

# Boundedness of Littlewood Paley Operators Relative to non Isotropic Dilations

メタデータ	言語: eng 出版者: 公開日: 2018-09-10 キーワード (Ja): キーワード (En): 作成者: メールアドレス: 所属:
URL	<a href="http://hdl.handle.net/2297/00052127">http://hdl.handle.net/2297/00052127</a>

This work is licensed under a Creative Commons  
Attribution-NonCommercial-ShareAlike 3.0  
International License.



BOUNDEDNESS OF LITTLEWOOD-PALEY OPERATORS  
RELATIVE TO NON-ISOTROPIC DILATIONS

SHUICHI SATO, Kanazawa

Received June 28, 2017. Published online August 6, 2018.

*Abstract.* We consider Littlewood-Paley functions associated with a non-isotropic dilation group on  $\mathbb{R}^n$ . We prove that certain Littlewood-Paley functions defined by kernels with no regularity concerning smoothness are bounded on weighted  $L^p$  spaces,  $1 < p < \infty$ , with weights of the Muckenhoupt class. This, in particular, generalizes a result of N. Rivière (1971).

*Keywords:* Littlewood-Paley function; non-isotropic dilation

*MSC 2010:* 42B25, 46E30

1. INTRODUCTION

Let  $P$  be an  $n \times n$  real matrix such that

$$\langle Px, x \rangle \geq \langle x, x \rangle \quad \forall x \in \mathbb{R}^n,$$

where  $\langle x, y \rangle = x_1 y_1 + \dots + x_n y_n$  is the inner product in  $\mathbb{R}^n$  with  $x = (x_1, \dots, x_n)$ ,  $y = (y_1, \dots, y_n)$ . Define a dilation group  $\{\delta_t\}_{t>0}$  on  $\mathbb{R}^n$  by  $\delta_t = t^P = \exp((\log t)P)$ . Let

$$(1.1) \quad g_\psi(f)(x) = \left( \int_0^\infty |f * \psi_t(x)|^2 \frac{dt}{t} \right)^{1/2}$$

be the Littlewood-Paley function on  $\mathbb{R}^n$ , where  $\psi_t(x) = t^{-\gamma} \psi(\delta_t^{-1}x)$  with  $\gamma$  being trace  $P$  and  $\psi$  a function in  $L^1(\mathbb{R}^n)$  such that

$$(1.2) \quad \int_{\mathbb{R}^n} \psi(x) dx = 0.$$

---

The author is partly supported by Grant-in-Aid for Scientific Research (C) No. 25400130, Japan Society for the Promotion of Science.

When  $P = E$  (the identity matrix) and  $g_\psi$  is defined with  $\psi_t(x) = t^{-n}\psi(t^{-1}x)$  in (1.1), it is known that if we further assume that  $|\psi(x)| \leq C(1 + |x|)^{-n-\varepsilon}$  for some  $\varepsilon > 0$ , then

$$(1.3) \quad \|g_\psi(f)\|_p \leq C_p \|f\|_p, \quad 1 < p < \infty,$$

where  $\|f\|_p = \|f\|_{L^p}$  (see [16] and [5], [8], [10], [11], [17] for related results; also [6] for a result on the homogeneous groups including the Heisenberg group). We refer to [1] for an earlier result, which requires in addition certain regularity on  $\psi$  to get (1.3). The reverse inequality of (1.3) also holds if a certain non-degeneracy condition on  $\psi$  is further assumed (see [13], Theorem 3.8, and also [19]).

In this note we shall prove, in particular, a result analogous to (1.3) in the case of the general dilation group  $\{\delta_t\}$  (see Theorem 1.1 below). To establish the generalization, we need an analogue of Lemma 3 of [16] (Lemma 3.4). It is to be noted that to prove Lemma 3.4 similarly to Lemma 3 of [16] we have a difficulty in the general case, which does not occur even in the case where  $P$  is diagonal. As can be seen in the proof of Lemma 3.4, this will be overcome by a method different from that of [16].

It is known that  $|\delta_t x|$  is strictly increasing as a function of  $t$  on  $\mathbb{R}_+ = (0, \infty)$  for  $x \neq 0$ , where  $|x| = \langle x, x \rangle^{1/2}$ . Define a norm function  $r(x)$ ,  $x \neq 0$ , to be the unique positive real number  $t$  such that  $|\delta_{t^{-1}} x| = 1$ , while let  $r(0) = 0$ . Then  $r(\delta_t x) = tr(x)$  for all  $t > 0$  and  $x \in \mathbb{R}^n$ . Further, the following properties of  $r(x)$  and  $\delta_t$  are known (see [3], [4]):

- (a)  $r(x + y) \leq r(x) + r(y)$ ,  $r(-x) = r(x)$ ;
- (b)  $r(x) \leq 1$  if and only if  $|x| \leq 1$ ;
- (c) if  $|x| \leq 1$ ,  $|x| \leq r(x)$ ;
- (d)  $|x| \geq r(x)$  for  $|x| \geq 1$ ;
- (e) if  $t \geq 1$ ,  $|\delta_t x| \geq t|x|$  for all  $x \in \mathbb{R}^n$ ;
- (f)  $|\delta_t x| \leq t|x|$  for all  $x \in \mathbb{R}^n$  if  $0 < t \leq 1$ .

