Simulation of hyperbolic mean curvature flow with an obstacle

Dissertation Abstract

Simulation of hyperbolic mean curvature flow with an obstacle

障害物を伴う双曲型平均曲率流の数値解析

Graduate School of
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Abstract

We treat an interface motion with an obstacle according to the hyperbolic mean curvature flow. In order to realize this motion, we follow the approximation method that is the so-called Hyperbolic MBO (HMBO) algorithm. We modify the scheme to treat the obstacle problem. The HMBO algorithm requires us to solve the wave equation. Therefore, we modify the wave equation based on the hyperbolic obstacle problem.

Moreover, we investigate the behaviour of the interface when it hits the obstacle. We consider two cases of the interface motion based on the choice of the initial curve. In the first case, we consider that the initial curve is a closed curve and the obstacle is located inside the curve. The interface stops moving and lies on the obstacle after touching it, following the shape of the obstacle. For the second case, the initial curve is fixed at the boundary of the domain and the obstacle is below the curve. After touching the obstacle, the interface reflects and vibrates above the obstacle. We plot the points when the interface contacts with the obstacle at every time. We call it the free boundary shape. The slope of the free boundary shape approaches the free boundary condition.

1 Introduction

In this study, we treat the interface motion with an obstacle. We consider the interfacial motion problem which is the so-called hyperbolic mean curvature flow [2]. We suppose that the interfaces are given by a parametrized family of curves $\gamma : I \times [0, T) \to \mathbb{R}^2$, where $I = [a, b]$ and $T > 0$. Let us consider the region $P \subset \mathbb{R}^2$ which the boundary of P coincides with the interface. Also, the interface is considered as an oriented curve, such that the region P is on the left side of the curve. Then, the hyperbolic mean curvature flow is the problem to find $\gamma(s, t)$ satisfying

$$\frac{\partial^2 \gamma}{\partial t^2}(s, t) = -\kappa(s, t)\boldsymbol{n}(s, t), \quad \gamma(s, 0) = \gamma_0(s), \quad \frac{\partial \gamma}{\partial t}(s, 0) = v_0(s)\boldsymbol{n}_0(s), \quad (1)$$

with $s \in I$ and $t \in [0, T)$. Here, $\kappa(s, t)$ is the curvature, $\boldsymbol{n}(s, t)$ is the unit outer normal vector to the curve γ at a point (s, t) and the unit outer normal vector at $t = 0$ is denoted by $\boldsymbol{n}_0(s)$. This geometric evolution says that the normal acceleration of the interface is proportional to its curvature [3] and given by

$$a = -\kappa. \quad (2)$$

To realize this interface motion, we follow the approximation method introduced by Ginder and Svadlenka [1, 2] that is the so-called Hyperbolic MBO (HMBO) algorithm. In those papers, the authors construct the numerical scheme for computation the problem according to (1). In this work, we focus in the obstacle problem when the interface touches the obstacle. The aims of this work are to modify the scheme for the obstacle problem using the HMBO algorithm and investigate the behaviour of the interface when it touches the obstacle.

2 Interface model with an obstacle

We assume that the obstacle $O \subset \mathbb{R}^2$ is a convex open set. As we mentioned before, the interface is expressed by a parametrized curve $\gamma(s, t)$. Here, we suppose that γ is a Lipschitz function and $\gamma \in C^2(\{\gamma \in \bar{O}^c\})$. According to (1), we formally solve the following obstacle problem

$$\begin{cases} \dfrac{\partial^2 \gamma}{\partial t^2} = -\kappa \boldsymbol{n} & \text{if } \gamma \in \bar{O}^c, \\ \dfrac{\partial \gamma}{\partial t} \cdot \nu_{\partial O} \leq 0 & \text{if } \gamma \in \bar{O}, \\ \gamma \in O^c, & \\ \gamma(s,0) = \gamma_0(s), & \\ \dfrac{\partial \gamma}{\partial t}(s,0) = v_0(s)\boldsymbol{n}_0(s), & \end{cases} \tag{3}$$

with $\nu_{\partial O}$ is the unit outer normal vector to the obstacle. To simulate the interface motion with an obstacle, we apply the HMBO algorithm using the level set method.

