Dissertation

# Long-time behavior of the one-phase Stefan problem in periodic media and random media


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Date of Submission: March 8, 2018

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# Acknowledgement

First and foremost, I would like to express my sincere gratitude to my chief advisor, Professor Seiro Omata, for his constant support and guidance of my Ph.D. study. He has been giving me all the best opportunities to study and gain the knowledge through the time of doing research. I also wish to thank my other advisor and mentor, Assistant Professor Norbert Požár, for introducing me the topic of the one-phase Stefan problem as well as for his patience and encouragement in many invaluable discussions. I have the deepest respect and admiration for my both academic advisors.

I am also grateful to the Professors in Division of Mathematical and Physical Sciences, Graduate school of Natural Science and Technology, Kanazawa University, whose interesting lectures and talks that I had a chance to attend. I was especially impressed with the immense knowledge provided in the classes and seminars by Professor Masato Kimura and Professor Norihisa Ikoma. They all bring me a great academic environment and inspire me to the study of partial differential equations. Besides, I would like to thank all members in my laboratory for their friendly help during the time we worked together. I especially want to thank Yoshiho Akagawa, whom I bothered with many problems not only in studying but also in dealing with the procedures in Kanazawa University.

Also many thanks to my teachers in Department of Mathematics and Informatics, Hanoi National University of Education, who taught me from the beginning to love mathematics with its own beauty despite many difficulties and challenges.

I have received financial support from Vietnamese Government through the Ministry of Education and Training, VIED (Vietnam International Education Development) scholarship programs, project 911, for which I am very thankful.

I would like to acknowledge all my colleagues in Department of Mathematics, Faculty of Information Technology, Vietnam National University of Agriculture who supported me with my job when I left for studying.

Our life in Kanazawa would be much more difficult without the precious help of many Vietnamese and Japanese friends, to whom I would like to send warm thanks.

Last but not least, I would like to deeply thank my family for their support and sacrifice. My parents unconditionally love and assist me, my husband and my sons constantly understand and take the time for me to do my research. This thesis is dedicated to my beloved sons, Nguyen Tri Khang and Nguyen Tri Kien, with the hope that it encourages them to study in the future.

Kanazawa, March 8, 2018

# Chapter 1

# Introduction

## 1.1 Introduction

The mathematical theory of partial differential equations (PDEs) has a long history motivated by various physical phenomena such as sound, heat, fluid dynamics, elasticity, quantum mechanics, etc. However, in many recent developments, PDEs find their practical applications in other fields of science as well like quantum chemistry, chemical kinetics, biology, economics and financial mathematics, or computer science. Some PDEs also come from the pure mathematical problems of other branches. In a boundary-initial value problem, which is the classical subject in the theory of PDEs, the domain of the governing equation is fixed in space with specified data on the boundary and at the initial time. Such problems were well studied in both standard analytical and numerical solution techniques. The more recent trend in PDEs is to consider *free boundary problems* or *moving interface problems* with totally different features, namely, the space domains of the equations are separated by *free boundaries* which are neither fixed nor known a priori and need to be determined together with the solution.

Due to the difficulties from the unknown geometric information together with the nonlinear nature and the singularities of the moving boundaries, only some simplest free boundary problems have been showed to have classical solutions. It gave rise to the question of generalizing the notion of solutions and defining some kinds of *weak solutions*. The study of the regularity of the solution as well as the free boundary itself, once the unique weak solution exists, is also one of the most interesting topics in the field of free boundary problems. Beside the interests of the well-posedness of a PDE and the regularity of the solution, the homogenization problems for finding an average solution of equations with highly oscillating coefficients have received a lot of attention. Although the theory of homogenization was studied extensively for classical initial boundary value problems, there are still many open questions for homogenization of free boundary problems.

Among various types of free boundary problems, the Stefan problem is one of the most classical ones, which typically models the melting (the phase transition) of ice in contact with a water region due to heat conduction and an exchange of latent heat energy. This physical problem was formulated in a mathematical model by Slovene physicist and mathematician Josef Stefan (1835-1893) who treated the formation of ice in the polar seas (Stefan 1891), and was considered earlier by Lamé and Clapeyron (1831). The mathematical formulation of the problem consists of the heat equation in each phase, the solid and the liquid phase, and an additional condition at the free boundary, which is the so-called Stefan condition, that expresses the local velocity of a moving boundary. A very common simplification of the Stefan problem is the problem when we assume that the temperature is fixed at one of the phases (usually by assuming that the body of ice is maintained at temperature 0) and it is called the *one-phase Stefan problem*. The one-phase Stefan problem then contains only one heat equation in the liquid phase and a simpler form of the Stefan condition by eliminating one of the temperature gradients. The related *Hele-Shaw problem* is usually referred to in the literature as the quasi-stationary limit of the one-phase Stefan problem when the heat operator is replaced by the Laplace operator. This problem typically describes the flow of an injected viscous fluid between two parallel plates which form the so-called Hele-Shaw cell, or the flow in porous media.

The one-phase Stefan problem and Hele-Shaw problem in homogeneous media in dimension $n = 1$ have the explicit classical solution ( [49]). However, we cannot expect to have the classical solutions of both problems in dimension $n \geq 2$ due to the singularities on the free boundary which might develop in finite time. Thus there are several approaches to define a notion of solutions including the notion of weak solutions in the sense of distributions, the notion of variational inequalities solutions and the notion of viscosity solutions. We will use the notion of weak solutions based on the variational inequality formulation and the notion of viscosity solutions in our investigation. The well-posedness in a weak sense and regularity of these problems were studied in detail by many authors such as Friedman, Kinderlehrer, Rodrigues, Caffarelli, etc. (see [21, 51, 53, 7, 8, 34] and references therein). The problem is also well-posed in viscosity sense and the coincidence of two notions of solutions was obtained by Kim and Mellet [34, 38, 39]. Furthermore, the asymptotic behavior of solutions has gained some attentions in the literature. The asymptotic homogenization of the Hele-Shaw and the one-phase Stefan problem was given in [50, 38, 39]. The convergence of the Stefan problem to Hele-Shaw as $t \to \infty$ in homogeneous media was observed in [49]. Moreover, the long-time behavior of the related Hele-Shaw problem was studied in detail in [45].

In our recent work, we focus on the long-time behavior of the one-phase Stefan and Stefan-type problems in some inhomogeneous media in dimension $n \geq 2$. Using the technique of rescaling which is consistent with the evolution of the free boundary, we are able to show the homogenization of the free boundary velocity as well as the locally uniform convergence of the rescaled solution to a self-similar solution of the homogeneous Hele-Shaw problem with a point source for classical multi-dimensional one-phase Stefan problem. In the anisotropic case, when the heat operator is generalized by parabolic operators of divergence form, we also obtain the homogenization of the elliptic operator, where the rescaled solution now converges locally uniformly to a self-similar solution of the homogenized Hele-Shaw-type problem with a point source. Moreover, by viscosity solution methods, we furthermore deduce that the rescaled free boundary uniformly approaches that of the homogeneous Hele-Shaw problem with respect to the Hausdorff distance.

In this chapter, we state some notations for the convenient use later. In Chapter 2, we present some basic background of our problem, which motivated us to consider the long-time behavior of the solutions. In Chapter 3, we investigate the asymptotic behavior of the isotropic inhomogeneous one-phase Stefan problem for long times in periodic and random media. The main reference of Chapter 3 is a joint-work of the author with N. Požár [47]. The treatment of asymptotic long-time behavior of anisotropic inhomogeneous Stefan-type problems is presented in Chapter 4 with the main reference [48] which is another joint-work of the author with N. Požár. It turns out that when we replace our simple heat equation by more general parabolic operators of divergence form, the construction of barriers is more challenging. Thus in this case, we restrict our consideration to the problems in dimension $n \geq 3$ and in periodic media. Finally, Appendix A covers some basic techniques of the fundamental solution of a uniformly elliptic equation of divergence form used in our arguments.

## 1.2 Notation

We will use the following notations throughout this work.

For a set $A$, $A^c$ is its complement.

Given a nonnegative function $v$, we denote the positive set and free boundary of $v$ by

$$\Omega(v) := \{(x,t) : v(x,t) > 0\}, \quad \Gamma(v) := \partial\Omega(v),$$

and for fixed time $t$,

$$\Omega_t(v) := \{x : v(x,t) > 0\}, \quad \Gamma_t(v) := \partial\Omega_t(v).$$

$(f)_+$ is the positive part of $f$: $(f)_+ = \max(f, 0)$.

We will denote the general elliptic operator of divergence form and its rescaling as

$$\mathcal{L}u = D_i(a_{ij}D_ju), \quad \mathcal{L}^\lambda u = D_i(a_{ij}(\lambda^{1/n}x)D_ju),$$

where we have used the Einstein's summation convention.

# Chapter 2

# The one-phase Stefan problem and the Hele-Shaw problem

## 2.1 Mathematical modeling and classical formulation

### 2.1.1 The Stefan problem

As introduced above, the Stefan problem is a mathematical model, which typically describes the process of phase transitions, the melting or freezing, between solid (ice) and liquid (water) driven by the heat conduction and the exchange of latent heat. The problem has numerous applications in industry and technology such as the casting in manufacture of steel, the melting in thermal storage system, the evaporation of water, the drying of food, etc., see [1, 27, 58, 18, 59] . We begin with the most basic formulation of the Stefan problem to model the melting of a semi-infinite solid in contact with a liquid region containing a fixed source, for example a thin block of ice occupying the region $0 \leq x < \infty$ that melts due to the heating by a heat source at the fixed boundary $x = 0$ (see Fingure 2.1).

We assume that the temperature in the solid phase is a constant, say, the ice is maintained at temperature 0. At the fixed boundary, we prescribe a time dependent positive continuous boundary data $h(t)$. The moving phase-change boundary is described by $x = s(t)$. At time $t$, the liquid occupies the subset $\Omega(t) := \{x : 0 < x < s(t)\} \subset [0, \infty)$ and the free boundary is $\Gamma(t) := \{x = s(t)\}$. As time $t$ increases, $\Gamma(t)$

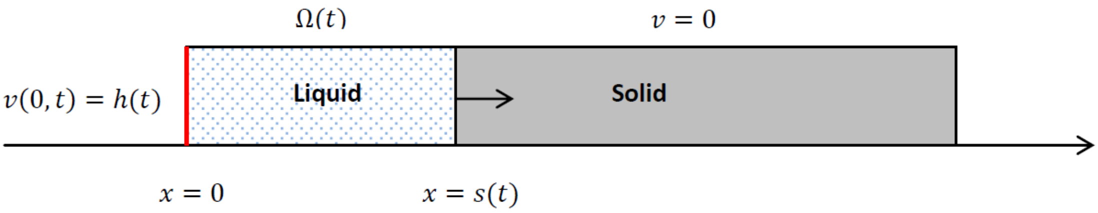


Figure 2.1: Semi-infinite slab melting from $x = 0$ due to a heat source $h(t)$ at $x = 0$

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travels from the left to the right and the liquid domain $\Omega(t)$ expands in the melting process. The classical formulation of this problem is the temperature distribution $v(x, t)$ in the liquid phase and the location of the free boundary $x = s(t)$. Even though there are two phases present, the problem is called a one-phase problem since the temperature is unknown only in the liquid phase.

1**D Stefan problem.** Find functions $x = s(t)$ and $v(x, t) : (0, \infty) \times [0, \infty) \to [0, \infty)$ satisfying

| | |
|---|---|
| **The liquid region** | $0 \leq x < s(t)$ |
| $\rho c v_t = k v_{xx}$, | The heat equation in $\Omega(t) \times (0, \infty)$, |
| $v(0, t) = h(t)$, | The fixed boundary data, $t > 0$, |
| $v(x, 0) = 0$, | The initial data, |
| **The free boundary** | $x = s(t)$ |
| $L\rho s'(t) = -k v_x(s(t), t)$, | The Stefan condition, |
| $s(0) = 0$, | The initial position of the interface, |
| $v(s(t), t) = 0$, | The continuity of temperature, |
| **The solid region**, | $s(t) < x < \infty$, |
| $v(x, t) = 0$, | The solid is maintained at temperature 0 for all $t > 0$. |

Here $\rho$ is the density, $c$ is the specific heat, $k$ is the thermal conductivity of the liquid and $L$ is the latent heat. In the liquid region, the temperature is governed by the standard heat diffusion. The Stefan condition is important to include the phase change to the model and can be understood by the energy conservation law as follows. Assume that the temperature depends only on the horizontal direction and let $A$ be a small portion of the interface at time $t = t_0$ having area $S$. At time $t_0$, the free boundary position is $s(t_0)$. As the solid melts, at time $t$, the boundary position is $s(t)$ and the portion $A$ has moved and formed a volume $V$. The energy we need to change the volume $V$ of solid into liquid from time $t_0$ to $t$ is

$$E_1 = L\rho V = L\rho S(s(t) - s(t_0)),$$

where $L$ is the latent heat, the energy required to change one unit mass of substance from solid to liquid. On the other hand, the energy delivered through the portion $A$ from time $t_0$ to $t$ can be computed as

$$E_2 = \int_{t_0}^{t} \int_A q \cdot \nu_{out} d\tau dS = S \int_{t_0}^{t} q \cdot \nu_{out} d\tau,$$

where $q$ is the heat flux density and $\nu_{out} = (1, 0, 0)$ is the unit outward normal vector. By Fourier's law of heat transfer, $q = -kDv$ where $k$ is a positive constant called the thermal conductivity of the liquid and $D$ is the gradient. Then putting it in $E_2$ we have

$$E_2 = S \int_{t_0}^{t} -kv_x(s(\tau), \tau) d\tau.$$

By the balance of energy, $E_1$ must equal $E_2$. Divide both sides by $(t - t_0)$ and take the limit as $t$ tends to $t_0$. With the help of the mean value theorem we have

$$L\rho s'(t_0) = -kv_x(s(t_0), t_0).$$

The phenomenon of solidification is formulated similarly. This problem can model the phenomenon in two or three dimensional space where $v$ is a function of two or three variables. The derivation of the Stefan condition on the interface is similar with noting that $V = SV_{\nu_{\mathrm{out}}}(t-t_0)$, here $V_{\nu_{\mathrm{out}}}$ is the outward normal velocity, $\nu_{\mathrm{out}}$ is the outward unit normal vector of the moving boundary and then we have

$$L\rho V_{\nu_{\mathrm{out}}} = -kDv \cdot \nu_{\mathrm{out}} \text{ on } \{x = s(t)\}.$$

Since the free boundary is a level set of $v$ then $\nu_{\mathrm{out}} = -\frac{Dv}{|Dv|}$ and $V_{\nu_{\mathrm{out}}} = \frac{v_t}{|Dv|}$ and we have an alternative form of the Stefan condition which is sometimes more useful for analytical treatment as

$$v_t = \frac{k}{\rho L} |Dv|^2.$$

Without loss of generality, we can assume that the constants are 1.

The mathematical problem is naturally generalized to an arbitrary dimension $n \geq 1$ and is still called the one-phase Stefan problem. Thus, the one-phase Stefan problem that we usually refer to is the following problem.

**The multi-dimensional one-phase Stefan problem.** Let $n \geq 1, K \subset \mathbb{R}^n$ be a compact set. The one-phase Stefan problem (on an exterior domain) is to find a function $v(x,t) : \mathbb{R}^n \times [0,\infty) \to [0,\infty)$ and a set $\{v > 0\}$ satisfying

$$\begin{cases} v_t - \Delta v = 0 & \text{in } \{(x,t) : v(x,t) > 0, x \in \mathbb{R}^n \backslash K\}, \\ v = h & \text{on } K, \\ v_t = |Dv|^2 & \text{on } \partial\{v > 0\}, \\ v(x,0) = v_0(x) & \text{on } \mathbb{R}^n, \end{cases} \tag{2.1}$$

where $v_0$ and $h = h(x,t)$ are given functions.

The one-phase Stefan problem can be generalized in many situations. First, if we assume that the temperature can vary in both phases, then we have the so-called two-phase Stefan problem. The derivation is analogous with an additional heat equation in the second phase and a little more complicated form of the Stefan condition. Since we only focus on the one-phase Stefan problem in our work, we will not introduce the two-phase problem and refer to [54, 53, 28] for more details. Moreover, if we assume that the constants in the model are now some nonnegative smooth functions, we can have some more complex models. For example, if we take $\rho = c = k = 1$ and $L = 1/g(x)$, where $g(x) > 0$, then we will have the Stefan condition of the form $v_t = g(x)|Dv|^2$. This problem is the one-phase Stefan problem with an inhomogeneous latent heat of phase transition, which is the subject for investigation of the next chapter. Furthermore, if in addition we assume that the heat diffuses in an anisotropic body, then the thermal conductivity coefficients vary through space and time, the heat flux vector $q$ is expressed as $q = -KDv$, with matrix $K = (k_{ij}(x,t))$. Then the heat equation in the positive set becomes a more general parabolic equation of divergence form $v_t - \operatorname{div}(KDv) = 0$ and the Stefan condition becomes $v_t = g(x)KDv \cdot Dv$. In Chapter 4, we will deal with this type of the one-phase Stefan problem with some more assumptions on the coefficients, say, $K$ is a symmetric, bounded, uniformly elliptic matrix, independent of time. Finally, if we consider the problem with zero specific heat, i.e., $c = 0$ then the heat operator simplifies into the Laplace operator and the problem is usually referred to the (one-phase) Hele-Shaw problem, which will be introduced in the next section.

### 2.1.2 The Hele-Shaw problem as the quasi-stationary limit of the Stefan problem

The classical Hele-Shaw problem is a two-dimensional mathematical model that typically describes the flow of an injected viscous fluid in the thin gap between two parallel plates, see Fig.2.2.

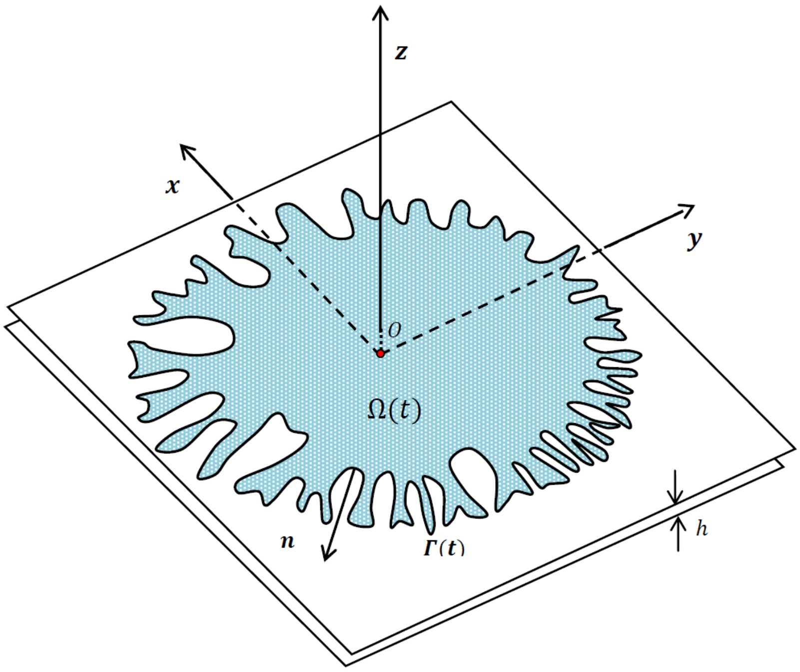


Figure 2.2: Hele-Shaw flow between two parallel flat plates separated by a small gap of width $h$ with the infinite pressure at the origin.

The governing equations of this problem are derived by gap-averaging the three-dimensional Navier-Stokes equations as in [40, 29]. We will sketch some of main features of this problem here. Let us consider the flow of a Newtonian, incompressible, inviscid fluid, driven by the singularity of a point source at the origin. Assume that at time $t$, the fluid occupies a domain $\Omega(t)$ in the $(x, y)$-plane with free boundary $\Gamma(t) := \partial\Omega(t)$. If the injected fluid is slow enough for the flow to be approximately stationary and the gap between two parallel plates $h$ is small enough, following [40, 29], the averaged velocity over the gap $\overline{\mathbf{v}} := \frac{1}{h}\int_0^h \mathbf{v}dz$ satisfies

$$\overline{\mathbf{v}} = -\frac{h^2}{12\mu}Dp \quad \text{in } \Omega(t) \tag{2.2}$$

away from singularity, where $p$ is the pressure of the fluid, $h$ is the distance between two plates and $\mu$ is the dynamic viscosity of the fluid. Moreover, by the incompressibility of the flow, $\operatorname{div} \overline{\mathbf{v}} = 0$ and thus we have

$$\Delta p = 0 \quad \text{in } \Omega(t) \backslash \{0\}.$$

We also need to specify the boundary condition on the moving interface $\Gamma(t)$. If we neglect the surface tension effects, the dynamic boundary condition is given by

$$p = 0 \quad \text{on } \Gamma(t).$$

The kinematic boundary condition states that the fluid particles on the interface remain on the interface for all time by the condition

$$V_{\nu_{\text{out}}} = \overline{\mathbf{v}} \cdot \nu_{\text{out}} \quad \text{on } \Gamma(t),$$

where $\nu_{\text{out}}$ is the unit outward normal vector on $\Gamma(t)$. By (2.2), on $\Gamma(t)$, $V_{\nu_{\text{out}}}$ can be written as

$$V_{\nu_{\text{out}}} = -kDp \cdot \nu_{\text{out}}, \tag{2.3}$$

where $k = \frac{h^2}{12\mu}$. Similar to the Stefan condition formulation, since the free boundary is a level set of $p$, (2.3) can also be expressed as $p_t = k|Dp|^2$. This problem is usually generalized to an arbitrary dimension. If $k$ is a constant then we have a homogeneous problem. We also have a flow in an inhomogeneous medium when $k$ is given by a function and in this case, we will have a finger shape interface as in Figure 2.2. In our work, the limit problem is the homogeneous the Hele-Shaw problem with a point source formally given as follows.

**The multi-dimensional Hele-Shaw problem with a point source.** Let $n \geq 1$, the Hele-Shaw problem with a point source is to find a function $p(x,t) : \mathbb{R}^n \times [0,\infty) \to [0,\infty)$ and a set $\{p > 0\}$ satisfying the free boundary problem

$$\begin{cases} \Delta p = 0 & \text{in } \{p > 0\}, \\ p_t = |Dp|^2 & \text{on } \partial\{p > 0\}, \\ p(x,0) = 0 & \text{on } \mathbb{R}^n, \\ \lim\limits_{|x|\to 0} \dfrac{p}{\Phi} = C, \end{cases} \tag{2.4}$$

where $\Phi$ is the fundamental solution of the Laplace equation and $C$ is a constant.

From the mathematical point of view, the Hele-Shaw problem can be regarded as the one-phase Stefan problem when the interface moves slowly, the flow is approximately stationary and the specific heat $c$ is negligible. Indeed, if in the Hele-Shaw model, instead of the point source, we consider the movement under a fixed source $K$ with a prescribed boundary data $h(t)$ and assume that at the initial time, the pressure is given by some function $p_0$ then we recover the following one-phase Stefan problem with zero specific heat if the pressure $p$ of the fluid is regarded as the temperature of the liquid:

$$\begin{cases} \Delta p = 0 & \text{in } \{p > 0\} \backslash K, \\ p = h & \text{on } K, \\ p_t = |Dp|^2 & \text{on } \partial\{p > 0\}, \\ p(x, 0) = p_0 & \text{on } \mathbb{R}^n, \end{cases} \tag{2.5}$$

where $p_0$ and $h = h(t)$ are given functions. This problem is also called a Hele-Shaw-type problem. In some cases, the temperature and the free boundary in the one-phase Stefan problem depend continuously on $c$. Thus, the free boundary of the Stefan problem approaches that of the Hele-Shaw problem as $c \to 0$. Moreover, as $t \to \infty$ the diffusion in the process usually reaches the *steady-state* and the heat equation in the Stefan problem loses the first term $v_t$. Thus, we also expect to get the coincidence between the solutions as well as the free boundaries of these two models in the asymptotic limit when $t \to \infty$.

The asymptotic convergence of the Stefan problem to Hele-Shaw is indeed the consideration in [49], where the authors analyzed the asymptotic behavior of weak solutions of both models in the multi-dimensional case $n \geq 2$. They explained the asymptotic behavior of the solutions in term of the *near-field limit*, i.e., the limit of the solutions at a fixed point $x$ in the space as time $t \to \infty$, and the *far-field limit*, i.e., the development in the region close to the free boundary. In the near-field limit setting, the results in [49] stated that in both cases, the solutions converge to the solution of the Dirichlet exterior problem for the Laplacian while in the far-field limit, they converge to the solution of the Hele-Shaw problem with a point source. The authors also showed that the free boundaries approach a sphere as $t \to \infty$ with a precise asymptotic growth rate. The subjects of the study in [49] are the classical Stefan problem and Hele-Shaw problem in homogeneous isotropic media. Their results give rise to the question: Do the results hold for the inhomogeneous anisotropic case?

The conclusions for the near-field limit can be automatically extended for the inhomogeneous case and also for the anisotropic case with some simple modifications. However, the developments for the far-field limit will be more complicated, since in [49] the analysis of the far-field limit is based on an appropriate rescaling and in an inhomogeneous setting, the homogenization problems for the free boundary velocity and the elliptic operator appear in the scaling limit. This question is partially answered by Požár in [45] where the author proved that in an inhomogeneous medium, the rescaled solution of the Hele-Shaw problem locally uniformly converges to the solution of a homogeneous Hele-Shaw problem with a point source and the free boundary also converges to a sphere with respect to the Hausdorff distance. We will extend this result to the Stefan problem with an inhomogeneous latent heat in Chapter 3 and that with an inhomogeneous latent heat and anisotropic conductivity in Chapter 4.

In the Section 2.3 below, we will give a brief introduction of a homogenization problem and how it is related to our investigation. Before that we will recall some notions of solutions of the one-phase Stefan problem used in our work.

## 2.2 Notion of solutions

In this section, we will recall some notions of solutions of the one-phase Stefan problem (2.1) for the space dimension $n \geq 2$. We consider the problem (2.1) with the initial data $v_0$ satisfying

$$\begin{aligned} &v_0 \in C^2(\overline{\Omega_0 \backslash K}), v_0 > 0 \text{ in } \Omega_0, v_0 = 0, \text{ on } \Omega_0^c := \mathbb{R}^n \setminus \Omega_0, \text{ and } v_0 = 1 \text{ on } K, \\ &|Dv_0| \neq 0 \text{ on } \partial\Omega_0, \text{ for some bounded domain } \Omega_0 \supset K. \end{aligned} \tag{2.6}$$

### 2.2.1 Classical solutions

Let $G(t) := \Omega_t(v) \times \{t\}$ and $Q_T := \bigcup_{0<t<T} G(t)$.

**Definition 2.1.** A classical solution of the one-phase Stefan problem in $[0,T]$ is a pair $(v(x,t), \Gamma(t))$ with $v \in C(\overline{Q_T}) \bigcap C^{2,1}(Q_T), Dv \in C(\overline{Q_T} \backslash G(0)))$ and $\Gamma(t) \in C^1((0,T]) \cap C([0,T])$ that satisfy (2.1).

In case the space dimension is $n = 1$, the existence and uniqueness of the classical solution of the Stefan problem with monotone free boundary exits globally in time for various kinds of boundary and initial data, see [19,54]. However, the situation in multi-dimensional space is much more complicated. As observed in [28, 20, 31], the singularities of the free boundary might develop in finite time such as the merging of water regions that move independently or the closing of an ice region to a point, or a piece of ice on melting, may break into two, etc. Thus, we do not expect that the classical solution exists for all time, even if the data are smooth. Nevertheless, the short time existence of the classical solution of (2.1) was established by Hanzawa in [30] for some sufficiently smooth compatible boundary and initial data.

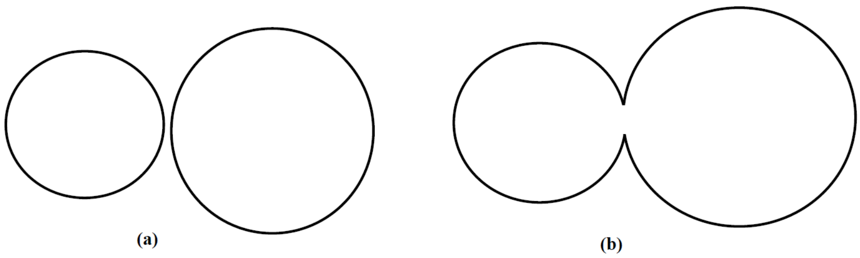


Figure 2.3: Singularities on the free boundary. (a)Two disc-shaped water regions that move independently, (b)Merging of the two disc at a later time.

The lack of classical solutions of the one-phase Stefan problem motivated to the study of *weak* solutions. In our work, we will use the following notions of weak solutions and viscosity solutions.

### 2.2.2 Weak solutions

In 1981, Elliot and Janovský introduced a notion of weak solution to the Hele-Shaw problem by taking integration in time of the classical solution and transforming the problem into an elliptic variational inequality. Following this approach, it was observed later by Duvaut [14] that we also can formulate the one-phase Stefan problem as a parabolic variational inequality. This method was then developed by Friedman and Kinderlehrer [21], Caffarelli [7, 8] and many other authors.

To motivate this method, let us suppose that $(v(x,t), \Gamma(t))$ is a classical solution of the Stefan problem (2.1) and introduce $u(x,t) := \int_0^t v(x,s)ds$. Fix $R, T > 0$ and set $B = B_R(0)$, $D = B\backslash K$. Following [21], it can be shown that, if $R$ is large enough (depending on $T$), then the function $u$ solves a variational problem. Indeed, since the free boundary $\Gamma(t)$ is $C^1((0,T])$ and $v_t > 0$ on $\Gamma(t)$, we represent the positive domain $\Omega(t)$ by $\Omega(t) = \{x : s(x) < t\}$ for some nonnegative function $s$ such that $\Omega_0 := \{x : v_0(x) > 0\} = \{x : s(x) = 0\}$ . From the definition of $u$ we have if $x \in \Omega_0^c$ then $s(x) > 0$ and

$$u(x,t) = \begin{cases} 0 & \text{if } 0 \le t \le s(x), \\ \displaystyle\int_{s(x)}^{t} v(x,s)ds & \text{if } s(x) < t \le T. \end{cases}$$

Now direct computation gives

$$\begin{aligned} u_{x_i} &= \int_{s(x)}^{t} v_{x_i}(x,s)ds - s_{x_i} v(x,s(x)) = \int_{s(x)}^{t} v_{x_i}(x,s)ds, \\ u_{x_i x_i} &= \int_{s(x)}^{t} v_{x_i x_i}(x,s)ds - s_{x_i} v_{x_i}(x,s(x)) \end{aligned} \tag{2.7}$$

for $x \in \Omega_0^c, \quad t > s(x)$.

Since $v$ satisfies (2.1) in classical sense then $V_{\nu_{\text{out}}} = |Dv|$. On the other hand, since the positive domain of $v$ is represented by $s(x)$ then $V_{\nu_{\text{out}}} = \frac{1}{|Ds|}$ and therefore $|Dv||Ds| = 1$, the vectors $Dv$ and $Ds$ are parallel (and parallel to $\nu_{\text{out}}$) but in opposite directions then we have $Dv \cdot Ds = -1$. In view of (2.1) and (2.7) we have

$$\begin{aligned} \Delta u(x,t) &= \int_{s(x)}^{t} v_s(x,s)ds + 1 \\ &= v(x,t) + 1 \\ &= u_t(x,t) + 1. \end{aligned}$$

Similarly, if $x \in \Omega_0$ then for all $0 < t \le T$, $u(x,t) = \int_0^t v(x,s)ds$, $s(x) = 0$ and

$$\begin{aligned} \Delta u(x,t) &= \int_0^t v_s(x,s)ds \\ &= v(x,t) - v(x,0) \\ &= u_t(x,t) - v_0(x). \end{aligned}$$

Define

$$f(x) := \begin{cases} v_0(x) & \text{if } x \in \Omega_0, \\ -1 & \text{if } x \in \Omega_0^c. \end{cases}$$

Then finally $u$ satisfies the nonlinear parabolic problem

$$\begin{cases} u > 0, \\ (u_t - \Delta u)(\varphi - u) = f(\varphi - u), \end{cases} \quad \text{for any } \varphi \in \mathcal{K}(t), x \in D, s(x) < t < T$$

and

$$\begin{cases} u = 0, \\ (u_t - \Delta u)(\varphi - u) = 0 \geq -\varphi = f(\varphi - u), ] \end{cases} \text{for any } \varphi \in \mathcal{K}(t), x \in D, 0 < t < s(x).$$

Here we set $\mathcal{K}(t) = \{\varphi \in H^1(D), \varphi \geq 0, \varphi = 0 \text{ on } \partial B, \varphi = t \text{ on } K\}$. We use the standard notation for Sobolev spaces $H^k$, $W^{k,p}$.