Similarly, we can also consider a norm function  $r^*(x)$  associated with the dilation group  $\{\delta_t^*\}_{t>0}$ , where  $\delta_t^*$  denotes the adjoint of  $\delta_t$ ; we have properties of  $r^*(x)$  and  $\delta_t^*$  analogous to those of  $r(x)$  and  $\delta_t$  mentioned above. A polar coordinates expression for the Lebesgue measure

$$\int_{\mathbb{R}^n} \varphi(x) dx = \int_0^\infty \int_{S^{n-1}} \varphi(\delta_t \theta) t^{\gamma-1} \mu(\theta) d\sigma(\theta) dt$$

will be used in the following, where  $\mu$  is a strictly positive  $C^\infty$  function on the unit sphere  $S^{n-1} = \{x: |x| = 1\}$  and  $d\sigma$  is the Lebesgue surface measure on  $S^{n-1}$ . See also [14], [21] for relevant results.

We consider a pointwise majorant of  $\psi$  defined as

$$(1.4) \quad \mathcal{H}_\psi(x) = h(r(x)) = \sup_{r(y) \geq r(x)} |\psi(y)|$$

and recall two seminorms from [16]:

$$(1.5) \quad \mathcal{B}_\varepsilon(\psi) = \int_{|x|>1} |\psi(x)||x|^\varepsilon dx \quad \text{for } \varepsilon > 0,$$

$$(1.6) \quad \mathcal{C}_u(\psi) = \left( \int_{|x|<1} |\psi(x)|^u dx \right)^{1/u} \quad \text{for } u > 1.$$

Let

$$B(x, t) = \{y \in \mathbb{R}^n : r(x - y) < t\}$$

be the ball with respect to  $r$  in  $\mathbb{R}^n$  with center  $x$  and radius  $t$ ; such a ball is also called an  $r$ -ball. We say that a weight function  $w$  belongs to the weight class  $A_p$ ,  $1 < p < \infty$ , of Muckenhoupt if the quantity

$$[w]_{A_p} = \sup_B \left( |B|^{-1} \int_B w(x) dx \right) \left( |B|^{-1} \int_B w(x)^{-1/(p-1)} dx \right)^{p-1}$$

is finite, where the supremum is taken over all  $r$ -balls  $B$  in  $\mathbb{R}^n$  and  $|B|$  denotes the Lebesgue measure of  $B$ . Let  $M(f)$  be the Hardy-Littlewood maximal function defined as

$$M(f)(x) = \sup_{x \in B} |B|^{-1} \int_B |f(y)| dy,$$

where the supremum is taken over all  $r$ -balls  $B$  in  $\mathbb{R}^n$  containing  $x$ . Then we define the class  $A_1$  to be the family of weight functions  $w$  such that  $M(w) \leq Cw$  almost everywhere. The infimum of all such  $C$  is defined to be  $[w]_{A_1}$ . We refer to [2], [12] for relevant results on  $A_p$ .

We denote by  $L_w^p$  the weighted  $L^p$  space of all functions  $f$  satisfying

$$\|f\|_{L_w^p} = \|f\|_{p,w} = \left( \int_{\mathbb{R}^n} |f(x)|^p w(x) dx \right)^{1/p} < \infty.$$

In this note, we shall prove the following.

**Theorem 1.1.** *Let  $\psi \in L^1(\mathbb{R}^n)$ . Let  $\mathcal{H}_\psi$ ,  $\mathcal{B}_\varepsilon(\psi)$  and  $\mathcal{C}_u(\psi)$  be as in (1.4), (1.5) and (1.6), respectively. We assume that  $\psi$  satisfies (1.2) and that*

- (1) *there exists  $\varepsilon > 0$  such that  $\mathcal{B}_\varepsilon(\psi) < \infty$ ;*
- (2) *there exists  $u > 1$  such that  $\mathcal{C}_u(\psi) < \infty$ ;*
- (3)  *$\mathcal{H}_\psi \in L^1(\mathbb{R}^n)$ .*

*Then  $g_\psi$  defined by (1.1) is bounded on  $L_w^p$  for all  $p \in (1, \infty)$  and  $w \in A_p$ .*

Special cases of Theorem 1.1 are treated in [14], Theorem (1.4), page 269. Theorem 1.1 follows from the next result.

**Theorem 1.2.** *Let  $\psi \in L^1(\mathbb{R}^n)$ . Suppose that  $\psi$  satisfies (1.2) and (1), (2) of Theorem 1.1. We also assume that there exist a non-negative, non-increasing function  $h$  on  $\mathbb{R}_+$  with  $h(r(x)) \in L^1(\mathbb{R}^n)$  and a non-negative function  $\Omega$  in  $L^q(S^{n-1})$  for some  $q$ ,  $2 \leq q \leq \infty$ , such that*

$$|\psi(x)| \leq h(r(x))\Omega(x'), \quad x' = \delta_{r(x)^{-1}}x.$$

Then

- (1)  $g_\psi$  is bounded on  $L_w^p$  if  $p > q'$  and  $w \in A_{p/q'}$ , where  $q'$  denotes the exponent conjugate to  $q$ ;
- (2)  $g_\psi$  is bounded on  $L_w^2$  and  $L_{w^{-1}}^2$  if  $w \in A_{2/q'}$ .

If we assume that  $\psi$  is compactly supported, then we have the following results.

