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PRINCIPLES FOR CONSTITUTIVE EQUATIONS AND EXPRESSIONS OF ANISOTROPY IN SOIL MATERIALS

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ABSTRACT

Much attention has been paid to anisotropic behavior of soils, such as the inherent anisotropy and the stress induced anisotropy. Nevertheless, they often appear in the literature without any clear definitions. In the theory of finite deformations, the concept of isotropy is well established and is clearly distinguishable from the objectivity. However, in the infinitesimal theory, which is usually assumed in soil mechanics, they seem to be in confusion and, as a result, even simple anisotropy is not clearly defined. In this paper, we therefore first discuss the general principles of constitutive equations in finite theories and in infinitesimal theories to make a clear distinction between the objectivity and the isotropy. One of our most important results is that, if reference vectors or tensors are employed, in infinitesimal theories, the objectivity requires that anisotropic materials can be represented by isotropic functions. Using a reference tensor, we finally give a clear definition of the inherent anisotropy and the stress induced anisotropy. We then examine their definitions employing a concrete example and show how the principles derived are useful, including the failure condition and the materials of differential type.

Key words : anisotropy, constitutive equation of soil, (induced anisotropy), (inherent anisotropy), (principle of objectivity), (principle of reference indifference) (**IGC** : D 6)

INTRODUCTION

Most soils in natural states more or less exhibit anisotropic behavior. Anisotropy in soils is, however, one of the complicated characteristics to which much attention has been paid in the literature. It is well known, for instance, that the deformation and strength of sand is considerably affected by a preferred orientation of particles developed during deposition (Oda, 1972; Arthur and Menzies, 1972). Similar anisotropy can also be observed in varved glacial lake deposits containing alternating layers of silt and clay. Such anisotropic behavior resulting from the natural structures of soil is, in general, called the inherent anisotropy.

On the other hand, it is also well known

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that the undrained shear strength of anisotropically consolidated clays varies with the direction of shear or direction of principal stress (Bjerrum, 1973; Ladd and Foott, 1974). Hansen and Gibson (1949) showed that such a phenomenon results from the anisotropic initial stress state. We may therefore consider that the strength of soils depends not only on the natural structures of soil but also on the initially anisotropic stress states. The latter is often called the stress induced anisotropy. The stress induced anisotropy is, therefore, essentially a different concept from the inherent anisotropy.

Precise definitions of these concepts, however, have not been given in the papers. Kanatani (1982) defined the concept of an isotropy and an induced anisotropy. However, his definitions are not very clear since he seems to confuse the isotropy with the objectivity. In fact, in the theories of finite deformation, several principles such as the isotropy and the objectivity are well known and well established. In infinitesimal theories, which are usually assumed in soil mechanics, these principles, however, have not been clarified and have been used confusedly.

As a result, in infinitesimal theories, even the definition of a simple anisotropy is not clear. We thus first have to discuss such general principles to make a clear distinction between the isotropy and the objectivity not only in finite theories but also in infinitesimal theories. This is one reason why we spend the first two parts on these principles.

In the first part we present brief discussions about the principles for constitutive equations for finite deformation theories: 1) Reference Indifference, 2) Objectivity, and, 3) Material symmetry (anisotropy) and isotropy. Then, employing reference vectors, we demonstrate the use of the principles and show that anisotropic materals can be represented by isotropic functions.

In the second part, using reference vectors, we first give a brief discussion about their principles in infinitesimal theories, and show that, even in infinitesimal theories, anisotropic materials can be represented by isotropic functions. Using such principles, we then demonstrate a derivation of a classical anisotropic material. Furthermore, we show that there may exist an elastic material in which an infinitesimal rotation is included.

In the last part, defining a reference tensor, we discuss our final goal: precise definitions of the initial anisotropy in soil materials such as the inherent anisotropy and the stress induced anisotropy. We then examine their definitions by using a concrete example of dilatancy of clays and show how the principles derived in the previous parts are useful. We also discuss examples of the failure condition and the material of differential type.