In conclusion, we have transformed the classical problem (2.1) into the following variational inequality problem.

**Variational inequality problem.** Find $u \in L^2(0,T;H^2(D))$ such that $u_t \in L^2(0,T;L^2(D))$ and

$$\begin{cases} u(\cdot,t) \in \mathcal{K}(t), \qquad 0 < t < T, \\ (u_t - \Delta u)(\varphi - u) \geq f(\varphi - u), \quad \text{a.e.}(x,t) \in B \times (0,T) \text{ for any } \varphi \in \mathcal{K}(t), \\ u(x,0) = 0 \text{ in } D. \end{cases} \tag{2.8}$$

Note that $u(x,t)$ is independent of the choice of $B$ as long as $R$ is large enough [39, Lemma 3.6]. If $v$ is a classical solution of (2.1) then $u$ is a solution of (2.8), but the inverse statement is not valid in general. However, we have the following result [21,51].

**Theorem 2.2** (Existence and uniqueness of the variational problem)**.** *If $v_0$ satisfies (2.6), then the problem (2.8) has a unique solution satisfying*

$$u \in L^\infty(0,T;W^{2,p}(D)), \qquad 1 \leq p \leq \infty,$$
$$u_t \in L^\infty(D \times (0,T)),$$

*and*

$$\begin{cases} u_t - \Delta u \geq f, \quad u \geq 0, \\ u(u_t - \Delta u - f) = 0 \end{cases} \text{a.e. in } D \times (0,\infty).$$

We will thus say that if $u$ is a solution of (2.8), then $u_t$ is a *weak solution* of the corresponding Stefan problem (2.1). The theory of variational inequalities for an obstacle problem is well developed, for more details, we refer to [21,51,38]. We now collect some useful results on the weak solutions from [21,51].

**Proposition 2.3.** *The unique solution $u$ of (2.8) satisfies*

$$0 \leq u_t \leq C \ a.e. D \times (0,T),$$

*where $C$ is a constant depending on $f$. In particular, $u$ is Lipschitz with respect to $t$ and $u$ is $C^\alpha(D)$ with respect to $x$ for all $\alpha \in (0,1)$. Furthermore, if $0 \leq t < s \leq T$, then $u(\cdot,t) < u(\cdot,s)$ in $\Omega_s(u)$ and also $\Omega_0 \subset \Omega_t(u) \subset \Omega_s(u)$.*

**Lemma 2.4** (Comparison principle for weak solutions)**.** *Suppose that $f \leq \hat{f}$. Let $u, \hat{u}$ be solutions of (2.8) for respective $f, \hat{f}$. Then $u \leq \hat{u}$, moreover,*

$$\theta \equiv \frac{\partial u}{\partial t} \leq \frac{\partial \hat{u}}{\partial t} \equiv \hat{\theta}.$$

*Remark* 2.5. Regularity of $\theta$ and its free boundary has been studied quite extensively, including Caffarelli and Friedman (see [7, 8, 22]). It is known that a weak solution is classical as long as $\Gamma_t(u)$ has no singularity. The smoothness criterion (see [7, 22], [49, Proposition 2.4]) immediately leads to the following corollary.

**Corollary 2.6.** *Radially symmetric weak solutions of the Stefan problem* (2.1) *are smooth classical solutions.*

*Remark* 2.7. If we consider the problem with an inhomogeneous latent of phase transition $L = 1/g(x)$ and an anisotropic diffusion $K = (k_{ij})$, then as shown in Section 2.1.1, the governing equation is a parabolic equation of divergence form

$$v_t - D_i(k_{ij} D_j v) = 0$$

in the positive domain and Stefan condition on the free boundary is given by

$$\frac{v_t}{|Dv|} = g k_{ij} D_j v \nu_i.$$

Here $D$ is the space gradient, $D_i$ is the partial derivative with respect to $x_i$, $v_t$ is the partial derivative of $v$ with respect to time variable $t$ and $\nu = \nu(x,t)$ is inward spatial unit normal vector of $\partial\{v > 0\}$ at point $(x,t)$ and we use the Einstein summation convention.

We can define a weak solution of this problem similarly to the homogeneous isotropic case. We only need to replace the heat operator by a more general parabolic operator of divergence form $D_i(k_{ij} D_j)$ and change the form of $f$ as

$$f(x) := \begin{cases} v_0(x) & \text{if } x \in \Omega_0 \\ -\dfrac{1}{g(x)} & \text{if } x \in \Omega_0^c. \end{cases}$$

All computations and almost all results remain to be valid. There are only some issues concerning the regularity of $\theta$ in the anisotropic case. Furthermore, we do not have classical radially symmetric solutions in the anisotropic case, which will lead to some difficulties in constructing barriers for our arguments.

### 2.2.3 Viscosity solutions

The second notion of solutions we will use are the viscosity solutions introduced in [34]. The main results in this work include the well-posedness of the Stefan problem (2.1) and a comparison principle for viscosity solutions. We will recall here some important ideas of viscosity solutions taken from [34, 39].

First, for any nonnegative function $w(x,t)$ we define:

$$w_\star(x,t) := \liminf_{(y,s)\to(x,t)} w(y,s), \qquad w^\star(x,t) := \limsup_{(y,s)\to(x,t)} w(y,s).$$

We will consider solutions in the space-time cylinder $Q := (\mathbb{R}^n \backslash K) \times [0,\infty)$.

**Definition 2.8.** A nonnegative upper semicontinuous function $v(x,t)$ defined in $Q$ is a viscosity subsolution of (2.1) if the following hold:

a) For all $T \in (0,\infty)$, the set $\overline{\Omega(v)} \cap \{t \leq T\} \cap Q$ is bounded.

b) For every $\phi \in C^{2,1}_{x,t}(Q)$ such that $v - \phi$ has a local maximum in $\overline{\Omega(v)} \cap \{t \leq t_0\} \cap Q$ at $(x_0,t_0)$, the following holds:

   i) If $v(x_0,t_0) > 0$, then $(\phi_t - \Delta\phi)(x_0,t_0) \leq 0$.

   ii) If $(x_0,t_0) \in \Gamma(v)$, $|D\phi(x_0,t_0)| \neq 0$ and $(\phi_t - \Delta\phi)(x_0,t_0) > 0$, then

$$(\phi_t - g(x_0)|D\phi|^2)(x_0,t_0) \leq 0. \tag{2.9}$$

**Definition 2.9.** A nonnegative lower semicontinuous function $v(x,t)$ defined in $Q$ is a viscosity supersolution of (2.1) if for every $\phi \in C^{2,1}_{x,t}(Q)$ such that $v - \phi$ has a local minimum in $\overline{\Omega(v)} \cap \{t \leq t_0\} \cap Q$ at $(x_0,t_0)$, the following holds:

a) If $v(x_0,t_0) > 0$, then $(\phi_t - \Delta\phi)(x_0,t_0) \geq 0$.

b) If $(x_0,t_0) \in \Gamma(v)$, $|D\phi(x_0,t_0)| \neq 0$ and $(\phi_t - \Delta\phi)(x_0,t_0) < 0$, then

$$(\phi_t - g(x_0)|D\phi|^2)(x_0,t_0) \geq 0. \tag{2.10}$$

Now let $v_0$ be a given initial condition with support $\Omega_0$ and free boundary $\Gamma_0 = \partial\Omega_0$, we can define viscosity subsolution and supersolution of (2.1) with corresponding initial data and boundary data.

**Definition 2.10.** A viscosity subsolution of (2.1) in $Q$ is a viscosity subsolution of (2.1) in $Q$ with initial data $v_0$ and boundary data 1 if:

a) $v$ is upper semicontinuous in $\bar{Q}, v = v_0$ at $t = 0$ and $v \leq 1$ on $\Gamma$,

b) $\overline{\Omega(v)} \cap \{t = 0\} = \overline{\{x : v_0(x) > 0\}}$.

**Definition 2.11.** A viscosity supersolution of (2.1) in $Q$ is a viscosity supersolution of (2.1) in $Q$ with initial data $v_0$ and boundary data 1 if $v$ is lower semicontinuous in $\bar{Q}, v = v_0$ at $t = 0$ and $v \geq 1$ on $\Gamma$,

And finally we can define viscosity solutions.

**Definition 2.12.** The function $v(x, t)$ is a viscosity solution of (2.1) in $Q$ (with initial data $v_0$ and boundary data 1) if $v$ is a viscosity supersolution and $v^\star$ is a viscosity subsolution of (2.1) in $Q$ (with initial data $v_0$ and boundary data 1).

*Remark* 2.13. By a standard argument, if $v$ is the classical solution of (2.1) then it is a viscosity solution of that problem in $Q$ with initial data $v_0$ and boundary data 1.

The existence and uniqueness of a viscosity solution as well as its properties were studied in great detail in [34]. One important feature of viscosity solutions is that they satisfy a comparison principle for strictly separated initial data.

**Definition 2.14.** We say that a pair of functions $u_0, v_0 : \overline{D} \to [0, \infty)$ are (strictly) separated and denote by $u_0 \prec v_0$ in $D \subset \mathbb{R}^n$ if:

a) $\overline{\{u_0 > 0\}} \cap \overline{D}$ is compact and

b) $u_0(x) < v_0(x)$ in $\overline{\{u_0 > 0\}} \cap \overline{D}$

**Theorem 2.15** (Comparison principle, [34, 39])**.** *Let $v_1, v_2$ be respectively viscosity subsolution and supersolution of (2.1) in $Q$. If $v_1 \prec v_2$ on the parabolic boundary of $Q$, then $v_1(\cdot, t) \prec v_2(\cdot, t)$ in $Q$.*

One of the main tool we will use in this work is the following Theorem about coincidence of weak and viscosity solutions. Following [39] we can state that:

**Theorem 2.16** (Theorem 3.1, [39]). *Assume that $v_0$ satisfies (2.6). Let $u(x,t)$ be the unique solution of (2.8) in $B \times [0,T]$ and let $v(x,t)$ be the solution of*

$$\begin{cases} v_t - \Delta v = 0 & \text{in } \Omega(u)\backslash K, \\ v = 0 & \text{on } \Gamma(u), \\ v = 1 & \text{in } K, \\ v(x,0) = v_0(x). \end{cases} \tag{2.11}$$

*Then the following hold:*

*a)* $v(x,t)$ *is a viscosity solution of* (2.1) *in* $B \times [0,T]$ *with initial data* $v(x,0) = v_0(x)$.

*b)* $u(x,t) = \int_0^t v(x,s)ds$

*Proof.* See proof of Theorem 3.1, [39]. □

*Remark* 2.17. We want to clarify the definition of a solution $v$ when $\Omega(u)$ is not smooth. Since $u$ is continuous and $\Omega(u)$ is bounded at all times (Lemma 3.6, [39]) then the existence of solution of (2.11) is provided by Perron's method as follows:

$$v = \sup\{w | w_t - \Delta w \leq 0 \text{ in } \Omega(u), w \leq 0 \text{ on } \Gamma(u), w \leq 1 \text{ in } K, w(x,0) \leq v_0(x)\}.$$

It may happen that $v$ is not continuous through its boundary in general. However, from the regularity results of Caffarelli and Friedman, we know that in our case $u_t$ is continuous in space and time, and we would have $v = u_t$.

The coincidence of weak and viscosity solutions gives us a more general comparison principle:

**Lemma 2.18** (Corollary 3.12, [39]). *Let $v^1$ and $v^2$ be, respectively, a viscosity subsolution and super solution of the Stefan problem* (2.1) *with continuous initial data $v_0^1 \leq v_0^2$ and boundary data 1. In addition, suppose that $v_0^1$(or $v_0^2$) satisfies condition (2.6). Then $v_\star^1 \leq v^2$ and $v^1 \leq (v^2)^\star$ in $\mathbb{R}^n \backslash K \times [0,\infty)$.*

*Remark* 2.19. Similar to the notion of weak solutions, we can define a viscosity solution of the one-phase Stefan problem with an inhomogeneous latent heat $L = \frac{1}{g(x)}$ and an anisotropic conductivity $K = (k_{ij})$ as in Remark 2.7. For this problem, the definition of a viscosity subsolution (resp. a viscosity supersolution) is analogous to Definition 2.8 (resp. Definition 2.9), when we replace the Laplace operator by $D_i(k_{ij}D_j)$ and (2.9) (resp. (2.10)) by

$$(\phi_t - g a_{ij} D_j \phi \nu_i |D\phi|)(x_0, t_0) \leq 0,$$

(resp. the inequality $\geq$), where $\nu$ is inward spatial unit normal vector of $\partial\{v > 0\}$. All the other definitions and results follow the same as in the homogeneous isotropic case.

## 2.3 Homogenization and long-time behavior problems

Homogenization theory is the study of partial differential equations with rapidly oscillating coefficients and extract homogenized equations. Many problems in physics, mechanics or chemistry are processes in inhomogeneous media with a fine microscopic structure. For example, we consider the steady state of the heat flow in a periodic anisotropic medium which can be modeled by an elliptic equation of divergence form

$$\operatorname{div}(A(x)Du(x)) = f(x) \quad \text{in } \Omega,$$

where $\Omega$ is an bounded domain in $\mathbb{R}^n$, $A(x)$ is a nonnegative, bounded, periodic matrix with period 1 and $f$ is a smooth function. Now we look at the process from far away, the period of the medium will be much smaller than the length scale of $\Omega$, say, period $\varepsilon$, we will then be concerned with the equation with rapidly oscillating coefficients

$$\operatorname{div}\left(A\left(\frac{x}{\varepsilon}\right) Du^\varepsilon(x)\right) = f(x) \quad \text{in } \Omega.$$

When the scale of the microscopic periodic structure is very small in comparison with the scale of the domain under consideration, the medium has homogenized (or macroscopic) characteristics, see Figure 2.4 below.

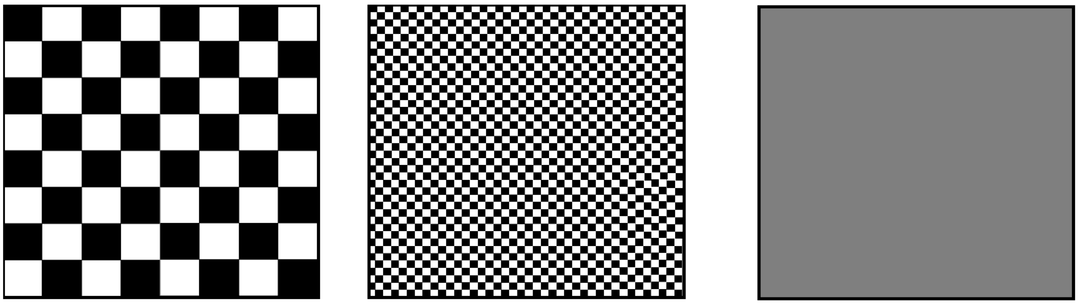

Figure 2.4: Checkerboard medium with self-averaging property

Thus we expect that as $\varepsilon \to 0$, $u^\varepsilon \to u^0$ in an appropriate sense and $u^0$ is a solution of a homogenized (or effective) equation

$$\operatorname{div}\left(A^0 Du^0\right) = f \quad \text{in } \Omega,$$

where $A^0$ is a homogenized matrix with constant coefficients. The main concerns of the homogenization theory is to obtain the convergence (usually in a weak sense) of the solution $u^\varepsilon$ to $u^0$ and to construct the homogenized matrix $A^0$ ($A^0$ is a constant matrix in many cases). The homogenization was first developed for periodic structures and then generalized for other media with self averaging properties (almost periodic, stationary ergodic). The operator type in a homogenization problem also can be more general, time independent or dependent, linear or nonlinear. We would like to refer to [6, 32, 11, 12, 13, 57, 7, 8] for more details.

The homogenization problems appear in our investigation as the following observation. Using the ideas in [49, 45], we will analyze the asymptotic behavior of the solution of the one-phase Stefan problem (2.1) in *far region* and *large time* by using an appropriate rescaling, i.e.,

$$v^\lambda(x,t) := \lambda^{(n-2)/n} v(\lambda^{1/n}x, \lambda t) \qquad\qquad \text{if } n \geq 3,$$

and the corresponding rescaling for variational solutions

$$u^\lambda(x,t) := \lambda^{-2/n} u(\lambda^{1/n}, \lambda t) \qquad\qquad \text{if } n \geq 3$$

(see Section 3.1.1 for $n = 2$). It is a natural rescaling since as shown in [49], the position of the free boundary expands with the rate $|x| \sim t^{1/n}$. The rescaling can be intuitively understood as looking at the solution from far away and for a very long time. The difference between our setting and that in [49] is that instead of considering $L$ to be a constant, we study the one-phase Stefan problem with an inhomogeneous latent heat $L(x) = \frac{1}{g(x)}$. Then the rescaled viscosity solution satisfies the free boundary velocity law

$$V^\lambda_{\nu_{\text{out}}} = g(\lambda^{1/n}x)|Dv^\lambda|.$$

Let us assume that the latent heat of phase transition $L(x) = 1/g(x)$ has averaging property, then it will average out as $\lambda \to \infty$ and the velocity become homogenized as

$$V_{\nu_{\text{out}}} = \frac{1}{\langle 1/g \rangle}|Dv|,$$

where $\langle 1/g \rangle$ represents the "average" of $1/g$. The convergence process of the rescaled normal velocity to a limit with a homogeneous coefficient is a homogenization of the free boundary velocity. Moreover, if the Laplace operator is replaced by a periodic elliptic operator of divergence form $\mathcal{L}$, then we see that $v^\lambda$ satisfies

$$\begin{aligned} \lambda^{(2-n)/n} v_t^\lambda - \mathcal{L}^\lambda v^\lambda = 0 \qquad & \qquad \text{in } \Omega(v^\lambda) \backslash K^\lambda, \\ V^\lambda_{\nu_{\text{out}}} = g(\lambda^{1/n}x) a_{ij}(\lambda^{1/n}x) D_j v^\lambda \nu_i & \quad \text{on } \Gamma(v^\lambda), \end{aligned}$$

where $\mathcal{L}, \mathcal{L}^\lambda$ were defined in Section 1.2 and $\nu_{\text{out}}, \nu$ are the outward and inward unit normal vector on $\Gamma(v^\lambda)$ respectively. As $\lambda \to \infty$, the parabolic operator becomes elliptic and the homogenization of the elliptic operator is also expected beside the homogenization of the free boundary normal velocity.

The rescaling for the variational solution is similar, the rescaled variational solution satisfies a variational inequality

$$(\lambda^{\frac{2-n}{n}} u_t^\lambda - \mathcal{L}^\lambda u^\lambda)(\varphi - u^\lambda) \geq f^\lambda(x)(\varphi - u^\lambda) \tag{2.12}$$

a.e. $(x, t) \in \mathbb{R}^n \times (0, \infty)$, for any $\varphi \in \mathcal{K}^\lambda(t)$, where

$$f^\lambda(x) := f(\lambda^{1/n}x) = \begin{cases} v_0(\lambda^{1/n}x) & \text{if } x \in \Omega_0^\lambda, \\ -1/g(\lambda^{1/n}x) & \text{if } x \in (\Omega_0^\lambda)^c, \end{cases}$$

$\Omega_0^\lambda = \Omega_0/\lambda^{1/n}$ and $\mathcal{K}^\lambda(t) = \{\varphi \in H^1(\mathbb{R}^n), \varphi \geq 0, \varphi = \lambda^{\frac{n-2}{n}} t \text{ on } K^\lambda\}$. As $\lambda \to \infty$, we also expect to have the homogenization of the variational inequality and the elliptic asymptotic convergence of the rescaled parabolic operator. Therefore, the homogenization processes are in particular extremely important to our problem.

Even though the homogenization has a long history, most of the results were obtained for boundary value problems (or initial boundary value problems) in a bounded domain. There are very few results for the homogenization of a free boundary problem. Fortunately, the homogenization of the one-phase Stefan problem was studied by Rodrigues [50] for a periodic setting and generalized for a stationary ergodic setting by Kim and Mellet [39]. Kim and Mellet also obtained similar results for the Hele-Shaw problem earlier in [38]. However, our situation is considerably different when the domain of rescaled variational solution $u^\lambda$ changes as $\lambda \to \infty$. In fact, due to the rescaling, the fixed domain $K$ shrinks to the origin and the solution gets singularity in the limit. Thus we need to characterize the singularity of solution as $|x| \to 0$. For the isotropic inhomogeneous case in Chapter 3, this task can be done by following directly the barrier arguments in [45] for the Hele-Shaw problem. In anisotropic inhomogeneous case of Chapter 4, since we cannot use the classical radially symmetric solution as barriers, we will construct some ones to modify the treatment in Chapter 3. Moreover, in the anisotropic case, we need to deal with the homogenization problem of the elliptic operator in the domain which together with the specified boundary data, change as $\lambda$ varies. Therefore, the known results on the homogenization of the one-phase Stefan problem cannot be applied directly. We will solve this problem with the help of the $\Gamma$-convergence techniques, which are classical approaches for homogenization of nonlinear variational problems, with some adaptations for our problem. Another characteristic of our problem which is significantly different with the previous work is the fact that we need to combine the homogenization problems with the long-time behavior problem. This kind of work was done for the Hele-Shaw problem in [45], however, the arguments in [45] rely on the very useful monotonicity of solutions while we do not have this property in the Stefan problem. We will instead use a weaker monotonicity stated later. In addition, the rescaled parabolic equation becomes elliptic when $\lambda \to \infty$, which also causes some issues in analyzing the convergence of the rescaled free boundary to the free boundary of the homogenized limit equation.

In our work, we will assume that the medium in the one-phase Stefan problem is periodic or stationary ergodic over a probability space $(A, \mathcal{F}, P)$. We recall [10, 38, 39] that a random variable $g(x, \cdot) : A \to \mathbb{R}$ is said to be *stationary ergodic* if it satisfies the following two conditions:

1. The distribution of the random variable $g(x, \cdot)$ is independent of $x$ (this property is referred as stationary). This can be expressed more precisely by the existence, for each $x \in \mathbb{R}^n$, of a measure-preserving transformation $\tau_x : A \to A$ such that:

$$g(x + x', \omega) = g(x, \tau_{x'}\omega) \quad \text{for all } x' \in \mathbb{R}^n \text{ and } \omega \in A.$$

2. The underlying transformation $\tau_x$ is ergodic, that is, if $B \subset A$ and $\tau_x B = B$ for all $x \in \mathbb{R}^n$, then $P(B) = 0$ or $P(B) = 1$.

This probabilistic setting, we can think about a random checkerboard for instance, is a general extension of the notions of periodicity for a function to have some self-averaging behavior. In particular, we will make use of the following important application of the subadditive ergodic theorem.(see [38, Lemma 4.1])

**Lemma 2.20** (cf. [38, Section 4, Lemma 7], see also [45])**.** *For given $g$ satisfying (2), there exists a constant, denoted by $\langle 1/g \rangle$, such that if $\Omega \subset \mathbb{R}^n$ is a bounded measurable set and if $\{u^\varepsilon\}_{\varepsilon>0} \subset L^2(\Omega)$ is a family of functions such that $u^\varepsilon \to u$ strongly in $L^2(\Omega)$ as $\varepsilon \to 0$, then*

$$\lim_{\varepsilon \to 0} \int_\Omega \frac{1}{g(x/\varepsilon, \omega)} u^\varepsilon(x) dx = \int_\Omega \left\langle \frac{1}{g} \right\rangle u(x) dx \ \ a.e. \ \omega \in A.$$

# Chapter 3

# Long-time behavior of the one-phase Stefan problem in periodic and random media

We consider the one-phase Stefan problem in periodic and random media in a dimension $n \geq 2$. The aim of this chapter is to understand the behavior of the solutions and their free boundaries when time $t \to \infty$. The contents of this chapter is based on the work that previously appeared in [47].

Let $K \subset \mathbb{R}^n$ be a compact set with sufficiently regular boundary, for instance $\partial K \in C^{1,1}$, and assume that $0 \in \operatorname{int} K$. The one-phase Stefan problem (on an exterior domain) with inhomogeneous latent heat of phase transition is to find a function $v(x,t) : \mathbb{R}^n \times [0,\infty) \to [0,\infty)$ that satisfies the free boundary problem

$$\begin{cases} v_t - \Delta v = 0 & \text{in } \{v > 0\} \backslash K, \\ v = 1 & \text{on } K, \\ V_\nu = g(x)|Dv| & \text{on } \partial\{v > 0\}, \\ v(x,0) = v_0 & \text{on } \mathbb{R}^n, \end{cases} \tag{3.1}$$

where $D$ and $\Delta$ are respectively the spatial gradient and Laplacian, $v_t$ is the partial derivative of $v$ with respect to time variable $t$, $V_\nu$ is the normal velocity of the *free boundary* $\partial\{v > 0\}$. $v_0$ and $g$ are given functions, see below. Note that the results in this chapter can be trivially extended to general time-independent positive continuous boundary data, 1 is taken only to simplify the exposition.

As introduced in Chapter 2, the one-phase Stefan problem is a mathematical model of phase transitions between a solid and a liquid. A typical example is the melting of a body of ice maintained at temperature 0, in contact with a region of water. The unknowns are the temperature distribution $v$ and its free boundary $\partial\{v(\cdot,t) > 0\}$, which models the ice-water interface. Given an initial temperature distribution of the water, the diffusion of heat in a medium by conduction and the exchange of latent heat will govern the system. In this chapter, we consider an inhomogeneous medium where the latent heat of phase transition, $L(x) = 1/g(x)$, and hence the velocity law depends on position. The related Hele-Shaw problem is usually referred to in the literature as the quasi-stationary limit of the one-phase Stefan problem when the heat operator is replaced by the Laplace operator. This problem typically describes the flow of an injected viscous fluid between two parallel plates which form the so-called Hele-Shaw cell, or the flow in porous media.

In this chapter, we assume that the function $g$ satisfies the following two conditions, which guarantee respectively the well-posedness of (3.1) and averaging behavior as $t \to \infty$:

1. $g$ is a Lipschitz function in $\mathbb{R}^n$, $m \le g \le M$ for some positive constants $m$ and $M$.
2. $g(x)$ has some averaging properties so that Lemma 2.20 applies, for instance, one of the following holds:
   a) $g$ is a $\mathbb{Z}^n$-periodic function,
   b) $g(x,\omega) : \mathbb{R}^n \times A \to [m, M]$ is a stationary ergodic random variable over a probability space $(A, \mathcal{F}, P)$.

For a detailed definition and overview of stationary ergodic media, we refer to [45,39] and the references therein.

Throughout most of the chapter we will assume that the initial data $v_0$ satisfies

$$\begin{aligned}
&v_0 \in C^2(\overline{\Omega_0 \backslash K}), v_0 > 0 \text{ in } \Omega_0, v_0 = 0, \text{ on } \Omega_0^c := \mathbb{R}^n \setminus \Omega_0, \text{ and } v_0 = 1 \text{ on } K,\\
&|Dv_0| \neq 0 \text{ on } \partial\Omega_0, \text{ for some bounded domain } \Omega_0 \supset K.
\end{aligned} \tag{3.2}$$

This will guarantee the existence of both the weak and viscosity solutions below and their coincidence, as well as the weak monotonicity (3.35). However, the asymptotic limit, Theorem 3.1, is independent of the initial data, and therefore the result applies to arbitrary initial data as long as the (weak) solution exists, satisfies the comparison principle, and the initial data can be approximated from below and from above by data satisfying (3.2). For instance, $v_0 \in C(\mathbb{R}^n)$, $v_0 = 1$ on $K$, $v_0 \geq 0$, $\operatorname{supp} v_0$ compact is sufficient.

The Stefan problem (3.1) does not necessarily have a global classical solution in $n \geq 2$ as singularities of the free boundary might develop in finite time. As shown in Chapter 2, the classical approach to define a generalized solution is to integrate $v$ in time and introduce $u(x,t) := \int_0^t v(x,s)ds$ [5, 14, 21, 15, 51, 53, 52]. If $v$ is sufficiently regular, then $u$ solves the variation inequality

$$\begin{cases} u(\cdot,t) \in \mathcal{K}(t), \\ (u_t - \Delta u)(\varphi - u) \geq f(\varphi - u) \text{ a.e } (x,t) \text{ for any } \varphi \in \mathcal{K}(t), \end{cases} \tag{3.3}$$

where $\mathcal{K}(t)$ is a suitable functional space specified in Section 2.2.2 and $f$ is

$$f(x) = \begin{cases} v_0(x), & v_0(x) > 0, \\ -\dfrac{1}{g(x)}, & v_0(x) = 0. \end{cases} \tag{3.4}$$

This parabolic inequality always has a global unique solution $u(x,t)$ for initial data satisfying (3.2) [21, 51, 53, 52]. The corresponding time derivative $v = u_t$, if it exists, is then called a *weak solution* of the Stefan problem (3.1). The main advantage of this definition is that the powerful theory of variational inequalities can be applied for the study of the Stefan problem, and as was observed in [50, 38, 39] yields homogenization of (3.3).

More recently, the notion of viscosity solutions of the Stefan problem was introduced and well-posedness was established by Kim [34]. Since this notion relies on the comparison principle instead of the variational structure, it allows for more general, fully nonlinear parabolic operators and boundary velocity laws. Moreover, the pointwise viscosity methods seem more appropriate for studying the behavior of the free boundaries. The natural question whether the weak and viscosity solutions coincide was answered positively by Kim and Mellet [39] whenever the weak solution exists. In this chapter we will use the strengths of both the weak and viscosity solutions to study the behavior of the solution and its free boundary for large times.

The homogeneous version of this problem, i.e, when $g \equiv const$, was studied by Quirós and Vázques in [49]. They obtained the result on the long-time convergence of weak solution of the one-phase Stefan problem to the self-similar solution of the Hele-Shaw problem. The homogenization of this type of problem was considered by Rodrigues in [50] and by Kim-Mellet in [38, 39]. The long-time behavior of solution of the Hele-Shaw problem was studied in detail by Požár in [45]. In particular, the rescaled solution of the inhomogeneous Hele-Shaw problem converges to the self-similar solution of the Hele-Shaw problem with a point-source, formally

$$\begin{cases} -\Delta v = C\delta & \text{in } \{v > 0\}, \\ v_t = \dfrac{1}{\langle 1/g \rangle} |Dv|^2 & \text{on } \partial\{v > 0\}, \\ v(\cdot, 0) = 0, \end{cases} \tag{3.5}$$

where $\delta$ is the Dirac $\delta$-function, $C$ is a constant depending on $K$ and $n$, and the constant $\langle 1/g \rangle$ will be properly defined later. Moreover, the rescaled free boundary uniformly approaches a sphere.

Here we extend the convergence result to the Stefan problem in the inhomogeneous medium. Since the asymptotic behavior of radially symmetric solutions of the Hele-Shaw and the Stefan problem are similar and the solutions are bounded, we can take the limit $t \to \infty$ and obtain the convergence for rescaled solutions and their free boundaries. However, solutions of the Hele-Shaw problem have a very useful monotonicity in time, which is missing in the Stefan problem. This makes some steps more difficult. We instead take advantage of a weak monotonicity property (3.35), which holds for regular initial data satisfying (3.2) and then show the convergence result for general initial data using the uniqueness of the limit and the comparison principle. Moreover, the heat operator is not invariant under the rescaling, unlike the Laplace operator. The rescaled parabolic equation becomes elliptic when $\lambda \to \infty$, which causes some issues when applying parabolic Harnack's inequality, for instance. Following [49, 45] we use the natural rescaling of solutions of the form

$$v^\lambda(x,t) := \lambda^{(n-2)/n} v(\lambda^{1/n} x, \lambda t) \qquad \text{if } n \geq 3,$$

and the corresponding rescaling for variational solutions

$$u^\lambda(x,t) := \lambda^{-2/n} u(\lambda^{1/n}, \lambda t) \qquad \text{if } n \geq 3$$

(see Section 3.1.1 for $n = 2$). Then the rescaled viscosity solution satisfies the free boundary velocity law

$$V_\nu^\lambda = g(\lambda^{1/n}x)|Dv^\lambda|.$$

Heuristically, if $g$ has some averaging properties, such as in condition (2), the free boundary velocity law should homogenize as $\lambda \to \infty$. Since the latent heat of phase transition $1/g$ should average out, the homogenized velocity law will be

$$V_\nu = \frac{1}{\langle 1/g \rangle}|Dv|,$$

where $\langle 1/g \rangle$ represents the "average" of $1/g$. More precisely, the quantity $\langle 1/g \rangle$ is the constant in the subadditive ergodic theorem such that

$$\int_{\mathbb{R}^n} \frac{1}{g(\lambda^{1/n}x, \omega)} u(x)dx \to \int_{\mathbb{R}^n} \left\langle \frac{1}{g} \right\rangle u(x)dx \text{ for all } u \in L^2(\mathbb{R}^n), \text{ for a.e. } \omega \in A.$$

In the periodic case, it is just the average of $1/g$ over one period. Since we always work with $\omega \in A$ for which the convergence above holds, we omit it from the notation in the rest of the chapter.