**Theorem 1.3.** *Let  $\psi \in L^1(\mathbb{R}^n)$  with (1.2). Suppose that  $\psi$  is compactly supported. Then we have the following.*

- (1) *If  $\psi \in L^q(\mathbb{R}^n)$  for some  $q \geq 2$ , then*
  - ( $\alpha$ )  $g_\psi$  is bounded on  $L_w^p$  for  $p > q'$ ,  $w \in A_{p/q'}$ ;
  - ( $\beta$ )  $g_\psi$  is bounded on  $L_w^2$  and  $L_{w^{-1}}^2$  for  $w \in A_{2/q'}$ .
- (2) *If  $\psi \in L^q(\mathbb{R}^n)$  for some  $q \in (1, 2]$ , then  $g_\psi$  is bounded on  $L_w^2$  and  $L_{w^{-1}}^2$  if  $w^{q'/2} \in A_1$ .*

**Corollary 1.4.** *Let  $\psi$  be a function of compact support in  $L^1(\mathbb{R}^n)$  with (1.2). If we further suppose that  $\psi \in L^q(\mathbb{R}^n)$  for some  $q \in (1, 2]$ , then  $g_\psi$  is bounded on  $L^p(\mathbb{R}^n)$  if  $0 < 1/p < 1/2 + 1/q'$ .*

When  $P = E$ , part (2) of Theorem 1.3 is due to [8]; so is Corollary 1.4 for  $p \in (1, 2)$ , where its optimality is also shown.

We denote by  $\widehat{\psi}$  the Fourier transform defined as

$$\widehat{\psi}(\xi) = \mathcal{F}(\psi)(\xi) = \int_{\mathbb{R}^n} \psi(x)e^{-2\pi i \langle x, \xi \rangle} dx.$$

Let  $\mathbb{Z}$  denote the set of integers and  $\mathcal{S}(\mathbb{R}^n)$  the Schwartz space of rapidly decreasing smooth functions on  $\mathbb{R}^n$ .

To prove the theorems above we apply the following result.

**Proposition 1.5.** *Let  $v \in A_2$ . Let  $\psi \in L^1(\mathbb{R}^n)$  with (1.2). If we further assume that*

$$(1.7) \quad \int_1^2 |\widehat{\psi}(\delta_t^* \xi)|^2 dt \leq C \min(|\xi|^\varepsilon, |\xi|^{-\varepsilon}) \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}$$

with some  $\varepsilon \in (0, 1)$  and

$$(1.8) \quad \sup_{k \in \mathbb{Z}} \int_{\mathbb{R}^n} \int_1^2 |\psi_{t2^k} * f(x)|^2 dt v(x) dx \leq C_v \|f\|_{L_v^2}^2 \quad \forall f \in \mathcal{S}(\mathbb{R}^n),$$

then  $g_\psi$  is bounded on  $L_v^2$ .

**Remark 1.6.** It is known and also can be seen from our proof of Corollary 1.4 for  $p \in (1, 2]$  below that  $g_\psi$  is bounded on  $L^p$  for every  $p$  with  $1 < p \leq 2$  if  $g_\psi$  is bounded on  $L_{w-1}^2$  for all  $w \in A_1$ . This can be applied to  $g_\psi$  of Theorem 1.2 and part (1) of Theorem 1.3.

**Remark 1.7.** Our results in this note may be stated also in terms of the non-isotropic dilations of [21] as in [18], which readers would infer easily.

In Section 2 we shall prove Proposition 1.5 by applying a well-known discrete parameter Littlewood-Paley decomposition with respect to the dilation group  $\{\delta_t^*\}$ . The method is analogous to the one in the isotropic case of [16]. We include the proof for the sake of completeness.

We shall prove Theorem 1.2 in Section 3. To prove part (2) we shall apply Proposition 1.5. Let

$$(1.9) \quad J_\varepsilon(\psi) = \sup_{|\xi|=1} \iint_{\mathbb{R}^n \times \mathbb{R}^n} |\psi(x)\psi(y)| |\langle \xi, P(x-y) \rangle|^{-\varepsilon} dx dy.$$

To use Proposition 1.5, we need to show  $J_\varepsilon(\psi) < \infty$  for some  $\varepsilon > 0$  under the assumptions of Theorem 1.2. A proof of this may be similar to the one for the case  $P = E$ , which is given in [16], if  $P$  is diagonal. However, in the general case, we need different methods that can be found in the proof of Lemma 3.4 below, as mentioned above. If  $J_\varepsilon(\psi) < \infty$ , we can deduce the decay estimate  $\int_1^2 |\widehat{\psi}(\delta_t^* \xi)|^2 dt \leq C |\xi|^{-\varepsilon}$ ,  $\varepsilon > 0$ , from certain trigonometric integral estimates in [18] (Lemma 3.2). We shall also apply an observation of [8] concerning duality in proving part (2). Part (1) of Theorem 1.2 follows from part (2) by the extrapolation theorem of Rubio de Francia [15].

In Section 4 we shall prove Theorem 1.3 and Corollary 1.4. Part (1) of Theorem 1.3 will be shown along the lines of the proof of Theorem 1.2 by using the compactness of the support of  $\psi$ . We shall apply methods of Duoandikoetxea [8] in proving the

weighted boundedness of part (2). To prove Corollary 1.4 for  $p \in (1, 2)$ , we shall apply Theorem 1.3 (2) and methods of [8]. The proof for  $p \in [2, \infty)$  will be given by adapting methods of [11], which is based on the use of vector valued inequalities.

