-2}\|U_\alpha^*(g)\|_{L^2(\mathbb{R}^n)}^2 \lesssim \beta^{-2}\|g\|_{L^2(\mathbb{R}^n)}^2 \lesssim \beta^{-p}\|f\|_{L^p(\mathbb{R}^n)}^p.$$

This is the desired estimate.

Next we estimate $U_\alpha^*(b)$. Let $y \in \mathbb{R}^n$. As in [8], the symbol $y \sim Q_j$ means that $y$ is contained in some $Q_i$ which touches or coincides with $Q_j$, roughly speaking, $y \sim Q_j$ means that $y$ is not much far away from $Q_j$ than $\mathrm{diam}(Q_j)$, otherwise we say $y \nsim Q_j$. Then, for any $x \in \mathbb{R}^n$, we have $U_\alpha^*(b)(x) \le N_1(x) + N_2(x)$, where

$$N_1(x) := \left[\int_0^\infty \int_{\mathbb{R}^n} \left|\sum_{\{j:\ y\nsim Q_j\}} \Psi_t * f_j(y)\right|^2 \chi(t^{-1}|y-x|)t^{-n}\,dy\,\frac{dt}{t}\right]^{1/2},$$

$$N_2(x) := \left[\int_0^\infty \int_{\mathbb{R}^n} \left|\sum_{\{j:\ y\sim Q_j\}} \Psi_t * f_j(y)\right|^2 \chi(t^{-1}|y-x|)t^{-n}\,dy\,\frac{dt}{t}\right]^{1/2},$$

$\Psi$ is as in (2.4), $f_j$ is as in Lemma 4.1(ii) and $\chi$ denotes the characteristic function of the unit ball $B(\vec{0},1)$.

We first estimate $N_1$. Notice that, for $\Phi$ satisfying Assumption 1.2 and $\Psi$ being as in (2.4),

$$|\Psi(x)| \lesssim |x|^{-n+\alpha}(1+|x|)^{-2} =: L(x), \quad \forall\, x \in \mathbb{R}^n. \tag{4.4}$$

Indeed, we know that, when $|x| < 3t_0$,

$$\inf_{x\in B(\vec{0},3t_0)} |x|^{-n+\alpha}(1+|x|)^{-2} = |3t_0|^{-n+\alpha}(1+|3t_0|)^{-2} \sim 1.$$

On the other hand, since $\Phi$ is bounded, it follows that, for any $x \in \mathbb{R}^n$,

$$\begin{aligned} |\Phi * I_\alpha(x)| &\le \int_{\mathbb{R}^n} |\Phi(x-z)| I_\alpha(z)\,dz \lesssim \int_{B(x,t_0)} |z|^{-n+\alpha}\,dz \\ &\lesssim \int_{B(\vec{0},|x|+t_0)} |z|^{-n+\alpha}\,dz \lesssim (|x|+t_0)^\alpha \lesssim t_0^\alpha \sim 1. \end{aligned} \tag{4.5}$$

Combining these estimates, we find that, when $|x| < 3t_0$,

$$|\Phi * I_\alpha(x)| \lesssim 1 \lesssim |x|^{-n+\alpha}(1+|x|)^{-2}.$$

When $|x| > 3t_0$, since $\Phi$ is radial, from the Taylor expansion, we deduce that, for any $x \in \mathbb{R}^n$,

$$\begin{aligned} |\Phi * I_\alpha(x) - I_\alpha(x)| &= \left| \int_{B(\vec{0},t_0)} \Phi(z)[I_\alpha(x-z) - I_\alpha(x)]\,dz \right| \\ &= \left| \int_{B(\vec{0},t_0)} \Phi(z) \left[ \nabla I_\alpha(x)\cdot(-z) + \sum_{|\beta|=2} \frac{D^\beta I_\alpha(x-\theta z)}{\beta!}(-z)^\beta \right] dz \right| \\ &= \left| \sum_{|\beta|=2} \frac{1}{\beta!} \int_{B(\vec{0},t_0)} \Phi(z) D^\beta I_\alpha(x-\theta z)(-z)^\beta\,dz \right| \\ &\lesssim \int_{B(\vec{0},t_0)} |x-\theta z|^{\alpha-n-2}\,dz \lesssim |x|^{-n+\alpha-2} \lesssim |x|^{-n+\alpha}(1+|x|)^{-2}, \end{aligned}$$

where $\theta \in (0,1)$. This proves (4.4).