Mathematical Symbols

Light-face letters indicate scalars; bold-face letters indicate vectors or tensors; we use standard indicial notation and Cartesian coordinates:

If $\boldsymbol{A}$, $\boldsymbol{B}$ are second order tensors and $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ are vectors,

$\boldsymbol{A}^T$ is the transpose of $\boldsymbol{A}$

$\boldsymbol{A}^{-1}$ is the inverse of $\boldsymbol{A}$

$$(\boldsymbol{AB})_{ij}=A_{ik}B_{kj}$$

$$\boldsymbol{A}\cdot\boldsymbol{B}=tr(\boldsymbol{AB}^T)=A_{ij}B_{ij}$$

$$\|\boldsymbol{A}\|=\sqrt{tr(\boldsymbol{A}^2)}$$

$$(\boldsymbol{Ab})_i=A_{ij}b_j$$

$$\boldsymbol{a}\cdot\boldsymbol{b}=a_ib_i$$

$$(\boldsymbol{a}\otimes\boldsymbol{b})_{ij}=a_ib_j \text{ thus } (\boldsymbol{a}\otimes\boldsymbol{b})\boldsymbol{c}=(\boldsymbol{b}\cdot\boldsymbol{c})\boldsymbol{a}$$

If $\boldsymbol{H}$ is a forth order tensor and $\boldsymbol{a}^\alpha$ ($\alpha=1, 2, 3, 4$) are vectors

$$(\boldsymbol{H}[\boldsymbol{A}])_{ij}=H_{ijkl}A_{kl}$$

$$(\boldsymbol{a}^1\otimes\boldsymbol{a}^2\otimes\boldsymbol{a}^3\otimes\boldsymbol{a}^4[\boldsymbol{A}])_{ij}=a_i^1a_j^2a_k^3a_l^4A_{kl}$$

PRINCIPLES FOR CONSTITUTIVE EQUATIONS AND EXPRESSIONS OF ANISOTROPIC MATERIALS

Here we first present brief discussions about the principles for constitutive equations in finite deformations (in detail, see e.g., Truesdell and Noll, 1965; Leigh, 1968: Gurtin, 1981). Then, introducing reference

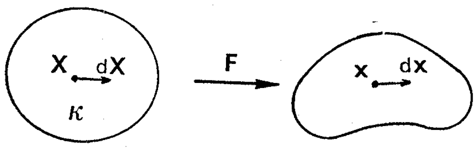

Fig. 1. Deformation of a body

vectors, we demonstrate their principles and show that the constitutive relations for anisotropic materials can be represented by isotropic functions.

Reference Indifference and Material Indifference (Objectivity)

Consider two particles, which are in a sufficiently small distance, X and $X+dX$ in the reference state (or reference configuration) κ. Denoting the positions of these particles in current state by x and $x+dx$ respectively, a tensor F defined by

$$dx=FdX \quad \text{or} \quad F=\frac{\partial x}{\partial X} \qquad (1)$$

is called the *deformation gradient* (Fig. 1). We consider here a material which is called the *simple material*

$$T(t)=\underset{s=0}{\overset{\infty}{f_{\kappa}}}(F(t-s)) \qquad (2)$$

or briefly,

$$T=f_{\kappa}(F_s)$$

where F_s is the history up to time t of the deformation gradient and T is *Cauchy stress tensor*. As shown in Eq. (1), the deformation gradient F depends on the reference state κ, and so clearly the form of the functional f in Eq. (2) depends on κ.
For the same current state but a different reference state $\bar{\kappa}$. we have

$$T=f_{\bar{\kappa}}(\bar{F}_s), \quad \bar{F}_s=\frac{\partial x_s}{\partial \bar{X}}$$

Since

$$\bar{F}_s=\frac{\partial x_s}{\partial X}\frac{\partial X}{\partial \bar{X}}$$

we have

$$f_{\kappa}(F_s)=f_{\bar{\kappa}}(F_s G^{-1}), \qquad (3)$$

where

$$G:\kappa\to\bar{\kappa} \qquad G=\frac{\partial \bar{X}}{\partial X}$$

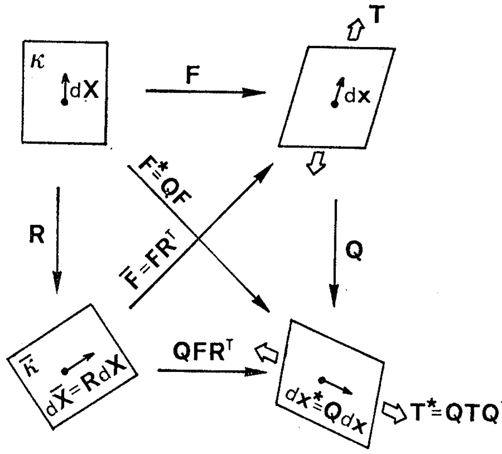

Fig. 2. Reference indifference and Objectivity

is the transformation gradient from a reference state κ to a reference state $\bar{\kappa}$. In Fig. 2, we take simply a rotation R for G and F for F_s. Thus $\bar{F}=FG^{-1}=FR^{-1}=FR^T$. We call Eq. (3) the *principle of reference indifference*, or simply the *reference indifference.*

When an arbitrary rigid motion is superposed on a current state in the form

$$x^*=c+Qx,$$

the deformation gradient and Cauchy stress tensor are transformed as (Fig. 2)

$$F^*=QF \qquad (4)$$

$$T^*=QTQ^T \qquad (5)$$

where c is an arbitrary vector, Q is an orthogonal tensor corresponding to the rigid rotation in the superposed rigid motion. Of course, both c and Q are generally functions of time.