## 2. PROOF OF PROPOSITION 1.5

Let  $f \in \mathcal{S}(\mathbb{R}^n)$ . We take  $\Phi \in \mathcal{S}(\mathbb{R}^n)$  such that  $\text{supp}(\Phi) \subset \{1/2 \leq r^*(\xi) \leq 2\}$  and

$$\sum_{j=-\infty}^{\infty} \Phi(\delta_{2^j}^* \xi) = 1 \quad \text{for } \xi \neq 0.$$

Let

$$\widehat{D_j(f)}(\xi) = \Phi(\delta_{2^j}^* \xi) \hat{f}(\xi), \quad j \in \mathbb{Z}.$$

We use the decomposition

$$f * \psi_t(x) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} D_{j+k}(f * \psi_t)(x) \chi_{[2^k, 2^{k+1})}(t) = \sum_{j=-\infty}^{\infty} A_j(x, t),$$

where  $\chi_E$  denotes the characteristic function of a set  $E$  and

$$A_j(x, t) = \sum_{k=-\infty}^{\infty} D_{j+k}(f * \psi_t)(x) \chi_{[2^k, 2^{k+1})}(t).$$

We have

$$g_\psi(f)(x) \leq \sum_{j=-\infty}^{\infty} S_j(f)(x)$$

with

$$S_j(f)(x) = \left( \int_0^\infty |A_j(x, t)|^2 \frac{dt}{t} \right)^{1/2}.$$

Note that

$$\|S_j(f)\|_2^2 = \sum_{k=-\infty}^{\infty} \int_{\mathbb{R}^n} \int_{2^k}^{2^{k+1}} |D_{j+k}(f * \psi_t)(x)|^2 \frac{dt}{t} dx.$$

Thus, if we set  $U_j = \{2^{-1-j} \leq r^*(\xi) \leq 2^{1-j}\}$ , Plancherel's theorem and (1.7) imply

$$\begin{aligned} \|S_j(f)\|_2^2 &\leq \sum_{k=-\infty}^{\infty} C \int_{U_{j+k}} \left( \int_{2^k}^{2^{k+1}} |\widehat{\psi}(\delta_t^* \xi)|^2 \frac{dt}{t} \right) |\hat{f}(\xi)|^2 d\xi \\ &\leq \sum_{k=-\infty}^{\infty} C \int_{U_{j+k}} \min(|\delta_{2^k}^* \xi|^\varepsilon, |\delta_{2^k}^* \xi|^{-\varepsilon}) |\hat{f}(\xi)|^2 d\xi. \end{aligned}$$

Applying the  $\delta_t^*$  analogues of (e), (f) of Section 1, we have

$$\min(|\delta_{2^k}^* \xi|^\varepsilon, |\delta_{2^k}^* \xi|^{-\varepsilon}) \leq C 2^{-\varepsilon|j|} \quad \text{for } \xi \in U_{j+k}.$$

Thus

$$(2.1) \quad \|S_j(f)\|_2^2 \leq C 2^{-\varepsilon|j|} \sum_{k=-\infty}^{\infty} \int_{U_{j+k}} |\hat{f}(\xi)|^2 d\xi \leq C 2^{-\varepsilon|j|} \|f\|_2^2,$$

where the last inequality follows from the Plancherel theorem and the fact that  $\sum \chi_{U_j}(\xi) \leq C$  with a constant  $C$  independent of  $\xi \in \mathbb{R}^n$ .

While, by (1.8) we see that

$$\begin{aligned} \|S_j(f)\|_{2,v}^2 &= \sum_{k=-\infty}^{\infty} \int_{\mathbb{R}^n} \int_{2^k}^{2^{k+1}} |D_{j+k}(f) * \psi_t(x)|^2 \frac{dt}{t} v(x) dx \\ &\leq \sum_{k=-\infty}^{\infty} C \int_{\mathbb{R}^n} |D_{j+k}(f)(x)|^2 v(x) dx. \end{aligned}$$

Since  $v \in A_2$ , by the  $L_v^2$  boundedness of the discrete Littlewood-Paley operator, we have

$$\sum_{k=-\infty}^{\infty} \int_{\mathbb{R}^n} |D_{j+k}(f)(x)|^2 v(x) dx \leq C \|f\|_{2,v}^2.$$

Thus

$$(2.2) \quad \|S_j(f)\|_{2,v}^2 \leq C \|f\|_{2,v}^2.$$

Interpolation with change of measures between the estimates (2.1) and (2.2) implies that

$$(2.3) \quad \|S_j(f)\|_{2,v^\theta} \leq C 2^{-\varepsilon(1-\theta)|j|/2} \|f\|_{2,v^\theta}$$

with  $\theta \in (0, 1)$ . Choosing  $\theta$  close to 1 such that  $v^{1/\theta} \in A_2$ , by (2.3) we have

$$\|S_j(f)\|_{2,v} \leq C 2^{-\varepsilon(1-\theta)|j|/2} \|f\|_{2,v}.$$

From this and the triangle inequality it follows that

$$\|g_\psi(f)\|_{2,v} \leq \sum_{j=-\infty}^{\infty} \|S_j(f)\|_{2,v} \leq C \|f\|_{2,v}.$$

This completes the proof of Proposition 1.5.



### 3. PROOF OF THEOREM 1.2

We first state some results needed to show Theorem 1.2 from Proposition 1.5.

**Lemma 3.1.** *Let  $\psi \in L^1(\mathbb{R}^n)$ . Suppose that  $\psi$  satisfies (1.2) and condition  $\mathcal{B}_\varepsilon(\psi) < \infty$  for some  $\varepsilon \in (0, 1]$ . Then*

$$\int_1^2 |\widehat{\psi}(\delta_t^* \xi)|^2 dt \leq C |\xi|^{2\varepsilon} \quad \text{for } \xi \in \mathbb{R}^n.$$

*Proof.* We have  $|\widehat{\psi}(\xi)| \leq C |\xi|^\varepsilon$  for all  $\xi \in \mathbb{R}^n$  (see [16] for a proof). By this and the  $\delta_t^*$  analogue of (f) of Section 1, we have the estimates as claimed.  $\square$

**Lemma 3.2.** *Let  $L$  be the degree of the minimal polynomial of  $P$ . Then for  $\eta, \zeta \in \mathbb{R}^n$  we have*

$$\left| \int_1^2 \exp(i \langle \delta_t^* \eta, \zeta \rangle) \frac{dt}{t} \right| \leq C |\langle \eta, P\zeta \rangle|^{-1/L}$$

for some positive constant  $C$  independent of  $\eta$  and  $\zeta$ .