Notice that, if $y \nsim Q_j$, then $\sup_{z\in Q_j} |y-z| \lesssim \inf_{z\in Q_j} |y-z|$, which implies that

$$\sup_{z\in Q_j} L_t(y-z) = t^{-\alpha} \sup_{z\in Q_j} |y-z|^{-n+\alpha} \left(1+\frac{|y-z|}{t}\right)^{-2} \lesssim \inf_{z\in Q_j} L_t(y-z), \tag{4.6}$$

where $L$ is as in (4.4). Therefore, by $\operatorname{supp} f_j \subset Q_j$, the Hölder inequality, and Lemma 4.1(ii), we know that

$$\left| \sum_{\{j:\ y\nsim Q_j\}} \Psi_t * f_j(y) \right| \lesssim \sum_{\{j:\ y\nsim Q_j\}} \left[ \sup_{z\in Q_j} L_t(y-z) \right] \int_{\mathbb{R}^n} |f_j(z)|\,dz$$

$$\lesssim \sum_{\{j:\ y\nsim Q_j\}} \left[\inf_{z\in Q_j} L_t(y-z)\right] \beta|Q_j| \lesssim \beta \int_{\mathbb{R}^n} L(z)\,dz \lesssim \beta.$$

Thus, for any $x \in \mathbb{R}^n$, we have

$$[N_1(x)]^2 \lesssim \beta \int_0^\infty \int_{\mathbb{R}^n} \left| \sum_{\{j:\ y\nsim Q_j\}} \Psi_t * f_j(y) \right| \chi(t^{-1}|y-x|)t^{-n}\,dy\,\frac{dt}{t}$$

and hence

$$\int_{\mathbb{R}^n} [N_1(x)]^2\,dx \lesssim \beta \int_0^\infty \int_{\mathbb{R}^n} \left| \sum_{\{j:\ y\nsim Q_j\}} \Psi_t * f_j(y) \right| dy\,\frac{dt}{t}. \tag{4.7}$$

Let $z_j$ be the center of $Q_j$. Using Lemma 4.4 and the Hölder inequality, we have

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}^n} \left| \sum_{\{j:\ y\nsim Q_j\}} \Psi_t * f_j(y) \right| dy\,\frac{dt}{t} \\
&\le \sum_j \int_0^\infty \int_{y\nsim Q_j} |\Psi_t * f_j(y)|\,dy\,\frac{dt}{t} \\
&\le \sum_j \int_0^\infty \int_{y\nsim Q_j} \left| \int_{Q_j} [\Psi_t(y-z) - \Psi_t(y-z_j)]\, f_j(z)\,dz \right| dy\,\frac{dt}{t} \\
&\le \sum_j \int_{Q_j} \left[ \int_0^\infty \int_{y\nsim Q_j} |\Psi_t(y-z) - \Psi_t(y-z_j)|\,dy\,\frac{dt}{t} \right] |f_j(z)|\,dz \\
&\lesssim \sum_j \int_{Q_j} |f_j(z)|\,dz \lesssim \sum_j \left( \int_{Q_j} |f_j(z)|^p\,dz \right)^{\frac{1}{p}} |Q_j|^{1-\frac{1}{p}} \\
&\lesssim \sum_j \beta|Q_j| \lesssim \beta^{1-p}\|f\|_{L^p(\mathbb{R}^n)}^p,
\end{aligned}$$

where the penultimate and the last inequalities follow from (ii) and (i) of Lemma 4.1, respectively. By this and (4.7), we conclude that $\|N_1\|_{L^2(\mathbb{R}^n)}^2 \lesssim \beta^{2-p}\|f\|_{L^p(\mathbb{R}^n)}^p$. Therefore, for any $\beta \in (0,\infty)$, we have

$$|\{x \in \mathbb{R}^n :\ N_1(x) > \beta\}| \le \beta^{-2}\|N_1\|_{L^2(\mathbb{R}^n)}^2 \lesssim \beta^{-p}\|f\|_{L^p(\mathbb{R}^n)}^p. \tag{4.8}$$

Next we estimate $N_2$. Let $\Omega := \cup_j Q_j$. Then, since $|\Omega| \lesssim \beta^{-p}\|f\|_{L^p(\mathbb{R}^n)}^p$, to prove estimate (4.8) with $N_2$ in place of $N_1$, it suffices to show that, for any $\beta \in (0,\infty)$,

$$\int_{\mathbb{R}^n\setminus\Omega} [N_2(x)]^2\,dx \lesssim \beta^{2-p}\|f\|_{L^p(\mathbb{R}^n)}^p. \tag{4.9}$$