3 Numerical method

3.1 The original HMBO algorithm

In this case, we consider that the interface will evolve up to a time T. We take the time step be $\Delta t = \frac{T}{M}$, where M is a positive integer and $0 < \Delta t \ll 1$. Then, the approximation method is as follows:

1. For $k = 0$, we assume that the initial curve is Γ_0. Then, construct the signed distance function to Γ_0 to be $d_0(x)$. We find $u : \Omega \times (0, \Delta t) \to \mathbb{R}, \Omega \subset \mathbb{R}^2$, satisfying

$$\begin{cases} u_{tt}(x,t) = \Delta u(x,t) & \text{in } \Omega \times (0, \Delta t), \\ u(x,t) = d_0(x) & \text{on } \partial\Omega \times (0, \Delta t), \\ u(x,0) = d_0(x) & \text{in } \Omega, \\ u_t(x,0) = 0 & \text{in } \Omega. \end{cases} \tag{4}$$

   For simplicity, we restrict the initial velocity $u_t(x,0) = 0$. Then, we define the zero level set of $u(x, \Delta t)$ as Γ_1 and compute the signed distance function to Γ_1.

2. For $k = 1, 2, \ldots, M-1$, repeat the following steps

(a) Solve the equation

$$\begin{cases} u_{tt}(x,t) = 2\Delta u(x,t) & \text{in } \Omega \times (0,\Delta t), \\ u(x,t) = d_k(x) & \text{on } \partial\Omega \times (0,\Delta t), \\ u(x,0) = 2d_k(x) - d_{k-1}(x) & \text{in } \Omega, \\ u_t(x,0) = 0 & \text{in } \Omega. \end{cases} \tag{5}$$

(b) Update the surrounded region and the interface using the zero level set of the solution to (5):

$$E_{k+1} = \{x \in \Omega | u(x,\Delta t) > 0\}$$
$$\Gamma_{k+1} = \partial E_{k+1}.$$

(c) Compute the signed distance function to Γ_{k+1}.

We apply the finite difference approximation to solve the wave equation in the HMBO algorithm. Hence, u_{tt} and Δu are approximated by the central difference. Here, the domain Ω is a subset of $\mathbb{R}^2$, $\Omega = (a,b) \times (a,b)$. Each (a,b) is divided into N equal intervals, that is $h = \Delta x = \Delta y = \frac{b-a}{N}$. So, we have $x_i = a + ih, y_j = a + jh$, for $i,j = 0,\dots,N$. Also, the time $(0,\Delta t)$ is divided into m equal intervals, that is $\Delta\tau = \frac{\Delta t}{m}$ and $\tau_l = l\Delta\tau, l = 0,\dots,m$.

3.2 The modification of the HMBO algorithm for the obstacle problem

3.2.1 The hyperbolic obstacle problem

In [5], the author describes the hyperbolic obstacle problem as the string vibration with an obstacle. Here, the shape of the string is described by the graph of a scalar function $v : \hat{\Omega} \times [0,T) \equiv \hat{\Omega}_T \to \mathbb{R}$, where $\hat{\Omega} \subset \mathbb{R}^n$ and the obstacle is the graph of a fixed function $\psi : \hat{\Omega} \to \mathbb{R}$. In [5], the author considered that ψ is the zero function. When the energy conservation law holds, the stationary points of the following action functional describe the motion of the string: $J(v) = \int_0^T \int_{\hat{\Omega}} ((v_t)^2 - |\nabla v|^2)\chi_{\{v>\psi\}}\, dxdt$, where $\chi_{\{v>\psi\}}$ is the characteristic function of the set $\{(x,t) \in \hat{\Omega}_T | v(x,t) > \psi(x)\}$. By calculating the first variation, we get the weak formulation for the wave-type equation

$$v_{tt} = \Delta v \quad \text{in} \quad \hat{\Omega}_T \cap \{v > \psi\}. \tag{6}$$

From the inner variation, we obtain the free boundary condition [5]

$$|\nabla v|^2 - (v_t)^2 = 0 \quad \text{on} \quad \hat{\Omega}_T \cap \partial\{v > \psi\}. \tag{7}$$

From (6) and (7), we can derive the equation [4]

$$\chi_{\overline{\{v>\psi\}}} v_{tt} = \Delta v \quad \text{in} \quad \hat{\Omega}_T.$$