By Eqs. (4) and (5), we have

$$Qf_{\kappa}(F_s)Q^T=f_{\kappa}(Q_sF_s). \qquad (6)$$

Eq. (6) is also a condition that any constitutive function f has to satisfy for any deformation gradient F and an arbitrary rotation Q. It is called the *principle of material indifference* or the *principle of material objectivity.* In what follows we call this simply the *objectivity.*

Material Symmetries and Isotropy

Suppose that, at a material point X in

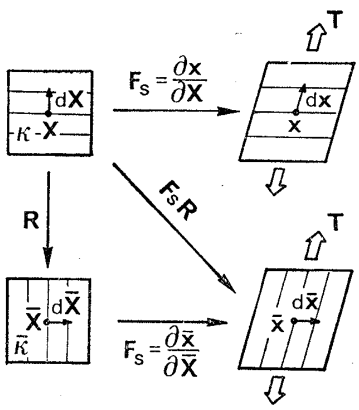

Fig. 3 Material symmetry and isotropy

a solid, there are two reference states κ and $\bar{\kappa}$ which differ by a rotation, such that the constitutive functionals with respect to these reference states, that is, f_κ and $f_{\bar{\kappa}}$, are the same. We have

$$f_\kappa(F_s)=f_{\bar{\kappa}}(F_s) \qquad (7)$$

for all deformation gradient histories F_s. We note here that we have two different current states as well as the different reference states κ and $\bar{\kappa}$ but such that $F_s=\partial x/\partial X=\partial\bar{x}/\partial\bar{X}$ (see Fig. 3). The physical meaning of Eq. (7) is that the mechanical response of the material at X in κ is indistinguishable from its response after it has been rotated to $\bar{\kappa}$.

By replacing F_s by F_sG in Eq. (3) with G a rotation R and using Eq. (7) we have

$$f_\kappa(F_s)=f_\kappa(F_sR) \qquad (8)$$

for all F_s and some rotations R. A material property in Eq. (7) or Eq. (8) is called the *material symmetries.* If a material has a reference state in which Eq. (8) is satisfied for arbitrary rotations R, the material is said to be *isotropic,* otherwise we call generally *anisotropic.*

Expressions of Anisotropic Materials by Reference Vectors

To represent anisotropic solids by tensorial expressions, we employ the unit vectors b, which are, in general, linearly independent ; these vectors indicate directions intrinsically chosen in the undeformed material, such as the directions of the axes of symmetry (i. e. principal axes of anisotropy). Assignment of such vectors, therefore, results from a priori knowledge concerning the nature of the undeformed reference state. Here and in what follows we call b the *reference vectors.*

Then consider an *elastic material* in the form

$$T=f(F,b). \qquad (9)$$

We note that Eq. (9) is a special case of the simple material of Eq. (2) and the dependency of f on the reference state κ in Eq. (2) is replaced by the reference vectors b. Since b are material elements in a reference state, $\bar{b}$ rotated by R are given by

$$\bar{b}=Rb$$

Thus the principle of reference indifference in Eq. (3) requires Eq. (9) to be

$$f(F,b)=f(FR^T,Rb) \qquad (10)$$

for all F, b and all rotations R since $G=R$ in Eq. (3) and $R^{-1}=R^T$. By objectivity in Eq. (6), we must have

$$Qf(F,b)Q^T=f(QF,b) \qquad (11)$$

for all F, b and all rotations Q.

By replacing F by QF and R by Q in Eq. (10) and then using Eq. (11), we have

$$Qf(F,b)Q^T=f(QFQ^T,Qb). \qquad (12)$$

Eq. (12) shows that the constitutive function f is isotropic in F and b. (If improper rotations are excluded from rotations, f is called hemitropic. In this paper, however, when we say rotations, they are regarded as full rotations,)

On the other hand, the material symmetry in Eq. (7) is rewritten as

$$f(F,b)=f(F,Rb) \qquad (13)$$

for all F, b and some rotations R. If the material is isotropic, Eq. (13) must be satisfied for all rotations R. Then, by Cauchy Theorem, unit vectors b are eliminated and Eq. (12) requires f to be isotropic in F. We note that it is not a result of the isotropy but a result of the objectivity and the reference indifference that requires the function f to be isotropic in F and b;

the isotropy merely requires the function $\boldsymbol{f}$ to be independent of $\boldsymbol{b}$.