Notice that

$$[N_2(x)]^2 = \int_0^\infty \int_\Omega \left| \sum_{\{j:\ y\sim Q_j\}} \Psi_t * f_j(y) \right|^2 \chi(t^{-1}|y-x|)t^{-n}\,dy\,\frac{dt}{t},$$

because $y \nsim Q_j$ for any $j$ if $y \in \mathbb{R}^n \setminus \Omega$. Also, we find that, for any $y \in \mathbb{R}^n$,

$$\Psi_t * f(y) = t^{-\alpha}[f * I_\alpha * \Phi_t(y) - f * I_\alpha(y)]$$

and, if $y \in Q_j$, then

$$\sum_{\{m:\ y \sim Q_m\}} f_m = \sum_{\{m:\ z_j \sim Q_m\}} f_m,$$

where $z_j$ denotes the center of $Q_j$. Put $f^j := \sum_{\{m: z_j \sim Q_m\}} f_m$. Then, if $x \in \mathbb{R}^n \setminus \Omega$, we have

$$\begin{aligned}
[N_2(x)]^2 &= \sum_j \int_0^\infty \int_{Q_j} |\Phi_t * I_\alpha * f^j(y) - I_\alpha * f^j(y)|^2 \chi(t^{-1}|y-x|) t^{-n-2\alpha}\, dy\, \frac{dt}{t} \\
&\lesssim \sum_j \int_0^\infty \int_{Q_j} |M(I_\alpha * f^j)(y)|^2 \chi(t^{-1}|y-x|) t^{-n-2\alpha}\, dy\, \frac{dt}{t} \\
&\sim \sum_j \int_{Q_j} |M(I_\alpha * f^j)(y)|^2 |x-y|^{-n-2\alpha}\, dy \\
&\lesssim \sum_j |x-z_j|^{-n-2\alpha} \|M(I_\alpha * f^j)\|^2_{L^2(\mathbb{R}^n)} \\
&\lesssim \sum_j |x-z_j|^{-n-2\alpha} \|I_\alpha * f^j\|^2_{L^2(\mathbb{R}^n)},
\end{aligned} \tag{4.10}$$

where $M$ denotes the Hardy-Littlewood maximal operator as in (1.19). Notice that

$$\|I_\alpha * f^j\|^2_{L^2(\mathbb{R}^n)} \lesssim \|f^j\|^2_{L^p(\mathbb{R}^n)} \lesssim \beta^2 \sum_{\{m:\ z_j \sim Q_m\}} |Q_m|^{2/p} \lesssim \beta^2 |Q_j|^{2/p},$$

where the last inequality follows from the geometry of the Whitney decomposition (see part (v) of Lemma 4.1). Then, by (4.10), we know that

$$\begin{aligned}
\int_{\mathbb{R}^n \setminus \Omega} [N_2(x)]^2\, dx &\lesssim \beta^2 \sum_j |Q_j|^{2/p} \int_{\mathbb{R}^n \setminus \Omega} |x-z_j|^{-n-2\alpha}\, dx \\
&\lesssim \beta^2 \sum_j |Q_j|^{2/p} \int_{\mathbb{R}^n \setminus Q_j} |x-z_j|^{-n-2\alpha}\, dx \\
&\lesssim \beta^2 \sum_j |Q_j|^{2/p} |Q_j|^{-2\alpha/n} \lesssim \beta^2 \sum_j |Q_j| \lesssim \beta^{2-p} \|f\|^p_{L^p(\mathbb{R}^n)},
\end{aligned}$$

which proves (4.9). Combining all above arguments, we prove that $U^*_\alpha$ is bounded from $L^p(\mathbb{R}^n)$ to $WL^p(\mathbb{R}^n)$.

Now we consider the operator $\widetilde{G}^*_{\alpha,\lambda} \circ I_\alpha$. The proof is similar to the above proof for $U^*_\alpha$. Indeed, for any $x \in \mathbb{R}^n$, let

$$\widetilde{N}_1(x) := \left\{ \int_0^\infty \int_{\mathbb{R}^n} \left| \sum_{\{j:\, y \nsim Q_j\}} \Psi_t * f_j(y) \right|^2 \left( \frac{t}{t+|x-y|} \right)^{\lambda n} t^{-n}\, dy\, \frac{dt}{t} \right\}^{1/2}$$

and

$$\widetilde{N}_2(x) := \left\{ \int_0^\infty \int_{\mathbb{R}^n} \left| \sum_{\{j:\, y\sim Q_j\}} \Psi_t * f_j(y) \right|^2 \left( \frac{t}{t+|x-y|} \right)^{\lambda n} t^{-n}\, dy\, \frac{dt}{t} \right\}^{1/2}.$$