Hence, we introduce the problem as

$$\begin{cases} \chi_{\overline{\{v>\psi\}}} v_{tt} = \Delta v & \text{in } \hat{\Omega}_T, \\ v(x,0) = f_0(x) & \text{on } \hat{\Omega}, \\ v_t(x,0) = g_0(x) & \text{on } \hat{\Omega}, \\ v(x,t) = p(x,t) & \text{on } \partial\hat{\Omega}_T, \text{ with } p(x,0) = f_0(x) \text{ on } \partial\hat{\Omega}, \end{cases} \tag{8}$$

where the first equation is understood in the sense of distributions. When the string touches the obstacle, the solution v also satisfies

$$\begin{cases} v \geq \psi, \quad \Delta v \geq v_{tt} & \text{in } \hat{\Omega}_T, \\ \Delta v = v_{tt} & \text{on } \hat{\Omega}_T \cap \{v > \psi\} \end{cases}$$

in the sense of distributions.

We solve (8) using the finite difference approximation. Consider $\hat{\Omega} \subset \mathbb{R}$, $\hat{\Omega} = (a,b)$ is divided into N equal intervals, so we have $h = \frac{b-a}{N}$ and $x_i = a + ih, i = 0, \ldots, N$. For $t_k = k\Delta t, k = 0, \ldots, M$, we approximate v_{tt} and Δv by central difference. Also, the characteristic function is defined by [4]

$$\chi_{\{v>\psi\}}(x_i, t_k) = \begin{cases} 1 & \text{if } v_{i-1}^k > \psi_{i-1} \text{ or } v_i^k > \psi_i \text{ or } v_{i+1}^k > \psi_{i+1}, \\ 0 & \text{otherwise}, \end{cases}$$

where $v_i^k = v(x_i, t_k)$ and $\psi_i = \psi(x_i)$. Hence, we get the scheme

$$\begin{cases} v_i^{k+1} = 2v_i^k - v_i^{k-1} + \frac{\Delta t^2}{h^2}(v_{i+1}^k + v_{i-1}^k - 2v_i^k), & \text{if } \chi_{\{v>\psi\}}(x_i, t_k) = 1, \\ v_i^{k+1} = \psi_i, & \text{if } \chi_{\{v>\psi\}}(x_i, t_k) = 0, \end{cases} \tag{9}$$

for $k = 0, \ldots, M-1$ and $i = 1, \ldots, N-1$.

3.2.2 The HMBO algorithm for the obstacle problem

In the HMBO algorithm, the interface is expressed as the zero level set of a function $u : \Omega \times (0, \Delta t) \to \mathbb{R}$. Similarly, the obstacle is also represented by the zero level set of a fixed function. We define $w : \Omega \to \mathbb{R}$ such that $\{x \in \Omega | w(x) = 0\}$ is the obstacle. Let $\mu : \Omega \to \mathbb{R}$ be the signed distance function to the obstacle. To treat the obstacle problem, we follow the discretization of the hyperbolic obstacle problem given in scheme (9) for solving equations (4) and (5). Let $u^l_{i,j} = u(x_i, y_j, \tau_l)$ and $\mu_{i,j} = \mu(x_i, y_j)$. Then, we have the following scheme

$$\begin{cases} u^{l+1}_{i,j} = 2u^l_{i,j} - u^{l-1}_{i,j} + c^2 \frac{\Delta\tau^2}{h^2}(u^l_{i+1,j} + u^l_{i-1,j} + u^l_{i,j+1} + u^l_{i,j-1} - 4u^l_{i,j}), \\ \qquad \text{if } \chi_{\{u>\mu\}}(x_i, y_j, \tau_l) = 1, \\ u^{l+1}_{i,j} = \mu_{i,j}, \\ \qquad \text{if } \chi_{\{u>\mu\}}(x_i, y_j, \tau_l) = 0, \end{cases} \tag{10}$$

for $l = 0, \dots, m-1$ and $i, j = 1, \dots, N-1$. The constant $c^2 = 1$ for equation (4) and $c^2 = 2$ for equation (5). Here, we define

$$\chi_{\{u>\mu\}}(x_i, y_j, \tau_l) = \begin{cases} 1 & \text{if } u^l_{i,j} > \mu_{i,j}, \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

By implementing this scheme, we obtain

$$u(x,t) \geq \mu(x) \quad \text{for} \quad (x,t) \in \Omega \times (0,T).$$

However, scheme (10) is still developed. We will investigate the results using this scheme.