This is from [18]. Let  $J_\varepsilon(\psi)$  be as in (1.9). Applying Lemma 3.2, we have the next result.

**Lemma 3.3.** *Let  $\psi \in L^1(\mathbb{R}^n)$ . Suppose that  $J_\varepsilon(\psi) < \infty$  with  $\varepsilon \in (0, 1/L]$ . Then*

$$\int_1^2 |\widehat{\psi}(\delta_t^* \xi)|^2 dt \leq C |\xi|^{-\varepsilon} \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}.$$

*Proof.* We write

$$\int_1^2 |\widehat{\psi}(\delta_t^* \xi)|^2 dt = \iint_{\mathbb{R}^n \times \mathbb{R}^n} \psi(x) \overline{\psi(y)} \int_1^2 \exp(-2\pi i \langle \delta_t^* \xi, x - y \rangle) dt dx dy.$$

Then Lemma 3.2 implies that

$$\int_1^2 |\widehat{\psi}(\delta_t^* \xi)|^2 dt \leq C \iint_{\mathbb{R}^n \times \mathbb{R}^n} |\psi(x) \psi(y)| \min(1, |\langle \xi, P(x - y) \rangle|^{-1/L}) dx dy.$$

We easily see that the right-hand side is bounded by  $C J_\varepsilon(\psi) |\xi|^{-\varepsilon}$  for  $\varepsilon \in (0, 1/L]$ . This completes the proof.  $\square$

A sufficient condition for  $J_\varepsilon(\psi) < \infty$  is given in the next result.

**Lemma 3.4.** *Let  $h$  be a non-negative function on  $\mathbb{R}_+$  and set  $H(x) = h(r(x))$ . Suppose that  $H \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ . Let  $\Omega$  be a non-negative function in  $L^v(S^{n-1})$  for some  $v > 1$ . For a non-negative function  $F$  on  $\mathbb{R}^n$  we assume that  $\mathcal{C}_u(F) < \infty$  for some  $u > 1$  and that*

$$F(x) \leq h(r(x))\Omega(x') \quad \text{for } |x| > 1.$$

Then we have  $J_\varepsilon(F) < \infty$  for  $\varepsilon < \min(1/u', 1/(2v'))$ .

**Proof.** We define

$$L_\varepsilon(f, g; \xi) = \iint_{\mathbb{R}^n \times \mathbb{R}^n} f(x)g(y)|\langle \xi, P(x-y) \rangle|^{-\varepsilon} dx dy,$$

where  $f, g$  are non-negative functions and  $\xi \in S^{n-1}$ . We write  $F = E + G$ , where  $E(x) = F(x)$  if  $|x| < 1$  and  $E(x) = 0$  otherwise. We note that

$$L_\varepsilon(F, F; \xi) = L_\varepsilon(E, E; \xi) + 2L_\varepsilon(E, G; \xi) + L_\varepsilon(G, G; \xi).$$

To prove the lemma it suffices to show that

$$(3.1) \quad \sup_{\xi \in S^{n-1}} L_\varepsilon(E, E; \xi) < \infty,$$

$$(3.2) \quad \sup_{\xi \in S^{n-1}} L_\varepsilon(E, G; \xi) < \infty,$$

$$(3.3) \quad \sup_{\xi \in S^{n-1}} L_\varepsilon(G, G; \xi) < \infty$$

when  $\varepsilon < \min(1/u', 1/(2v'))$ .

By Hölder inequality and rotation, we see that

$$L_\varepsilon(E, E; \xi) \leq C^{\mathcal{C}_u(F)^2} \left( \iint_{|x| < 1, |y| < 1} |x_1 - y_1|^{-\varepsilon u'} dx dy \right)^{1/u'} < \infty,$$

which proves (3.1).

Let  $\zeta = P^*\xi$ . Then if  $Y = E$  or  $Y = G$ , we see that

$$L_\varepsilon(Y, G; \xi) \leq \int_{\mathbb{R}^n} Y(x) \left( \int_{\mathbb{R}^n} G(y) |\langle \zeta, x-y \rangle|^{-\varepsilon} dy \right) dx.$$

We prove

$$(3.4) \quad \sup_{x \in \mathbb{R}^n, \xi \in S^{n-1}} \int_{\mathbb{R}^n} G(y) |\langle \zeta, x-y \rangle|^{-\varepsilon} dy < \infty$$

if  $\varepsilon < 1/(2v')$ , which will imply (3.2) and (3.3) since  $Y \in L^1$ .

If  $n = 1$ , we have

$$\begin{aligned} \int_{\mathbb{R}} G(y) |\langle \zeta, x - y \rangle|^{-\varepsilon} dy &\leq C \int_{|x-y|>1} G(y) |x - y|^{-\varepsilon} dy + C \int_{|x-y|\leq 1} G(y) |x - y|^{-\varepsilon} dy \\ &\leq C \|H\|_1 + C \|H\|_{\infty} \int_{|y|\leq 1} |y|^{-\varepsilon} dy, \end{aligned}$$

which proves (3.4) when  $n = 1$ .