This yields the first main result of this chapter, Theorem 3.12, on the homogenization of the obstacle problem (3.3) for the rescaled solutions, with the correct singularity of the limit function at the origin, and therefore the locally uniform convergence of variational solutions. To prove the second main result in Theorem 3.16 on the locally uniform convergence of viscosity solutions and their free boundaries, we use pointwise viscosity solution arguments. In summary, we will show the following theorem.

**Theorem 3.1.** *For almost every $\omega \in A$, the rescaled viscosity solution $v^\lambda$ of the Stefan problem (3.1) converges locally uniformly to the unique self-similar solution $V$ of the Hele-Shaw problem (3.5) in $(\mathbb{R}^n \setminus \{0\}) \times [0, \infty)$ as $\lambda \to \infty$, where $C$ depends only on $n$, the set $K$ and the boundary data 1. Moreover, the rescaled free boundary $\partial\{(x,t) : v^\lambda(x,t) > 0\}$ converges to $\partial\{(x,t) : V(x,t) > 0\}$ locally uniformly with respect to the Hausdorff distance.*

It is a natural question to consider more general linear divergence form operators $\sum_{i,j} \partial_{x_i}(a_{ij}(x)\partial_{x_j}\cdot)$ instead of the Laplacian in (3.1) so that the variational structure is preserved. This was indeed the setting considered in [39], with $g \equiv 1$

and appropriate free boundary velocity law adjusted for the operator above. In the limit $\lambda \to \infty$, we expect that the rescaled solutions $v^\lambda$ to converge to the unique solution of the Hele-Shaw type problem with a point source with the homogenized non-isotropic operator with coefficients $\bar{a}_{i,j}$. This question is partially answered in the next chapter.

## Context and open problems

In recent years, there have been significant developments in the homogenization theory of partial differential equations like Hamilton-Jacobi and second order fully nonlinear elliptic and parabolic equations that have been made possible by the improvements of the viscosity solutions techniques, see for instance the classical [16, 57, 10, 9] to name a few.

A common theme of these results is finding (approximate) correctors and use the perturbed test function method to establish the homogenization result in the periodic case, or use deeper properties in the random case, such as the variational structure of the Hamilton-Jacobi equations or the strong regularity results for elliptic and parabolic equations, including the ABP inequality.

One of the goals of this chapter is to illustrate the powerful combination of variational and viscosity solution techniques for some free boundary problems that have a variational structure. By viscosity solution techniques we mean specifically pointwise arguments using the comparison principle.

Unfortunately, when the variational structure is lost, for instance, when the free boundary velocity law is more general as in the problem with contact angle dynamics $V_\nu = |Dv| - g(x)$ so that the motion is non-monotone [36, 37], or even simple time-dependence $V_\nu = g(x,t)|Dv|$ [46], the comparison principle is all that is left. Even in the periodic case, the classical correctors as solutions of a cell problem are not available. This is in part the consequence of the presence of the free boundary on which the operator is strongly discontinuous. [35, 36, 46] use a variant of the idea that appeared in [10] to replace the correctors by solutions of certain obstacle problems. However, the analysis of these solutions requires rather technical pointwise arguments since there are almost no equivalents of the regularity estimates for elliptic equations. An important tool in [46] to overcome this was the large scale Lipschitz regularity of the free boundaries of the obstacle problem solutions (called cone flatness there) that allows for the control of the oscillations of the free boundary in the homogenization limit.

For the reasons above, the homogenization of free boundary problems is rather challenging and there are still many open problems. Probably the most important one is the homogenization of free boundary problems of the Stefan and Hele-Shaw type that do not admit a variational structure, such as those mentioned above, in random environments. Currently there is no known appropriate stationary subadditive quantity to which we could apply the subadditive ergodic theorem to recover the homogenized free boundary velocity law, for instance. Other tools like concentration inequalities have so far not yielded an alternative.

Another important problem are the optimal convergence rates of the free boundaries in the Hausdorff distance. The techniques used in this chapter do not provide this information, however viscosity techniques were used to obtain non-optimal algebraic convergence rates in [37]. It is an interesting question what the optimal rate in the periodic case is, even for problems like (3.1). The large scale Lipschitz estimate from [46] could possibly directly give only $\varepsilon|\log \varepsilon|^{1/2}$-rate for velocity law with $g(x/\varepsilon)$, but there are some indications that a rate $\varepsilon$ might be possible.

### Outline

This chapter is organized as follows: In Section 3.1, we introduce the rescaling and state some results for radially symmetric solutions. In Section 3.2, we recall the limit obstacle problem and prove the locally uniform convergence of rescaled variational solutions. In Section 3.3, we focus on treating the locally uniform convergence of viscosity solutions and their free boundaries.

## 3.1 Preliminaries

### 3.1.1 Rescaling

Let $v$ be the solution of the one-phase Stefan problem (3.1) and $u$ be the solution of the corresponding variational inequality (3.3), the definitions as well as the relationship of $v$ and $u$ was introduced in Chapter 2. We will use the following rescaling of solutions as in [45].

#### 3.1.1.1 For $n \geq 3$

For $\lambda > 0$ we use the rescaling

$$v^\lambda(x,t) = \lambda^{\frac{n-2}{n}} v(\lambda^{\frac{1}{n}} x, \lambda t), \qquad u^\lambda(x,t) = \lambda^{-\frac{2}{n}} u(\lambda^{\frac{1}{n}} x, \lambda t).$$

If we define $K^\lambda := K/\lambda^{\frac{1}{n}}$ and $\Omega_0^\lambda := \Omega_0/\lambda^{\frac{1}{n}}$ then $v^\lambda$ satisfies the problem

$$\begin{cases} \lambda^{\frac{2-n}{n}} v_t^\lambda - \Delta v^\lambda = 0 & \text{in } \Omega(v^\lambda)\backslash K^\lambda, \\ v^\lambda = \lambda^{\frac{n-2}{n}} & \text{on } K^\lambda, \\ v_t^\lambda = g^\lambda(x)|Dv^\lambda|^2 & \text{on } \Gamma(v^\lambda), \\ v^\lambda(\cdot,0) = v_0^\lambda, & \text{on } \Omega_0^\lambda\backslash K^\lambda, \end{cases} \tag{3.6}$$

where $g^\lambda(x) = g(\lambda^{\frac{1}{n}} x)$. And the rescaled $u^\lambda$ satisfies the obstacle problem

$$\begin{cases} u^\lambda(\cdot,t) \in \mathcal{K}^\lambda(t), & \\ (\lambda^{\frac{2-n}{n}} u_t^\lambda - \Delta u^\lambda)(\varphi - u^\lambda) \geq f^\lambda(x)(\varphi - u^\lambda) & \text{a.e } (x,t) \in \mathbb{R}^n \times (0,\infty) \\ & \text{for any } \varphi \in \mathcal{K}^\lambda(t), \\ u^\lambda(x,0) = 0, & \end{cases} \tag{3.7}$$

where

$$f^\lambda(x) := f(\lambda^{1/n} x) = \begin{cases} v_0(\lambda^{1/n} x) & \text{if } x \in \Omega_0^\lambda, \\ -\dfrac{1}{g(\lambda^{1/n} x)} & \text{if } x \in (\Omega_0^\lambda)^c. \end{cases}$$

$\mathcal{K}^\lambda(t) = \{\varphi \in H^1(\mathbb{R}^n), \varphi \geq 0, \varphi = \lambda^{\frac{n-2}{n}} t \text{ on } K^\lambda\}$.

#### 3.1.1.2 For n=2

For dimension $n = 2$, we use a different rescaling that preserve the singularity of logarithm:

$$v^\lambda(x,t) = \log \mathcal{R}(\lambda) v(\mathcal{R}(\lambda) x, \lambda t), \tag{3.8}$$

where $\mathcal{R}(\lambda)$ is the unique solution of:

$$\mathcal{R}^2 \log \mathcal{R} = \lambda.$$

Note that we have (see [45] for derivation):

$$\lim_{\lambda\to\infty}\frac{\mathcal{R}(\lambda)}{\mathcal{R}_\infty(\lambda)}=1 \qquad\qquad \mathcal{R}_\infty(\lambda)=\left(\frac{2\lambda}{\log\lambda}\right)^{1/2}. \tag{3.9}$$

If we define $K^\lambda := K\backslash\mathcal{R}(\lambda)$ and $\Omega_0^\lambda := \Omega_0\backslash\mathcal{R}(\lambda)$ then the rescaling $v^\lambda$ satisfies the following problem:

$$\begin{cases}\dfrac{1}{\log\mathcal{R}(\lambda)}v_t^\lambda-\Delta v^\lambda=0 & \text{in } \Omega(v^\lambda)\backslash K^\lambda,\\ v^\lambda=\log\mathcal{R}(\lambda) & \text{on } K^\lambda,\\ v_t^\lambda=g^\lambda(x)|Dv^\lambda|^2 & \text{on } \Gamma(v^\lambda),\\ v^\lambda(\cdot,0)=v_0^\lambda\end{cases} \tag{3.10}$$

where $g^\lambda(x)=g(\mathcal{R}(\lambda)x)$ derived in [45].

Consequently,we will use rescaling

$$u^\lambda(x,t)=\frac{\log\mathcal{R}(\lambda)}{\lambda}u(\mathcal{R}(\lambda)x,\lambda t). \tag{3.11}$$

And the rescaled $u^\lambda$ satisfies the obstacle problem:

$$\begin{cases} u^\lambda(\cdot,t)\in\mathcal{K}^\lambda(t),\\ \left(\dfrac{1}{\log\mathcal{R}(\lambda)}u_t^\lambda-\Delta u^\lambda\right)(\varphi-u^\lambda)\geq f(\mathcal{R}(\lambda)x)(\varphi-u^\lambda) \quad \text{a.e } (x,t)\in B_R\times(0,T)\\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for any } \varphi\in\mathcal{K}^\lambda(t),\\ u^\lambda(x,0)=0.\end{cases} \tag{3.12}$$

Where $\mathcal{K}^\lambda(t)=\{\varphi\in H^1(\mathbb{R}^n),\varphi\geq 0,\varphi=\log\mathcal{R}(\lambda)t \text{ on } K^\lambda\}$

*Remark* 3.2. We can take the admissible set $K^\lambda(t)$ as above due to the continuity with respect to the $H^1$ norm of all terms in the variational inequality and the fact that the variational solution $u$ has a compact support in space at every time.

### 3.1.2 Convergence of radially symmetric solutions

We will recall the results on the convergence of radially symmetric solutions of (3.1) as derived in [49]. First, we collect some useful facts of radial solution of the Hele-Shaw problem and then use a comparison to have the information of radial solution of the Stefan problem. The radially symmetric solution of the Hele-Shaw problem in the domain $|x| \geq a, t \geq 0$ is a pair of functions $p(x,t)$ and $R(t)$, where $p$ is of the form

$$p(x,t) = \begin{cases} \dfrac{Aa^{n-2}\left(|x|^{n-2} - R^{n-2}(t)\right)_+}{a^{2-n} - R^{2-n}(t)}, & n \geq 3, \\ \dfrac{A\left(\log \frac{R(t)}{|x|}\right)_+}{\log \frac{R(t)}{a}}, & n = 2, \end{cases} \tag{3.13}$$

and $R(t)$ satisfies a certain algebraic equation (see [49] for details).

This solution satisfies the boundary conditions and initial conditions

$$\begin{aligned} p(x,t) &= Aa^{2-n} && \text{for } |x| = a > 0, \\ p(x,t) &= 0 && \text{for } |x| = R(t), \\ R'(t) &= \frac{1}{L}|Dp| && \text{for } |x| = R(t), \\ R(0) &= b > a. \end{aligned} \tag{3.14}$$

Furthermore,

$$\begin{aligned} \lim_{t\to\infty} \frac{R(t)}{c_\infty t^{1/n}} &= 1, \qquad c_\infty = \left(\frac{An(n-2)}{L}\right)^{1/n} && \text{if } n \geq 3, \\ \lim_{t\to\infty} \frac{R(t)}{c_\infty \left(t/\log t\right)^{1/2}} &= 1, \qquad c_\infty = 2\sqrt{A/L} && \text{if } n = 2. \end{aligned}$$

In dimension $n = 2$, we will also use $\lim\limits_{t\to\infty} \frac{\log R(t)}{\log t} = \frac{1}{2}$.

The radial solution of the Stefan problem satisfies the corresponding conditions similar to (3.14) together with the initial data

$$\theta(x,0) = \theta_0(|x|) \text{ if } |x| \geq a. \tag{3.15}$$

The following results were shown in [49].

**Lemma 3.3** (cf. [49, Proposition 6.1])**.** *Let $p$ and $\theta$ be radially symmetric solutions to the Hele-Shaw problem and to the Stefan problem respectively, and let $\{|x| = R_p(t)\}, \{|x| = R_\theta(t)\}$ be the corresponding interfaces. If $R_p(0) > R_\theta(0), p(x,0) \geq \theta(x,0)$ and, moreover, $p(x,t) \geq \theta(x,t)$ on the fixed boundary, that is, for $|x| = a, t > 0$, then $p(x,t) \geq \theta(x,t)$ for all $|x| \geq a$ and $t \geq 0$.*

This immediately leads to an upper bound for the free boundary of radial solutions of Stefan problem, see Corollary 6.2, Theorem 6.4, Theorem 7.1 in [49].

**Lemma 3.4.** *Let $\{|x| = R(t)\}$ be the free boundary of a radial solution to the Stefan problem satisfying the corresponding conditions (3.14) and (3.15). There are constants $C, T > 0$, such that, for all $t \geq T$,*

$$R(t) \leq Ct^{1/n}, n \geq 3, \quad or \quad R(t) \leq C(t/\log t)^{1/2}, n = 2.$$

*Moreover, we have*

$$\lim_{t\to\infty} \frac{R(t)}{t^{1/n}} = (An(n-2)/L)^{1/n}, n \geq 3, \quad or \quad \lim_{t\to\infty} \frac{R(t)}{(t/\log t)^{1/2}} = 2\sqrt{A/L}, n = 2.$$

The solution of the Stefan problem (not restricted to the radially symmetric case) is bounded for all time.

**Lemma 3.5** (cf. [49, Lemma 6.3])**.** *Let $\theta$ be a weak solution of the Stefan problem for $n \geq 2$. There is a constant $C > 0$ such that, for all $t > 0$, $0 \leq \theta(x,t) \leq C|x|^{2-n}$.*

Next, we define the solution of the Hele-Shaw problem with a point source, which will appear as the limit function in our convergence results,

$$V(x,t) = V_{A,L}(x,t) = \begin{cases} A\left(|x|^{2-n} - \rho^{2-n}(t)\right)_+, & n \geq 3, \\ A\left(\log \frac{\rho(t)}{|x|}\right)_+, & n = 2, \end{cases} \tag{3.16}$$

where

$$\rho(t) = \rho_L(t) = R_\infty = \begin{cases} (An(n-2)t/L)^{1/n}, & n \geq 3, \\ (2At/L)^{1/2}, & n = 2. \end{cases}$$

It is the unique solution of the Hele-Shaw problem with a point source,

$$\begin{cases} \Delta v = 0 & \text{in } \Omega(v)\backslash\{0\}, \\ \lim_{|x|\to 0} \frac{v(x,t)}{|x|^{2-n}} = A, & n \geq 3, \quad \text{or} \quad \lim_{|x|\to 0} -\frac{v(x,t)}{\log(|x|)} = A, \quad n = 2, \\ v_t = \frac{1}{L}|Dv|^2 & \text{on } \partial\Omega(v), \\ v(x,0) = 0 & \text{in } \mathbb{R}^n\backslash\{0\}. \end{cases} \tag{3.17}$$

The asymptotic result for radial solutions of the Stefan problem follows from Theorem 6.5 and Theorem 7.2 in [49].

**Theorem 3.6** (Far field limit)**.** *Let $\theta$ be the radial solution of the Stefan problem satisfying the corresponding boundary and initial conditions (3.14), (3.15). Then*

$$\lim_{t\to\infty} t^{(n-2)/n}|\theta(x,t) - V(x,t)| = 0 \tag{3.18}$$

*uniformly on sets of form* $\{x \in \mathbb{R}^n : |x| \geq \delta t^{1/n}\}, \delta > 0$ *if* $n \geq 3$, *and*

$$\lim_{t\to\infty} \log \sqrt{\frac{2A}{L}}\mathcal{R}(t) \left| \theta(x,t) - \frac{A}{\log \sqrt{\frac{2A}{L}}\mathcal{R}(t)} \left( \log \sqrt{\frac{2A}{L}}\mathcal{R}(t) - \log|x| \right)_+ \right| = 0 \quad (3.19)$$

*uniformly on sets of form* $\{x \in \mathbb{R}^n : |x| \geq \delta\mathcal{R}(t)\}, \delta > 0$ *if* $n = 2$.

*Proof.* Following the proof of Theorem 6.5 in [49] with recalling that we assume $\theta = Aa^{2-n}$ for $|x| = a$, we immediately get the result for $n = 3$.

For $n = 2$, let $\mathcal{R}_1(t)$ be the solution of $\frac{\mathcal{R}_1^2}{2}\left(\log \mathcal{R}_1 - \frac{1}{2}\right) = \frac{At}{L}$ with $\lim_{t\to\infty} \frac{\mathcal{R}_1(t)}{\mathcal{R}(t)} = \sqrt{\frac{2A}{L}}$. Thus, we can replace $\mathcal{R}_1(t)$ in Theorem 7.2 in [49] by $\sqrt{\frac{2A}{L}}\mathcal{R}(t)$. □

Finally, we can improve Theorem 3.6 to have the following convergence result for rescaled radial solutions of the Stefan problem which holds up to $t = 0$.

**Lemma 3.7** (Convergence for radial case)**.** *Let* $\theta(x,t)$ *be a radial solution of the Stefan problem satisfying the corresponding boundary and initial conditions* (3.14) *and* (3.15)*. Then* $\theta^\lambda$ *converges locally uniformly to* $V_{A,L}$ *in the set* $(\mathbb{R}^n \backslash \{0\}) \times [0,\infty)$.

*Proof.* We will prove the uniform convergence in the sets $Q = \{(x,t) : |x| \geq \varepsilon, 0 \leq t \leq T\}$ for some $\varepsilon, T > 0$ and use notation $V = V_{A,L}$. We consider the case $n \geq 3$ first. Set $\xi = \lambda^{1/n}x, \tau = \lambda t$ then an easy computation leads to $V(x,t) = \lambda^{(n-2)/n}V(\xi,\tau)$. Let $t_0 = \rho^{-1}(\varepsilon/2)$. We split the proof into two cases:

(a) When $0 \leq t \leq t_0$: Clearly from the formula, we have $V(x,t) = 0$ in $\{(x,t) : |x| \geq \varepsilon, 0 \leq t \leq t_0\}$. Besides, for $\lambda$ large enough,

$$R_\lambda(t) = \frac{R(\lambda t)}{\lambda^{1/n}} \leq \frac{R(\lambda t_0)}{\lambda^{1/n}} < \rho(t_0) + \frac{\varepsilon}{2} = \varepsilon \text{ (due to Proposition 3.4).}$$

Thus, $\theta^\lambda = 0 = V$ in $\{(x,t) : |x| \geq \varepsilon, 0 \leq t \leq t_0\}$ for $\lambda$ large enough.

(b) When $t_0 \leq t \leq T$, we have:

$$|\theta^\lambda(x,t) - V(x,t)| = t^{(2-n)/n}\tau^{(n-2)/n}|\theta(\xi,\tau) - V(\xi,\tau)| \quad (3.20)$$

Since $t_0 \leq t \leq T$, $t^{(2-n)/n}$ is bounded. From Theorem 3.6, the right hand side of (3.20) converges to 0 uniformly in the sets $\{\xi \in \mathbb{R}^n : |\xi| \geq \delta\tau^{1/n}\} = \{x \in \mathbb{R}^n : |x| \geq \delta t^{1/n}\} \supset \{(x,t) : |x| \geq \varepsilon, t_0 \leq t \leq T\}$ for fixed $\varepsilon$ and $\delta > 0$ small enough and thus we obtain the convergence for $n \geq 3$.

For $n = 2$, we argue similar as the case $n \geq 3$, but noting that $\lim_{\lambda\to\infty} \frac{\mathcal{R}(\tau)}{\mathcal{R}(\lambda)} = t^{1/2}$ together with Theorem 3.6. □

### 3.1.3 Some more results for viscosity solutions

Following [45, 49], we also can state some results for viscosity solutions.

**Lemma 3.8.** *For $L = 1/m$ (resp. $L = 1/M$), with $m, M$ as in (1), let $\theta(x,t)$ be the radial solution of Stefan problem (3.1) satisfying boundary conditions (3.14) and initial condition (3.15) with $g(x) = 1/L$ and $a$ such that $B(0,a) \subset K$ (resp. $K \subset B(0,a)$). Then the function $\theta(x,t)$ is a viscosity subsolution (resp. supersolution) of the Stefan problem (3.1) in $Q$.*

*Proof.* The statement follows directly from properties of radially solutions and the fact that a classical solution is also a viscosity solution. □

Using viscosity comparison principle, we also can get the same estimates for free boundary as in Proposition 3.4 and boundedness for a **general viscosity solution**.

**Lemma 3.9.** *Let $v$ be a viscosity solution of (3.1). There exists $t_0 > 0$ and constant $C, C_1, C_2 > 0$ such that for $t \geq t_0$,*

$$C_1 t^{1/n} < \min_{\Gamma_t(v)} |x| \leq \max_{\Gamma_t(v)} |x| < C_2 t^{1/n} \qquad \text{if } n \geq 3,$$

$$C_1 \mathcal{R}(t) < \min_{\Gamma_t(v)} |x| \leq \max_{\Gamma_t(v)} |x| < C_2 \mathcal{R}(t) \qquad \text{if } n = 2,$$

*and for $0 \leq t \leq t_0$, $\max\limits_{\Gamma_t(v)} |x| < C_2$. Moreover, $0 \leq v(x,t) \leq C|x|^{2-n}$ for all $n \geq 2$.*

*Proof.* Argue as in [45] with using Lemma 3.4 and Lemma 3.5 above. □

We also have the near field limit and the asymptotic behavior result as in [49].

**Theorem 3.10** (Near-field limit)**.** *The viscosity solution $v(x,t)$ of the Stefan problem (3.1) converges to the unique solution $P(x)$ of the exterior Dirichlet problem*

$$\begin{cases} \Delta P = 0, & x \in \mathbb{R}^n \backslash K, \\ P = 1, & x \in \Gamma, \\ \lim\limits_{|x|\to\infty} P(x) = 0 \quad \text{if } n \geq 3, & \text{or} \quad P \text{ is bounded if } n = 2, \end{cases} \tag{3.21}$$

*as $t \to \infty$ uniformly on compact subsets of $\overline{K^c}$.*

*Proof.* See proof of Theorem 8.1 in [49]. □

**Lemma 3.11** (cf. [49, Lemma 4.5])**.** *There exists a constant $C_* = C_*(K,n)$ such that the solution $P$ of problem (3.21) satisfies $\lim\limits_{|x|\to\infty} |x|^{n-2} P(x) = C_*$.*

## 3.2 Uniform convergence of the rescaled variational solutions

### 3.2.1 Limit problem

We first recall the limit variational problem as introduced in [45] (see [45, section 5] for derivation and properties). Let $U_{A,L}(x,t) := \int_0^t V_{A,L}(x,s)\,ds$ then $U_{A,L}(x,t)$ has form:

$$U_{A,L}(x,t) = \begin{cases} \left[ At|x|^{2-n} + \dfrac{L}{2n}|x|^2 - \dfrac{1}{2}(Ant)^{2/n} \left(\dfrac{n-2}{L}\right)^{(2-n)/n} \right]_+ & \text{if } n \geq 3, \\ \left[ \dfrac{A}{2} t \log \dfrac{2At}{Le|x|^2} + \dfrac{L|x|^2}{4} \right]_+ & \text{if } n = 2. \end{cases} \tag{3.22}$$

For given $A, L > 0$, [45, Theorem 5.1] yields that $U_{A,L}(x,t)$ is the unique solution of the limit obstacle problem

$$\begin{cases} w \in \mathcal{K}_t, \\ a(w,\phi) \geq \langle -L, \phi \rangle, & \text{for all } \phi \in V, \\ a(w,\psi w) = \langle -L, \psi w \rangle & \text{for all } \psi \in W, \end{cases} \tag{3.23}$$

where $\mathcal{K}_t = \left\{ \varphi \in \bigcap_{\varepsilon > 0} H^1(\mathbb{R}^n \backslash B_\varepsilon) \cap C(\mathbb{R}^n \backslash B_\varepsilon) : \varphi \geq 0,\ \lim\limits_{|x|\to 0} \frac{\varphi(x)}{U_{A,L}(x,t)} = 1 \right\}$,

$$V = \left\{ \phi \in H^1(\mathbb{R}^n) : \phi \geq 0, \phi = 0 \text{ on } B_\varepsilon \text{ for some } \varepsilon > 0 \right\}, \tag{3.24}$$

$$W = V \cap C^1(\mathbb{R}^n), \tag{3.25}$$

and

$$a_\Omega(u,v) := \int_\Omega Du \cdot Dv dx, \quad \langle u, v \rangle_\Omega := \int_\Omega uv dx.$$

We omit the set $\Omega$ in the notation if $\Omega = \mathbb{R}^n$.

### 3.2.2 Uniform convergence of rescaled variational solutions

Now we are ready to prove the first main result, similar to Theorem 6.2 in [45].

**Theorem 3.12.** *Let $u$ be the unique solution of variational problem (2.8) and $u^\lambda$ be its rescaling. Let $U_{A,L}$ be the unique solution of limit problem (3.23) where $A = C_*$ as in Lemma 3.11, and $L = \langle 1/g \rangle$ as in Lemma 2.20. Then the functions $u^\lambda$ converges locally uniformly to $U_{A,L}$ as $\lambda \to \infty$ on $(\mathbb{R}^n \backslash \{0\}) \times [0, \infty)$.*

*Proof.* We argue as in [45]. Fix $T > 0$. By Lemma 3.9, we can bound $\Omega_t(u^\lambda)$ by $\Omega := B_\delta(0)$ for some $\delta > 0$, for all $0 \le t \le T$ and $\lambda > 0$. For some $\varepsilon > 0$, define $\Omega_\varepsilon := \Omega \backslash \overline{B(0,\varepsilon)}$, $Q_\varepsilon := \Omega_\varepsilon \times [0,T]$ . We will prove the convergence in $Q_\varepsilon$.

Let $v$ be the viscosity solution of the Stefan problem (3.1). We can find constants $0 < a < b$ such that $K \subset B_a(0)$ and $\overline{\Omega_0} \subset B_b(0)$. Set $L = 1/M$ and $A = \max v_0$. Choose radially symmetric smooth $\theta_0 \ge 0$ such that $\theta_0 \ge v_0$ on $\Omega_0 \setminus B_a(0)$ and $\theta_0 = 0$ on $\mathbb{R}^n \setminus B_b(0)$. The radial solution $\theta$ of the Stefan problem on $\mathbb{R}^n \setminus \overline{B_a(0)}$ with such parameters will be above $v$ by the comparison principle. Thus, for $\lambda$ large enough, the rescaled solutions satisfy

$$0 \le v^\lambda \le \theta^\lambda \text{ in } Q_{\varepsilon/2}.$$

On the other hand, by Lemma 3.7, $\theta^\lambda$ converges to $V_{A,L}$ as $\lambda \to \infty$ uniformly on $Q_{\varepsilon/2}$ and $V_{A,L}$ is bounded in $Q_{\varepsilon/2}$ and therefore for $\lambda$ large enough so that $(B_a(0))^\lambda := \frac{B_a(0)}{\lambda^{1/n}} \subset B_{\varepsilon/2}(0)$,

$$\|u_t^\lambda\|_{L^\infty(Q_{\varepsilon/2})} = \|v^\lambda\|_{L^\infty(Q_{\varepsilon/2})} \le C(\varepsilon). \tag{3.26}$$

Since $u^\lambda$ satisfies (3.7), we have

$$\Delta u^\lambda(\varphi - u^\lambda) \le \left(\lambda^{(2-n)/n} u_t^\lambda - f(\lambda^{1/n} x)\right)(\varphi - u^\lambda) \text{ a.e for any } \varphi \in \mathcal{K}^\lambda(t).$$

As $u_t^\lambda$ is bounded, $u^\lambda$ satisfies the elliptic obstacle problem

$$\Delta u^\lambda(\varphi - u^\lambda) \le \left(C\lambda^{(2-n)/n} - f(\lambda^{1/n} x)\right)(\varphi - u^\lambda)$$

a.e for any $\varphi \in \mathcal{K}^\lambda(t)$ such that $\varphi - u^\lambda \ge 0$.

Now we can use the standard regularity estimates for the obstacle problem (see [51, Proposition 2.2, chapter 5] for instance),

$$\|\Delta u^\lambda(\cdot,t)\|_{L^p(\Omega_{\varepsilon/2})} \le \left\| C\lambda^{(2-n)/n} - \frac{1}{g^\lambda} \right\|_{L^p(\Omega_{\varepsilon/2})} \le C_0 \text{ for all } 1 \le p \le \infty,$$

for all $\lambda$ large so that also $\Omega_0^\lambda \subset B_{\varepsilon/2}(0)$. Using (3.26) and $u^\lambda(x,t) = \int_0^t v^\lambda(x,s)ds$, we conclude $\|u^\lambda(\cdot,t)\|_{L^p(\Omega_{\varepsilon/2})}$ is bounded uniformly in $t \in [0,T]$ and $\lambda$ large.

Using elliptic interior estimate results for obstacle problem again (for example, [51, Theorem 2.5]), we can find constants $0 < \alpha < 1$ and $C_2$, independent of $t \in [0,T]$ and $\lambda \gg 1$, such that

$$\begin{aligned} \|u^\lambda(\cdot,t)\|_{W^{2,p}(\Omega_\varepsilon)} &\le C_2, \\ \|u^\lambda(\cdot,t)\|_{C^{0,\alpha}(\Omega_\varepsilon)} &\le C_2, \end{aligned} \quad \text{for all } 0 \le t \le T, \lambda \gg 1.$$

Moreover, using (3.26) again, we have $|u^\lambda(x,t) - u^\lambda(x,s)| \le C_3|t-s|$. Thus $u^\lambda$ is Hölder continuous in $x$ with $0 < \alpha < 1$ and Lipschitz continuous in $t$. In particular, $u^\lambda$ satisfies

$$\|u^\lambda\|_{C^{0,\alpha}(Q_\varepsilon)} \le C_4(C_2, C_3) \text{ for all } \lambda \ge \lambda_0.$$

The argument for case $n = 2$ is similar.

By the Arzelà-Ascoli theorem, we can find a function $\bar{u} \in C((\mathbb{R}^n \setminus \{0\}) \times [0,\infty))$ and a subsequence $\{u^{\lambda_k}\} \subset \{u^\lambda\}$ such that

$$u^{\lambda_k} \to \bar{u} \text{ locally uniformly on } (\mathbb{R}^n \setminus \{0\}) \times [0,\infty) \text{ as } k \to \infty,$$

Due to the compact embedding of $H^2$ in $H^1$, we have, $u^{\lambda_k}(\cdot,t) \to \bar{u}(\cdot,t)$ strongly in $H^1(\Omega_\varepsilon)$ for all $t \ge 0$, $\varepsilon > 0$.

To finish the proof, we need to show that the function $\bar{u}$ is the solution of limit problem (3.23) and then by the uniqueness of the limit problem, we deduce that the convergence is not restricted to a subsequence.