4 Numerical results

4.1 First case

For the numerical test, we consider a circle evolving by (1) with initial radius r_0 and initial velocity v_0. We give a fixed circle with a smaller radius as an obstacle. The circle will shrink before touching the obstacle and stop after touching it. Before touching the obstacle, the curve is the circle with radius $r(t)$ satisfying

$$r''(t) = -\frac{1}{r(t)}, \quad r(0) = r_0, \quad r'(0) = v_0. \tag{12}$$

In this part, we take the condition that is close to the case of a circle evolving by the hyperbolic mean curvature flow in [2], so we can compare both results. The error of the radius of the circle before touching the obstacle is obtained by the comparison between the result from the HMBO algorithm and the solution of (12) using Runge Kutta fourth order method. The L^2 error is

$$e = \sqrt{\Delta t \sum_{k=1}^{2^9} (r_r^k - r_n^k)^2},$$

where r_n is the maximum distance to the center from the HMBO algorithm result and r_r is the solution to (12). The error and the convergence order related to L^2 error are shown in the table below.

Table 1: Error and convergence order using the HMBO algorithm

N	e	convergence order
16	0.107927	-
32	0.0966846	0.159
64	0.0813308	0.249
128	0.0573378	0.504
256	0.0324356	0.822
512	0.0181595	0.837

From Table 1, as the mesh size decreases, the error value also decreases. Moreover, the L^2 error and its convergence order agree with the result of the circle case given in [2].

4.2 Second case

In this case, we consider the curve having small displacement, such that the interface motion by the hyperbolic mean curvature flow coincides with the wave equation. Therefore, we compare between the results of the HMBO algorithm for the obstacle problem and the hyperbolic obstacle problem using scheme (9). In this trial, the initial curves are given by a piece-wise linear function and a quadratic function with small displacement. Also, we set the position of an obstacle below the curve such that it is not too close to the curve but the curve motion can touch it.

4.2.1 Constant function

In this part, the obstacle is given by a constant function. The initial conditions and the obstacle for both schemes are described below. Here, $u(x, y, 0)$ and $w(x, y)$ are the initial value and the obstacle function for the HMBO algorithm, respectively. For scheme (9), the initial conditions are $v(x, 0)$ and $v_t(x, 0)$ with the boundary conditions are $v(-1, t) \equiv v(-1, 0)$ and $v(1, t) \equiv v(1, 0)$. Also, $\psi(x)$ represents the obstacle function. More precisely, we have

- Case 1 (Initial curve is a piece-wise linear function)
$$\left.\begin{array}{l} u(x,y,0) = -0.05|x| - y + 1.05 \\ w(x,y) = 0.975 - y \end{array}\right. \quad \text{on } \Omega,$$
$$\left.\begin{array}{l} v(x,0) = -0.05|x| + 1.05, \quad v_t(x,0) = 0 \\ \psi(x) = 0.975 \end{array}\right. \quad \text{on } \hat{\Omega}.$$
- Case 2 (Initial curve is a quadratic function)
$$\left.\begin{array}{l} u(x,y,0) = -0.05x^2 - y + 1.05 \\ w(x,y) = 0.975 - y \end{array}\right. \quad \text{on } \Omega,$$
$$\left.\begin{array}{l} v(x,0) = -0.05x^2 + 1.05, \quad v_t(x,0) = 0 \\ \psi(x) = 0.975 \end{array}\right. \quad \text{on } \hat{\Omega}.$$

From the simulation, the curve reflects after touching the obstacle and vibrates above the obstacle. We plot the points when the curve contacts with the obstacle at every time step. To get the contact points with the obstacle, we find the end points in both sides when the curve is close enough to the obstacle. Furthermore, we plot the contact points every time step for each mesh size using both schemes. These graphs are called the free boundary shape [5].

Moreover, we find the slope of the free boundary shape when the curve is going up for both sides. This slope represents the free boundary condition (7). According to this free boundary condition, we expect that the slope of the free boundary shape should be ± 1. The slopes of the free boundary shape using both schemes for each mesh size are given in the following tables.

By the curve motion, we consider that t^* denote the time when the curve starts going up.