Let  $n \geq 2$ . For  $x \in \mathbb{R}^n$ ,  $s > 0$ , let

$$I_{\varepsilon}(\zeta, x, s) = \int_{S^{n-1}} |\langle \zeta, x - \delta_s \omega \rangle|^{-\varepsilon} \Omega(\omega) \mu(\omega) d\sigma(\omega).$$

Then by Hölder inequality

$$I_{\varepsilon}(\zeta, x, s) \leq C (N_{\varepsilon v'}(\zeta, x, s))^{1/v'} \|\Omega\|_v,$$

where

$$N_{\varepsilon}(\zeta, x, s) = \int_{S^{n-1}} |\langle \zeta, x - \delta_s \omega \rangle|^{-\varepsilon} \mu(\omega) d\sigma(\omega).$$

Thus

$$\begin{aligned} (3.5) \quad \int_{\mathbb{R}^n} G(y) |\langle \zeta, x - y \rangle|^{-\varepsilon} dy &\leq \int_1^{\infty} h(s) s^{\gamma-1} I_{\varepsilon}(\zeta, x, s) ds \\ &\leq C \|\Omega\|_v \int_1^{\infty} h(s) s^{\gamma-1} (N_{\varepsilon v'}(\zeta, x, s))^{1/v'} ds. \end{aligned}$$

We shall see that

$$(3.6) \quad \sup_{x \in \mathbb{R}^n} N_{\varepsilon}(\zeta, x, s) \leq C |\delta_s^* \zeta|^{-\varepsilon}$$

if  $0 < \varepsilon < 1/2$ . Then (3.4) follows from (3.5) and (3.6). We can deduce (3.6) from Lemma 3.5 below. This completes the proof.  $\square$

The following result is used in the proof of Lemma 3.4.

**Lemma 3.5.** *Let  $n \geq 2$  and  $\eta \in S^{n-1}$ . Suppose that  $0 < \delta < 1/2$ . Then*

$$\sup_{a \in \mathbb{R}} \int_{S^{n-1}} |a - \langle \eta, \omega \rangle|^{-\delta} d\sigma(\omega) \leq C$$

with a constant  $C$  independent of  $\eta$ .

Proof. By rotation, we may assume that  $\eta = (1, 0, \dots, 0)$ . Thus we have to estimate

$$I(a) = \int_{S^{n-1}} |a - \omega_1|^{-\delta} d\sigma(\omega), \quad a \in \mathbb{R}.$$

We see that

$$I(a) = c_{n-2} \int_0^\pi |a - \cos \theta|^{-\delta} (\sin \theta)^{n-2} d\theta \leq c_{n-2} \int_0^\pi |a - \cos \theta|^{-\delta} d\theta,$$

where  $c_{n-2} = 2\pi^{(n-1)/2}/\Gamma((n-1)/2)$ . Thus the conclusion follows from an elementary fact that

$$\sup_{a \in \mathbb{R}} \int_0^\pi |a - \cos \theta|^{-\delta} d\theta < \infty \quad \text{if } 0 < \delta < 1/2.$$

□

Let

$$M_\Omega(f)(x) = \sup_{r>0} r^{-\gamma} \int_{r(y)<r} |f(x-y)| \Omega(\delta_{r(y)-1}y) dy,$$

where  $\Omega$  is a non-negative function on  $S^{n-1}$ .

We also need the following result (see [7]) for proving Theorem 1.2.

**Lemma 3.6.** *Let  $\Omega \in L^q(S^{n-1})$ ,  $\Omega \geq 0$ . Then  $M_\Omega$  is bounded on  $L_w^2$  if  $q \geq 2$  and  $w \in A_{2/q'}$ .*

Proof of Theorem 1.2. Applying the assumed pointwise majorization of  $\psi$ , we can prove that

$$\sup_{t>0} |\psi_t * f(x)| \leq C M_\Omega(f)(x)$$

as in [20], pages 63–64. Thus, Lemma 3.6 implies

$$(3.7) \quad \int_{\mathbb{R}^n} |\psi_t * f(x)|^2 v(x) dx \leq C \int_{\mathbb{R}^n} |f(x)|^2 v(x) dx$$

for  $v = w$  with  $w \in A_{2/q'}$  and also for  $v = w^{-1}$  with  $w \in A_{2/q'}$  by duality with a constant  $C$  independent of  $t > 0$ . From (3.7) we see that (1.8) holds for  $\psi$  of Theorem 1.2 for  $v = w$  and  $v = w^{-1}$  with  $w \in A_{2/q'}$ .

Also by Lemma 3.4 we have  $J_\varepsilon(\psi) < \infty$  for  $\varepsilon < \min(1/u', 1/(2q'))$ . Here we note that the function  $h$  of Theorem 1.2 is bounded on  $[1, \infty)$ . Thus (1.7) follows from Lemmas 3.1 and 3.3.

Therefore Proposition 1.5 can be applied to get the boundedness of  $g_\psi$  on  $L_w^2$  and  $L_{w^{-1}}^2$  with  $w \in A_{2/q'}$ , since  $A_{2/q'} \subset A_2$ . Thus part (2) of Theorem 1.2 follows. Also, by the boundedness of  $g_\psi$  on  $L_w^2$ ,  $w \in A_{2/q'}$ , and the extrapolation theorem of Rubio de Francia [15] we have part (1). □

#### 4. PROOFS OF THEOREM 1.3 AND COROLLARY 1.4

**P r o o f** of Theorem 1.3. To prove  $(\alpha)$  and  $(\beta)$  of part (1) of Theorem 1.3, similarly to the proof of Theorem 1.2 in Section 3, it suffices to show that  $\psi$  satisfies conditions (1.7) and (1.8) for  $v = w$ ,  $v = w^{-1}$  with  $w \in A_{2/q'}$ .