**Lemma 3.13** (cf. [45, Lemma 6.3])**.** *For each $t \ge 0$, $\bar{w} := \bar{u}(\cdot,t)$ satisfies*

$$a(\bar{w},\phi) \ge \langle -L, \phi\rangle \textit{ for all } \phi \in V, \tag{3.27}$$

$$a(\bar{w},\psi\bar{w}) = \langle -L, \psi\bar{w}\rangle \textit{ for all } \psi \in W, \tag{3.28}$$

*where $L = \langle 1/g\rangle$ as in Lemma 2.20 and $V, W$ as in (3.24) and (3.25).*

*Proof.* Consider $n \ge 3$. Following the techniques in [45], fix $t \in [0,T]$ and denote $w^k := u^{\lambda_k}(\cdot,t)$. Take $\phi \in V$ first. Analogously to Remark 3.2, we only need to prove the inequality for functions $\phi$ with compact support, the conclusion for general function $\phi$ in $V$ will follow by the continuity of all terms in the inequality. There exists $k_0 > 0$ such that for all $k \ge k_0$, $\Omega_0^{\lambda_k} \subset B_\varepsilon(0)$ and $\phi = 0$ on $B_\varepsilon(0)$. Set $\varphi^k = \phi + w^k \in \mathcal{K}^{\lambda_k}(t)$. Substitute the function $\varphi^k$ into the rescaled equation (3.7) and integrate both sides and integrate by parts, which yields

$$a(w^{\lambda_k},\phi) \ge -\lambda_k^{(2-n)/n}\left\langle u_t^{\lambda_k}(\cdot,t),\phi\right\rangle + \left\langle -\frac{1}{g^{\lambda_k}},\phi\right\rangle.$$

The linear functional $w \mapsto a(w,\phi)$ is bounded in $H^1$. Recalling Lemma 2.20 and that $u_t^{\lambda_k}$ is bounded, since $w^k \to \bar{w}$ strongly in $H^1$ as $k \to \infty$, we can send $\lambda_k \to \infty$ and obtain (3.27).

Now take $\psi \in W$. As above, we assume that $\psi$ has compact support, and without loss of generality we can also assume that $0 \leq \psi \leq 1, \psi = 0$ on $B_\varepsilon(0)$ (otherwise consider $\frac{\psi}{\max_{\mathbb{R}^n} \psi}$ instead). Take $k_0$ such that $\Omega_0^{\lambda_k} \subset B_\varepsilon(0)$ for all $k \geq k_0$. Since $\psi \in W$ then $\psi \bar{w} \in V$. As above we have $a(\bar{w}, \psi \bar{w}) \geq \langle -L, \psi \bar{w} \rangle$. Moreover, consider $\varphi^k = (1-\psi) w^k \in \mathcal{K}^{\lambda_k}(t)$, $k \geq k_0$. Then,

$$a(w^k, \psi w^k) = -a(w^k, \varphi^k - w^k) \leq \left\langle -\frac{1}{g^{\lambda_k}}, \psi w^k \right\rangle - \lambda_k^{(2-n)/n} \left\langle u_t^{\lambda_k}(\cdot, t), \psi w^k \right\rangle.$$

Again using Lemma 2.20, boundedness in $L^\infty(\mathbb{R}^n)$ of $w^k$ and $u_t^{\lambda_k}$, the lower semi-continuity in $H^1$ of the map $w \mapsto a(w, \psi w)$, and the fact that $w^k \to \bar{w}$ strongly in $H^1$ as $k \to \infty$ we can conclude the equality (3.28).

Again, $n = 2$ is similar. □

Finally, the next lemma establishes that the singularity of $\bar{u}$ as $|x| \to 0$ is correct.

**Lemma 3.14** (cf. [45, Lemma 6.4])**.** *We have*

$$\lim_{|x| \to 0} \frac{\bar{u}(x,t)}{U_{C_*,L}}(x,t) = 1$$

*for every $t \geq 0$, where $L = \langle 1/g \rangle$ as in Lemma 2.20 and $C_*$ as in Lemma 3.11.*

*Proof.* Let $C_*$ as in Lemma 3.11 and fix a $\varepsilon > 0$. By Lemma 3.11, there exists $a$ large enough such that

$$\left| \frac{P(x)}{|x|^{2-n}} - C_* \right| < \frac{\varepsilon}{2}$$

for all $x, |x| \geq a$. In particular,

$$\left| \frac{P(x)}{a^{2-n}} - C_* \right| < \frac{\varepsilon}{2}.$$

Consider the Stefan problem in the set $\Omega_a := \{|x| \geq a\}, \Omega_a \subset \mathbb{R}^n \backslash K$ for $a$ large enough. The fixed boundary $\{|x| = a\}$ is a compact subset of $\mathbb{R}^n \backslash K$. Then by Theorem 3.10, there exists $t_0 > 0$ such that for all $t \geq t_0$:

$$\left| \frac{v(x,t)}{a^{2-n}} - \frac{P(x)}{a^{2-n}} \right| < \frac{\varepsilon}{2}.$$

Thus by triangle inequality we have for all $t \geq t_0$, for all $x$ such that $|x| = a$,

$$\left| \frac{v(x,t)}{a^{2-n}} - C_* \right| \leq \varepsilon.$$

Now let $\theta_+, \theta_-$ be respective radial solutions of the Stefan problem satisfying boundary and initial data

$$\frac{\theta_\pm}{a^{2-n}} = C_* \pm \varepsilon \text{ on } |x| = a, \qquad \theta_-(x,t_0) < v(x,t_0) < \theta_+(x,t_0),$$
$$R_{\theta_+}(t_0) = \max_{x \in \Gamma_{t_0}(v)} |x|, \qquad R_{\theta_-}(t_0) = \min_{x \in \Gamma_{t_0}(v)} |x|,$$
$$R'_{\theta_\pm}(t) = \frac{1}{L_\pm}|D\theta|, \qquad L_+ = \frac{1}{M}, L_- = \frac{1}{m}$$

where $M, m$ as in (1).

The comparison principle for viscosity solutions tell us that

$$\theta_+ \le v \le \theta_- \text{ for all } |x| \ge a, t \ge t_0.$$

Then the respective rescaling satisfies

$$\theta_+^\lambda \le v^\lambda \le \theta_-^\lambda \text{ in } \{(x,t) : \lambda^{1/n}|x| \ge |a|, \lambda \ge t_0/t\} \text{ for } n \ge 3,$$

or

$$\theta_+^\lambda \le v^\lambda \le \theta_-^\lambda \text{ in } \{(x,t) : \mathcal{R}(\lambda)|x| \ge |a|, \lambda \ge t_0/t\} \text{ for } n = 2.$$

Note that Lemma 3.7 gives us the locally uniform convergence of $\theta_\pm^\lambda$ to solutions of the Hele-Shaw problem with a point source $V_\pm := V_{C_* \pm \varepsilon, L_\pm}$ on $(\mathbb{R}^n \backslash \{0\}) \times [0, \infty)$ as $\lambda \to \infty$. Applying the Baiocchi transform of $v^\lambda$ we get, for $\lambda^{1/n}|x| \ge a$ (or $\mathcal{R}(\lambda)|x| \ge a$) and $\lambda \ge t_0/t$:

$$\begin{aligned} u^\lambda(x,t) &= \int_0^t v^\lambda(x,s)ds \\ &\le \int_0^{t_0/\lambda} v^\lambda(x,s)ds + \int_{t_0/\lambda}^t \theta_+^\lambda(x,s)ds. \end{aligned}$$

We see that:

- Similar to the explanation before, for every $\lambda$ , function $v^\lambda$ is bounded in the set $\{(x,t) : \lambda^{1/n}|x| \ge a$ (or $\mathcal{R}(\lambda)|x| \ge a$) and $0 \le s \le t_0/\lambda\}$, the first term of right hand side tends to 0 as $\lambda \to \infty$ ,

- By Lemma 3.7, $\theta_+^\lambda \to V_+$ uniformly in $(\mathbb{R}^n \backslash \{0\}) \times [0, \infty)$ as $\lambda \to \infty$ then by the dominated convergence theorem, the second term converges to $\int_0^t V_+(x,s)ds$ as $\lambda \to \infty$.

Using the same argument we can find the lower bound for $u^\lambda$ and ve have

$$\int_0^t V_-(x,s)ds \le \liminf_{\lambda\to\infty} u^\lambda(x,t) \le \bar{u}(x,t) \le \limsup_{\lambda\to\infty} u^\lambda(x,t) \le \int_0^t V_+(x,s)ds \quad (3.29)$$

for all $(x,t) \in (\mathbb{R}^n \backslash \{0\}) \times [0,\infty)$.

Consider the case $n \ge 3$, dividing (3.29) by $|x|^{2-n}$ and taking the limit when $|x| \to 0$ we have

$$\begin{aligned} \liminf_{|x|\to 0} \frac{1}{|x|^{2-n}} \int_0^t V_-(x,s)ds \quad &\le \liminf_{|x|\to 0} \frac{\bar{u}(x,t)}{|x|^{2-n}} \\ &\le \limsup_{|x|\to 0} \frac{\bar{u}(x,t)}{|x|^{2-n}} \le \limsup_{|x|\to 0} \frac{1}{|x|^{2-n}} \int_0^t V_+(x,s)ds. \end{aligned} \quad (3.30)$$

We know from Section 3.2.1 that

$$\int_0^t V_\pm(x,s)ds = U_\pm(x,t) := U_{C_*\pm\varepsilon, L_\pm}(x,t),$$

and $U_\pm$ have explicit form as in (3.22) with respective constants then

$$\lim_{|x|\to 0} \frac{1}{|x|^{2-n}} \int_0^t V_\pm(x,s)ds = \lim_{|x|\to 0} \frac{U_\pm(x,t)}{|x|^{2-n}} = (C_* \pm \varepsilon)t.$$

We do the same way for the case when $n = 2$, just replace $|x|^{2-n}$ by $-\log|x|$ and obtain the similar result. Now since $\varepsilon > 0$ is arbitrary, we can take the limit when $\varepsilon \to 0+$ to get

$$\lim_{|x|\to 0} \frac{\bar{u}(x,t)}{|x|^{2-n}} = C_* t = \lim_{|x|\to 0} \frac{U_{C_*,L}(x,t)}{|x|^{2-n}}$$

which finish the proof of Lemma 3.14. □

This finishes the proof of Theorem 3.12 . □

## 3.3 Uniform convergence of the rescaled viscosity solutions and free boundaries

In this section, we will deal with the convergence of $v^\lambda$ and their free boundaries. Let $v$ be a viscosity solution of the Stefan problem (3.1) and $v^\lambda$ be its rescaling. Let

$V = V_{C_*,L}$ be the solution of Hele-Shaw problem with a point source as in (3.16), where $C_*$ is the constant of Lemma 3.11 and $L = \langle 1/g \rangle$ as in Lemma 2.20.

We define the half-relaxed limits in $\{|x| \neq 0, t \geq 0\}$:

$$v^*(x,t) = \limsup_{(y,s),\lambda \to (x,t),\infty} v^\lambda(y,s), \qquad v_*(x,t) = \liminf_{(y,s),\lambda \to (x,t),\infty} v^\lambda(y,s),$$

*Remark* 3.15. $V$ is continuous in $\{|x| \neq 0, t \geq 0\}$, therefore $V_* = V = V^*$.

To complete Theorem 3.1, we prove a result similar to [45, Theorem 7.1.]

**Theorem 3.16.** *The rescaled viscosity solution $v^\lambda$ of the Stefan problem (3.1) converges locally uniformly to $V = V_{C_*,\langle 1/g \rangle}$ in $(\mathbb{R}^n \backslash \{0\}) \times [0,\infty)$ as $\lambda \to \infty$ and*

$$v_* = v^* = V.$$

*Moreover, the rescaled free boundary $\{\Gamma(v^\lambda)\}_\lambda$ converges to $\Gamma(V)$ locally uniformly with respect to the Hausdorff distance.*

To prepare for the proof of Theorem 3.16, we need to collect some results which are similar to the ones in [39] and [45] with some adaptations to our case. All the results for $n \geq 3$ we have in this section can be obtained for $n = 2$ by using limit $\frac{1}{\log \mathcal{R}(\lambda)} \to 0$ as $\lambda \to \infty$. Thus, from here on we only consider case $n \geq 3$, the results for $n = 2$ are omitted.

### 3.3.1 Some necessary technical results

**Lemma 3.17** (cf. [39, Lemma 3.9])**.** *The viscosity solution $v$ of the Stefan problem (3.1) is strictly positive in $\Omega(u)$, satisfies $\Omega(v) = \Omega(u)$ and $\Gamma(v) = \Gamma(u)$.*

**Lemma 3.18.** *Let $v^\lambda$ be a viscosity solution of the rescaled problem* (3.6)*. Then $v^*(\cdot,t)$ is subharmonic in $\mathbb{R}^n \setminus \{0\}$ and $v_*(\cdot,t)$ is superharmonic in $\Omega_t(v_*)\backslash\{0\}$ in viscosity sense.*

*Proof.* We will prove the statement for subharmonic case using contradiction argument, the proof for superharmonic case is similar.

Assume that $v^*(\cdot,t_0)$ is not subharmonic in $\Omega_{t_0}(v^*)\backslash\{0\}$ in viscosity sense. Then there exists a function $\varphi \in C^2(\Omega_{t_0}(v^*)\backslash\{0\})$ that touches $v^*(\cdot,t_0)$ from above at $x_0$ in $B_r(x_0) := B(x_0,r) \subset \Omega_{t_0}(v^*)\backslash\{0\}$ and $\Delta\varphi(x) < 0$ in $B_r(x_0)$.

Consider a smooth perturbation of $\varphi$, we can assume that there exists some small constants $\delta, r > 0$ such that $\varphi \geq 0$ in $B_r(x_0)$ and

$$\varphi(x) < v^*(x, t_0) - \delta, \qquad \text{for all } x \in B_\delta(x_0), \tag{3.31}$$

$$\varphi(x) > \max_{\overline{B_r(x_0)}} v^*(y, t_0) + \delta, \qquad \text{for all } x \in \partial B_r(x_0). \tag{3.32}$$

From (3.32) and the fact that $v^*$ is upper semicontinuous function we have there exists $t_1 < t_0$ such that $\varphi(x) > \max_{\overline{B_r(x_0)} \times [t_1, t_0]} v^*(y, t)$ for all $x \in \partial B_r(x_0)$. Indeed, assume that for every $t_1 < t_0$, there exists $x_0 \in \partial B_r(x_0)$ such that $\varphi(x_0) \leq \max_{\overline{B_r(x_0)} \times [t_1, t_0]} v^*(y, t)$. Choose a sequence $\{t_1^n = t_0 - \frac{1}{n}\}$, for each $t_1^n$, there exists $x_n \in \partial B_r(x_0)$ and $(y_n, t_n) \in \overline{B_r(x_0)} \times [t_1, t_0]$ such that $\varphi(x_n) \leq v^*(y_n, t_n)$. Since all the sequences are bounded, there exists subsequences $\{x_{n_k}\}$ converges to $x^* \in \partial B_r(x_0)$ and $\{(y_{n_k}, t_{n_k})\}$ converges to $(y_0, t_0) \in \overline{B_r(x_0)} \times \{t_0\}$. Taking lim sup of $\varphi(x_{n_k})$ and $v^*(y_{n_k}, t_{n_k})$ as $k \to \infty$ we have $\varphi(x^*) \leq v^*(y_0, t_0)$ which yields a contradiction with (3.32).

Let $Q(x_0, t_0) := B_r(x_0) \times (t_1, t_0)$ and $\Gamma_p := \partial_p Q(x_0, t_0)$ be the parabolic boundary of $Q(x_0, t_0)$. Consider function

$$\tilde{\varphi}(x, t) := \varphi(x) + \frac{t_0 - t}{t_0 - t_1} \left( \max_{\Gamma_p} v^*(y, t) \right).$$

We have on $\partial B_r(x_0) \times [t_1, t_0]$,

$$\tilde{\varphi}(x, t) > \varphi(x) > \max_{\Gamma_p} v^*(y, t),$$

and on $B_r(x_0) \times \{t_1\}$,

$$\tilde{\varphi}(x, t_1) = \varphi(x) + \max_{\Gamma_p} v^*(y, t) > \max_{\Gamma_p} v^*(x, t).$$

Therefore $\tilde{\varphi} > \max_{\Gamma_p} v^*(y, t)$ on the parabolic boundary $\partial_p Q(x_0, t_0)$ of $Q(x_0, t_0)$.

Moreover, for compact set $\Gamma_p$, for all $\varepsilon > 0$, there exists $\lambda_0 > 0$ such that

$$v^\lambda \leq \max_{\Gamma_p} v^* + \varepsilon \text{ on } \Gamma_p$$

for all $\lambda \geq \lambda_0$. Choose

$$\varepsilon = \min_{\Gamma_P} \left( \frac{\tilde{\varphi}(x, t) - \max_{\Gamma_p} v^*}{2} \right)$$

we have $v^\lambda < \tilde{\varphi}$ on $\Gamma_p$ for all $\lambda \geq \lambda_0$.

Besides, we have $\Delta\tilde{\varphi} = \Delta\varphi$ is strictly negative in $Q(x_0, t_0)$ and $\lambda_k^{(2-n)/n} \to 0$ as $k \to \infty$,

$$\tilde{\varphi}_t = \frac{1}{t_1 - t_0} \max_{\Gamma_p} v^*(y, t) \text{ is a finite constant.}$$

Then for $k$ large enough we have:

$$\lambda^{(2-n)/n} \tilde{\varphi}_t - \Delta\tilde{\varphi} > 0 \text{ in } Q(x_0, t_0).$$

Therefore $\tilde{\varphi}$ is a supersolution of elliptic equation in (3.6) in $Q(x_0, t_0)$ for $\lambda$ large enough and we have

$$\tilde{\varphi} \geq v^\lambda \text{ in } Q(x_0, t_0). \tag{3.33}$$

On the other hand, by the definition of lim sup, there exists a subsequence $\{v^{\lambda_k}\} \subset \{v^\lambda\}$, and $\{y_k\}, \{s_k\}$ such that

$$y_k \to x, s_k \to t_0 \text{ as } k \to \infty,$$
$$v^*(x, t_0) = \lim_{k \to \infty} v^{\lambda_k}(y_k, s_k).$$

Thus, for every $\delta > 0$, there exists $k_0$ such that $\left|v^{\lambda_k}(y_k, s_k) - v^*(x, t_0)\right| \leq \delta/2$ for all $k \geq k_0$. Let $\delta$ as in (3.31), since $\tilde{\varphi}$ is continuous, we have

$$v^{\lambda_k}(y_k, s_k) \geq v^*(x, t_0) - \delta/2 > \varphi(x) + \delta/2 \geq \tilde{\varphi}(y_k, t_k)$$

for some $x \in B_\delta(x_0)$ and $k$ large enough, which is in contradiction with (3.33). Therefore, for each $t$, $v^*(\cdot, t)$ is subharmonic in $\Omega_t(v^*)$ and $v^*(\cdot, t) = 0$ in $\Omega_t^c(v^*)\backslash\{0\}$. Such function is subharmonic in $\mathbb{R}^n\backslash\{0\}$ as we expected. □

The behavior of functions $v^*, v_*$ at the origin and their boundaries can be established by following the arguments in [45] and [39].

**Lemma 3.19** ($v^*$ and $v_*$ behave as $V$ at the origin)**.** *The functions $v^*, v_*$ have a singularity at 0 with:*

$$\lim_{|x| \to 0+} \frac{v_*(x, t)}{V(x, t)} = 1, \qquad \lim_{|x| \to 0+} \frac{v^*(x, t)}{V(x, t)} = 1, \text{ for each } t > 0. \tag{3.34}$$

*Proof.* See [45, Lemma 7.4]. □

**Lemma 3.20** (cf. [39, Lemma 5.4])**.** *Suppose that $(x_k, t_k) \in \{u^{\lambda_k} = 0\}$ and $(x_k, t_k, \lambda_k) \to (x_0, t_0, \infty)$. Then:*

*a)* $U(x_0, t_0) = 0$,

*b) If* $x_k \in \Gamma_{t_k}(u^{\lambda_k})$ *then* $x_0 \in \Gamma_{t_0}(U)$,

*where* $U = U_{C_*,L}$ *is the limit function in Theorem 3.12.*

*Proof.* See proof of [39, Lemma 5.4]. □

The rest of the convergence proof in [45] relies on the monotonicity of the solutions of the Hele-Shaw problem in time. Since the Stefan problem lacks this monotonicity, we will show that sufficiently regular initial data satisfies a weak monotonicity below. The convergence result for general initial data will then follow by the uniqueness of the limit and the comparison principle.

**Lemma 3.21.** *Suppose that* $v_0$ *satisfies* (3.2)*. Then there exist* $C \geq 1$ *independent of* $x$ *and* $t$ *such that*

$$v_0(x) \leq Cv(x,t) \text{ in } \mathbb{R}^n \backslash K \times [0, \infty). \tag{3.35}$$

*Proof.* Let $\gamma_1 := \min_{\partial\Omega_0} |Dv_0|, \gamma_2 := \max_{\partial\Omega_0} |Dv_0|$. Note that $0 < \gamma_1 \leq \gamma_2 < \infty$. For given $\varepsilon > 0$, let $w$ be the solution of boundary value problem

$$\begin{cases} \Delta w = 0 & \text{in } \Omega_0 \backslash K, \\ w = \varepsilon & \text{on } K, \\ w = 0 & \text{on } \Omega_0^c. \end{cases}$$

For $x$ close to $\partial\Omega_0$ we have $v_0(x) \geq \frac{\gamma_1}{2}\text{dist}(x, \partial\Omega_0)$. Since $\gamma_1 > 0$, $v_0 > 0$ in $\Omega_0$ and $\partial\Omega_0$ has a uniform ball condition, we can choose $\varepsilon > 0$ small enough such that $w \leq v_0$ in $\mathbb{R}^n \backslash K$. By Hopf's Lemma, $\gamma_w := \min_{\partial\Omega_0} |Dw| > 0$. It is clear that $w$ is a classical subsolution of the Stefan problem (3.1) and the comparison principle yields

$$w \leq v \text{ in } (\mathbb{R}^n \backslash K) \times [0, \infty). \tag{3.36}$$

Now assume that (3.35) does not hold, that is, for every $k \in \mathbb{N}$, there exists $(x_k, t_k) \in \mathbb{R}^n \backslash K \times [0, \infty)$ such that

$$\frac{1}{k} v_0(x_k) > v(x_k, t_k). \tag{3.37}$$

Clearly $x_k \in \Omega_0$. $\{t_k\}$ is bounded by Theorem 3.10 since $v_0$ is bounded. Therefore, there exists a subsequence $(x_{k_l}, t_{k_l})$ and a point $(x_0, t_0)$ such that $(x_{k_l}, t_{k_l}) \to (x_0, t_0)$.

Since $v_0$ is bounded, we get $v(x_0,t_0) \le 0$ and thus $x_0 \in \partial\Omega_0$ by (3.36). Consequently, for $k_l$ large enough,

$$w(x_{k_l}) \ge \frac{1}{2}\gamma_w \mathrm{dist}(x_{k_l}, \partial\Omega_0) = \left(\frac{\gamma_w}{4\gamma_2}\right) 2\gamma_2 \mathrm{dist}(x_{k_l}, \partial\Omega_0) \ge \frac{\gamma_w}{4\gamma_2} v_0(x_{k_l}).$$

Combine this with (3.36) and (3.37) to obtain

$$\frac{1}{k_l} v_0(x_{k_l}) > \frac{\gamma_w}{4\gamma_2} v_0(x_{k_l})$$

for every $k_l$ large enough, which yields a contradiction since $v_0(x_{k_l}) > 0$. $\square$

Some of the following lemmas will hold under the condition (3.35).

**Lemma 3.22.** *Let $u$ be the solution of the variational problem* (2.8)*, and $v$ be the associated viscosity solution of the Stefan problem, and suppose that* (3.35) *holds. Then*

$$u(x,t) \le Ctv(x,t). \tag{3.38}$$

*Proof.* The statement follows from checking that $\tilde{u} := Ctv$ is a supersolution of the heat equation in $\Omega(u)$ and the classical comparison principle. Indeed, $\tilde{u}_t - \Delta\tilde{u} = Cv + Ct(v_t - \Delta v) \ge v_0 \ge f = u_t - \Delta u$ in $\Omega(u)$ by (3.35). $\square$

**Lemma 3.23** (cf. [39, Lemma 5.5])**.** *The function $v_*$ satisfies $\Omega(V) \subset \Omega(v_*)$. In particular $v_* \ge V$.*

*Proof.* Assume that the inclusion does not hold, there exists $(x_0,t_0) \in \Omega(V)$ and $v_*(x_0,t_0) = 0$. By (3.35) and Lemma 3.22, there exists $C > 1$ such that $u(x,t) \le Ctv(x,t)$. This inequality is preserved under the rescaling, $u^\lambda(x,t) \le Ctv^\lambda(x,t)$ in $(\mathbb{R}^n \backslash K^\lambda) \times [0,\infty)$. Taking $\liminf^*$ of both sides gives the contradiction $0 < U(x_0,t_0) \le Ct_0 v_*(x_0,t_0) = 0$.

The inequality $v_* \ge V$ follows from the elliptic comparison principle as $v_*$ is superharmonic in $\Omega(v_*) \setminus \{0\}$ by Lemma 3.18 and behaves as $V$ at the origin by Lemma 3.19. $\square$

**Lemma 3.24.** *There exists constant $C > 0$ independent of $\lambda$ such that for every $x_0 \in \overline{\Omega_{t_0}(u^\lambda)}$ and $B_r(x_0) \cap \Omega_0^\lambda = \emptyset$ for some $r$, for every $\lambda$ large enough we have*

$$\sup_{x \in \overline{B_r(x_0)}} u^\lambda(x,t_0) > Cr^2.$$

*Proof.* Follow the arguments in [38, Lemma 3.1] while noting that since $u_t^\lambda$ is bounded then, for $\lambda$ large enough, $u^\lambda$ is a strictly subharmonic function in $\Omega_{t_0}(u^\lambda) \setminus \overline{\Omega_0^\lambda}$. □

**Corollary 3.25.** *There exists a constant $C_1 = C_1(n, M, \lambda_0)$ such that if $(x_0, t_0) \in \Omega(v^\lambda)$ and $B_r(x_0) \cap \Omega_0^\lambda = \emptyset$ and $\lambda \geq \lambda_0$, we have*

$$\sup_{B_r(x_0)} v^\lambda(x, t_0) \geq \frac{C_1 r^2}{t_0}.$$

*Proof.* The inequality follows directly from Lemma 3.22 and Lemma 3.24. □

**Lemma 3.26** (cf. [39, Lemma 5.6 ii])**.** *We have the following inclusion:*

$$\Gamma(v^*) \subset \Gamma(V).$$

*Proof.* Argue as in [39, Lemma 5.6 ii] together with using Lemma 3.20 and Lemma 3.24 above. □

Now we are ready to prove Theorem 3.16.

### 3.3.2 Proof of Theorem 3.16

*Proof.* ***Step 1***. We prove the convergence of viscosity solutions and the free boundaries under the conditions (3.2) and (3.35) first.

Lemma 3.9 yields that $\Omega_t(v^*)$ is bounded at all time $t > 0$. Since $\Omega(V)$ is simply connected set, Lemma 3.26 implies that

$$\overline{\Omega(v^*)} \subset \overline{\Omega(V)} \subset \Omega(V_{C^*+\varepsilon, L}) \text{ for all } \varepsilon > 0.$$

We see from Lemma 3.18, $v^*(\cdot, t)$ is a subharmonic function in $\mathbb{R}^n \setminus \{0\}$ for every $t > 0$ and $\lim_{|x| \to 0} \frac{v^*(x,t)}{V(x,t)} = 1$ for all $t \geq 0$ by Lemma 3.19, comparison principle yields $v^*(x, t) \leq V_{C_*+\varepsilon, L}(x, t)$ for every $\varepsilon > 0$.

By Lemma 3.23, $V(x, t) \leq v_*$ and letting $\varepsilon \to 0^+$ we obtain by continuity

$$V(x, t) \leq v_*(x, t) \leq v^*(x, t) \leq V(x, t).$$

Therefore, $v_* = v^* = V$ and in particular, $\Gamma(v_*) = \Gamma(v^*) = \Gamma(V)$.

Now we need to show the uniform convergence of the free boundaries with respect to the Hausdorff distance. Fix $0 < t_1 < t_2$ and denote:

$$\Gamma^\lambda := \Gamma(v^\lambda) \cap \{t_1 \leq t \leq t_2\}, \qquad \Gamma^\infty := \Gamma(V) \cap \{t_1 \leq t \leq t_2\},$$

a $\delta$-neighborhood of a set $A$ in $\mathbb{R}^n \times \mathbb{R}$ is

$$U_\delta(A) := \{(x,t) : \operatorname{dist}((x,t),A) < \delta\}.$$

We need to prove that for all $\delta > 0$, there exists $\lambda_0 > 0$ such that:

$$\Gamma^\lambda \subset U_\delta(\Gamma^\infty) \quad \text{and} \quad \Gamma^\infty \subset U_\delta(\Gamma^\lambda), \qquad \forall \lambda \geq \lambda_0. \tag{3.39}$$

We prove the first inclusion in (3.39) by contradiction. Suppose therefore that we can find a subsequence $\{\lambda_k\}$ and a sequence of points $(x_k,t_k) \in \Gamma^{\lambda_k}$ such that $\operatorname{dist}((x_k,t_k),\Gamma^\infty) \geq \delta$. Since $\Gamma^\lambda$ is uniformly bounded in $\lambda$ by Lemma 3.9, there exists a subsequence $\{(x_{k_j},t_{k_j})\}$ which converge to a point $(x_0,t_0)$. By Lemma 3.20, $(x_0,t_0) \in \Gamma(U) = \Gamma(V)$. Moreover, since $t_1 \leq t_{k_j} \leq t_2$ then $t_1 \leq t_0 \leq t_2$ and therefore, $(x_0,t_0) \in \Gamma^\infty$, a contradiction.

The proof of the second inclusion in (3.39) is more technical. We prove a pointwise result first. Suppose that there exists $\delta > 0, (x_0,t_0) \in \Gamma^\infty$ and $\{\lambda_k\}, \lambda_k \to \infty$, such that $\operatorname{dist}((x_0,t_0),\Gamma^{\lambda_k}) \geq \dfrac{\delta}{2}$ for all $k$. Then there exists $r > 0$ such that $D_r(x_0,t_0) := B(x_0,r) \times [t_0 - r, t_0 + r]$ satisfies either:

$$D_r(x_0,t_0) \subset \{v^{\lambda_k} = 0\} \text{ for all } k, \tag{3.40}$$

or after passing to a subsequence,

$$D_r(x_0,t_0) \subset \{v^{\lambda_k} > 0\} \text{ for all } k. \tag{3.41}$$

If (3.40) holds, clearly $V = v_* = 0$ in $D_r(x_0,t_0)$ which is in a contradiction with the assumption that $(x_0,t_0) \in \Gamma^\infty$.

Thus we assume that (3.41) holds. In $D_r(x_0,t_0)$, $v^{\lambda_k}$ solves the heat equation $\lambda^{(2-n)/n} v_t^{\lambda_k} - \Delta v^{\lambda_k} = 0$. Set

$$w^k(x,t) := v^{\lambda_k}(x, \lambda_k^{(2-n)/n} t)$$

then $w^k > 0$ in $D_r^w(x_0,t_0) := B(x_0,r) \times [\lambda_k^{(n-2)/n}(t_0 - r), \lambda_k^{(n-2)/n}(t_0 + r)]$ and $w^k$ satisfies $w_t^k - \Delta w^k = 0$ in $D_r^w(x_0,t_0)$. Since $\lambda_k^{(n-2)/n}\frac{r}{2} \to \infty$ as $k \to \infty$, by Harnack's inequality for the heat equation, for fixed $\tau > 0$ there exists a constant $C_1 > 0$ such that for each $t \in [t_0 - \frac{r}{2}, t_0 + \frac{r}{2}]$ and $\lambda_k$ such that $\tau < \lambda_k^{(n-2)/n}\frac{r}{4}$ we have

$$\sup_{B(x_0,r/2)} w^k(\cdot, \lambda_k^{(n-2)/n} t - \tau) \leq C_1 \inf_{B(x_0,r/2)} w^k(\cdot, \lambda_k^{(n-2)/n} t).$$

This inequality together with Corollary 3.25 yields:

$$\frac{C_2 r^2}{t-\lambda_k^{(2-n)/n}\tau} \le \sup_{B(x_0,r/2)} v^{\lambda_k}(\cdot, t-\lambda_k^{(2-n)/n}\tau) \le C_1 \inf_{B(x_0,r/2)} v^{\lambda_k}(\cdot,t)$$

for all $t \in [t_0 - \frac{r}{2}, t_0 + \frac{r}{2}]$, $\lambda_k \ge \lambda_0$ large enough, where $C_2$ only depends on $n, M, \lambda_0$. Taking the limit when $\lambda_k \to \infty$, the uniform convergence of $\{v^{\lambda_k}\}$ to $V$ gives $V > 0$ in $B(x_0, \frac{r}{2}) \times [t_0 - \frac{r}{2}, t_0 + \frac{r}{2}]$, which is a contradiction with $(x_0, t_0) \in \Gamma^\infty \subset \Gamma(V)$.