To make the above idea clear, we consider two typical expressions of anisotropic materials:

$$\boldsymbol{T}=\boldsymbol{g}(\boldsymbol{B},\boldsymbol{a}), \quad \boldsymbol{a}=\boldsymbol{F}\boldsymbol{b}, \tag{14}$$

$$\boldsymbol{S}=\boldsymbol{h}(\boldsymbol{C},\boldsymbol{b}), \tag{15}$$

where $\boldsymbol{B}=\boldsymbol{F}\boldsymbol{F}^T$ (left Cauchy-Green strain tensor), $\boldsymbol{C}=\boldsymbol{F}^T\boldsymbol{F}$ (right Cauchy-Green strain tensor) and $\boldsymbol{S}=(\det\boldsymbol{F})\boldsymbol{F}^{-1}\boldsymbol{T}(\boldsymbol{F}^{-1})^T$ (second Piola-Kirchhoff stress tensor).

It is not difficult to show that $\boldsymbol{T}$, $\boldsymbol{B}$, and $\boldsymbol{a}$ are independent of the rotation of reference state but transformed to $\boldsymbol{Q}\boldsymbol{T}\boldsymbol{Q}^T$, $\boldsymbol{Q}\boldsymbol{B}\boldsymbol{Q}^T$, $\boldsymbol{Q}\boldsymbol{a}$ respectively under the rotation $\boldsymbol{Q}$ of current state. Conversly, $\boldsymbol{S}$, $\boldsymbol{C}$, and $\boldsymbol{b}$ are independent of the rotation of current state, but transformed to $\boldsymbol{R}\boldsymbol{S}\boldsymbol{R}$, $\boldsymbol{R}\boldsymbol{C}\boldsymbol{R}$ and $\boldsymbol{R}\boldsymbol{b}$ respectively under the rotation $\boldsymbol{R}$ of reference state. Thus, in Eq. (14), the principle of reference indifference is automatically satisfied and the objectivity requires the function $\boldsymbol{g}$ to be isotropic in $\boldsymbol{B}$ and $\boldsymbol{a}$. In Eq. (15), the objectivity is automatically satisfied and the reference indifference requires the function $\boldsymbol{h}$ to be isotropic in $\boldsymbol{C}$ and $\boldsymbol{b}$. If the material is isotropic, $\boldsymbol{g}$ and $\boldsymbol{h}$ are independent of $\boldsymbol{a}$ and $\boldsymbol{b}$ respectively. Note again that it is not the isotropy but the objectivity or the reference indifference resectively that requires the function $\boldsymbol{g}$ and $\boldsymbol{h}$ to be isotropic.

PRINCIPLES FOR CONSTITUTIVE EQUATIONS IN INFINITESIMAL THEORIES

Objectivity and Symmetries in infinitesimal Theories

Let $\boldsymbol{u}$ denote the displacement from the reference state

$$\boldsymbol{u}=\boldsymbol{x}-\boldsymbol{X} \tag{16}$$

and $\boldsymbol{H}$ the *gradient of displacement*

$$\boldsymbol{H}=\partial\boldsymbol{u}/\partial\boldsymbol{X}=\boldsymbol{F}-1. \tag{17}$$

The *infinitesimal strain tensor* $\boldsymbol{e}$ is defined by a symmetric part of $\boldsymbol{H}$

$$\boldsymbol{e}=(\boldsymbol{H}+\boldsymbol{H}^T)/2 \tag{18}$$

and the skew part

$$\boldsymbol{r}=(\boldsymbol{H}-\boldsymbol{H}^T)/2 \tag{19}$$

is called the *infinitesimal rotation.*

By infinitesimal theories we mean, briefly, that both the strain and the displacement from the reference state are negligibly small. We can not therefore distinguish between reference and current states; in other words, we must consider the same rigid rotation in the reference state whenever we consider an arbitrary rigid rotation in a current state. This means that we must consider $\boldsymbol{R}\equiv\boldsymbol{Q}$ in Fig. 2. If we assume, in infinitesimal theories, that the strain is small but the displacement is not necessarily small, it is not difficult to show that even a linear constitutive function, such as $\boldsymbol{T}=\boldsymbol{E}[\boldsymbol{e}]$ with $\boldsymbol{E}$ being a forth order elastic constant, does not satisfy the objectivity. (See, in detail, Casey and Naghdi, 1980)