The estimation of $\widetilde{N}_1$ is similar to that of $N_1$ due to the fact that

$$\int_{\mathbb{R}^n} \left( \frac{t}{t+|x-y|} \right)^{\lambda n} dx < \infty;$$

while the estimation of $\widetilde{N}_2$ is similar to that of $N_2$ because, if $\lambda > \frac{2}{p}$, then, for any $x$, $y \in \mathbb{R}^n$ with $x \neq y$,

$$\begin{aligned}
&\int_0^\infty \left( \frac{t}{t+|x-y|} \right)^{\lambda n} t^{-n-2\alpha-1}\, dt \\
&\quad = \int_0^{|x-y|} \left( \frac{t}{t+|x-y|} \right)^{\lambda n} t^{-n-2\alpha-1}\, dt + \int_{|x-y|}^\infty \left( \frac{t}{t+|x-y|} \right)^{\lambda n} t^{-n-2\alpha-1}\, dt \\
&\quad \le \int_0^{|x-y|} \frac{|x-y|^{\lambda n-n-2\alpha-1}}{(t+|x-y|)^{\lambda n}}\, dt + \int_{|x-y|}^\infty \frac{t^{\lambda n-n-2\alpha-1}}{(2|x-y|)^{\lambda n}}\, dt \lesssim |x-y|^{-n-2\alpha}.
\end{aligned}$$

With these estimates and repeating the argument used for the estimation of $U_\alpha^*$, we conclude that $\widetilde{G}^*_{\alpha,\lambda} \circ I_\alpha$ is also bounded from $L^p(\mathbb{R}^n)$ to $WL^p(\mathbb{R}^n)$. This proves Theorem 1.5(i).

Now we consider the case $\alpha = 2$, namely, we prove Theorem 1.5(ii). To show

$$\widetilde{U}(f, C_0\Delta f) \in WL^p(\mathbb{R}^n) \quad \text{with} \quad p = \frac{2n}{n+4},$$

it suffices to show that $V^*$ in (3.16) is of weak type $(p,p)$. This proof is also similar to the above proof of $U_\alpha^*$, and the differences lie in (4.4), (4.6) and the estimation of $N_2$. Indeed, letting $K$ be as in Theorems 1.3 and 1.4, similarly to (4.4), when $\alpha = 2$, for any $x \in \mathbb{R}^n$, we have

$$|K(x)| \lesssim L(x),$$

where, for any $x \in \mathbb{R}^n$,

$$L(x) := |x|^{-n+2}(1+|x|^{\frac{3}{2}})^{-2}. \tag{4.11}$$

Indeed, when $|x| > 3t_0$, by the Taylor expansion and an argument similar to that used for the proof of (3.12), we know that

$$\begin{aligned}
&|\Phi * I_2(x) - I_2(x)| \\
&\quad = \left| \int_{B(\vec{0},t_0)} \Phi(z)[I_2(x-z) - I_2(x)]\, dz \right| \\
&\quad = \left| \int_{B(\vec{0},t_0)} \Phi(z) \left[ \nabla I_2(x)\cdot(-z) + \sum_{|\beta|=2} \frac{D^\beta I_2(x)}{\beta!}(-z)^\beta + \sum_{|\beta|=3} \frac{D^\beta I_2(x-\theta z)}{\beta!}(-z)^\beta \right] dz \right|
\end{aligned}$$

$$\lesssim |x|^{-n-1} \lesssim L(x),$$

where $\theta \in (0,1)$. When $|x| < 3t_0$, similarly to (4.5), we obtain

$$|K(x)| \lesssim |x|^{-n+2} \lesssim L(x).$$

Hence, (4.6) holds true for $L$ as in (4.11). To obtain an estimate analog to $N_2$, we employ the fact that, for any $t \in (0,\infty)$ and $y \in \mathbb{R}^n$,

$$|t^2\Phi_t * f^j(y)| = \left|\int_{\mathbb{R}^n} t^{2-n}\Phi\big(\frac{y-z}{t}\big) f^j(z)\,dz\right| \lesssim \int_{\mathbb{R}^n} |y-z|^{2-n}|f^j(z)|\,dz \sim |I_2 * |f^j|(y)|,$$

where the second inequality is deduced from the facts that $\Phi$ is bounded and $n > 4$ (due to $p := \frac{2n}{n+4} > 1$ in the assumption). The remainder of the proof for the case $\alpha = 2$ is similar to the proof for the case $\alpha \in (0,2)$, the details being omitted.