We have proved that every point of $\Gamma^\infty$ belongs to all $U_{\delta/2}(\Gamma^\lambda)$ for sufficiently large $\lambda$. Therefore the second inclusion in (3.39) follows from the compactness of $\Gamma^\infty$.

This concludes the proof of Theorem 3.16 when (3.35) holds.

***Step 2.*** For general initial data, we will find upper and lower bounds for the initial data for which (3.35) holds, and use the comparison principle. For instance, assume that $v_0 \in C(\mathbb{R}^n)$, $v_0 \ge 0$, such that $\operatorname{supp} v_0$ is bounded, $v_0 = 1$ on $K$.

Choose smooth bounded domains $\Omega_0^1, \Omega_0^2$ such that $K \subset \Omega_0^1 \subset \overline{\Omega_0^1} \subset \operatorname{supp} v_0 \subset \Omega_0^2$. Let $v_0^1, v_0^2$ be two functions satisfying (3.2) with positive domains $\Omega_0^1, \Omega_0^2$, respectively, and $v_0^1 \le v_0 \le v_0^2$. If necessary, that is, when $v_0$ is not sufficiently regular at $\partial K$, we may perturb the boundary data for $v_0^1$, $v_0^2$ on $K$ as $1-\varepsilon$ and $1+\varepsilon$, respectively, for some $\varepsilon \in (0,1)$.

Let $v_1, v_2$ be respectively the viscosity solution of the Stefan problem (3.1) with initial data $v_0^1, v_0^2$. By the comparison principle, we have $v_1 \le v \le v_2$ and after rescaling $v_1^\lambda \le v^\lambda \le v_2^\lambda$. By Step 1, we see that $v_1^\lambda \to V_{C_{*,1-\varepsilon},L}$ and $v_2^\lambda \to V_{C_{*,1+\varepsilon},L}$. Since $C_{*,1\pm\varepsilon} \to C_*$ as $\varepsilon \to 0$ by [49, Lemma 4.5], we deduce the local uniform convergence of $v^\lambda \to V = V_{C_*,L}$.

The convergence of free boundaries follows from the ordering $\Omega(v_1) \subset \Omega(v) \subset \Omega(v_2)$ and the convergence of free boundaries of $V_{C_{*,1\pm\varepsilon},L}$ to the free boundary of $V_{C_*,L}$ locally uniformly with respect to the Hausdorff distance. $\square$

# Chapter 4

# Long-time behavior of one-phase Stefan-type problems with anisotropic diffusion in periodic media

We consider an anisotropic one-phase Stefan-type problem with periodic coefficients in a dimension $n \geq 3$. Our purpose is to investigate the asymptotic behavior of the solution of the following problem (4.1) and its free boundary as time $t \to \infty$. The results in this chapter, which also appeared in the main reference [48], are the generalizations of our previous work in Chapter 3 for the isotropic case.

Let $K \subset \mathbb{R}^n$ be a compact set and $0 \in \operatorname{int} K$. Furthermore, assume that $K$ has a sufficiently regular boundary, for instance $\partial K \in C^{1,1}$. The one-phase Stefan-type problem is to find a function $v(x,t) : \mathbb{R}^n \times (0,\infty) \to [0,\infty)$ satisfying

$$\begin{cases} v_t - D_i(a_{ij}D_j v) = 0 & \text{in } \{v > 0\} \backslash K, \\ v = 1 & \text{on } K, \\ \dfrac{v_t}{|Dv|} = g a_{ij} D_j v \nu_i & \text{on } \partial\{v > 0\}, \\ v(x,0) = v_0 & \text{on } \mathbb{R}^n, \end{cases} \tag{4.1}$$

where $D$ is the space gradient, $D_i$ is the partial derivative with respect to $x_i$, $v_t$ is the partial derivative of $v$ with respect to time variable $t$ and $\nu = \nu(x,t)$ is inward spatial unit normal vector of $\partial\{v > 0\}$ at point $(x,t)$. Here we use the Einstein summation convention.

We prescribe the Dirichlet boundary data 1 on the fixed source $K$ and an initial temperature distribution of water $v_0$. Note that the results in this chapter apply to a more general time-independent positive fixed boundary data, the constant function 1 is taken only to simplify the notation. We also specify an inhomogeneous medium with the latent heat of phase transition $L(x) = \frac{1}{g(x)}$ and an anisotropic diffusion with the thermal conductivity coefficients given by $a_{ij}(x)$. The unknowns here are the temperature distribution in the water $v$ and the water-ice interface $\partial\{v > 0\}$, which is the so-called free boundary. Since the free boundary is a level set of $v$, the outward normal velocity of the moving interface is given by $\frac{v_t}{|Dv|}$. The free boundary condition thus says that the interface moves outward with the velocity $ga_{ij}D_jv\nu_i$ in the normal direction. Note that we can also rewrite the free boundary condition as

$$v_t = ga_{ij}D_jvD_iv. \tag{4.2}$$

Throughout this chapter, we will consider the problem under the following assumptions. The matrix $A(x) = (a_{ij}(x))$ is assumed to be symmetric, bounded, and uniformly elliptic, i.e., there exits some positive constants $\alpha$ and $\beta$ such that

$$\alpha|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \beta|\xi|^2 \text{ for all } x \in \mathbb{R}^n \text{ and } \xi \in \mathbb{R}^n. \tag{4.3}$$

Moreover, we are interested in the problems with highly oscillating coefficients that guarantees averaging behavior in the scaling limit, in particular $a_{ij}$ and $g$ are

$$\begin{aligned}&\text{Lipschitz functions in } \mathbb{R}^n, m \leq g \leq M \text{ for some positive constants } m \text{ and } M,\\ &\mathbb{Z}^n\text{-periodic functions.}\end{aligned} \tag{4.4}$$

From the ellipticity (4.3) and the boundedness of $g$, we also have

$$m\alpha|\xi|^2 \leq ga_{ij}\xi_i\xi_j \leq M\beta|\xi|^2 \text{ for all } x \in \mathbb{R}^n \text{ and } \xi \in \mathbb{R}^n. \tag{4.5}$$

Furthermore, we assume the following initial data throughout most of the chapter,

$$\begin{aligned}&v_0 \in C^2(\overline{\Omega_0 \backslash K}), v_0 > 0 \text{ in } \Omega_0, v_0 = 0, \text{ on } \Omega_0^c := \mathbb{R}^n \setminus \Omega_0, \text{ and } v_0 = 1 \text{ on } K,\\ &|Dv_0| \neq 0 \text{ on } \partial\Omega_0, \text{ for some bounded domain } \Omega_0 \supset K.\end{aligned} \tag{4.6}$$

As in Chapter 3, this assumption on the initial data guarantees the well-posedness of the Stefan problem (4.1) and the coincidence of weak and viscosity solutions below, as well as a very useful weak monotonicity (4.28). However, the asymptotic limit, Theorem 4.1 is independent of the initial data, therefore we are able to apply the results for more general initial data. In particular, it is sufficient if the initial data guarantees the existence of the (weak) solution satisfying the comparison principle, and the initial data can be approximated from below and from above by data satisfying (4.6). For instance, $v_0 \in C(\mathbb{R}^n), v_0 = 1$ on $K, v_0 \geq 0, \operatorname{supp} v_0$ compact is enough.

As seen in Chapter 2, the global classical solution of the Stefan problem (4.1) in multi-dimensional space is not expected to exist due to the singularities on the free boundary which might appear in finite time. In our consideration, we continue to use the notions of weak solutions and viscosity solutions introduced in Chapter 2, which were well developed in the literature. Recall that a weak solution is defined by taking the integral in time of the classical solution $v$ and looking at the equation that the new function $u(x,t) := \int_0^t v(x,s)ds$ satisfies. It turns out that if $v$ is sufficiently regular, then $u(\cdot,t)$ solves the obstacle problem (see [5, 14, 21, 15, 51, 53, 52])

$$\begin{cases} u(\cdot,t) \in \mathcal{K}(t), \\ \left(u_t - D_i(a_{ij}D_j u)\right)(\varphi - u) \geq f(\varphi - u) \text{ a.e } (x,t) \text{ for any } \varphi \in \mathcal{K}(t), \end{cases} \tag{4.7}$$

where $\mathcal{K}(t)$ is a suitable functional space specified in Section 2.2.2 and $f$ is

$$f(x) = \begin{cases} v_0(x), & v_0(x) > 0, \\ -\dfrac{1}{g(x)}, & v_0(x) = 0. \end{cases} \tag{4.8}$$

This formulation interprets the Stefan problem as a fixed domain problem and allows us to apply the well-known results in the general variational inequality theory. Indeed, the obstacle problem (4.7) has a global unique solution $u$ for the initial data (4.6). If the corresponding time derivative $v = u_t$ exists, it is called a *weak solution* of the Stefan problem (4.1). Moreover, the homogenization of this problem was also observed based on the approach of homogenization of variational inequalities, see [50, 38, 39]. In a different consideration, Kim introduced the notion of viscosity solutions of the Stefan problem as well as proved the global existence and uniqueness results in [34]. The analysis of viscosity solutions relies on the comparison principle and point-wise arguments, which is more appropriate to study the behavior of the free boundaries. The notions of weak and viscosity solutions were first introduced for the classical homogeneous isotropic Stefan problem where $g(x) = 1$ and the parabolic operator is the simple heat operator, however, it is natural to define the same notions for our Stefan problem (4.1) and obtain the analogous results as observed in [50, 39]. Moreover, the notion of viscosity solutions is also applicable for more general, fully nonlinear parabolic operators and boundary velocity laws since it does not require the variational structure. One interesting result obtained in [39] is that the weak and the viscosity solutions of (4.1) coincide whenever the weak solution exists, thus we will use the strengths of both weak and viscosity solutions to study our problem.

The historical story of the study of the asymptotic and large time behavior of solutions of the one-phase Stefan problem, as we mentioned in Chapter 3, observed the work of Quirós and Vázquez [49] on the convergence of the one-phase Stefan problem to Hele-Shaw in homogeneous isotropic case, the homogenization of the Stefan problem of type (4.1) by Rodrigues [50] and Kim and Mellet [39]. Dealing directly with the long-time behavior of the solutions in inhomogeneous media, the work of the Požár in [45], and then the results in Chapter 3 showed the convergence in appropriate rescaling of solutions of both the Hele-Shaw problem and the Stefan problem to the self-similar solution of the Hele-Shaw problem with a point-source in the isotropic setting. The convergence of the rescaled free boundary is also obtained, and it uniformly approaches a sphere.

In this chapter, we extend the previous results in Chapter 3 to the anisotropic case, where the heat operator is replaced by more general linear parabolic operators of divergence form. This was indeed the setting considered in [50] for periodic homogenization problem and in [39] for more general random media. In this setting, the variational structure is preserved, thus we are still able to use the notions of weak solutions as well as viscosity solutions and their coincidence. However, the main difficulties come from the loss of radially symmetric solutions which were used as barriers in the isotropic case and the homogenization problems appear not only for velocity law but also for the elliptic operators. To overcome the first difficulty, we will construct barriers for our problem from the fundamental solution of the corresponding elliptic equation of divergence form. Unfortunately, even though the unique fundamental solution of this elliptic equation exists for $n \geq 2$, its behavior in the case dimension $n = 2$ and dimension $n \geq 3$ are significantly different. Moreover, we need to make use of a very useful gradient estimate (4.13) for the fundamental solution, which only holds for the periodic structure. Therefore, we will restrict our problem into the problem in periodic media and dimension $n \geq 3$. Following [49,45] and Chapter 3, we use the rescaling of solutions as

$$v^\lambda(x,t) := \lambda^{(n-2)/n} v(\lambda^{1/n} x, \lambda t), \quad u^\lambda(x,t) := \lambda^{-2/n} u(\lambda^{1/n} x, \lambda t).$$

Using this rescaling we can deduce the uniform convergence of the rescaled solution to a limit function away from the origin. In the limit, the fixed domain $K$ shrinks to the origin due to the rescaling, and the rescaled solutions develop a singularity at the origin as $\lambda \to \infty$. Moreover, in a periodic setting, the elliptic operator and velocity law should homogenize as $\lambda \to \infty$, and therefore heuristically, the limit function should be the self-similar solution (under the corresponding rescaling) of the following Hele-Shaw type problem with a point-source

$$\begin{cases} -q_{ij} D_{ij} v = C\delta & \text{in } \{v > 0\}, \\ \qquad v_t = \dfrac{1}{\langle 1/g \rangle} q_{ij} D_i v D_j v & \text{on } \partial\{v > 0\}, \\ v(\cdot, 0) = 0, \end{cases} \tag{4.9}$$

where $\delta$ is the Dirac $\delta$-function, $q_{ij}$ are constants satisfying a uniform ellipticity with some elliptic coefficients, $C$ is a constant depending on $K, n, q_{ij}$ and the boundary data 1, and the constant $\langle 1/g \rangle$ is the average value of the latent heat $L(x) = \frac{1}{g(x)}$. Similarly, the limit variational solution should satisfy the corresponding limit obstacle problem.

The first main result of this chapter, Theorem 4.15, is the locally uniform convergence of the rescaled variational solution to the solution of the limit obstacle problem. Using the constructed barriers, we are able to prove that the limit function has the correct singularity as $|x| \to 0$. Moreover, from the construction of the barriers, we also obtain the growth rate of the free boundary, more precisely, the free boundary expands with the rate of $t^{1/n}$ when $t$ is large enough, which is the same with the isotropic case. The aim is then to prove the homogenization effects of the rescaling to our problem. The shrinking of the fixed domain $K$ in the rescaling also makes our current situation slightly different from the standard classical homogenization problem of variational inequalities, where the domain of observation and the boundary condition are usually fixed. In addition, we also need to show that the rescaled parabolic operator becomes elliptic when $\lambda \to 0$. We will use the notion of the Γ-convergence introduced by De Giorgi and homogenization techniques developed by G. Dal Maso and L. Modica in [11, 12, 13] to our problem. The issue here is that we need to modify the $\Gamma$-convergence sequence in order to use the integration by part formula for the variational inequality. This task will be done with the help of a *cut-off function* and the *fundamental estimate* for the $\Gamma$-convergence. Note that our techniques is applicable not only for the periodic case but also for the random case, thus we expect to extend our results to the problem in random media in future.

As the last step, we will use the coincidence of the weak and viscosity solutions of the problem (4.1) and the viscosity arguments to obtain the locally uniform convergence of the rescaled viscosity solution and its free boundary to the asymptotic profile in the second main result, Theorem 4.24. Fortunately, all the viscosity arguments of the isotropic case can be adapted for the anisotropic case. Therefore, the proof is similar to the the proof of Chapter 3, Theorem 3.16, where we make use of a weak monotonicity (4.28) together with the comparison principle. An important point in the proof of Chapter 3, Theorem 3.16 is that we need to apply Harnack's inequality and we can do the same way here since the rescaled elliptic operator does not change the constant in Harnack's inequality. As the arguments require only some simple modifications, we will skip the proofs of some lemmas and refer to Chapter 3 for more details.

In summary, we will show the following theorem.

**Theorem 4.1.** *The rescaled viscosity solution $v^\lambda$ of the Stefan-type problem (4.1) converges locally uniformly to the unique self-similar solution $V$ of the Hele-Shaw type problem (4.9) in $(\mathbb{R}^n \backslash \{0\}) \times [0, \infty)$ as $\lambda \to \infty$, where $q_{ij}$ are constants satisfying a uniform ellipticity, $C$ depends only on $q_{ij}, n$, the set $K$ and the boundary data 1. Moreover, the rescaled free boundary $\partial\{(x,t) : v^\lambda(x,t) > 0\}$ converges to $\partial\{(x,t) : V(x,t) > 0\}$ locally uniformly with respect to the Hausdorff distance.*

As mentioned above, almost all of the arguments in our recent work hold for ergodic-stationary random case. However, in this situation, we lose a very important point-wise gradient estimate (4.13) for the fundamental solution of the corresponding elliptic equation to construct the barriers. In fact, for non-periodic coefficients, even though the optimal bounds for gradient continue to hold for a bounded domain, it cannot hold in the large scale when $|x - y| \to \infty$. The results in [42, 25] tell us that for random stationary coefficients satisfying a logarithmic Sobolev inequality, we can have similar bounds for gradient in local square average forms. This result cannot be upgraded to the point-wise bounds since there is no regularity to control the square average integral as in [25, Remark 3.7], however, it suggested the possibility to modify our approach to the random case. Another question is the completeness of the present results for the dimension $n = 2$. Since the unique (up to an addition of a constant) fundamental solution of the corresponding elliptic equation exists and the gradient estimates also hold in 2D case, we expect to obtain analogous results as in this chapter. The essential reason that it remains open is the lack of homogenization result for the fundamental solution (Green's function) in 2D, which is of an independent interest. This issue is under the investigation by the authors.

### Outline:

In Section 4.1, we recall some basic facts of the fundamental solution of the corresponding elliptic equation. The rescaling is introduced and we discuss the convergence of the fundamental solution in the rescaling limit. The core of this section is the construction of a subsolution and a supersolution of the Stefan problem (4.1) in Subsection 4.1.3. Moreover, we state some limit problems before giving the proofs of the main results in later sections. Section 4.2 is our major work, where we prove the locally uniform convergence of the rescaled variational solutions. In Section 4.3, we deal with the locally uniform convergence of viscosity solutions and their free boundaries.

## 4.1 Preliminaries

### 4.1.1 The fundamental solution of linear elliptic equation

Note that we use the notation of elliptic operators $\mathcal{L}, \mathcal{L}^\lambda$ as introduced in Section 1.2 and consider $\mathcal{L}^\varepsilon := D_j(a_{ij}(x/\varepsilon)D_i)$.

In this section, we will recall some important facts about the fundamental solution of the self-adjoint uniformly elliptic second order linear equation in divergence form

$$-\mathcal{L}u = 0, \tag{4.10}$$

in dimension $n \geq 3$, where $\mathcal{L}$ is defined as in Section 1.2 and $a_{ij}(x)$ satisfy (4.3) and (4.4). This fundamental solution will be used to construct barriers for the Stefan problem. The facts of the fundamental solution are proved in more detail in the Appendix A.

We will take the definition of the fundamental solution of (4.10) as Green's function in the whole space introduced in [41, 2].

**Definition 4.2.** We say that $G : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is the fundamental solution (Green's function) of (4.10) if $G(\cdot, y)$ is the weak (distributional) solution of $-\mathcal{L}G(\cdot, y) = \delta_y$, where $\delta_y$ is the Dirac measure at $y$, i.e.,

$$\int_{\mathbb{R}^n} a_{ij} D_j G(\cdot, y) D_i \varphi dx = \varphi(y), \quad \forall y \in \mathbb{R}^n, \qquad \forall \varphi \in C_0^\infty(\mathbb{R}^n),$$

and $\lim_{|x-y|\to\infty} G(x, y) = 0$.

The existence and uniqueness of the fundamental solution were given by the remark following [41, Corollary 7.1] or more precisely by [2, Theorem 1].

**Theorem 4.3** (cf. [2, Theorem 1])**.** *Assume that $n \geq 3$, $a_{ij}(x)$ satisfy* (4.3) *and* (4.4)*. Then, there exists a unique fundamental solution $G(x, y)$ of* (4.10) *such that $G(\cdot, y) \in H^1_{loc}(\mathbb{R}^n \backslash \{y\}) \cap W^{1,p}_{loc}(\mathbb{R}^n), p < \frac{n}{n-1}$, and for some constant $C > 0$ we have*

$$C^{-1}|x - y|^{2-n} \leq G(x, y) \leq C|x - y|^{2-n}, \quad \forall x, y \in \mathbb{R}^n. \tag{4.11}$$

*Remark* 4.4. Note that in any bounded domain $U$ of $\mathbb{R}^n \backslash \{0\}$, $G(\cdot, y)$ satisfies all the properties of a weak solution of a uniformly elliptic equation. The fundamental solution of (4.10) also has the following properties (for more details, see [41, 43, 24]):

- $G(x, y) = G(y, x)$
- $G(\cdot, y) \in C^{1,\alpha}(U)$ for some $\alpha > 0$.
- The function $u(x) = \int_{\mathbb{R}^n} G(x, y)f(y)dy$ is a weak solution in $H^1_{\text{loc}}(\mathbb{R}^n)$ of the equation $-\mathcal{L}u = f$ for any $f \in C_0^\infty(\mathbb{R}^n)$.
- When the coefficients $a_{ij}$ are constants, the fundamental solution can be given explicitly as

$$G^0(x, y) := \frac{1}{(n-2)\alpha_n\sqrt{|A|}} \left( \sum_{ij} (A^{-1})_{ij}(x_i - y_i)(x_j - y_j) \right)^{(2-n)/2}, \tag{4.12}$$

where $(A^{-1})_{ij}$ are the elements of the inverse matrix of $(a_{ij})$, $|A|$ is the determinant of $(a_{ij})$ and $\alpha_n$ is the volume of the unit ball in $\mathbb{R}^n$.

Moreover, in a periodic setting, the results in [2, Proposition 5] give bounds on the gradient of the fundamental solution.

**Lemma 4.5** (cf. [2, Proposition 5]). *If $n \geq 2$ and $A$ is periodic then the fundamental solution of (4.10) satisfies the following gradient estimates:*

$$\exists C > 0, \qquad \forall x \in \mathbb{R}^n, \qquad \forall y \in \mathbb{R}^n, \qquad |D_x G(x,y)| \leq \frac{C}{|x-y|^{n-1}}, \tag{4.13}$$

$$\exists C > 0, \qquad \forall x \in \mathbb{R}^n, \qquad \forall y \in \mathbb{R}^n, \qquad |D_y G(x,y)| \leq \frac{C}{|x-y|^{n-1}}. \tag{4.14}$$

Using the technique of $G$-convergence, the authors in [60] established results on the homogenization and the asymptotic behavior of the fundamental solution of (4.10). We refer to [60, 13] for the definition of $G$-convergence and more details of the homogenization problem.

**Lemma 4.6** (cf. [60, Theorem 2, Chapter III]). *Let $n \geq 3$, $A$ satisfy (4.3), (4.4) and $G^\varepsilon(x,y)$ be the fundamental solution of*

$$-\mathcal{L}^\varepsilon u = 0. \tag{4.15}$$

*Then $G^\varepsilon$ converges locally uniformly to $G^0$ in $\mathbb{R}^{2n} \backslash \{x = y\}$ as $\varepsilon \to 0$, where $G^0(x,y)$ is the fundamental solutions of*

$$-\mathcal{L}^0 u = 0, \tag{4.16}$$

*and $\mathcal{L}^0$ is a uniform elliptic operator with constant coefficients. Moreover, if we denote $G(x,y)$ as the fundamental solution of (4.10), then we will have an asymptotic expression*

$$G(x,y) = G_0(x,y) + |x-y|^{2-n}\theta(x,y), \tag{4.17}$$

*where $\theta(x,y) \to 0$ as $|x-y| \to \infty$ uniformly on the set $\{|x| + |y| < a|x-y|\}$, $a$ is any fixed positive constant.*

### 4.1.2 Rescaling

Let $v$ be the solution of the one-phase Stefan problem (4.1) and $u$ be the solution of the corresponding variational inequality (4.7), the definitions as well as the relationship of $v$ and $u$ was introduced in Chapter 2.

#### 4.1.2.1 Rescaling for $n \geq 3$

Following the idea in [49, 45], for $\lambda > 0$ and $n \geq 3$ we will use the rescaling of solutions as

$$v^\lambda(x,t) = \lambda^{\frac{n-2}{n}} v(\lambda^{\frac{1}{n}} x, \lambda t).$$

If we define $K^\lambda := K/\lambda^{\frac{1}{n}}$ and $\Omega_0^\lambda := \Omega_0/\lambda^{\frac{1}{n}}$ then as derived in [45], $v^\lambda$ is a solution of the problem

$$\begin{cases} \lambda^{\frac{2-n}{n}} v_t^\lambda - \mathcal{L}^\lambda v^\lambda = 0 & \text{in } \Omega(v^\lambda)\backslash K^\lambda, \\ v^\lambda = \lambda^{\frac{n-2}{n}} & \text{on } K^\lambda, \\ \dfrac{v_t^\lambda}{|Dv^\lambda|} = g^\lambda(x) a_{ij}^\lambda(x) D_j v^\lambda \nu_i & \text{on } \Gamma(v^\lambda), \\ v^\lambda(\cdot,0) = v_0^\lambda & \text{in } \mathbb{R}^n, \end{cases} \tag{4.18}$$

where $v_0^\lambda(x) = \lambda^{\frac{n-2}{n}} v_0(\lambda^{1/n}x)$ and $g^\lambda(x) = g(\lambda^{\frac{1}{n}}x), a_{ij}^\lambda(x) = a_{ij}(\lambda^{1/n}x)$.

Also as in [45], we will use the corresponding rescaling of weak solutions

$$u^\lambda(x,t) = \lambda^{-\frac{2}{n}} u(\lambda^{\frac{1}{n}}x, \lambda t).$$

The rescaled $u^\lambda$ satisfies the obstacle problem:

$$\begin{cases} u^\lambda(\cdot,t) \in \mathcal{K}^\lambda(t), \qquad 0 < t < \infty, \\ (\lambda^{\frac{2-n}{n}} u_t^\lambda - \mathcal{L}^\lambda u^\lambda)(\varphi - u^\lambda) \geq f^\lambda(x)(\varphi - u^\lambda) & \text{a.e. } (x,t) \in \mathbb{R}^n \times (0,\infty) \\ & \text{for any } \varphi \in \mathcal{K}^\lambda(t), \\ u^\lambda(x,0) = 0, \end{cases} \tag{4.19}$$

where $\mathcal{K}^\lambda(t) = \{\varphi \in H^1(\mathbb{R}^n), \varphi \geq 0, \varphi = \lambda^{\frac{n-2}{n}} t \text{ on } K^\lambda\}$ and $f^\lambda(x) = f(\lambda^{1/n}x)$.

*Remark* 4.7. We can take the admissible set $K^\lambda(t)$ as above due to the continuity with respect to the $H^1$ norm of all terms in the variational inequality and the fact that the variational solution $u$ has a compact support in space at every time. Note that for any fixed time $t$, the admissible set $\mathcal{K}^\lambda(t)$ depends on $\lambda$.

#### 4.1.2.2 Convergence of the rescaled fundamental solution

By Lemma 4.6, we have the following convergence result on the rescaled fundamental solution.

**Lemma 4.8.** *Let $G$ be the fundamental solution of (4.10) in dimension $n \geq 3$. Consider the rescaling*

$$G^\lambda(x,y) = \lambda^{\frac{n-2}{n}} G(\lambda^{1/n}x, \lambda^{1/n}y).$$

*Then $G^\lambda$ is the fundamental solution of*

$$-\mathcal{L}^\lambda u = 0, \tag{4.20}$$

*and $|G^\lambda(x,y) - G^0(x,y)| \to 0$ uniformly on every compact subset of $\mathbb{R}^{2n} \backslash \{(x,x) \in \mathbb{R}^{2n}\}$ where $G^0$ is the fundamental solution of (4.16).*

*Proof.* We will show that $G^\lambda$ is the fundamental solution of (4.20), then the result follows directly from Lemma 4.6 with $\varepsilon = \lambda^{-1/n}$.

For simplicity, we will check that $G^\lambda$ satisfies the definition of the fundamental solution of (4.20) for fixed $y = 0$, $F(x) = G(x,0)$ and $F^\lambda(x) := \lambda^{(n-2)/n} F(\lambda^{1/n}x)$.

Indeed, we have $D_j F^\lambda(x) = \lambda^{(n-1)/n} D_j F(\lambda^{1/n}x)$. Take a function $\varphi \in C_0^\infty(\mathbb{R}^n)$, then

$$\begin{aligned}
\int_{\mathbb{R}^n} a_{ij}^\lambda(x) D_j F^\lambda(x) D_i \varphi(x) dx &= \int_{\mathbb{R}^n} \lambda^{(n-1)/n} a_{ij}(\lambda^{1/n}x) D_j F(\lambda^{1/n}x) D_i \varphi(x) dx \\
&= \int_{\mathbb{R}^n} \lambda^{-1/n} a_{ij}(y) D_j F(y) D_i \varphi(\lambda^{-1/n}y) dy \\
&= \int_{\mathbb{R}^n} a_{ij}(y) D_j F(y) D_i \tilde{\varphi}(y) dy \\
&= \tilde{\varphi}(0) = \varphi(0),
\end{aligned}$$

where $\tilde{\varphi}(y) = \varphi(\lambda^{-1/n}y)$. Moreover, $F^\lambda$ satisfy the estimate (4.11) since $F$ has this property. Hence, by definition, $F^\lambda$ is the fundamental solution of (4.20). □

*Remark* 4.9. The rate of this convergence as well as the rate of convergence for derivatives were also derived in [4].

### 4.1.3 Construction of a sub-solution and a super-solution from a fundamental solution

The main goal of this section is to construct a sub-solution and a super-solution of (4.1) from a fundamental solution of the elliptic equation so that we can use them as barriers to track the behavior of the support of a solution of (4.1).

From now on, we will let $\mathcal{L}^0$ be the limit of the operators of $\mathcal{L}^\lambda$ as in Lemma 4.6 and consider the fundamental solutions of (4.10), (4.20) and (4.16) with pole at the origin as

$$F(x) := G(x,0), \quad F^\lambda(x) := G^\lambda(x,0) = \lambda^{(n-2)/n} F(\lambda^{1/n}x), \qquad F^0(x) := G^0(x,0)$$

respectively. Note that $F^0$ is preserved under the rescaling by (4.12).

#### 4.1.3.1 Construction of a supersolution

Define

$$\theta(x,t) := [C_1 F(x) - C_2 t^{(2-n)/n}]_+,$$

where $C_1, C_2$ are non-negative constants chosen later. It easily follows that in $\{\theta > 0\}\backslash\{0\}$,

$$\begin{aligned}
\theta_t(x,t) &= \frac{C_2(n-2)}{n} t^{(2-2n)/n} \geq 0,\\
D\theta(x,t) &= C_1 DF(x),\\
\mathcal{L}\theta &= 0,\\
\theta_t - \mathcal{L}\theta &\geq 0.
\end{aligned}$$

Due to estimates (4.11) and (4.13), there exists a constant $C$ such that

$$\begin{aligned}
C^{-1}|x|^{2-n} \leq F(x) &\leq C|x|^{2-n},\\
|DF(x)| &\leq C|x|^{1-n}.
\end{aligned} \tag{4.21}$$

Then for $(x,t) \in \partial\{\theta > 0\}$ we have

$$C_2 t^{(2-n)/n} = C_1 F(x) \geq C_1 C^{-1}|x|^{2-n},$$

which yields

$$t^{1/n} \leq \left(\frac{C_1}{CC_2}\right)^{1/(2-n)} |x|.$$

Thus on $\partial\{\theta > 0\}$,

$$\theta_t = \frac{C_2(n-2)}{n} t^{(2-2n)/n} \geq \frac{n-2}{n}\left(\frac{C_1}{C}\right)^{\frac{2-2n}{2-n}} C_2^{\frac{n}{2-n}} |x|^{2-2n}.$$

Fix any $t_0 > 0$. We can choose $C_1$ large enough and $C_2$ small enough such that

$$\theta_t \geq M\beta C_1^2 C^2 |x|^{2-2n} \geq M\beta |D\theta|^2 \text{ on } \partial\{\theta > 0\}, \tag{4.22}$$

$$\theta > 1 \text{ on } K \text{ and } \theta(x,t_0) > v(x,t_0), \tag{4.23}$$

where $\alpha, \beta$ are constants from (4.3). By (4.5), $\theta_t \geq a_{ij} D_j\theta D_i\theta$ on $\partial\{\theta > 0\}$ and by (4.2), $\theta$ is a supersolution of (4.1) in $\mathbb{R}^n \times [t_0, \infty)$.