Consider an elastic material in infinitesimal theories in the form

$$\boldsymbol{\sigma}=\boldsymbol{f}(\boldsymbol{e},\boldsymbol{b}),$$

where $\boldsymbol{\sigma}$ is the *infinitesimal stress tensor.*

When an arbitrary rigid motion (or replacement) is superposed on a current state by

$$\boldsymbol{x}^*=\boldsymbol{c}+\boldsymbol{Q}\boldsymbol{x},$$

infinitesimal deformations require the same replacement for the reference state

$$\boldsymbol{X}^*=\boldsymbol{c}+\boldsymbol{Q}\boldsymbol{X}.$$

We note that, therefore, $\boldsymbol{Q}$ and $\boldsymbol{c}$ in infinitesimal theories must be constant in time since they are independent of time in the reference state. Reference vectors are then transformed as

$$\boldsymbol{b}^*=\boldsymbol{Q}\boldsymbol{b}$$

and, by Eqs. (16)–(18),

$$\boldsymbol{e}^*=\boldsymbol{Q}\boldsymbol{e}\boldsymbol{Q}^T.$$

Then the objectivity trivially requires

$$\boldsymbol{\sigma}^*=\boldsymbol{Q}\boldsymbol{\sigma}\boldsymbol{Q}^T,$$

that is, $\boldsymbol{f}$ is isotropic in $\boldsymbol{e}$ and $\boldsymbol{b}$:

$$\boldsymbol{f}(\boldsymbol{Q}\boldsymbol{e}\boldsymbol{Q}^T,\boldsymbol{Q}\boldsymbol{b})=\boldsymbol{Q}\boldsymbol{f}(\boldsymbol{e},\boldsymbol{b})\boldsymbol{Q}^T. \tag{20}$$

We note that, in infinitesimal theories, the principle of reference indifference is indistinguishable from the objectivity since the reference state is indistinguishable from the current state.

The material symmetry in Eq. (7) or (13) is written as

$$\boldsymbol{f}(\boldsymbol{e},\boldsymbol{b})=\boldsymbol{f}(\boldsymbol{e},\boldsymbol{R}\boldsymbol{b}) \tag{21}$$

for all $\boldsymbol{e}$ and $\boldsymbol{b}$ and some rotations $\boldsymbol{R}$. By replacing $\boldsymbol{Q}$ by $\boldsymbol{R}^T$ and $\boldsymbol{b}$ by $\boldsymbol{Rb}$ in Eq. (20) and then using Eq. (21), Eq. (21) is rewritten as

$$\boldsymbol{R}^T\boldsymbol{f}(\boldsymbol{e},\boldsymbol{b})\boldsymbol{R}=\boldsymbol{f}(\boldsymbol{R}^T\boldsymbol{e}\boldsymbol{R},\boldsymbol{b}).$$

Applications of the Principles

Here we demonstrate several applications of the principles derived in the previous section.

Let $\boldsymbol{b}$ be a reference vector, that is, the mechanical response is assumed to be independent of arbitrary rotations about the vector. Since the objectivity in Eq. (20) requires $\boldsymbol{f}$ to be isotropic in $\boldsymbol{e}$ and $\boldsymbol{b}$, a polynomial isotropic tensor function $\boldsymbol{\sigma}=\boldsymbol{f}(\boldsymbol{e},\boldsymbol{b})$ has the representation (see Smith, 1971):

$$\begin{aligned}\boldsymbol{\sigma}=&\alpha_1\boldsymbol{1}+\alpha_2\boldsymbol{e}+\alpha_3\boldsymbol{e}^2+\alpha_4\boldsymbol{b}\otimes\boldsymbol{b}\\&+\alpha_5(\boldsymbol{b}\otimes\boldsymbol{e}\boldsymbol{b}+\boldsymbol{e}\boldsymbol{b}\otimes\boldsymbol{b})\\&+\alpha_6(\boldsymbol{b}\otimes\boldsymbol{e}^2\boldsymbol{b}+\boldsymbol{e}^2\boldsymbol{b}\otimes\boldsymbol{b})\end{aligned} \tag{22}$$

where $\alpha_1, \cdot\cdot, \alpha_6$ are scalar-valued functions of $\{tr\boldsymbol{e},\ tr\boldsymbol{e}^2,\ tr\boldsymbol{e}^3,\ \boldsymbol{b}\cdot\boldsymbol{e}\boldsymbol{b},\ \boldsymbol{b}\cdot\boldsymbol{e}^2\boldsymbol{b}\}$. Eq. (22) is a general expression of nonlinear transversaly isotropic elastic material in infinitesimal theories.

Let $\boldsymbol{b}^\alpha$ $(\alpha=1,2,3)$ be three orthogonal reference vectors. Then, similarly to Eq. (22),

$$\boldsymbol{\sigma}=\boldsymbol{f}(\boldsymbol{e},\boldsymbol{b}^\alpha) \tag{23}$$

can be represented by an isotropic function in $\boldsymbol{e}$ and $\boldsymbol{b}^\alpha$. However, we consider here an alternative approach in order to compare with classical expressions of anisotropic materials.