Finally, the proof of $\widetilde{G}^*_\lambda(f, C_0\Delta f) \in WL^p(\mathbb{R}^n)$ can also be proved like $\widetilde{G}^*_{\alpha,\lambda} \circ I_\alpha$, via replacing $\Psi_t$ therein by $K_t$, the details being omitted again. This finishes the proof of Theorem 1.5. □

**Acknowledgements.** Fan Wang would like to express his deep thanks to Ziyi He for some helpful discussions on this article.

## References

[1] R. Alabern, J. Mateu and J. Verdera, A new characterization of Sobolev spaces on $\mathbb{R}^n$, Math. Ann. 354 (2012), 589-626.

[2] F. Dai, A. Gogatishvili, D. Yang and W. Yuan, Characterizations of Sobolev spaces via averages on balls, Nonlinear Anal. 128 (2015), 86-99.

[3] F. Dai, A. Gogatishvili, D. Yang and W. Yuan, Characterizations of Besov and Triebel-Lizorkin spaces via averages on balls, J. Math. Anal. Appl. 433 (2016), 1350-1368.

[4] F. Dai, J. Liu, D. Yang and W. Yuan, Littlewood-Paley characterizations of fractional Sobolev spaces via averages on balls, Proc. Roy. Soc. Edinburgh Sect. A. (to appear) or arXiv: 1511.07598.

[5] E. Di Nezza, G. Palatucci and E. Valdinoci, Hitchhiker's guide to the fractional Sobolev spaces, Bull. Sci. Math. 136 (2012), 521-573.

[6] J. Duoandikoetxea, Fourier Analysis, Graduate Studies in Mathematics 29, American Mathematical Sociaty Providence, Rhode Island, 2000.

[7] C. Fefferman and E. M. Stein, Some maximal inequalities, Amer. J. Math. 93 (1971), 107-115.

[8] C. Fefferman, Inequalities for strongly singular convolution operators, Acta Math. 124 (1970), 9-36.

[9] J. García-Cuerva and J. L. Rubio de Francia, Weighted Norm Inequalities and Related Topics, North-Holland Mathematics Studies 116, Notas de Matemática [Mathematics Notes] 104, North-Holland Publishing Co., Amsterdam, 1985.

[10] L. Grafakos, Classical Fourier Analysis, Third edition, Graduate Texts in Mathematics 249, Springer, New York, 2014.
[11] L. Grafakos, Modern Fourier Analysis, Third edition, Graduate Texts in Mathematics 250, Springer, New York, 2014.
[12] P. Hajłasz, Sobolev spaces on an arbitrary metric space, Potential Anal. 5 (1996), 403-415.
[13] Z. He, D. Yang and W. Yuan, Littlewood-Paley charaterizations of second-order Sobolev spaces via averages on balls, Canadian Math. Bull. 59 (2016), 104-118.
[14] J. Heinonen, P. Koskela, N. Shanmugalingam and J. Tyson, Sobolev Spaces On Metric Measure Spaces: An Approach Based On Upper Gradients, Cambridge Studies in Advanced Mathematics 44, Cambridge University Press, Cambridge, 2015
[15] J. Hu, A note on Hajłasz-Sobolev spaces on fractals, J. Math. Anal. Appl. 280 (2003), 91-101.
[16] S. Sato, Littlewood-Paley operators and Sobolev spaces, Illinois J. Math. 58 (2014), 1025-1039
[17] N. Shanmugalingam, Newtonian spaces: an extension of Sobolev spaces to metric measure spaces, Rev. Mat. Iberoam. 16 (2000), 243-279.
[18] H. Triebel, Theory of Function Spaces, Birkhäuser Verlag, Basel, 1983.
[19] T. Ullrich, Continuous characterizaations of Besov-Lizorkin-Triebel spaces and new interpretation as coorbits, J. Funct. Spaces Appl. 2012, Art. ID 163213, 47 pp.
[20] D. Yang, New characterizations of Hajłasz-Sobolev spaces on metric spaces, Sci. China Ser. A 46 (2003), 675-689.

Shuichi Sato

Department of Mathematics, Faculty of Education, Kanazawa University, Kanazawa 920-1192, Japan

*E-mail*: `shuichi@kenroku.kanazawa-u.ac.jp`

Fan Wang, Dachun Yang (Corresponding author) and Wen Yuan

School of Mathematical Sciences, Beijing Normal University, Laboratory of Mathematics and Complex Systems, Ministry of Education, Beijing 100875, People's Republic of China

*E-mails:* `fanwang@mail.bnu.edu.cn` (F. Wang)
`dcyang@bnu.edu.cn` (D. Yang)
`wenyuan@bnu.edu.cn` (W. Yuan)