#### 4.1.3.2 Construction of a subsolution

Let $h(x)$ be the barrier constructed in [39, Appendix A] with $\mathcal{L}h = n, Dh(x) = (A(x))^{-1}x$ and let $c, \tilde{c}$ be non-negative constants such that

$$c|x|^2 \le h(x) \le \tilde{c}|x|^2. \tag{4.24}$$

Consider the following function with non-negative constants $c_1, c_2, c_3$:

$$\theta(x,t) := \left[c_1 F(x) + \frac{c_2 h(x)}{t} - c_3 t^{(2-n)/n}\right]_+ \chi_{E(t)} \tag{4.25}$$

where

$$E(t) := \{x : F_b'(|x|, t) < 0\}, \quad F_b(r,t) := Cc_1 r^{2-n} + \frac{c_2 \tilde{c} r^2}{t} - c_3 t^{(2-n)/n},$$

$C, \tilde{c}$ are constants as in (4.21), (4.24) and $F_b'(r,t)$ is the derivative of $F_b(r,t)$ with respect to $r$. We claim that we can choose constants $c_1, c_2, c_3, t_0$ such that $\theta$ is a subsolution of (4.1) for $t \in [t_0, \infty)$. The differentiation of $\theta$ on the set $\{\theta > 0\}\backslash\{0\}$ leads to

$$\begin{aligned} D\theta(x,t) &= c_1 DF(x) + \frac{c_2 A^{-1}(x)x}{t}, \\ \mathcal{L}\theta(x,t) &= \frac{c_2 n}{t}, \\ \theta_t(x,t) &= -\frac{c_2 h(x)}{t^2} + \frac{c_3(n-2)}{n} t^{(2-2n)/n} = t^{(2-2n)/n}\left[\frac{c_3(n-2)}{n} - \frac{c_2 h(x)}{t^{2/n}}\right], \end{aligned}$$

$$\theta_t(x,t) - \mathcal{L}\theta(x,t) = t^{(2-2n)/n}\left[\frac{c_3(n-2)}{n} - \frac{c_2 h(x)}{t^{2/n}} - \frac{c_2 n}{t^{(2-n)/n}}\right] < 0 \text{ when t is large enough.} \tag{4.26}$$

Thus, we can choose $t_0$ large enough such that $\theta_t - \mathcal{L}\theta < 0$ for $t \ge t_0$.

Now we will prove the continuity of $\theta$. We have

$$0 \le \theta(x,t) \le [F_b(|x|,t)]_+ \chi_{E(t)} =: F_b^+(x,t), \tag{4.27}$$

hence $\Omega_t(\theta) \subset \Omega_t(F_b^+)$ for all $t$. We see that

$$F_b'(r,t) = Cc_1(2-n)r^{1-n} + \frac{2c_2\tilde{c}r}{t} < 0 \Leftrightarrow r < \left(\frac{Cc_1(n-2)}{2c_2\tilde{c}}\right)^{1/n} t^{1/n} =: r_0(t) \tag{4.28}$$

or $E(t) = \{x : |x| < r_0(t)\}$. We have $\theta$ is continuous in time and for each time $t$, $\theta(\cdot,t)$ is continuous in $E(t), \theta(\cdot,t) = 0$ on $(E(t))^c$. We will show that we can choose the constants such that $\theta(\cdot,t)$ is continuous through boundary of $E(t)$ for all $t$. Indeed, for $x_0 \in \partial E(t)$,

$$F_b(|x_0|,t) = F_b(r_0(t),t) = C_{F_b} t^{(2-n)/n},$$

where $C_{F_b} = (Cc_1)^{2/n}(c_2\tilde{c})^{(n-2)/n}\left[\left(\frac{n-2}{2}\right)^{2/n}\frac{n}{n-2}\right] - c_3$ . We can choose $c_1, c_2, c_3$ such that $C_{F_b} < 0$ then $F_b(|x_0|, t) < 0$ for all $t$. Since $F_b(\cdot, t)$ is continuous at $x_0$, there exists a small neighborhood $B(x_0, \varepsilon(t))$ of $x_0$ such that in that neighborhood, $F_b(|x|, t) < 0$ and therefore $F_b^+(x, t) = 0$ and by (4.27), $\theta(x, t) = 0$ for $x \in B(x_0, \varepsilon(t))$. Thus $\theta(\cdot, t)$ is continuous at $x_0$ and therefore it is continuous in $\mathbb{R}^n$. Note that $C_{F_b} < 0$ if and only if

$$c_3 \geq C_0(c_1)^{2/n}(c_2)^{(n-2)/n}, \tag{4.29}$$

where $C_0$ is a constant depending only on $n, C, \tilde{c}$.

We finally need to show that we can choose suitable constants such that $\theta$ satisfies the sub-inequality on the free boundary.

We first note that $\theta(x, t) \geq \tilde{\theta}(x, t) := \left[Cc_1|x|^{2-n} - c_3t^{(2-n)/n}\right]_+$ then $\Omega(\tilde{\theta}) \subset \Omega(\theta)$, or more precisely, there exists a constant $\tilde{C}$ such that

$$|x| \geq \tilde{C}t^{1/n} \text{ for all } (x, t) \in \partial\{\theta > 0\}. \tag{4.30}$$

By (4.26) we have

$$\begin{aligned}
\theta_t &\leq c_3t^{(2-2n)/n},\\
|D\theta|^2 &= c_1^2|DF(x)|^2 + \frac{2c_1c_2}{t}DF(x)\cdot A^{-1}x + \frac{c_2^2}{t^2}|A^{-1}x|^2,\\
&\geq \frac{2c_1c_2}{t}DF(x)\cdot A^{-1}x + \frac{c_2^2}{t^2}|A^{-1}x|^2.
\end{aligned}$$

Since $A$ is a symmetric bounded matrix satisfying the ellipticity (4.3), then these properties also hold for $A^{-1}$ and $A^{-2}$ with some other constants. Hence,

$$\begin{aligned}
|D\theta|^2 &\geq \frac{c_2^2}{t^2}\tilde{\alpha}|x|^2 - \frac{2c_1c_2}{t}C_A|DF(x)||x| && \text{for some } \tilde{\alpha}, C_A > 0\\
&\geq \frac{c_2^2}{t^2}\tilde{\alpha}|x|^2 - \frac{2c_1c_2}{t}CC_A|x|^{2-n} && (\text{ by (4.21)})\\
&\geq \left(c_2^2\tilde{\alpha}\tilde{C}^2 - 2c_1c_2CC_A\tilde{C}^{2-n}\right)t^{(2-2n)/n} && (\text{ by (4.30)}).
\end{aligned}$$

We want to choose $c_1, c_2, c_3$ such that $\theta_t \leq m\alpha|D\theta|^2$ on $\partial\{\theta > 0\}$, which will hold if

$$c_3 \leq m\alpha\left(c_2^2\tilde{\alpha}\tilde{C}^2 - 2c_1c_2CC_A|\tilde{C}^{2-n}\right) =: C_0^1c_2^2 - C_0^2c_1c_2, \tag{4.31}$$

where $C_0^1, C_0^2$ are fixed positive constants. Then by (4.15), $\theta_t \leq ga_{ij}D_j\theta D_i\theta$ on $\partial\{\theta > 0\}$.

The conditions (4.29) and (4.31) hold if we choose some suitable $c_1, c_2, c_3$, for example, fix any $c_1 > 0$, choose $c_2$ large enough such that

$$C_0(c_1)^{2/n}(c_2)^{(n-2)/n} < C_0^1 c_2^2 - C_0^2 c_1 c_2.$$

Note that the above inequality holds for $c_2$ large enough since for fixed $c_1 > 0$, the right hand side tends to $\infty$ as $c_2 \to \infty$ faster than the left hand side. Then (4.29) and (4.31) hold for any $c_3$ which is between these two numbers. Fix $t_0$ such that $\theta_t - \mathcal{L}\theta < 0$ in $\{\theta > 0\}$ for chosen $c_2, c_3$ and $t \geq t_0$. Choosing a smaller $c_1$ if it is needed, we can assume that the support of $\theta(\cdot, t_0)$ is contained in $\Omega_{t_0}(v), \theta(x, t_0) \leq v(x, t_0)$ and $\theta < 1$ in $K$. Thus, with the help of (4.2), we see that $\theta$ is a subsolution of the Stefan problem (4.1) for that choice of constants.

#### 4.1.3.3 Some results on the barriers for the Stefan problem (4.1)

As the construction above, we can use the functions of the form

$$\theta(x,t) := [C_1 F(x) - C_2 t^{(2-n)/n}]_+ \tag{4.32}$$

where $C_1, C_2 > 0$ as the barriers for the Stefan problem (4.1). As our purpose is to study the asymptotic behavior, we first observe the convergence of the rescaled barriers.

**Lemma 4.10.** *Let $\theta$ be a function of the form (4.32) and $\theta^\lambda := \lambda^{(n-2)/n}\theta(\lambda^{1/n}x, \lambda t)$. Then $\theta^\lambda \to \theta^0$ locally uniformly in $(\mathbb{R}^n \backslash \{0\}) \times [0, \infty)$, where*

$$\theta^0(x,t) := [C_1 F^0(x) - C_2 t^{(2-n)/n}]_+. \tag{4.33}$$

*Proof.* We have

$$\theta^\lambda(x,t) = [C_1 F^\lambda(x) - C_2 t^{(2-n)/n}]_+,$$

where $F^\lambda(x) = \lambda^{(n-2)/n}F(\lambda^{1/n}x)$ . By Lemma 4.8, $F^\lambda \to F^0$ locally uniformly in $\mathbb{R}^n \backslash \{0\}$ and the lemma follows. □

Moreover we will also need to know the integration of the barriers in time on the way to analyze the weak solution of the Stefan problem (4.1).

**Lemma 4.11.** *Let $\Theta(x,t) := \int_0^t \theta(x,s)ds$. Then $\Theta(x,t)$ has form*

$$\Theta(x,t) = \left[C_1 F(x) t - \frac{C_2 n}{2} t^{2/n} + o(F(x))\right]_+, \quad \text{as } |x| \to 0. \tag{4.34}$$

*Proof.* We can derive (4.34) simply by integrating the function $\theta$ of the form (4.32). Since $\theta$ has the form (4.32), we see that

$$\begin{aligned} &\theta > 0 \quad \text{if } t > s(x), \\ &\theta = 0 \quad \text{if } t \le s(x), \end{aligned} \qquad \text{where } s(x) = \left(\frac{C_1}{C_2}F(x)\right)^{n/(2-n)}.$$

Thus,

$$\Theta(x,t) = \begin{cases} 0, & t \le s(x), \\ \displaystyle\int_{s(x)}^{t} (C_1F(x) - C_2 s^{(2-n)/n})ds, & t > s(x). \end{cases}$$

When $t > s(x)$,

$$\begin{aligned} \Theta(x,t) &= C_1F(x)t - \frac{C_2 n}{2}t^{2/n} - C_1F(x)s(x) + \frac{C_2 n}{2}(s(x))^{2/n} \\ &= C_1F(x)t - \frac{C_2 n}{2}t^{2/n} + \frac{n-2}{2}\frac{(C_1)^{2/(2-n)}}{(C_2)^{n/(2-n)}}(F(x))^{2/(2-n)} \\ &= C_1F(x)t - \frac{C_2 n}{2}t^{2/n} + C(F(x))^{2/(2-n)}. \end{aligned}$$

Since $F(x)$ has a singularity at $x = 0$ (by (4.11)) then $C(F(x))^{2/(2-n)} = o(F(x))$ as $|x| \to 0$ which completes the proof. $\square$

From these barriers, we can obtain the rate of expanding support for viscosity solutions.

**Lemma 4.12.** *Let $n \ge 3$ and $v$ be a viscosity solution of (4.1). There exists $t_0 > 0$ and constants $C, C_1, C_2 > 0$ such that for $t \ge t_0$,*

$$C_1 t^{1/n} \le \min_{\Gamma_t(v)} |x| \le \max_{\Gamma_t(v)} |x| < C_2 t^{1/n}$$

*and for $0 \le t \le t_0$,*

$$\max_{\Gamma_t(v)} |x| < C_2.$$

*Moreover,*

$$0 \le v(x,t) \le C|x|^{2-n}.$$

*Proof.* We figure out the boundedness for $v(x,t)$ first. Let $F(x)$ be the fundamental solution of elliptic equation (4.10) as in section 4.1.3 then $\hat{\theta} = CF(x)$ is a stationary solution of the equation $v_t - \mathcal{L}v = 0$. Its integration in time is also a solution of variational inequality problem with $\hat{f} = CF(x)$. If we take $C$ large enough then $\hat{f} \ge f$ and $\hat{\theta} \ge 1$ on $K$. Applying the comparison principle for variational problem ( [49, Proposition 2.2]) we have $v(x,t) \le CF(x) \le \tilde{C}|x|^{2-n}$ (by (4.11)). The boundedness of the support of $v(\cdot,t)$ at all times has been proved in [39, Lemma 3.6.]

Consider $\theta_1, \theta_2$ which are a subsolution and a supersolution of the Stefan problem (4.1) for $t \ge t_0$ as constructed in Section 4.1.3.1 and 4.1.3.2. The bounds on the support of $v$ for $t \ge t_0$ follow directly from the behavior of the support of $\theta_1, \theta_2$. $\square$

### 4.1.4 Limit problems

The expected limit problem is the corresponding Hele-Shaw type problem with a point source.

#### 4.1.4.1 Limit problem for $v^\lambda$

We expect $v^\lambda$ to converge to a solution of

$$\begin{cases} q_{ij}D_{ij}v = 0 & \text{in } \{v>0\}, \\ \dfrac{v_t}{|Dv|} = (1/L)q_{ij}D_j v \nu_i & \text{on } \partial\{v>0\}, \\ \lim\limits_{|x|\to 0} \dfrac{v}{F^0} = C, & \\ v(x,0) = 0 & \text{in } \mathbb{R}^n \backslash \{0\}, \end{cases} \tag{4.35}$$

where $C, L$ are positive constants, $q_{ij}$ are constants of the operator $\mathcal{L}^0$ and $F^0$ is the fundamental solution of (4.16).

Since $Q := (q_{ij})$ is symmetric and positive definite, we can write $Q = P^2$, where $P$ is a symmetric positive definite matrix. Let $\tilde{v}(x,t) := v(Px,t)$. A direct computation then shows that equation (4.35) becomes the classical Hele-Shaw problem with a point source for function $\tilde{v}$,

$$\begin{cases} \Delta\tilde{v} = 0 & \text{in } \{\tilde{v}>0\}, \\ \tilde{v}_t = (1/L)|D\tilde{v}|^2 & \text{on } \partial\{v>0\}, \\ \lim\limits_{|x|\to 0} \dfrac{\tilde{v}}{|x|^{2-n}} = C, & \\ \tilde{v}(x,0) = 0 & \text{in } \mathbb{R}^n \backslash \{0\}. \end{cases} \tag{4.36}$$

The problem (4.36) has a unique classical solution $\tilde{V}$ which is given explicitly (see Chapter 3, [45] for instance). Thus (4.35) has unique classical solution $V(x,t) := \tilde{V}(P^{-1}x,t)$, which is continuous in $(\mathbb{R}^n \backslash \{0\}) \times [0,\infty)$.

#### 4.1.4.2 Limit problem for $u^\lambda$

Assume that $V = V_{C,L}$ is the classical solution of (4.35) above and

$$U(x,t) := \int_0^t V(x,s)ds. \tag{4.37}$$

It is known that the time integral of classical Hele-Shaw problem with a point source (4.36) satisfies an obstacle problem derived in [45]. Following [45] and using a change variables again, we see that $U$ uniquely solves the following problem, which is our limit variational problem:

$$\begin{cases} w \in \mathcal{K}_t, \\ q(w,\phi) \geq \langle -L, \phi \rangle, & \forall \phi \in W_1, \\ q(w,\psi w) = \langle -L, \psi w \rangle, & \forall \psi \in W_2, \end{cases} \tag{4.38}$$

where

$$\mathcal{K}_t = \left\{ \varphi \in \bigcap_{\varepsilon>0} H^1(\mathbb{R}^n \backslash B_\varepsilon) \cap C(\mathbb{R}^n \backslash B_\varepsilon) : \varphi \geq 0, \lim_{|x|\to 0} \frac{\varphi(x)}{F^0(x)} = Ct \right\},$$

$$W_1 = \left\{ \phi \in H^1(\mathbb{R}^n \backslash B_\varepsilon) : \phi \geq 0, \phi = 0 \text{ on } B_\varepsilon \text{ for some } \varepsilon > 0 \right\}, \tag{4.39}$$

$$W_2 = W_1 \cap C^1(\mathbb{R}^n). \tag{4.40}$$

We also use the standard notation for the bilinear form on $H^1$ and inner product in $L^2$, in particular

$$a_\Omega(u,v) := \int_\Omega a_{ij} D_j u D_i v dx, \quad \langle u, v \rangle_\Omega := \int_\Omega uv dx.$$

We omit the set $\Omega$ in the notation if $\Omega = \mathbb{R}^n$, $q(u,v)$ is defined analogously when $a_{ij}$ are replaced by $q_{ij}$.

#### 4.1.4.3 Near-field limit

Using the boundedness results of Lemma 4.12, we have the following general near-field limit adapted for viscosity solutions and the asymptotic behavior result for solution of limit problem as in [49].

**Theorem 4.13** (Near-field limit)**.** *The viscosity solution $v(x,t)$ of the Stefan problem (4.1) converges to the unique solution $P(x)$ of the exterior Dirichlet problem*

$$\begin{cases} D_j(a_{ij}D_iP) = 0, & x \in \mathbb{R}^n\backslash K, \\ P = 1, & x \in K, \\ \lim_{|x|\to\infty} P(x) = 0 \end{cases} \tag{4.41}$$

*as $t \to \infty$ uniformly on compact subsets of $\overline{K^c}$.*

*Proof.* Follow the arguments in the proof of [49, Lemma 8.4] and note that by Lemma 4.12, the support of $v$ expands to the whole space as time $t \to \infty$. $\square$

The results on the isolated singularity of solutions of linear elliptic equation in [56] allow us to deduce the asymptotic behavior of $P$ as $|x| \to \infty$.

**Lemma 4.14.** *There exists a constant $C_* = C_*(K)$ such that the solution $P$ of problem (4.41) satisfies*

$$\lim_{|x|\to\infty} \frac{P(x)}{F(x)} = C_*$$

*where $F(x)$ is a fundamental solution of elliptic equation $D_j(a_{ij}D_iv) = 0$ in $\mathbb{R}^n$.*

*Proof.* Lemma 4.14 is a direct corollary of [56, Theorem 5]. The arguments follow the same techniques as in [49, Lemma 4.3] using a general Kelvin transform and Green's function for linear elliptic equations. Following [49, Lemma 4.3], it can also be shown that the constant $C_*$ depends continuously on the data of the fixed boundary $\Gamma$. We will make a detail proof of this lemma in the Appendix A. $\square$

## 4.2 Uniform convergence of the rescaled variational solutions

Our first main result is the uniform convergence of the rescaled variational solutions, which is similar to Chapter 3, Theorem 3.12.

**Theorem 4.15.** *Let $u$ be the unique solution of variational problem (2.8) and $u^\lambda$ be its rescaling. Let $U_{A,L}$ be the unique solution of limit problem (4.38) where $A = C_*$ as in Lemma 4.14, and $L = \langle 1/g \rangle$ as in Lemma 2.20. Then the functions $u^\lambda$ converge locally uniformly to $U_{A,L}$ as $\lambda \to \infty$ on $(\mathbb{R}^n\backslash\{0\}) \times [0,\infty)$.*

The classical homogenization results of variational inequalities are usually stated for a fixed bounded domain. Since our admissible set $\mathcal{K}^\lambda(t)$ defined in Section 4.1.2 changes with $\lambda$, we will need to refine the proof. We will use the techniques of $\Gamma$-convergence introduced in [13] and [39]. Note that these techniques can be applied not only for periodic case but also for stationary ergodic coefficients over a probability space $(A, \mathcal{F}, P)$.

### 4.2.1 $\Gamma$-convergence of functionals

We recall some basic concepts and results of the $\Gamma$-convergence which are taken from [13]. Let $\Omega$ be a bounded open set in $\mathbb{R}^n$. Consider the functional

$$J^\lambda(u, \Omega) := \begin{cases} \displaystyle\int_\Omega a_{ij}(\lambda^{1/n}x) D_i u D_j u dx & \text{if } u \in H^1(\Omega), \\ \infty & \text{otherwise.} \end{cases} \tag{4.42}$$

By [13, Chapter 8], we can define the $\Gamma$-convergence of a sequence of functionals as follows.

**Definition 4.16.** Let X be a metric space. A sequence of functionals $F_h$ is said to $\Gamma(X)$-converge to $F$ if the following conditions are satisfied:

(i) For every $u \in X$ and for every sequence $(u_h)$ converging to $u$ in $X$, we have

$$F(u) \le \liminf_{h\to 0} F_h(u_h).$$

(ii) For every $u \in X$, there exists a sequence $(u_h)$ converging to $u$ in $X$, such that

$$F(u) = \lim_{h\to 0} F_h(u_h).$$

From [13, 39], we have that the $\Gamma$-convergence of $J^\lambda$ is equivalent to the $G$-convergence of elliptic operator $\mathcal{L}^\lambda$ and a crucial result on Gamma-convergence of $J^\lambda$ as follows.

**Theorem 4.17** (cf. [39, Theorem 4.3])**.** *The functionals $J^\lambda$ $\Gamma(L^2)$-converge as $\lambda \to \infty$ to a functional $J^0$, where $J^0$ is a quadratic functional of the form*

$$J^0(u) := \begin{cases} \displaystyle\int_\Omega q_{ij} D_i u D_j u dx & \textit{if } u \in H^1(\Omega), \\ \infty & \textit{otherwise.} \end{cases}$$

*Here $q_{ij}$ are the constants coefficients of the limit operator $\mathcal{L}^0$ as in Lemma 4.6.*

To deal with the Dirichlet boundary condition, we need to use the *cut-off function* and the *fundamental estimate.*

**Definition 4.18.** [13, Definition 18.1] Let $\mathcal{A}$ be the class of all open subsets of $\Omega$ and $A', A'' \in \mathcal{A}$ with $A' \Subset A''$. We say that a function $\varphi : \mathbb{R}^n \to \mathbb{R}$ is a *cut-off function* between $A'$ and $A''$ if $\varphi \in C_0^\infty(A''), 0 \leq \varphi \leq 1$ on $\mathbb{R}^n$, and $\varphi = 1$ in a neighborhood of $\overline{A'}$ .

**Definition 4.19.** [13, Definition 18.2] Let $F : L^p(\Omega) \times \mathcal{A} \to [0, \infty]$ be a non-negative functional. We say that $F$ satisfies the *fundamental estimate* if for every $\varepsilon > 0$ and for every $A', A'', B \in \mathcal{A}$, with $A' \Subset A''$, there exists a constant $M > 0$ with the following property: for very $u, v \in L^p(\Omega)$, there exists a cut-off function $\varphi$ between $A'$ and $A''$, such that

$$\begin{aligned} F(\varphi u + (1-\varphi)v, A' \cup B) &\leq (1+\varepsilon)(F(u, A'') + F(v, B)) \\ &\quad + \varepsilon(\|u\|^p_{L^p(S)} + \|v\|^p_{L^p(S)} + 1) + M\|u - v\|^p_{L^p(S)}, \end{aligned} \tag{4.43}$$

where $S = (A'' \backslash A') \cap B$. Moreover, if $\mathcal{F}$ is a class of non-negative functionals on $L^p(\Omega) \times \mathcal{A}$, we say that the fundamental estimate holds uniformly in $\mathcal{F}$ if each element $F$ of $\mathcal{F}$ satisfies the fundamental estimate with $M$ depending only on $\varepsilon, A', A'', B$, while $\varphi$ may depend also on $F, u, v$.

The result in [13, Theorem 19.1] provides a wide class of integral functionals uniformly satisfying the fundamental estimate.

**Theorem 4.20.** *[13, Theorem 19.1] Let $c_1, c_2, c_3, c_4$ be real numbers with $c_i \geq 0$, and let $\sigma : \mathcal{A} \to [0, \infty]$ be a superadditive increasing function with $\sigma(A) < \infty$ for every $A \Subset \Omega$. Denote by $\mathcal{F} = \mathcal{F}(p, c_1, c_2, c_3, c_4, \sigma)$ the class of all local functionals $F : L^p(\Omega) \times \mathcal{A} \to [0, \infty]$ for which there exists a function $a \in L^1_{loc}(\Omega)$ and two non-negative Borel functions*

$$f : \Omega \times \mathcal{R} \times \mathbb{R}^n \to [0, \infty) \quad \textit{and} \quad g : \Omega \times \mathbb{R}^n \to [0, \infty)$$

*(depending on $F$) such that*

$$\textit{(i)}\;\; F(u, A) = \begin{cases} \displaystyle\int_A f(x, u(x), Du(x))dx, & \textit{if } u \in W^{1,1}_{loc}(A), \\ \infty, & \textit{otherwise,} \end{cases}$$

*(ii)* $g(x,\xi) \leq f(x,s,\xi) \leq c_1 g(x,\xi) + c_2|s|^p + a(x)$,

*(iii)* $0 \leq g(x,\xi) \leq c_3|\xi|^p + a(x)$,

*(iv)* $g(x,\cdot)$ *is convex on* $\mathbb{R}^n$,

*(v)* $g(x,2\xi) \leq c_4(2g(x,\xi) + a(x))$,

*(vi)* $\int_A a(x)dx \leq \sigma(A)$,

*for every* $u \in L^p(\Omega), A \in \mathcal{A}, x \in \Omega, s \in \mathbb{R}, \xi \in \mathbb{R}^n$. *Then the fundamental estimate holds uniformly in the class* $\mathcal{F}$.

Note that each functional $F : L^p(\Omega) \times \mathcal{A} \to [0,\infty]$ of the form

$$F(u,A) = \begin{cases} \displaystyle\int_A f(x, Du(x))dx, & \text{if } u \in W^{1,1}_{loc}(A), \\ \infty, & \text{otherwise,} \end{cases}$$

with

$$f(x,\xi) = \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j, \quad \text{and} \qquad 0 \leq \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \leq \beta|\xi|^2, \forall x \in \Omega, \forall \xi \in \mathbb{R}^n$$

belongs to the class $\mathcal{F} = \mathcal{F}(p,1,0,\beta,2,\sigma)$, with $\sigma$ is the usual Lebesgue measure in $n$-dimension. For each functional in $\mathcal{F}$, we can choose $a = 0, g(x,\xi) = |\xi|^2$ and then all the conditions from (i) to (vi) in Theorem 4.20 hold. Thus for every $F \in \mathcal{F}$, there exists the cut-off function $\varphi$ such that (4.43) hold with constant $M$ does not depend on $F$. In particular, our functional $J^\lambda$ belongs to $\mathcal{F}$ and it guarantees the existence the cut-off function $\xi^\lambda$ with constant M independent of $\lambda$.

### 4.2.2 Uniform convergence of the rescaled variational solutions

Now we are ready to prove Theorem 4.15.

*Proof of Theorem 4.15.* Fix $T > 0$. By Lemma 4.12, we can bound $\Omega_t(u^\lambda)$ by $\Omega := B_\delta(0)$ for some $\delta > 0$, for all $0 \leq t \leq T$ and $\lambda > 0$. For some $\varepsilon > 0$, define $\Omega_\varepsilon := \Omega\backslash\overline{B(0,\varepsilon)}$, $Q_\varepsilon := \Omega_\varepsilon \times [0,T]$ . We will prove the convergence in $Q_\varepsilon$.

We argue the same way as in the proof of Chapter 3, Theorem 3.12. Using the boundedness of $u^\lambda, u^\lambda_t$ and the standard regularity estimates for an elliptic obstacle problem which hold uniformly in $\lambda$, we obtain a uniform Hölder estimate for $u^\lambda$. Then by the Arzelà-Ascoli theorem and diagonalization argument, we can find a function $\bar{u} \in C((\mathbb{R}^n \setminus \{0\}) \times [0,\infty))$ and a subsequence $\{u^{\lambda_k}\} \subset \{u^\lambda\}$ such that

$$u^{\lambda_k} \to \bar{u} \text{ locally uniformly on } (\mathbb{R}^n \setminus \{0\}) \times [0,\infty) \text{ as } k \to \infty,$$
$$u^{\lambda_k}(\cdot,t) \to \bar{u}(\cdot,t) \text{ strongly in } H^1(\Omega_\varepsilon) \text{ for all } t \geq 0, \varepsilon > 0.$$

To finish the proof, we need to show that the function $\bar{u}$ is the solution of the limit problem (4.38) and then by the uniqueness of the limit problem, we deduce that the convergence is not restricted to a subsequence. Firstly we show that $\bar{u}$ has the correct singularity by the following lemma.

**Lemma 4.21.** *We have*

$$\lim_{|x|\to 0} \frac{\overline{u}(x,t)}{U_{C_*,L}(x,t)} = 1$$

*for every $t \geq 0$, where $L = \langle 1/g \rangle$ as in Lemma 2.20 and $C_*$ as in Lemma 4.14.*

*Proof.* Let $C_*$ as in Lemma 3.11 and $F$ be the fundamental solution of (4.10) as in Section 4.1.3. Fix $\varepsilon > 0$. By Lemma 3.11, there exists $a$ large enough such that

$$\left| \frac{P(x)}{F(x)} - C_* \right| < \frac{\varepsilon}{2}, \quad \text{in } \{|x| \geq a\}. \tag{4.44}$$

In particular, (4.44) holds for every $x, |x| = a$.

Consider the Stefan problem in the set $\Omega_a := \{|x| \geq a\}, K \subset \Omega_a$ for $a$ large enough. The fixed boundary $\{|x| = a\}$ is a compact subset of $\mathbb{R}^n \setminus K$. Then by Theorem 3.10, there exists $t_0 > 0$ such that for all $t \geq t_0$,

$$\left| \frac{v(x,t)}{F(x)} - \frac{P(x)}{F(x)} \right| < \frac{\varepsilon}{2}, \quad \text{for all } x, |x| = a.$$

Thus by triangle inequality we have for all $t \geq t_0$, for all $x$ such that $|x| = a$,

$$\left| \frac{v(x,t)}{F(x)} - C_* \right| < \varepsilon.$$

Let $\Phi(x,t)$ be the fundamental solution of parabolic equation

$$u_t - \mathcal{L}u = 0. \tag{4.45}$$

As shown in [19, 3], such unique fundamental solution exists and satisfies

$$N^{-1} t^{-\frac{n}{2}} e^{-\frac{N|x|^2}{t}} \leq \Phi(x,t) \leq N t^{-\frac{n}{2}} e^{-\frac{|x|^2}{Nt}} \tag{4.46}$$

for some $N > 0$. We consider $\theta_1, \theta_2$ as follows:

$$\theta_1(x,t) := \left[ (C_* - \varepsilon)F(x) + \frac{c_2 h(x)}{t} - c_3 t^{(2-n)/n} \right]_+ \chi_{E(t)},$$
$$\theta_2(x,t) := (C_* + \varepsilon)F(x) + C_2\Phi(x,t),$$

where $E(t), h(x)$ are defined as in Section 4.1.3.2. We will show that we can choose the coefficients such that $\theta_1$ is a subsolution and $\theta_2$ is a supersolution of (3.1) in $\{|x| \geq a\} \times \{t \geq t_0\}$ for some $t_0$. Since we fix the first coefficient of $\theta_1$ and $\theta_2$, we need to be more careful to check the boundary and initial conditions.

Note that on the set $\{|x| = a\}$, $\theta_1 \to (C_* - \varepsilon)F(x)$ and $\theta_2 \to (C_* + \varepsilon)F(x)$ uniformly as $t \to \infty$. Thus we can choose a large time $t_0$ such that $\theta_1 \leq v \leq \theta_2$ on $\{|x = a|\} \times \{t \geq t_0\}$. By (4.28), we can choose $c_2$ large enough such that $\operatorname{supp}\theta_1(\cdot, t_0) \subset E(t_0) \subset B_a(0)$ and then $\theta_1(\cdot, t_0) \leq v(\cdot, t_0)$ in $\{|x| \geq a\}$. Following Section 4.1.3.2, by choosing larger $c_2, t_0$ if necessary and $c_3$ satisfying (4.29), (4.31) then $\theta_1$ is a subsolution of (3.1) in $\{|x| \geq a\} \times \{t \geq t_0\}$.