Since $\boldsymbol{b}^\alpha$ are three unit orthogonal vectors, we may put

$$e_{\alpha\beta}=\boldsymbol{b}^\alpha\cdot\boldsymbol{e}\boldsymbol{b}^\beta \tag{24}$$

so that Eq. (23) yields

$$\boldsymbol{\sigma}=\bar{\boldsymbol{f}}(e_{\alpha\beta},\boldsymbol{b}^\alpha).$$

Then, noting that

$$e_{\alpha\beta}{}^*=\boldsymbol{Q}\boldsymbol{b}^\alpha\cdot(\boldsymbol{Q}\boldsymbol{e}\boldsymbol{Q}^T)\boldsymbol{Q}\boldsymbol{b}^\beta=e_{\alpha\beta}$$

we use the objectivity (Eq. (20)) so that

$$\boldsymbol{Q}\bar{\boldsymbol{f}}(e_{\alpha\beta},\boldsymbol{b}^\alpha)\boldsymbol{Q}^T=\bar{\boldsymbol{f}}(e_{\alpha\beta},\boldsymbol{Q}\boldsymbol{b}^\alpha). \tag{25}$$

Eq. (25) requires $\bar{\boldsymbol{f}}$ to be isotropic in $\boldsymbol{b}^\alpha$ and then has the representation (see Smith, 1971):

$$\boldsymbol{\sigma}=\tilde{f}_{\alpha\beta}(e_{\gamma\delta})\boldsymbol{b}^\alpha\tilde{\otimes}\boldsymbol{b}^\beta \tag{26}$$

where $\tilde{\otimes}$ stands for the symmetric part of $\otimes$. Eq. (26) is a general tensorial representation of nonlinear anisotropic elastic materials in infinitesimal theories. If $\tilde{f}$ in Eq. (26) are linear in $\boldsymbol{e}$, we have

$$\boldsymbol{\sigma}=E_{\alpha\beta\gamma\delta}e_{\gamma\delta}\boldsymbol{b}^\alpha\tilde{\otimes}\boldsymbol{b}^\beta$$

or using Eq. (24),

$$\boldsymbol{\sigma}=E_{\alpha\beta\gamma\delta}\boldsymbol{b}^\alpha\otimes\boldsymbol{b}^\beta\otimes\boldsymbol{b}^\gamma\otimes\boldsymbol{b}^\delta[\boldsymbol{e}] \tag{27}$$

where $E_{\alpha\beta\gamma\delta}$ are constant with $E_{\alpha\beta\gamma\delta}=E_{\beta\alpha\gamma\delta}=E_{\alpha\beta\delta\gamma}$.

We then have a classical expression of linear anisotropic elastic materials in the component form of Eq. (27) with respect to $\boldsymbol{b}$

$$\sigma_{\alpha\beta}=E_{\alpha\beta\gamma\delta}e_{\gamma\delta}$$

or, in arbitrary fixed Cartesian coordinates,

$$\sigma_{ij}=E_{\alpha\beta\gamma\delta}e_{\gamma\delta}b_i{}^\alpha b_j{}^\beta.$$

Finally, we introduce a new type of elastic material:

$$\boldsymbol{\sigma}=\boldsymbol{f}(\boldsymbol{e},\boldsymbol{r})$$

in which an infinitesimal rotation is included. Since $\boldsymbol{f}$ is independent of the reference state, it is clearly isotropic. Since $\boldsymbol{r}^*=\boldsymbol{Q}\boldsymbol{r}\boldsymbol{Q}^T$ by Eq. (17) and (19), the objectivity requires $\boldsymbol{f}$ to be isotropic in the symmetric tensor $\boldsymbol{e}$ and the skew symmetric tensor $\boldsymbol{r}$ in the form

$$\boldsymbol{Q}\boldsymbol{f}(\boldsymbol{e},\boldsymbol{r})\boldsymbol{Q}^T=\boldsymbol{f}(\boldsymbol{Q}\boldsymbol{e}\boldsymbol{Q}^T,\boldsymbol{Q}\boldsymbol{r}\boldsymbol{Q}^T).$$