To prove (1.7), we first note that  $\mathcal{B}_1(\psi) < \infty$ . Without loss of generality, we may assume that  $\text{supp}(\psi) \subset B(0, 1)$  in what follows. By applying Lemma 3.4 we can see that  $J_\varepsilon(\psi) < \infty$  if  $\varepsilon < 1/q'$ . Thus, by Lemmas 3.1 and 3.3 we have (1.7).

Next we deal with condition (1.8). Let  $q > 2$ . Then we have

$$\sup_{t>0} |\psi_t * f(x)| \leq C M(|f|^{q'})(x)^{1/q'}$$

by Hölder inequality and it implies (3.7) with  $v = w$ ,  $w \in A_{2/q'}$ . To prove (3.7) when  $q = 2$  and  $v = w \in A_1$ , we may assume that  $t = 1$  by dilation invariance. Now applying Schwarz inequality, we have

$$|\psi * f(x)|^2 \leq \|\psi\|_2^2 \int_{r(y)<1} |f(x-y)|^2 dy.$$

Therefore we see that

$$\int |\psi * f(x)|^2 w(x) dx \leq \|\psi\|_2^2 \int |f(y)|^2 \left( \int_{r(x-y)<1} w(x) dx \right) dy.$$

Since  $w \in A_1$ , we have

$$\int_{r(x-y)<1} w(x) dx \leq C[w]_{A_1} w(y) \quad \text{a.e.},$$

and hence

$$\int |\psi * f(x)|^2 w(x) dx \leq C \|\psi\|_2^2 \int |f(y)|^2 w(y) dy.$$

So (3.7) holds with  $v = w$ ,  $w \in A_{2/q'}$ ,  $q \geq 2$ . Inequality (3.7) for  $v = w^{-1}$  follows by duality. Thus, we have (1.8) for  $v = w$  and  $v = w^{-1}$  with  $w \in A_{2/q'}$ ,  $q \geq 2$ .

We next prove part (2). We apply ideas of [8]. To prove (1.8) for  $v = w$  and  $v = w^{-1}$  with  $w^{q'/2} \in A_1$ , it suffices to show estimate (3.7) for  $v = w$  and  $v = w^{-1}$  under the condition  $w^{q'/2} \in A_1$  with a constant  $C$  independent of  $t > 0$ . If we have (3.7) for  $v = w$ , then the result for  $v = w^{-1}$  follows by duality. To prove (3.7) for  $v = w$  with  $w^{q'/2} \in A_1$ , we may assume that  $t = 1$  by dilation invariance as

above. Now, by applying Hölder inequality twice and the condition  $w^{q'/2} \in A_1$ , we have

$$\begin{aligned}
\int |f * \psi(x)|^2 w(x) dx &\leq \int \left( \|\psi\|_q^q \int |f(x-y)|^2 |\psi(y)|^{2-q} dy \right) w(x) dx \\
&= \|\psi\|_q^q \int |f(y)|^2 \left( \int |\psi(x-y)|^{2-q} w(x) dx \right) dy \\
&\leq \|\psi\|_q^q \int |f(y)|^2 \|\psi\|_q^{2-q} \left( \int_{r(y-x)<1} w(x)^{q'/2} dx \right)^{2/q'} dy \\
&\leq C \|\psi\|_q^2 \int |f(y)|^2 w(y) dy.
\end{aligned}$$

Thus, since condition (1.7) also holds when  $q \in (1, 2]$ , Proposition 1.5 implies that  $g_\psi$  is bounded on  $L_w^2$  and  $L_{w^{-1}}^2$  if  $w^{q'/2} \in A_1$ . This completes the proof of Theorem 1.3.  $\square$

**P r o o f** of Corollary 1.4. The proof of [11], Corollary 3 (ii) can be adapted to prove the boundedness of  $g_\psi$  on  $L^p$  when  $2 \leq p < \infty$  and  $\psi \in L^q$  for some  $q \in (1, \infty)$ , since the proof is based on vector valued inequalities derived from the boundedness on  $L^r$ ,  $1 < r < \infty$ , of the maximal function

$$N_\psi(f)(x) = \sup_{k \in \mathbb{Z}} |p_{2^k} * f(x)|, \quad p(x) = \int_1^2 |\psi_t(x)| dt/t$$

and the boundedness follows by the methods of [9] together with some Fourier transform estimates in the present context:

$$\begin{aligned}
\left| \int_1^2 \mathcal{F}(|\psi|)(\delta_t^* \xi) dt \right| &\leq C |\xi|^{-\varepsilon} \quad \text{for some } \varepsilon > 0, \\
\left| \int_1^2 \mathcal{F}(|\psi|)(\delta_t^* \xi) dt - \|\psi\|_1 \right| &\leq C |\xi|^\varepsilon \quad \text{for some } \varepsilon > 0,
\end{aligned}$$

which can be shown by the methods of this note.