Fix the time $t_0$ such that $\theta_1$ is a subsolution of (3.1) in $\{|x| \geq a\} \times \{t \geq t_0\}$ as above. By (4.11) and (4.46), $\theta_2 > 0$ in $\mathbb{R}^n$. Moreover, since $F(x)$ and $\Phi(x,t)$ are the fundamental solutions of (4.10) and (4.45) respectively, then $(\theta_2)_t - \mathcal{L}\theta_2 = 0$ in $\mathbb{R}^n$. If we choose $C_2$ large enough then $\theta_2(\cdot, t_0) > v(\cdot, t_0)$ and $\theta_2$ is a super solution of (3.1) in $\{|x| \geq a\} \times \{t \geq t_0\}$.

By comparison principle, $\theta_1 \leq v \leq \theta_2$. Moreover,

$$\theta_1(x,t) \geq \tilde{\theta}_1(x,t) := \left[ (C_* - \varepsilon)F(x) - c_3 t^{(2-n)/n} \right]_+.$$

Therefore $\tilde{\theta}_1^\lambda \leq v^\lambda \leq \theta_2^\lambda$.

Noting that $\Phi^\lambda(x,t) := \lambda^{(n-2)/n}\Phi(\lambda^{1/n}x, \lambda t) \to 0$ uniformly as $\lambda \to \infty$ by (4.46), then by Lemma 4.10, $\tilde{\theta}_1^\lambda, \theta_2^\lambda$ converges locally uniformly to $\theta_1^0, \theta_2^0$ of the form

$$\theta_1^0(x,t) := \left[ (C_* - \varepsilon)F^0(x) - c_3 t^{(2-n)/n} \right]_+,$$
$$\theta_2^0(x,t) := (C_* + \varepsilon)F^0(x),$$

where $F^0$ is the fundamental solution of $-\mathcal{L}^0 u = 0$, $\mathcal{L}^0$ is the limit of the operators $\mathcal{L}^\lambda$ as in Lemma 4.6. Applying the same method as in [45] we have

$$\int_0^t \theta_1^0(x,s)ds \leq \overline{u}(x,t) \leq \int_0^t \theta_2^0(x,s)ds. \tag{4.47}$$

By Lemma 4.11 we obtain

$$\left[(C_* - \varepsilon)F^0(x)t - \frac{c_3 n}{2} t^{2/n} + o(F^0(x))\right]_+ \le \overline{u}(x,t) \le (C_* + \varepsilon)F^0(x)t$$

as $|x| \to 0$. Dividing both sides of by $F^0(x)$ and taking the limit as $|x| \to 0$ we get

$$(C_* - \varepsilon)t \le \liminf_{|x|\to 0} \frac{\overline{u}(x,t)}{F^0(x)} \le \limsup_{|x|\to 0} \frac{\overline{u}(x,t)}{F^0(x)} \le (C_* + \varepsilon)t.$$

Since $\varepsilon > 0$ is arbitrary, we have the correct singularity by sending $\varepsilon$ to 0. $\square$

Finally, we will check that the limit function $\bar{u}$ satisfies the inequality and equality in (4.38).

**Lemma 4.22.** *For each $0 \le t \le T$, $\overline{w} = \overline{u}(\cdot, t)$ satisfies*

$$q(\overline{w}, \phi) \ge \langle -L, \phi \rangle, \qquad \forall \phi \in W_1, \tag{4.48}$$

$$q(\overline{w}, \psi\overline{w}) = \langle -L, \psi\overline{w} \rangle, \qquad \forall \psi \in W_2, \tag{4.49}$$

*where $L = \langle 1/g \rangle$ and $W_1, W_2$ were defined as in Section 4.1.4.2.*

*Proof.* Fix $t \in [0,T]$ and take any $\phi \in W_1$. By continuity, we can choose $\phi$ with a compact support contained in $\Omega := B(0,R)\backslash\overline{B(0,\varepsilon_0)}$ for some $R, \varepsilon_0$. Let $w^k(x) := u^{\lambda_k}(x,t)$ and $\overline{\varphi} := \overline{w} + \phi \in H^1(\mathbb{R}^n)$. By Theorem 4.17, there exists a sequence $\{\varphi^k\}$ that converges strongly in $L^2(\Omega)$ to $\overline{\varphi}$ such that

$$J^{\lambda_k}(\varphi^k, \Omega) \to J^0(\overline{\varphi}, \Omega). \tag{4.50}$$

We will show that we can modify $\varphi^k$ into $\tilde{\varphi}^k$ such that $\tilde{\varphi}^k \in \mathcal{K}^{\lambda_k}(t)$ and all the convergences are preserved.

First, we see that $J^0(\bar{\varphi}, \Omega) < \infty$ since $\bar{\varphi} \in H^1(\Omega)$. By (4.50), $J^{\lambda_k}(\varphi^k, \Omega) < \infty$ and hence $\varphi^k \in H^1(\Omega)$ when $k$ is large enough.

Next, we need to modify $\varphi^k$ so that the boundary condition on $K^{\lambda_k}$ is satisfied. Since $\overline{\varphi} \in H^1(\Omega)$, for every $\varepsilon > 0$, there exists a compact set $A(\varepsilon) \subset \Omega$ such that $\operatorname{supp}\phi \subset A(\varepsilon)$ and

$$\int_{\Omega\backslash A(\varepsilon)} |D\overline{\varphi}|^2\, dx < \varepsilon. \tag{4.51}$$

Let $A'(\varepsilon), A''(\varepsilon)$ such that $A(\varepsilon) \subset A'(\varepsilon) \Subset A''(\varepsilon) \Subset \Omega$ and $B(\varepsilon) = \Omega\backslash A(\varepsilon)$. By [13, Theorem 19.1], the fundamental estimate (4.43) holds uniformly in the class of all functionals of the form (4.42). Thus there exists a constant $M \ge 0$ independent of $\lambda_k$ and a sequence of cut off functions $\xi_\varepsilon^k \in C_0^\infty(A''(\varepsilon)), 0 \le \xi_\varepsilon^k \le 1, \xi_\varepsilon^k = 1$ in a neighborhood of $\overline{A'(\varepsilon)}$ such that

$$\begin{aligned} J^{\lambda_k}(\xi_\varepsilon^k \varphi^k + (1-\xi_\varepsilon^k)(w^k+\phi), \Omega) \le & (1+\varepsilon)(J^{\lambda_k}(\varphi^k, A''(\varepsilon)) + J^{\lambda_k}(w^k+\phi, B(\varepsilon))) \\ & + \varepsilon(\|\varphi^k\|^2_{L^2(\Omega)} + \|w^k+\phi\|^2_{L^2(\Omega)} + 1) \\ & + M\|\varphi^k - w^k - \phi\|^2_{L^2(\Omega)}. \end{aligned} \tag{4.52}$$

Define

$$\varphi_\varepsilon^k(x) := \begin{cases} \xi_\varepsilon^k(x)\varphi^k(x) + (1-\xi_\varepsilon^k(x))(w^k(x)+\phi(x)) & \text{if } x \in \Omega, \\ w^k(x) & \text{if } x \notin \Omega. \end{cases}$$

Then $\varphi_\varepsilon^k \in H^1(\mathbb{R}^n), \|\varphi_\varepsilon^k - \bar\varphi\|_{L^2(\Omega)} \le \|\varphi^k - \bar\varphi\|_{L^2(\Omega)} + \|w^k + \phi - \bar\varphi\|_{L^2(\Omega)} \to 0$ as $k \to \infty$ and $\varphi_\varepsilon^k - w^k$ has compact support in $\Omega$.

By ellipticity (4.3) we have

$$J^{\lambda_k}(w^k + \phi, B) \le \beta \int_{B(\varepsilon)} |D(w^k+\phi)|^2\, dx. \tag{4.53}$$

By (4.51), choose the sequence $\varepsilon_n := \frac{1}{n}$ and denote $\varphi_n^k := \varphi_{\varepsilon_n}^k$. By (4.52), (4.53), the convergences $\varphi_n^k \to \overline{\varphi}$ in $L^2(\Omega)$ and $w^k \to \overline{w}$ in $H^1(\Omega)$ as $k \to \infty$, for each $n$ there exists $k_0(n)$ such that

$$\begin{cases} \|\varphi_n^k - \overline{\varphi}\|_{L^2(\Omega)} \le \min\left\{\dfrac{1}{n}, \dfrac{1}{Mn}\right\}, \\ \quad J^{\lambda_k}(\varphi_n^k, \Omega) \le \left(1+\dfrac{1}{n}\right)\left(J^0(\overline{\varphi}, \Omega) + \dfrac{\beta+1}{n}\right) + \dfrac{1}{n}\left(2\|\overline{\varphi}\|_{L^2(\Omega)} + \dfrac{1}{n} + 1\right) + \dfrac{2}{n}, \end{cases} \tag{4.54}$$

for every $k \ge k_0(n)$. We can choose $k_0(n)$ such that $k_0$ is an increasing function of $n$ and $k_0(n) \to \infty$ as $n \to \infty$. We will form a new sequence $\{\hat\varphi^k\}$ from the class of sequences $\{\varphi_n^k\}$. The idea is that for each $k$, we will choose an appropriate $n(k)$ and set $\hat\varphi^k := \varphi_{n(k)}^k$. We need to choose a suitable $n(k)$ such that $n(k) \to \infty$ and (4.54) holds for $\varphi_{n(k)}^k$ when $k$ is large enough. To this end it we we introduce an "inverse" of $k$ as

$$n(k) := \min\{j \in \mathbb{N} : k < k_0(j+1)\}.$$

$n(k)$ is well-defined, non-decreasing and tends to $\infty$ as $k \to \infty$. From the definition of $n(k)$ we see that if $k \ge k_0(2)$ then $n(k) \ge 2$ and $k_0(n(k)) \le k < k_0(n(k)+1)$ (otherwise $n(k)$ is not the minimum). Thus by (4.54) and definition of $\hat\varphi^k$ we have for all $k \geq k_0(2)$,

$$\begin{cases} \|\hat{\varphi}^k - \overline{\varphi}\|_{L^2(\Omega)} \leq \min\left\{\dfrac{1}{n(k)}, \dfrac{1}{Mn(k)}\right\}, \\ J^{\lambda_k}(\hat{\varphi}^k, \Omega) = J^{\lambda_k}(\varphi^k_{n(k)}, \Omega) \\ \qquad \leq \left(1 + \dfrac{1}{n(k)}\right)\left(J^0(\overline{\varphi}, \Omega) + \dfrac{\beta+1}{n(k)}\right) \\ \qquad + \dfrac{1}{n(k)}\left(2\|\overline{\varphi}\|_{L^2(\Omega)} + \dfrac{1}{n(k)} + 1\right) + \dfrac{2}{n(k)}. \end{cases}$$

Sending $k$ to $\infty$ we get

$$\begin{cases} \lim_{k\to\infty} \|\hat{\varphi}^k - \bar{\varphi}\|_{L^2(\Omega)} = 0, \\ \limsup_{k\to\infty} J^{\lambda_k}(\hat{\varphi}^k, \Omega) \leq J^0(\bar{\varphi}, \Omega). \end{cases}$$

On the other hand, by Theorem 4.17,

$$J^0(\bar{\varphi}, \Omega) \leq \liminf_{k\to\infty} J^{\lambda_k}(\hat{\varphi}^k, \Omega)$$

and thus we can conclude that $\hat{\varphi}^k \to \bar{\varphi}$ strongly in $L^2(\Omega)$ and $J^{\lambda_k}(\hat{\varphi}^k, \Omega) \to J^0(\bar{\varphi}, \Omega)$. Moreover, by the definitions of $\varphi^k_\varepsilon, \hat{\varphi}^k$, we also have $\hat{\varphi}^k \in H^1(\Omega)$ and $\hat{\varphi}^k - w^k$ has compact support in $\Omega$.

Now, following the argument in the proof of [39, Lemma 4.5], if we set $\tilde{\varphi}^k := |\hat{\varphi}^k|$ then $\tilde{\varphi}^k \in H^1(\Omega), \tilde{\varphi}^k \geq 0, \tilde{\varphi}^k = w^k$ in $\Omega^c \supset K^{\lambda_k}$ for $k$ large enough, and thus $\tilde{\varphi}^k \in \mathcal{K}^{\lambda_k}(t)$ for $k$ large enough. Moreover, $\tilde{\varphi}^k \to \bar{\varphi}$ in $L^2(\Omega)$ and $J^{\lambda_k}(\tilde{\varphi}^k, \Omega) \to J^0(\bar{\varphi}, \Omega)$.

Since $w^k, \tilde{\varphi}^k \in \mathcal{K}^{\lambda_k}(t)$ and $\mathrm{supp}(\tilde{\varphi}^k - w^k) \subset \Omega$, by (4.19) and integration by parts formula we have

$$a^{\lambda_k}_{\Omega}(w^k, \tilde{\varphi}^k - w^k) \geq -\lambda_k^{(2-n)/n}\left\langle u^{\lambda_k}_t, \tilde{\varphi}^k - w^k\right\rangle_\Omega + \left\langle -\frac{1}{g^{\lambda_k}}, \tilde{\varphi}^k - w^k\right\rangle_\Omega.$$

The inequality $a^{\lambda_k}(u, v-u) \leq \frac{1}{2}J^{\lambda_k}(v) - \frac{1}{2}J^{\lambda_k}(u)$ for any $u, v$ implies

$$\frac{1}{2}J^{\lambda_k}(\tilde{\varphi}^k, \Omega) \geq \frac{1}{2}J^{\lambda_k}(w^k, \Omega) - \lambda_k^{(2-n)/n}\left\langle u^{\lambda_k}_t, \phi^k\right\rangle_\Omega + \left\langle -\frac{1}{g^{\lambda_k}}, \phi^k\right\rangle_\Omega,$$

where $\phi^k := \tilde{\varphi}^k - w^k \to \phi$ in $L^2(\Omega)$. Taking the $\liminf$ as $k \to \infty$ and using the fact that $u^{\lambda_k}_t$ is bounded give

$$\frac{1}{2}J^0(\overline{\varphi}, \Omega) \geq \frac{1}{2}J^0(\overline{w}, \Omega) + \langle -L, \phi\rangle_\Omega. \tag{4.55}$$

This holds for any $\phi \in W_1$ and therefore also for $\delta\phi$, where $0 < \delta < 1$. Replacing $\phi$ in (4.55) by $\delta\phi$ we have

$$\begin{aligned} \frac{1}{2} J^0(\bar{w} + \delta\phi, \Omega) &\geq \frac{1}{2} J^0(\bar{w}, \Omega) + \langle -L, \delta\phi \rangle \\ \Leftrightarrow \frac{1}{2}\left[J^0(\bar{w}, \Omega) + 2\delta q_\Omega(\bar{w}, \phi) + \delta^2 J^0(\phi)\right] &\geq \frac{1}{2} J^0(\bar{w}, \Omega) + \langle -L, \delta\phi \rangle . \end{aligned}$$

Dividing both sides by $\delta$ and sending $\delta \to 0$ we obtain

$$q_\Omega(\overline{w}, \phi) \geq \langle -L, \phi \rangle_\Omega .$$

Since $\operatorname{supp} \phi \in \Omega$, we conclude that (4.48) holds in $\mathbb{R}^n$.

Now take $\psi \in W_2$. As above, we assume that $\psi$ has a compact support contained in $\Omega$, and without loss of generality we can also assume that $0 \leq \psi \leq 1, \psi = 0$ on $B_\varepsilon(0)$ (otherwise consider $\frac{\psi}{\max_{\mathbb{R}^n} \psi}$ instead). Since $\psi \in W_2$ then $\psi\overline{w} \in W_1$ and (4.48) holds for $\psi\bar{w}$, we have $q(\overline{w}, \psi\overline{w}) \geq \langle -L, \psi\overline{w} \rangle$. For the reverse inequality, define $\overline{\varphi} := (1-\psi)\overline{w} \in H^1(\Omega)$. Arguing as before, we can choose $\tilde{\varphi}^k \in \mathcal{K}^{\lambda_k}(t)$ such that $\tilde{\varphi}^k \to \overline{\varphi}$ in $L^2(\Omega)$, $J^{\lambda_k}(\tilde{\varphi}^k, \Omega) \to J^0(\overline{\varphi}, \Omega)$. Again, since $w^k, \tilde{\varphi}^k \in \mathcal{K}^{\lambda_k}(t)$, by (4.19) and the inequality $a^{\lambda_k}(u, v-u) \leq \frac{1}{2} J^{\lambda_k}(v) - \frac{1}{2} J^{\lambda_k}(u)$ we have

$$\frac{1}{2} J^{\lambda_k}(\tilde{\varphi}^k, \Omega) \geq \frac{1}{2} J^{\lambda_k}(w^k, \Omega) - \lambda_k^{(2-n)/n} \left\langle u_t^{\lambda_k}, \tilde{\varphi}^k - w^k \right\rangle_\Omega + \left\langle -\frac{1}{g^{\lambda_k}}, \tilde{\varphi}^k - w^k \right\rangle_\Omega ,$$

Taking $\liminf$ as $k \to \infty$ and arguing the same as in the proof of (4.48) we get

$$\begin{aligned} q_\Omega(\overline{w}, \overline{\varphi} - \overline{w}) &\geq \langle -L, \overline{\varphi} - \overline{w} \rangle_\Omega \\ \Leftrightarrow -q_\Omega(\overline{w}, \psi\overline{w}) &\geq - \langle -L, \psi\overline{w} \rangle_\Omega \\ \Leftrightarrow q_\Omega(\overline{w}, \psi\overline{w}) &\leq \langle -L, \psi\overline{w} \rangle_\Omega . \end{aligned}$$

Thus we have $q(\overline{w}, \psi\overline{w}) = \langle -L, \psi\overline{w} \rangle$ for every $\psi \in W_2$. □

This completes the proof of Theorem 4.15. □

## 4.3 Uniform convergence of the rescaled viscosity solutions and free boundaries

In this section, we will deal with the convergence of $v^\lambda$ and their free boundaries. Let $v$ be the viscosity solution of the Stefan problem (4.1) and $v^\lambda$ be its rescaling.

Let $V = V_{C_*,L}$ be the solution of Hele-Shaw problem with a point source (4.35), where $C_*$ is the constant of Lemma 4.14 and $L = \langle 1/g \rangle$ as in Lemma 2.20.

We define the half-relaxed limits of $v^\lambda$ in $\{|x| \neq 0, t \geq 0\}$:

$$v^*(x,t) = \limsup_{(y,s),\lambda \to (x,t),\infty} v^\lambda(y,s), \qquad v_*(x,t) = \liminf_{(y,s),\lambda \to (x,t),\infty} v^\lambda(y,s),$$

*Remark* 4.23. $V$ is continuous in $\{|x| \neq 0, t \geq 0\}$, therefore $V_* = V = V^*$.

We will prove a result similar to Chapter 3, Theorem 3.16.

**Theorem 4.24.** *Let $n \geq 3$. The rescaled viscosity solution $v^\lambda$ of the Stefan problem (4.1) converges locally uniformly to $V = V_{C_*,\langle 1/g \rangle}$ in $(\mathbb{R}^n \backslash \{0\}) \times [0, \infty)$ as $\lambda \to \infty$ and*

$$v_* = v^* = V.$$

*Moreover, the rescaled free boundary $\{\Gamma(v^\lambda)\}_\lambda$ converges to $\Gamma(V)$ locally uniformly with respect to the Hausdorff distance.*

All the viscosity arguments used in Chapter 3, Section 3.3 can be applied in our anisotropic case with some minor adaptations. Therefore, we will omit some of the proofs and refer to Chapter 3, Section 3.3 and [38, 39, 45] for more details. Let us give a brief review of the techniques in the spirit of Chapter 3, Section 3.3 as follows.

1. We first prove the convergence of the rescaled viscosity solution and their free boundary under the condition (4.6).
   - By the regularity of the initial data $v_0$ as in (4.6), we deduce a weak monotonicity of the solution $v$.
   - Using the weak monotonicity and point-wise arguments with comparison principles, we then show the convergences for regular initial data.
2. For general initial data, we will find upper and lower regular bounds for initial data and use a comparison principle together with the uniqueness of limit function to have the conclusion.

We will state the necessary results here with some remarks on the adaptations for the anisotropic case.

### 4.3.1 Some necessary technical results

First, we have the correct singularity of $v^*$ and $v_*$ at the origin, which can be established similarly to Lemma 4.21.

**Lemma 4.25** (cf. Chapter 3, Lemma 3.19, $v^*$ and $v_*$ behave as $V$ at the origin)**.** *The functions $v^*, v_*$ have a singularity at 0 with*

$$\lim_{|x|\to 0+} \frac{v_*(x,t)}{V(x,t)} = 1, \qquad \lim_{|x|\to 0+} \frac{v^*(x,t)}{V(x,t)} = 1, \text{ for each } t > 0. \tag{4.56}$$

*Proof.* Arguing as in the proof of Lemma 4.21. □

We will also make use of an uniform estimate on $u^\lambda$ and the convergence of boundary points deduced from convergence of variational solutions.

**Lemma 4.26** (cf. [38, Lemma 3.1])**.** *There exists constant $C > 0$ independent of $\lambda$ such that for every $x_0 \in \overline{\Omega_{t_0}(u^\lambda)}$ and $B_r(x_0) \cap \Omega_0^\lambda = \emptyset$ for some $r$, for every $\lambda$ we have*

$$\sup_{x \in \overline{B_r(x_0)}} u^\lambda(x, t_0) > Cr^2.$$

*Proof.* We will prove the statement for $x_0 \in \Omega_{t_0}(u^\lambda)$ first, the results then follows by continuity of $u^\lambda$. Since $B_r(x_0) \cap \Omega_0^\lambda = \emptyset$ then $u^\lambda$ satisfies

$$\lambda^{(2-n)/n} u_t^\lambda - \mathcal{L}^\lambda u^\lambda = -\frac{1}{g} \text{ in } \{u^\lambda > 0\} \cap (B_r(x_0) \times \{t = t_0\})$$

and $\lambda^{(2-n)/n} \to 0$ as $\lambda \to \infty$, $u_t^\lambda(\cdot, t_0)$ is bounded, $-\dfrac{1}{g} \leq -\dfrac{1}{M}$ then there exists a positive constant $C_0(n, M, \lambda_0)$ such that $-\mathcal{L}^\lambda u^\lambda \leq -C_0$ in $\{u^\lambda > 0\} \cap (B_r(x_0) \times \{t = t_0\})$ for $\lambda \geq \lambda_0$ large enough.

Define

$$w^\lambda(x) = u^\lambda(x, t_0) - \frac{C_0}{n} h^\lambda(x - x_0)$$

where $h^\lambda(x)$ is the barrier with quadratic growth corresponding to elliptic operator $\mathcal{L}^\lambda$ stated in Section 4.1.3.2. We have $\{w^\lambda > 0\} \cap B_r(x_0) \subset \{u^\lambda > 0\} \cap \{t = t_0\}$ and therefore, for all $\lambda \geq \lambda_0$,

$$-\mathcal{L}^\lambda w^\lambda \leq 0 \text{ in } \{w^\lambda > 0\} \cap B_r(x_0).$$

We see that $w^\lambda(x_0) > 0$, then maximum of $w^\lambda$ in $\overline{B_r(x_0)}$ is positive and by maximum principle, $w^\lambda$ attains the maximum on the boundary $\{w^\lambda > 0\} \cap \partial B_r(x_0)$

and therefore

$$\sup_{B_r(x_0)} u^\lambda(x,t_0) \geq \sup_{|x-x_0|=r} u^\lambda(x,t_0) > \inf_{|x-x_0|=r} \frac{C_0}{n} h^\lambda(x-x_0).$$

By the quadratic growth of $h^\lambda$, where the coefficients on the growth rate only depends on the elliptic constants, we have

$$\sup_{B_r(x_0)} u^\lambda(x,t_0) \geq Cr^2,$$

for some constant $C$ which does not depend on $\lambda$. □

**Lemma 4.27** (cf. [39, Lemma 5.4])**.** *Suppose that* $(x_k,t_k) \in \{u^{\lambda_k}=0\}$ *and* $(x_k,t_k,\lambda_k) \to (x_0,t_0,\infty)$. *Let* $U = U_{C_*,L}$ *be the limit function as in Theorem 4.15. Then:*

*a)* $U(x_0,t_0)=0$,

*b) If* $x_k \in \Gamma_{t_k}(u^{\lambda_k})$ *then* $x_0 \in \Gamma_{t_0}(U)$,

*Proof.* See proof of [39, Lemma 5.4]. □

The weak monotonicity in time of the solution of the Stefan problem is given by the following lemma.

**Lemma 4.28** ([cf. Chapter 3, Lemma 3.21, Lemma 3.22, Weak monotonicity)**.** *Let* $u$ *be the solution of the variational problem* (4.7)*, and* $v$ *be the associated viscosity solution of the Stefan problem* (4.1)*. Suppose that* $v_0$ *satisfies* (4.6)*. Then there exist* $C \geq 1$ *independent of* $x$ *and* $t$ *such that*

$$v_0(x) \leq Cv(x,t) \text{ and } u(x,t) \leq Ctv(x,t) \text{ in } \mathbb{R}^n \backslash K \times [0,\infty). \tag{4.57}$$

*Proof.* Following the same arguments as in Chapter 3, Lemma 3.21, Lemma 3.22, we obtain (4.57) simply by using elliptic operator $\mathcal{L}$ instead of the Laplace operator. □

Lemma 4.26 and Lemma 4.28 automatically give us a crucial uniform estimate on $v^\lambda$ and allow us to show the relationship between $v_*$, $v^*$ and $V$.

**Corollary 4.29.** *There exists a constant* $C_1 = C_1(n,M)$ *such that if* $(x_0,t_0) \in \Omega(v^\lambda)$ *and* $B_r(x_0) \cap \Omega_0^\lambda = \emptyset$*, we have for every* $\lambda$

$$\sup_{B_r(x_0)} v^\lambda(x,t_0) \geq \frac{C_1 r^2}{t_0}.$$

**Lemma 4.30.** *Let $v^\lambda$ be a viscosity solution of* (4.18). *Then the following statements hold.*

*i)* $v^*(\cdot, t)$ *is subsolution of* (4.16) *in* $\mathbb{R}^n \backslash \{0\}$ *and* $v_*(\cdot, t)$ *is supersolution of* (4.16) *in* $\Omega_t(v_*)\backslash\{0\}$ *in viscosity sense.*

*ii)* $\Omega(V) \subset \Omega(v_*)$ *and in particular* $v_* \geq V$.

*iii)* $\Gamma(v^*) \subset \Gamma(V)$.

*Proof.* i) follows from a standard viscosity argument with noting that we can take a sequence of test functions for rescaled elliptic equation that converges to the test function for (4.16) by classical homogenization results.

ii) See Chapter 3, Lemma 3.23, the conclusion holds by i), Lemma 4.56 and Lemma 4.28.

iii) See [39, Lemma 5.6 ii].

□

Now we are ready to prove Theorem 4.24.

### 4.3.2 Proof of Theorem 4.24

*Proof.* (See proof of Chapter 3, Theorem 3.16 for more details).

***Step 1***. We prove the convergence of viscosity solutions and the free boundaries under the conditions (4.6) and (4.57) first.

By Lemma 4.30, the correct singularity of $v^*$ from Lemma 4.56 and the comparison principle for elliptic equation (4.16) we have

$$V(x,t) \leq v_*(x,t) \leq v^*(x,t) \leq V_{C_*+\varepsilon,\langle 1/g\rangle}(x,t).$$

Let $\varepsilon \to 0$ we obtain $v_* = v^* = V$ by continuity and in particular, $\Gamma(v_*) = \Gamma(v^*) = \Gamma(V)$.

Now we need to show the uniform convergence of the free boundaries with respect to the Hausdorff distance. Fix $0 < t_1 < t_2$ and denote:

$$\Gamma^\lambda := \Gamma(v^\lambda) \cap \{t_1 \leq t \leq t_2\}, \qquad \Gamma^\infty := \Gamma(V) \cap \{t_1 \leq t \leq t_2\},$$

a $\delta$-neighborhood of a set $A$ in $\mathbb{R}^n \times \mathbb{R}$ is

$$U_\delta(A) := \{(x,t) : \operatorname{dist}((x,t), A) < \delta\}.$$

We need to prove that for all $\delta > 0$, there exists $\lambda_0 > 0$ such that:

$$\Gamma^\lambda \subset U_\delta(\Gamma^\infty) \quad \text{and} \quad \Gamma^\infty \subset U_\delta(\Gamma^\lambda), \qquad \forall \lambda \geq \lambda_0. \tag{4.58}$$

The first inclusion follows by contradiction argument, using Lemma 4.27 above.

For the second inclusion in (4.58), we will prove a pointwise result first. Suppose that there exists $\delta > 0$, $(x_0, t_0) \in \Gamma^\infty$ and $\{\lambda_k\}, \lambda_k \to \infty$, such that $\mathrm{dist}((x_0, t_0), \Gamma^{\lambda_k}) \geq \frac{\delta}{2}$ for all $k$. Then there exists $r > 0$ such that $D_r(x_0, t_0) := B(x_0, r) \times [t_0 - r, t_0 + r]$ satisfies either:

$$D_r(x_0, t_0) \subset \{v^{\lambda_k} = 0\} \text{ for all } k, \tag{4.59}$$

or after passing to a subsequence,

$$D_r(x_0, t_0) \subset \{v^{\lambda_k} > 0\} \text{ for all } k. \tag{4.60}$$

If (4.59) holds, clearly $V = v_* = 0$ in $D_r(x_0, t_0)$ which is in a contradiction with the assumption that $(x_0, t_0) \in \Gamma^\infty$. Thus we assume that (4.60) holds. Since the rescaled parabolic operator becomes elliptic in the limit, we will apply the Moser-Harnack's inequality for $v^{\lambda_k}$ in a shrinking domain of time by setting

$$w^k(x, t) := v^{\lambda_k}(x, \lambda_k^{(2-n)/n} t)$$

then $w^k > 0$ in $D_r^w(x_0, t_0) := B(x_0, r) \times [\lambda_k^{(n-2)/n}(t_0 - r), \lambda_k^{(n-2)/n}(t_0 + r)]$ and $w^k$ satisfies $w_t^k - \mathcal{L}^\lambda w^k = 0$ in $D_r^w(x_0, t_0)$. Since $\lambda_k^{(n-2)/n}\frac{r}{2} \to \infty$ as $k \to \infty$, by Moser-Harnack's inequality for the parabolic equation, for fixed $\tau > 0$ there exists a constant $C_1 > 0$ such that for each $t \in [t_0 - \frac{r}{2}, t_0 + \frac{r}{2}]$ and $\lambda_k$ such that $\tau < \lambda_k^{(n-2)/n}\frac{r}{4}$ we have

$$\sup_{B(x_0, r/2)} w^k(\cdot, \lambda_k^{(n-2)/n} t - \tau) \leq C_1 \inf_{B(x_0, r/2)} w^k(\cdot, \lambda_k^{(n-2)/n} t).$$

Note that $C_1$ depends only on $\tau$ and elliptic constants, and therefore does not depend on $\lambda_k$. This inequality together with Corollary 4.29 yields:

$$\frac{C_2 r^2}{t - \lambda_k^{(2-n)/n}\tau} \leq \sup_{B(x_0, r/2)} v^{\lambda_k}(\cdot, t - \lambda_k^{(2-n)/n}\tau) \leq C_1 \inf_{B(x_0, r/2)} v^{\lambda_k}(\cdot, t)$$

for all $t \in [t_0 - \frac{r}{2}, t_0 + \frac{r}{2}]$, $\lambda_k \geq \lambda_0$ large enough, where $C_2$ only depends on $n, M, \lambda_0$. Taking the limit when $\lambda_k \to \infty$, the uniform convergence of $\{v^{\lambda_k}\}$ to $V$ gives $V > 0$ in $B(x_0, \frac{r}{2}) \times [t_0 - \frac{r}{2}, t_0 + \frac{r}{2}]$, which is a contradiction with $(x_0, t_0) \in \Gamma^\infty \subset \Gamma(V)$.

We have proved that every point of $\Gamma^\infty$ belongs to all $U_{\delta/2}(\Gamma^\lambda)$ for sufficiently large $\lambda$. Therefore the second inclusion in (4.58) follows from the compactness of $\Gamma^\infty$. This concludes the proof of Theorem 4.24 when (4.57) holds.

***Step 2***. For general initial data, arguing as in step 2 of the proof of Chapter 3, Theorem 3.16, we are able to find upper and lower bounds for the initial data for which (4.57) holds. The comparison principle for viscosity solution of the Stefan problem (4.1) then yields the convergence since the limit function $V$ is unique, does not depend on the initial data.