Then $\boldsymbol{f}$ has the representation (see Smith, 1971):

$$\begin{aligned}\boldsymbol{\sigma}=&\alpha_1\boldsymbol{1}+\alpha_2\boldsymbol{e}+\alpha_3\boldsymbol{e}^2+\alpha_4\boldsymbol{r}^2+\alpha_5(\boldsymbol{e}\boldsymbol{r}-\boldsymbol{r}\boldsymbol{e})+\alpha_6\boldsymbol{r}\boldsymbol{e}\boldsymbol{r}\\&+\alpha_7(\boldsymbol{e}^2\boldsymbol{r}-\boldsymbol{r}\boldsymbol{e}^2)+\alpha_8(\boldsymbol{r}\boldsymbol{e}\boldsymbol{r}^2-\boldsymbol{r}^2\boldsymbol{e}\boldsymbol{r})\end{aligned} \tag{28}$$

where $\alpha_1, \cdots, \alpha_8$ are scalar-valued functions of $\{tr\boldsymbol{e},\ tr\boldsymbol{e}^2,\ tr\boldsymbol{e}^3,\ tr\boldsymbol{r}^2,\ tr(\boldsymbol{e}\boldsymbol{r}^2),\ tr(\boldsymbol{e}^2\boldsymbol{r}^2),\ tr(\boldsymbol{e}^2\boldsymbol{r}^2\boldsymbol{e}\boldsymbol{r})\}$. We note that in the anisotropic elastic material in finite theories defined by

$$T = f(F),$$

the rotation has been completely eliminated since, replacing Q by R and F by U in Eq. (12), we have

$$Rf(U)R^T = f(RUR^T) = f(V) = T$$

where U and V are the *right* and *left strech tensor* respectively defined by $F = RU = VR$. If we assume that σ in Eq. (28) is a first order function of e and r, we have a classical linear elastic material

$$\sigma = \lambda(tre)1 + 2\mu e$$

where λ and μ are constant and in which the infinitesimal rotation r is eliminated.

EXPRESSIONS OF ANISOTROPY IN SOIL MATERIALS

We are now in a position to state our precise definitions of anisotropy in soil materials. The discussions are confined to infinitesimal theories.

Inherent Anisotropy and Stress Induced Anisotropy

Previous discussions are concerned with the deformations from the undeformed state or unstressed state. In soil materials, our main concern is, however, the deformation from an initial state, which is already stressed, or experienced some deformations for a long period. In the laboratory, soils are often given preceding stress histories such as consolidation before we have an initial state. In order to distinguish between an undeformed reference state at $t=0$ and an initial reference state at $t=t_0$ (>0), we introduce the *reference tensor* defined by

$$K = \sum_{\alpha=1}^{3} C_\alpha b^\alpha \otimes b^\alpha \quad (C_\alpha : \text{scalars}) \qquad (29)$$

for the undeformed reference state and similarly K_0 for the initial reference state.

The constitutive equation of an initially stressed material is then assumed to be

$$e(t) = \mathop{f}_{s=0}^{s<t-t_0} (\sigma(t-s),\ \sigma(t_0), K_0) \qquad (30)$$

and

$$K_0 = \mathop{k}_{s=t-t_0}^{s=t} (\sigma(t-s), K) \qquad (31)$$

with $e(t)=0$ if $\sigma(t-s)=\sigma(t_0)$ for $0 \leq s \leq t-t_0$. We note that K_0 is a function of K and stress histories $\sigma(t-s)$ for $t-t_0 \leq s \leq t$. Here and in what follows, for convenience, we write

$$e = f(\sigma_s, \sigma_0, K_0) \qquad (32)$$

and

$$K_0 = k(\sigma_{s-}, K) \qquad (33)$$

for Eq. (30) and Eq. (31). (See Table 1. for Eq. (32)—Eq. (55))

When the reference state is rotated, the reference tensor in Eq. (29) is transformed as

$$K = \sum_{\alpha=1}^{3} C_\alpha b^\alpha \otimes b^\alpha = \sum_{\alpha=1}^{3} C_\alpha (Qb^\alpha) \otimes (Qb^\alpha)$$
$$= Q\left(\sum_{\alpha=1}^{3} C_\alpha b^\alpha \otimes b^\alpha\right) Q^T = QKQ^T$$

Thus the objectivity in infinitesimal theories requires

$$Qf(\sigma_s, \sigma_0, K_0)Q^T = f(Q\sigma_s Q^T, Q\sigma_0 Q^T, QK_0Q^T) \qquad (34)$$

and

$$Qk(\sigma_{s-}, K)Q^T = k(Q\sigma_{s-}Q^T, QKQ^T) \qquad (35)$$

for all σ, K and all rotations Q so that f and k are isotropic in $\{\sigma_s, \sigma_0, K_0\}$ and $\{\sigma_{s-}, K\}$ respectively.