Let us assume that  $1 < p < 2$  and apply the methods of [8]. Let  $s > q'/2$  and define  $M_s(f) = M(|f|^s)^{1/s}$ . Then  $M_s(|f|^{2-p})^{q'/2}$  is in  $A_1$  (we may assume that  $0 < M_s(|f|^{2-p}) < \infty$ ) and  $M_s$  is bounded on  $L^{p/(2-p)}$  if  $s < p/(2-p)$ . So by Hölder inequality and the  $L_{w^{-1}}^2$  boundedness of  $g_\psi$  of Theorem 1.3 (2) with  $w = M_s(|f|^{2-p})$ , if  $q'/2 < s < p/(2-p)$ , we have

$$\begin{aligned}
\int g_\psi(f)(x)^p dx &= \int g_\psi(f)(x)^p M_s(|f|^{2-p})(x)^{-p/2} M_s(|f|^{2-p})(x)^{p/2} dx \\
&\leq \left( \int g_\psi(f)(x)^2 M_s(|f|^{2-p})(x)^{-1} dx \right)^{p/2} \left( \int M_s(|f|^{2-p})(x)^{p/(2-p)} dx \right)^{1-p/2}
\end{aligned}$$

$$\begin{aligned} &\leq C \left( \int |f(x)|^2 M_s(|f|^{2-p})(x)^{-1} dx \right)^{p/2} \|f\|_p^{p(1-p/2)} \\ &\leq C \left( \int |f(x)|^2 |f(x)|^{p-2} dx \right)^{p/2} \|f\|_p^{p(1-p/2)} = C \|f\|_p^p. \end{aligned}$$

This completes the proof of Corollary 1.4. □

### References

- [1] *A. Benedek, A. P. Calderón, R. Panzone*: Convolution operators on Banach space valued functions. *Proc. Natl. Acad. Sci. USA* *48* (1962), 356–365. [zbl](#) [MR](#) [doi](#)
- [2] *A. P. Calderón*: Inequalities for the maximal function relative to a metric. *Stud. Math.* *57* (1976), 297–306. [zbl](#) [MR](#) [doi](#)
- [3] *A. P. Calderón, A. Torchinsky*: Parabolic maximal functions associated with a distribution. *Adv. Math.* *16* (1975), 1–64. [zbl](#) [MR](#) [doi](#)
- [4] *O. N. Capri*: On an inequality in the theory of parabolic  $H^p$  spaces. *Rev. Unión Mat. Argent.* *32* (1985), 17–28. [zbl](#) [MR](#)
- [5] *L. C. Cheng*: On Littlewood-Paley functions. *Proc. Am. Math. Soc.* *135* (2007), 3241–3247. [zbl](#) [MR](#) [doi](#)
- [6] *Y. Ding, S. Sato*: Littlewood-Paley functions on homogeneous groups. *Forum Math.* *28* (2016), 43–55. [zbl](#) [MR](#) [doi](#)
- [7] *J. Duoandikoetxea*: Weighted norm inequalities for homogeneous singular integrals. *Trans. Am. Math. Soc.* *336* (1993), 869–880. [zbl](#) [MR](#) [doi](#)
- [8] *J. Duoandikoetxea*: Sharp  $L^p$  boundedness for a class of square functions. *Rev. Mat. Complut.* *26* (2013), 535–548. [zbl](#) [MR](#) [doi](#)
- [9] *J. Duoandikoetxea, J. L. Rubio de Francia*: Maximal and singular integral operators via Fourier transform estimates. *Invent. Math.* *84* (1986), 541–561. [zbl](#) [MR](#) [doi](#)
- [10] *J. Duoandikoetxea, E. Seijo*: Weighted inequalities for rough square functions through extrapolation. *Stud. Math.* *149* (2002), 239–252. [zbl](#) [MR](#) [doi](#)
- [11] *D. Fan, S. Sato*: Remarks on Littlewood-Paley functions and singular integrals. *J. Math. Soc. Japan* *54* (2002), 565–585. [zbl](#) [MR](#) [doi](#)
- [12] *J. Garcia-Cuerva, J. L. Rubio de Francia*: *Weighted Norm Inequalities and Related Topics*. North-Holland Mathematics Studies 116, Notas de Matemática 104, North-Holland, Amsterdam, 1985. [zbl](#) [MR](#) [doi](#)
- [13] *L. Hörmander*: Estimates for translation invariant operators in  $L^p$  spaces. *Acta Math.* *104* (1960), 93–140. [zbl](#) [MR](#) [doi](#)
- [14] *N. Rivière*: Singular integrals and multiplier operators. *Ark. Mat.* *9* (1971), 243–278. [zbl](#) [MR](#) [doi](#)
- [15] *J. L. Rubio de Francia*: Factorization theory and  $A_p$  weights. *Am. J. Math.* *106* (1984), 533–547. [zbl](#) [MR](#) [doi](#)
- [16] *S. Sato*: Remarks on square functions in the Littlewood-Paley theory. *Bull. Aust. Math. Soc.* *58* (1998), 199–211. [zbl](#) [MR](#) [doi](#)
- [17] *S. Sato*: Estimates for Littlewood-Paley functions and extrapolation. *Integral Equations Oper. Theory* *62* (2008), 429–440. [zbl](#) [MR](#) [doi](#)
- [18] *S. Sato*: Estimates for singular integrals along surfaces of revolution. *J. Aust. Math. Soc.* *86* (2009), 413–430. [zbl](#) [MR](#) [doi](#)
- [19] *S. Sato*: Littlewood-Paley equivalence and homogeneous Fourier multipliers. *Integral Equations Oper. Theory* *87* (2017), 15–44. [zbl](#) [MR](#) [doi](#)
- [20] *E. M. Stein*: *Singular Integrals and Differentiability Properties of Functions*. Princeton Mathematical Series 30, Princeton University Press, Princeton, 1970. [zbl](#) [MR](#) [doi](#)

- [21] *E. M. Stein, S. Wainger*: Problems in harmonic analysis related to curvature. *Bull. Am. Math. Soc.* *84* (1978), 1239–1295.



*Author's address*: Sh u i c h i S a t o, Department of Mathematics, Faculty of Education, Kanazawa University, Kanazawa 920-1192, Japan, e-mail: shuichi@kenroku.kanazawa-u.ac.jp.