$\square$

# Appendix A

# The fundamental solution of an uniformly elliptic equation of divergence form

In this section, we will recall about the fundamental solution or Green's function of an uniformly second order elliptic equation of divergence form and some useful results used in our work. More specifically, as in Section 4.1.1, we consider a self-adjoint uniformly elliptic second order linear operator of divergence form $-\mathcal{L}$ in dimension $n \geq 3$, where $\mathcal{L}$ is defined as in Section 1.2 and $A(x) = (a_{ij}(x))$ is a symmetric, bounded matrix satisfying the ellipticity (4.3) as well as the highly oscillating property (2.20).

We define Green's function $g(x, y)$ of the operator $-\mathcal{L}$ on a bounded domain $\Omega \subset \mathbb{R}^n$ as the weak solution (in distributional sense), vanishing on $\partial\Omega$ of the equation

$$-\mathcal{L}g = \delta_y, \tag{A.1}$$

where $\delta_y$ is the Dirac measure at $y$. The basic facts of Green's function were first proved for elliptic operator of the form $-\mathcal{L}$ with symmetric bounded measurable coefficients in a bounded domain $\Omega$ in $\mathbb{R}^n, n \geq 3$, by Littman, Stampacchia and Weinberger in [41]. Their results were then studied extensively for more general elliptic operators with non-symmetric coefficients in [26], where the author proved the existence, the uniqueness and the bounds for Green's function in a bounded domain with the constants in the estimate are independent of the domain. These results are also obtained by compactness methods in [4]. More precisely, we have the following theorem taken from [26, Theorem 1.1]. Let us denote some notations of the weak $L^p$ spaces. Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ and $p \in [1, \infty]$. We define a Banach space $L^{p,\infty}$ as

$$L^{p,\infty}(\Omega) = \{f : \Omega \to \mathbb{R}, f \text{ measurable}, \|f\|_{L^{p,\infty}(\Omega)} < \infty\},$$

where

$$\|f\|_{L^{p,\infty}(\Omega)} = \sup_{t \geq 0} \left\{t\mu(\{x \in \Omega, |f(x)| > t\})^{1/p}\right\},$$

$\mu$ is the Lebesgue measure in $\mathbb{R}^n$. Note that (see [26, 2]), for any $0 < \varepsilon < p - 1$,

$$C(p, \varepsilon, \Omega)\|f\|_{L^{p-\varepsilon}(\Omega)} \leq \|f\|_{L^{p,\infty}(\Omega)} \leq \|f\|_{L^p(\Omega)}. \tag{A.2}$$

The explicit formula of $C(p, \varepsilon, \Omega)$ was given, see [26, 2] and references therein for more details.

**Theorem A.1** (cf. [26, Theorem 1.1])**.** *Assume that $A$ is a bounded, measurable and uniformly elliptic matrix. Then there exists a unique Green's function $g_R(x, y)$ of $-\mathcal{L}$ in the ball $B_R := B(0, R)$, i.e., the function $g_R : \Omega \times \Omega \to \mathbb{R}$ such that $g_R \geq 0$, $g_R(\cdot, y) \in W^{1,1}(\Omega) \cap H^1_{loc}(\Omega \backslash \{y\})$ satisfying (A.1) in $B_R$ and $g_R(\cdot, y) = 0$ if $|x| = R$. Moreover for each $y \in B_R$,*

$$\|g_R(\cdot, y)\|_{L^{\frac{n}{n-2},\infty}(B_R)} \leq C, \tag{A.3}$$

$$\|Dg_R(\cdot, y)\|_{L^{\frac{n}{n-1},\infty}(B_R)} \leq C, \tag{A.4}$$

*and for every $(x, y) \in B_R \times B_R$,*

$$C_1|x - y|^{2-n} \leq g_R(x, y) \leq C_2|x - y|^{2-n}, \tag{A.5}$$

*for some constants $C, C_1, C_2$.*

The optimal constants $C_1, C_2$ were given in the remark following [26, Theorem 1.1] as

$$C_1 = C(n)\left(\frac{\beta}{\alpha}\right)^{\frac{n-2}{n}}\left(1 + \log\left(\frac{\beta}{\alpha}\right)\right),$$
$$C_2 = c(n)^{1+\left(\frac{\beta}{\alpha}\right)^{1/2}},$$

which are independent of the domain $B_R$ as well as the pole $y$. The point-wise bounds for the gradient and second derivatives of Green's function $g_R$ were also established in [26], however, the constants in the estimates priori depend on the domain $B_R$.

The remark following [41, Corollary 7.1] says that we can define Green's function $G(x, y)$ in the whole space by taking the limit of Green's functions $g_R(x, y)$ in $B(0, R)$ as the radius $R \to \infty$. This Green's function has all the basic properties of Green's function in a bounded domain such as $G$ is symmetric, non-negative, the total mass of $G$ in the whole space is 1 and the convolution of $G$ with a $H^1$-function $\psi$ is the solution of the problem

$$\begin{cases} -\mathcal{L}u = \psi, \\ \lim\limits_{x\to\infty} u = 0. \end{cases}$$

This function is not in $H^1(\mathbb{R}^n)$ but it is in $W^{1,p}_{\text{loc}}(\mathbb{R}^n) \cap H^1_{\text{loc}}(\mathbb{R}^n \backslash \{y\})$ for every $p < \frac{n}{n-1}$. Moreover, the bounds (A.5) for Green's function $g_R$ hold uniformly in $\mathbb{R}^n$, therefore also hold for the limit function. These results are proved in detail by Anantharaman, Blanc and Legoll in [2]. Especially, in this paper the authors also addressed the question of the decay of the derivatives of $G$ at infinity. In this section, we show the proof of the existence and uniqueness results, Theorem 4.3, as well as the proof for the bounds of gradients, Lemma 4.5 taken from [2].

*Proof of Theorem 4.3.* These following arguments are given in the proof of [2, Theorem 1].

Let $R > 0$ and $g_R$ be Green's function of $-\mathcal{L}$ in $B_R$. If $R' > R$ then $g_{R'} \geq g_R$ in $B_R \times B_R$ by the maximum principle. Therefore $g_R$ is a non-decreasing function of $R$, bounded in every compact set in $\mathbb{R}^{2n} \backslash \{x = y\}$ by (A.5), which implies that $g_R$ converges to some function $G$ in $\mathbb{R}^{2n} \backslash \{x = y\}$ as $R \to \infty$. In addition, by (A.5), (A.2) and the Dominated Convergence Theorem, we also can deduce that $g_R \to G$ in $L^p_{\text{loc}}(\mathbb{R}^{2n})$ and $g_R(\cdot, y) \to G(\cdot, y)$ in $L^p_{\text{loc}}(\mathbb{R}^n)$, $p < \frac{n}{n-2}$. The limit function $G$ has all the basic properties of $g_R$ such as it is non-negative, symmetric and $\lim_{|x-y|\to\infty} G(x, y) = 0$. $G$ also satisfies the estimate (A.5).

We will check that the limit function $G$ satisfies (A.1) in $\mathbb{R}^n$ and $G(\cdot, y)$ belongs to the space $W^{1,1}(\mathbb{R}^n) \cap H^1_{\text{loc}}(\mathbb{R}^n \backslash \{y\})$. Let $\Omega \subset \mathbb{R}^n$ be any bounded domain. By (A.4) and (A.2), $Dg_R(\cdot, y)$ is bounded in $(L^p(\Omega))^n$ for every $R$ such that $\Omega \subset B_R$ and any $p < \frac{n}{n-1}$. Hence, extracting a subsequence if necessary, there exist a $T \in (L^p(\Omega))^n$ such that $Dg_R(\cdot, y) \rightharpoonup T$ weakly in $(L^p(\Omega))^n, p < \frac{n}{n-1}$. Since $g_R(\cdot, y) \to G(\cdot, y)$ in

$L^p(\Omega), p < \frac{n}{n-2}$ then $T = DG(\cdot, y)$. Now for every $\varphi \in C_0^\infty(\mathbb{R}^n)$, choose $\Omega$ such that the support of $\varphi$ is contained in $\Omega$. For every $R$ such that $\Omega \subset B_R$, since $g_R$ is Green's function of $-\mathcal{L}$ in $B_R$ then

$$\int_\Omega a_{ij}(x) D_i g_R(x,y) D_j \varphi(x) dx = \int_{B_R} a_{ij}(x) D_i g_R(x,y) D_j \varphi(x) dx = \varphi(y), \quad \forall y \in B_R. \tag{A.6}$$

Sending $R \to \infty$ in (A.6) we can conclude that $G$ satisfies (A.1) in distributional sense. Moreover, by (A.3) and (A.4), we see that $G(\cdot, y) \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$ for any $p < \frac{n}{n-1}$. The fact that $G(\cdot, y) \in H^1(\mathbb{R}^n \backslash \{y\})$ was obtained in the proof of Theorem A.1 in [26] where the arguments do not require the boundedness of the domain.

It remains to check the uniqueness of the function $G$, then the convergence of $g_R$ to $G$ is not restricted to a subsequence. Assume that $G_1, G_2$ are two Green's functions, then $H = G_1 - G_2$ is a solution of $-\mathcal{L}H(\cdot, y) = 0$ in $\mathbb{R}^n$ for any $y \in \mathbb{R}^n$. Fix a $y \in \mathbb{R}^n$. By [44, Theorem 4], $\sup_{|x-y|=r} H(\cdot, y) - \inf_{|x-y|=r} H(\cdot, y)$ must grow at least like a power of $r$ as $r \to \infty$ provided $H$ is not a constant. This is a contradiction with the growth of $G_1, G_2$ provided by (A.5). □

*Remark* A.2. All the arguments used in the proof of Theorem 4.3 are still valid for elliptic operators of the form $-\mathcal{L}$ with non-symmetric coefficients, the only needed assumptions here are the ellipticity (4.3) and the bounded measurable property of the coefficients.

*Remark* A.3. The existence and uniqueness of the fundamental solution of elliptic operator $-\mathcal{L}$ was also deduced in the case $n = 2$. This result was fist observed by Kenig and Ni in [33]. An alternative proof was also given in [2]. In particular, we have the following theorem.

*Theorem* A.4 (cf. [2, Theorem 2]). *Let $n = 2$ and $A$ is bounded, measurable and uniformly elliptic matrix. Then there exist a unique (up to the addition of a constant) Green's function $G$ of $-\mathcal{L}$ in $\mathbb{R}^n$ such that $G(\cdot, y) \in W^{1,p}_{loc}(\mathbb{R}^n) \cap H^1_{loc}(\mathbb{R}^n \backslash \{y\}), p < 2$ and*

$$\exists C > 0, \quad \forall (x,y) \in \mathbb{R}^{2n}, \quad |G(x,y)| \le C(1 + |\log|x-y||).$$

The proof of [2, Theorem 2] is based on an approach similar to the proof of Theorem 4.3 by defining first Green's function $g_R$ of $-\mathcal{L}$ in $B_R$, then search for a limit as $R \to \infty$. However, the estimates in Theorem A.1 cannot be applied since they hold only for case $n \geq 3$. Instead, in the proof of [2, Theorem 2], the authors used a gradient bound to construct a limit of $Dg_R$ and then checked that the limit is the gradient of Green's function in $\mathbb{R}^n$. The proof is more technical than the one for the case $n \geq 3$. For more details, see [2].

Now we will show the proof of the gradient estimates, Lemma 4.5 for the case $n \geq 3$. The result for the dimension $n = 2$ was also established by more complicated techniques again and is omitted in this work. Note that Lemma 4.5 requires stronger assumptions than Theorem 4.3 where it holds only for operator $-\mathcal{L}$ with bounded, not necessary symmetric coefficients satisfying (4.3) and (4.4).

*Proof of Lemma 4.5 for the case $n \geq 3$.* This is a part of the proof of [2, Proposition 5].

This lemma follows by applying an important $L^\infty$ estimate of the gradient for a solution of an uniformly elliptic equation with periodic coefficients in [4, Lemma 16] and the bound for Green's function provided by Theorem 4.3. More specifically, by [4, Lemma 16] we have

$$\forall x \in \mathbb{R}^n, \quad \forall y \in \mathbb{R}^n, \quad \forall r < |x-y|, \quad \|DG(\cdot, y)\|_{L^\infty(B(x,\frac{r}{2}))} \leq \frac{C}{r}\|G(\cdot, y)\|_{L^\infty(B(x,r))},$$

where $C$ only depends on the elliptic constants of the operator $-\mathcal{L}$ and the dimension $n$. Now, by (A.5) we have

$$\|DG(\cdot, y)\|_{L^\infty(B(x,\frac{r}{2}))} \leq \frac{C}{r} \sup_{z \in B(x,r)} \frac{1}{|z-y|^{n-2}}. \tag{A.7}$$

By the continuity of $DG(\cdot, y)$ away from $y$, (A.7) holds for the point-wise gradient. Let $r = \frac{|x-y|}{2}$, we have

$$|x-y| \leq |x-z| + |z-y| \leq \frac{1}{2}|x-y| + |z-y|.$$

This together with (A.7) imply

$$|DG(\cdot, y)| \leq \frac{2^{n-1}C}{|x-y|^{n-1}}$$

then (4.13) holds. Next, we will show (4.14). If the matrix $A$ is symmetric then $G(x,y) = G(y,x)$ and (4.14) automatically follows. Otherwise we see that $\tilde{G}(x,y) = G(y,x)$ is Green's function of the adjoint operator $-\mathcal{L}^* = -D_i(a_{ji}D_j)$, (see [26, Theorem 1.3]). Applying (4.13) to $\tilde{G}$ we get (4.14). $\square$

*Remark* A.5. The proof of [2, Proposition 5] also covers the gradient estimate for case $n = 2$. The basic idea is to use the relationship of Green's function $G$ in $2D$ with Green's function $\tilde{G}$ of operator $\tilde{\mathcal{L}}$ in $3D$, where $\tilde{\mathcal{L}}u = -\mathcal{L}u - u_{tt}$ and deduce the estimate for $G$ from the estimate for $\tilde{G}$.

Moreover, the authors in [2] also obtained the bounds for second derivatives of Green's function as follows.

*Proposition* A.6 (cf. [2, Proposition 7]). *Assume that the matrix $A$ in the operator $\mathcal{L}$ is a bounded, not necessary symmetric matrix with the coefficients satisfying* (4.3) *and* (4.4). *Then Green's function of the operator $-\mathcal{L}$ satisfies the following estimate:*

$$\exists C > 0, \quad \forall x \in \mathbb{R}^n, \quad y \in \mathbb{R}^n, \quad |D_x D_y G(x, y)| \leq \frac{C}{|x - y|^n}.$$

Note that similar estimates as in Lemma 4.5 and Proposition A.6 are well-known for Green's function in a bounded domain, see [26] for instance. The important point here is that these estimates continue to hold at infinity. However, as stated before, even though the bounds for Green's function hold for elliptic operators with any bounded, measurable coefficients, the bounds for its gradient require a stronger assumption of the regularity and periodicity of the coefficients.

In this work, we will refer to the fundamental solution of the elliptic operator $-\mathcal{L}$ as Green's function of $-\mathcal{L}$ in the whole space. We include here the proof of the asymptotic expansion of the fundamental solution (Green's function) in dimension $n \geq 3$, Lemma 4.17. This result was proved in [60, Chapter III,Theorem 2] using the techniques of $G$-convergence. It turns out that the asymptotic expansion is determined by the behavior of the fundamental solution of the corresponding $G$-convergence operator (see [60, Chapter III]). In a periodic (or stationary ergodic) setting, the standard homogenization results guarantee that the family of operator $\mathcal{L}^\varepsilon := D_j\left(a_{ij}(\varepsilon^{-1}x)D_i\right)$ has the $G$-limit is $\mathcal{L}^0 := q_{ij}D_{ij}$ in $\mathbb{R}^n$ where $q_{ij}$ are constants (see [60, 32]). We recall the definition of $G$-convergence in [60, 32] as follows.

Let $V$ be a Hilbert space, and let $V^*$ be the dual of $V$. Consider a sequence of linear operators $A_\varepsilon : V \to V^*$ that are uniformly coercive and uniformly bounded:

$$\langle A_\varepsilon u, u\rangle \geq \nu_1 \|u\|_V^2, \quad \nu_1 > 0,$$
$$\|A_\varepsilon u\|_{V^*} \leq \nu_2 \|u\|_V.$$

By the Lax-Milgram Lemma, any coercive bounded operator $A : V \to V^*$ has an inverse one $A^{-1} : V^* \to V$.

**Definition A.7** (cf. [60, Definition 2, §1, Chapter I]). A bounded operator $A_0 : V \to V^*$ satisfying the coerciveness inequality $\langle A_0 u, u\rangle \geq \nu_1 \|u\|^2$ is called the $G$-limit operator for the sequence $A_\varepsilon$ (and we write $A_\varepsilon \xrightarrow{G} A_0$), if for any $f \in V^*$

$$A_\varepsilon^{-1} f \rightharpoonup A_0^{-1} f \text{ weakly in } V.$$

Now we will show the proof of Lemma 4.17 taken from the proof of [60, Chapter III,Theorem 2].

*Proof of Lemma 4.17.* Since $A$ is periodic matrix, [60, Chapter II, Theorem 1] implies that $\mathcal{L}^\varepsilon$ $G$-converges to a uniform elliptic operator $\mathcal{L}^0 = q_{ij} D_{ij}$ with constant coefficients $q_{ij}$. Without loss of generality, we can assume that $\mathcal{L}^0 = \Delta$, otherwise we can use a change of coordinates to recover the general case. Let $G, G^\varepsilon, G_0$ be the fundamental solutions as in the assumption of Lemma 4.17. We will show the uniform convergence of $G^\varepsilon$ to $G_0$ first.

Let $\varphi \in C_0^\infty(\mathbb{R}^n)$, $Q$ be a bounded domain in $\mathbb{R}^n$. Define

$$u^\varepsilon(x) := \int_{\mathbb{R}^n} G^\varepsilon(x, y)\varphi(y)dy$$

then $u^\varepsilon$ is the $H^1_{\text{loc}}(\mathbb{R}^n)$ solution of the problem

$$\begin{cases} -\mathcal{L}^\varepsilon u^\varepsilon = \varphi & \text{in } \mathbb{R}^n, \\ \lim\limits_{|x|\to\infty} (x) = 0. \end{cases}$$

Since $|G^\varepsilon| \leq C_1 |x-y|^{2-n}$ by Theorem 4.3 then $|u^\varepsilon| \leq C_2 |x|^{2-n}$, where $C_2$ is independent of $\varepsilon$. Moreover, using an estimate for solution of an elliptic equation we have

$$\|u^\varepsilon\|_{H^1(Q')} \leq C(\|u^\varepsilon\|_{L^2(Q)} + 1),$$

where $C$ does not depend on $\varepsilon$. Therefore $u^\varepsilon$ is uniformly bounded in $H^1(Q')$ for any bounded domain $Q' \subset \mathbb{R}^n$. Hence, there exists a subsequence $u^{\varepsilon_k}$ such that $u^{\varepsilon_k} \rightharpoonup u_0$ weakly in $H^1(Q')$. Since the $G$-limit of $\mathcal{L}^\varepsilon$ in $\mathbb{R}^n$ is $\Delta$ then by the classical $G$-convergence results (see [60]), $u_0$ is the solution of

$$\begin{cases} -\Delta u_0 = \varphi & \text{in } \mathbb{R}^n, \\ |u_0(x)| \leq C_2 |x|^{2-n} & \text{in } \mathbb{R}^n. \end{cases}$$

By the uniqueness of $u_0$, the convergence is not restricted to a subsequence.

Now we have for every $\varphi(x), \psi(x) \in C_0^\infty(\mathbb{R}^n)$,

$$\begin{aligned}\int_{\mathbb{R}^{2n}} G^\varepsilon(x,y)\psi(x)\varphi(y)dxdy = \int_{\mathbb{R}^n} u_\varepsilon(x)\psi(x)dx \to \int_{\mathbb{R}^n} u_0(x)\psi(x)dx \\ = \int_{\mathbb{R}^{2n}} \Phi(x,y)\psi(x)\varphi(y)dxdy,\end{aligned} \tag{A.8}$$

where $\Phi$ is the fundamental solution of Laplace equation. Besides, $G^\varepsilon$ is uniformly bounded in any compact set in $\mathbb{R}^n\backslash\{0\}$. By elliptic regularity, $G^\varepsilon$ are uniformly Hölder continuous with respect to $\varepsilon$. Thus, by Arzelà-Ascoli, there exists a subsequence $G^{\varepsilon_k}$ that locally uniformly converges to a function $\hat{G}$ in $\mathbb{R}^n\backslash\{0\}$. Now by (A.8) and the fundamental lemma of calculus of variations, we have $\hat{G} = \Phi$. Since the limit function $\Phi$ is unique then ve conclude that $G^\varepsilon \to \Phi$ locally uniformly in $\mathbb{R}^n\backslash\{0\}$.

It remains to show the asymptotic expansion formula. Define

$$\theta(x,y) := \frac{G(x,y) - \Phi(x,y)}{|x-y|^{2-n}}. \tag{A.9}$$

Similar to the computations in the proof of Lemma 4.8, we see that $G^\varepsilon(x,y) = \varepsilon^{2-n}G(\frac{x}{\varepsilon},\frac{y}{\varepsilon})$ for every $\varepsilon > 0$. Therefore, for every $\varepsilon > 0$ we have

$$\theta(x,y) = \frac{\varepsilon^{n-2}G^\varepsilon(\varepsilon x,\varepsilon y) - \Phi(x,y)}{|x-y|^{2-n}} = \frac{\varepsilon^{n-2}\left(G^\varepsilon(\varepsilon x,\varepsilon y) - \Phi(\varepsilon x,\varepsilon y)\right)}{|x-y|^{2-n}}.$$

Fix a positive constant $a$ and let $\varepsilon = |x-y|^{-1}$, $(x',y') = (\varepsilon x,\varepsilon y)$. If $(x,y) \in A := \{|x| + |y| < a|x-y|\}$ then $\varepsilon > 0$ and

$$\begin{cases} |x'| + |y'| < a, \\ |x'-y'| = 1. \end{cases}$$

Moreover, $\theta(x,y) = \tilde{\theta}(x',y') = G^\varepsilon(x',y') - \Phi(x',y') \to 0$ uniformly in the set $\{|x'| + |y'| < a, |x'-y'| = 1\}$ as $\varepsilon \to 0$. Thus, $\theta$ converges uniformly to 0 as $|x-y| \to \infty$ in the set $A$. By (A.9) then we have an asymptotic expansion of $G$ as

$$G(x,y) = \Phi(x,y) + |x-y|^{2-n}\theta(x,y), \tag{A.10}$$

where $\theta(x,y) \to 0$ as $|x-y| \to \infty$, uniformly on the set $\{|x| + |y| < a|x-y|\}$, $a$ is any fixed positive constant. $\square$

Lastly, we would like to recall one of the crucial tools to analyze a solution of an elliptic equation of the form

$$-\mathcal{L}u = 0 \tag{A.11}$$

in a punctured domain $\Omega := \{0 < |x| < R\}$ or in a exterior domain $\Omega_e := \{|x| > R\}$, where $\mathcal{L}$ is the operator as considered at the beginning of this appendix. The method we mention here is a generalized Kelvin transformation, which allows us to take a uniformly elliptic equation (A.11) defined in an exterior domain $\Omega_e$ into another uniformly elliptic equation of the same form (with different coefficients) defined in a punctured domain, and vice versa. This transformation, similarly to the classical one, can be defined using the fundamental solution of operator $-\mathcal{L}$. Using this transformation, we are able to prove the asymptotic behavior of the solution of near-field limit problem, Lemma 4.14.

We recall a generalization of the Kelvin inversion transformation for Laplace's equation, which was established by Serrin and Weinberger in [56].

**Lemma A.8** (cf. [56, Section 3])**.** *Let $u$ and $w$ be two solutions of the elliptic equation (A.11) in a domain $\Omega$ with $w > 0$. Let $y_k = y_k(x_1, x_2, ..., x_n), k = 1, , 2, ..., n$, be a continuously differentiable one to one coordinate transformation with non-vanishing Jacobian $J = \det(\partial y_k/\partial x_i)$ and inverse $x_i = x_i(y_1, ..., y_n)$. Then the function*

$$v(y) = \frac{u(x(y))}{w(x(y))} \tag{A.12}$$

*is a solution of the elliptic equation*

$$D_l(\bar{a}_{kl} D_k v) = 0 \tag{A.13}$$

*in the image domain $\Omega'$, where*

$$\bar{a}_{kl} = \frac{w^2}{|J|} a_{ij} \frac{\partial y_k}{\partial x_i} \frac{\partial y_l}{\partial x_j}. \tag{A.14}$$

*Proof.* As shown in the proof of the first lemma of [56, Section 3], we need to check that $v$ is a weak solution of (A.13). Let $\varphi \in C_0^\infty(\Omega')$ then

$$\begin{aligned}
\int_{\Omega'} \bar{a}_{kl} v_k \varphi_l dy &= \int_{\Omega'} \frac{w^2}{|J|} a_{ij} \frac{\partial y_k}{\partial x_i} \frac{\partial y_l}{\partial x_j} v_k \varphi_l dy \\
&= \int_{\Omega} \frac{w^2}{|J|} a_{ij} \frac{\partial v}{\partial x_i} \frac{\partial \varphi}{\partial x_j} |J| dx \qquad \text{(By a change of variables and chain rule)} \\
&= \int_{\Omega} w^2 a_{ij} \left(\frac{u}{w}\right)_i \varphi_j dx \\
&= \int_{\Omega} a_{ij} \varphi_j (u_i w - u w_i) dx \\
&= \int_{\Omega} a_{ij} [(\varphi w)_j u_i - (\varphi u)_j w_i] dx \quad \text{(By symmetry of } a_{ij}\text{)} \\
&= 0.
\end{aligned}$$

The last equality follows from the fact that $\varphi = \varphi(y(x))$ is a function with compact support in $\Omega$, $u$ and $w$ are solutions of (A.13), and we also have $\varphi w, \varphi u \in C_0^\infty(\Omega)$. Therefore, $v$ is a solution of (A.13) in $\Omega'$. □

Next, we will show that if $w$ is taken as the fundamental solution of $-\mathcal{L}$ and the coordinate transformation is the inversion $y_k = \frac{x_k}{|x|^2}$ then the equation (A.13) is a uniformly elliptic equation with the elliptic constants that depend only on the elliptic constants of $\mathcal{L}$.

**Lemma A.9** (Generalized Kelvin transformation, cf. [56, Theorem 2, Theorem 3])**.** *Let $G$ be the fundamental solution of $-\mathcal{L}$ and $F(x) := G(x, 0)$. If $u$ is a solution of the uniformly elliptic equation* (A.11) *in $\Omega$ (resp. $\Omega_e$), then*

$$v(y) = \frac{u(y/|y|^2)}{F(y/|y|^2)} \tag{A.15}$$

*is a solution of the uniformly elliptic equation* (A.13) *in $\{|y| > R^{-1}\}$ (resp. $\{0 < y < R^{-1}\}$), and elliptic constants of* (A.13) *depends only on the elliptic constant of* (A.11)*.*

*Proof.* This lemma is a particular case of [56, Theorem 2, Theorem 3] and we include the proof from there.

Since $G$ is the fundamental solution of (A.11) and $F(x) = G(x, 0)$, by Theorem 4.3, there exists a positive constant $C$ depending only on the elliptic constants of $\mathcal{L}$ such that

$$C^{-1/2}|x|^{2-n} \le F(x) \le C^{1/2}|x|^{2-n}, \tag{A.16}$$

for every $x \in \mathbb{R}^n$, where $n \geq 3$.

By the inversion $y_k = \frac{x_k}{|x|^2}$, the domain $\Omega$ (resp. $\Omega_e$) is transformed into $\{|x| > R^{-1}\}$ (resp. $\{0 < |x| < R^{-1}\}$). Differentiating $y_k$ leads to

$$\frac{\partial y_k}{\partial x_i} = \frac{1}{|x|^2}\left(\delta_{ik} - \frac{2x_i x_k}{|x|^2}\right).$$

A simple computation also shows that the matrix $O := \left(\delta_{ik} - \frac{2x_i x_k}{|x|^2}\right)$ is a symmetric orthogonal matrix. Indeed, it is clear that $O$ is symmetric and

$$O^2 = I - \frac{4}{|x|^2}B + \frac{4}{|x|^4}B^2,$$

where $B = (x_i x_k)$. We have $B^2 = (b_{ik})$, where $b_{ik} = \sum_{s=1}^n x_i x_s x_s x_k = x_i x_k |x|^2$. Thus $B^2 = |x|^2 B$, then $O^2 = I$ and $|\det O| = 1$.

Now let $u$ be a solution of (A.11) in $\Omega$, $v$ as in (A.15). Since $F$ is a solution of (A.11) in $\mathbb{R}^n \backslash \{0\}$ then by Lemma A.9, $v$ is a solution of (A.13) in $\Omega' = \{|x| > R^{-1}\}$, where the new coefficients are defined by (A.14). We have

$$\begin{aligned}
|J| &= \frac{1}{|x|^{2n}},\\
\bar{a}_{kl} &= \frac{F^2}{|x|^{4-2n}} a_{ij}\left(\delta_{ik} - \frac{2x_i x_k}{|x|^2}\right)\left(\delta_{jl} - \frac{2x_j x_l}{|x|^2}\right).
\end{aligned}$$

By (A.16),

$$C^{-1} a_{ij}\left(\delta_{ik} - \frac{2x_i x_k}{|x|^2}\right)\left(\delta_{jl} - \frac{2x_j x_l}{|x|^2}\right) \leq \bar{a}_{kl} \leq C a_{ij}\left(\delta_{ik} - \frac{2x_i x_k}{|x|^2}\right)\left(\delta_{jl} - \frac{2x_j x_l}{|x|^2}\right)$$

and

$$C^{-1}\langle O\eta, AO\eta\rangle \leq \bar{a}_{kl}\eta_k\eta_l \leq C\langle O\eta, AO\eta\rangle.$$

By ellipticity (4.3),

$$C^{-1}\alpha|O\eta|^2 \leq \bar{a}_{kl}\eta_k\eta_l \leq C\beta|O\eta|.$$

Moreover, since $O$ is an orthogonal matrix then $|O\eta|^2 = |\eta|^2$ and we get

$$C^{-1}|\eta|^2 \leq \bar{a}_{kl}\eta_k\eta_l \leq C\beta|\eta|^2.$$

In conclusion, the function $v$ defined by (A.15) is a solution of the uniformly elliptic equation (A.13) with the elliptic constants depending only on the elliptic constants of (A.11). An analogous result holds for $u$ is a solution of (A.11) in $\Omega_e$ and $v$ is defined as in (A.15). $\square$

Now we are ready to prove the asymptotic behavior of the near-filed limit problem, Lemma 4.14.

*Proof of Lemma 4.14.* Let $P$ be the unique solution of the exterior Dirichlet problem (4.41). Consider the inversion mapping $I : x \mapsto \frac{x}{|x|^2}$. Let $\tilde{\Omega} = I(\mathbb{R}^n \backslash K)$. Thus $\tilde{\Omega}$ is a bounded domain and $\partial\tilde{\Omega} = I(\partial K)$. Let $\tilde{P}$ be the Kelvin transformation of $P$, i.e.,

$$\tilde{P}(x) = \frac{P(x/|x|^2)}{F(x/|x|^2)}.$$

By Lemma A.9, $\tilde{P}$ is the weak solution of

$$\begin{cases} D_i(\bar{a}_{ij} D_j \tilde{P}) = 0 & \text{in } \tilde{\Omega}, \\ \tilde{P}(x) = \dfrac{1}{F(x/|x|^2)} & \text{on } \partial\tilde{\Omega}. \end{cases}$$

By Lemma 4.12 we have $0 \leq P(x) \leq CF(x)$, thus $0 \leq \tilde{P}(x) \leq C$. Therefore $\tilde{P}$ is not singular at the origin and it is the regular solution of $D_i(\bar{a}_{ij} D_j \tilde{P}) = 0$ in $\tilde{\Omega} \cup \{0\}$ that satisfies the corresponding boundary condition. From that we can conclude that

$$\lim_{|x|\to\infty} \frac{P(x)}{F(x)} = \tilde{P}(0) = C_*.$$

The constant $C_*$ can be computed explicitly as

$$\tilde{P}(0) = \int_{\partial\tilde{\Omega}} \frac{1}{F(x/|x|^2)} \bar{a}_{ij}(x) D_i g_{\tilde{\Omega}}(x, 0) \nu_j(x) dS,$$

where $g_{\tilde{\Omega}}$ is Green's function of the elliptic operator of (A.13) in $\tilde{\Omega}$. The constant $C_*$ depends only on $K, n$ and the boundary condition applied on $K$. $\square$

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