Table 2: The slope of the free boundary shape for Case 1

N	Slope			
	HBMO algorithm		scheme (9)	
	right and left sides	t^*	right and left sides	t^*
128	$\pm$ 0.76	1.119	$\pm$ 1.01	1.556
256	$\pm$ 0.91	1.338	$\pm$ 1.01	1.521
512	$\pm$ 0.93	1.425	$\pm$ 1	1.518
1024	$\pm$ 0.99	1.454	$\pm$ 1.01	1.493

Table 3: The slope of the free boundary shape for Case 2

N	Slope			
	HBMO algorithm		scheme (9)	
	right and left sides	t^*	right and left sides	t^*
128	$\pm$ 0.75	1.167	$\pm$ 1	1.628
256	$\pm$ 0.9	1.392	$\pm$ 0.99	1.628
512	$\pm$ 0.93	1.478	$\pm$ 0.99	1.61
1024	$\pm$ 0.95	1.509	$\pm$ 0.99	1.606

4.2.2 Linear function

We simulate the curve motion with the obstacle given by a linear function using the HMBO algorithm and scheme (9). We take

- Case 3 (Initial curve is a piece-wise linear function)

$$\left.\begin{array}{l} u(x,y,0) = -0.05|x| - y + 1.05 \\ w(x,y) = 0.015x - y + 0.975 \end{array}\right. \quad \text{on } \Omega,$$

$$\left.\begin{array}{l} v(x,0) = -0.05|x| + 1.05, \quad v_t(x,0) = 0 \\ \psi(x,y) = 0.015x + 0.975 \end{array}\right. \quad \text{on } \hat{\Omega}.$$

We find the slope of the free boundary shape when the curve is going up. The tables below represent the slopes of the free boundary shape for both schemes.

Table 4: The slope of the free boundary shape for Case 3

N	Slope					
	HBMO algorithm			scheme (9)		
	right side	left side	t^*	right side	left side	t^*
128	-0.66	0.8	1.187	-1	1	1.652
256	-0.85	0.91	1.427	-1.01	1	1.65
512	-0.92	0.93	1.524	-0.99	1	1.621
1024	-0.94	0.94	1.557	-1	1	1.619

From all cases, the slopes of the free boundary shape obtained by using the HMBO algorithm and scheme (9) coincide as the mesh size becomes smaller. It means that the curve motion using both schemes gives similar results. Also, the slope of the free boundary shape approaches the free boundary condition.

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学位論文審査報告書（甲）

１．学位論文題目（外国語の場合は和訳を付けること。）

Simulation of hyperbolic mean curvature flow with an obstacle

（障害物を伴う双曲型平均曲率流の数値解析）

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３．審査結果の要旨（600～650 字）

Vita さんは、２０１４年１０月にインドネシア政府奨学生として自然科学研究科博士後期課程に入学した。入学以来双曲型自由境界問題の研究に取り組んできた。特に界面の平均曲率加速度運動に興味を持ち、この問題の数値解析的な研究に取り組んできた。界面の運動のうち平均曲率加速度流は Svadlenka-Ginder によって最近提案された問題である。従来から平均曲率流と呼ばれる問題が盛んに研究されてきている。これは、界面の運動速度がその平均曲率に比例するもので、解の特異点発生問題など様々な難問を突破して大きな分野として発展してきている。それに対して、加速度流は解の特異点や誤差などが伝播しうる波動型方程式を取り扱うことになり、格段に難しい問題となっている。

Vita さんは、これらの問題のうち、界面が障害物に制約される場合の数値計算法の開発を行った。彼女は、「界面が障害物に触れた場合ただちに静止する」という接触条件をもつモデルを導出し解析を行った。数値計算結果の検証のため、解析解が存在する場合についてのチェックを行うとともに、状況によっては類似の解を持つと考えられる双曲型自由境界問題の解とも比較している。この目標のため双曲型自由境界問題を解くためのアルゴリズムの開発も行っている。

Vita さんはこれらの結果を論文にまとめ、金沢大学サイエンスレポートに投稿して査読を受けている。本審査委員会は掲載が決定することを条件に、Vita さんの論文は博士（理学）の学位を授与するに相当であると判断した。

４．審査結果　(1) 判　定（いずれかに○印）　合　格　・　不合格

(2) 授与学位　博　士（　理　学　）