If the material is (*inherently*) *isotropic* at $t=0$, the material symmetry requires Eq. (33) to be

$$k(\sigma_{s-}, K) = k(\sigma_{s-}, QKQ^T) \qquad (36)$$

for all σ_{s-}, K, and all rotations Q. Or if K is an isotropic tensor

$$K = \lambda 1 \qquad (37)$$

with λ a scalar, the material is automatically (*inherently*) *isotropic* at $t=0$.

Thus we call the material the *inherent anisotropy* if

$$k(\sigma_{s-}, K) \neq k(\sigma_{s-}, QKQ^T) \qquad (38)$$

for some σ_{s-}, K, or some rotations Q and

$$K \neq \lambda 1 \qquad (39)$$

Similarly to Eqs. (36) and (37), the material is called the *initial isotropy* at $t=0$ if

Table 1. Inherent anisotropy and stress induced anisotropy

<table>
<tr><td colspan="3">$e=f(\sigma_s, \sigma_0, K_0)$ (32)
$K_0=k(\sigma_{s-}, K)$ (33)</td><td colspan="2">(f and k are isotropic functions in all variables) (34) (35)</td></tr>
<tr><td colspan="3">(Inherent) Isotropy
$k(\sigma_{s-}, K)=k(\sigma_{s-}, QKQ^T)$ (36)
or
$K=\lambda 1$ (37)</td><td colspan="2">Inherent anisotropy
$k(\sigma_{s-}, K)\neq k(\sigma_{s-}, QKQ^T)$ (38)
and
$K\neq\lambda 1$ (39)</td></tr>
<tr><td rowspan="2">Initial isotropy
$f(\sigma_s, \sigma_0, K_0)$
$=f(\sigma_s, \sigma_0, QK_0Q^T)$
(40)
or
$K_0=\beta 1$ (41)</td><td colspan="4">Initial anisotropy
$f(\sigma_s, \sigma_0, K_0)\neq f(\sigma_s, \sigma_0, QK_0Q^T)$ (42)
and
$K_0\neq\beta 1$ (43)</td></tr>
<tr><td colspan="2">(1)
Stress induced initial anisotropy
$K_0=k(\sigma_{s-})$ (44)
$\sigma_{s-}\neq p_{s-}1$</td><td>(2)
Inherent initial anisotropy
$K_0=k(K)$ (45)</td><td>(3)
Combined initial anisotropy
$K_0=k(\sigma_{s-}, K)$ (46)</td></tr>
<tr><td>(Example)</td><td colspan="4">$v_d=\delta\left\|K_0p_0\sigma_0^{-1}\left(\frac{\sigma}{p}-\frac{\sigma_0}{p_0}\right)\right\|$, $K_0=k(\sigma_c, K)$ (47), (48)</td></tr>
<tr><td>$K_0=\beta 1$</td><td colspan="2">$K_0=\lambda\frac{\sigma_c}{p_c}$ (50)</td><td>$K_0=K$</td><td>$K_0=k(\sigma_c, K)$ (48)</td></tr>
<tr><td>$v_d=\beta\delta\|p_0\sigma_0^{-1}(\eta-\eta_0)\|$
(54)</td><td colspan="2">$v_d=\lambda\delta\left\|\sigma_c\sigma_0^{-1}\frac{p_0}{p_c}\left(\frac{\sigma}{p}-\frac{\sigma_0}{p_0}\right)\right\|$
(51)</td><td></td><td rowspan="3">$v_d=\delta\left\|K_0p_0\sigma_0^{-1}\left(\frac{\sigma}{p}-\frac{\sigma_0}{p_0}\right)\right\|$
(47)</td></tr>
<tr><td>$\sigma_0=p_0 1$
$v_d=\beta\delta\|\eta\|$</td><td rowspan="2">$\sigma_0=\sigma_c$
$v_d=\lambda\delta\|\eta-\eta_0\|$
(53)</td><td rowspan="2">$\sigma_0\neq\sigma_c$
$(\sigma_0=p_0 1)$
$v_d=\lambda\delta\left\|\frac{\sigma_c}{p_c}\eta\right\|$
(52)</td><td rowspan="2">$\sigma_0=p_0 1$
$v_d=\delta\|K\eta\|$ (49)</td></tr>
<tr><td>$v_d=\delta(\|\eta\|^2-\|\eta_0\|^2)^{1/2}$
(55)</td></tr>
</table>

$$f(\boldsymbol{\sigma}_s, \boldsymbol{\sigma}_0, \boldsymbol{K}_0)=f(\boldsymbol{\sigma}_s, \boldsymbol{\sigma}_0, \boldsymbol{QK}_0\boldsymbol{Q}^T) \quad (40)$$

for all $\boldsymbol{\sigma}_s$, $\boldsymbol{\sigma}_0$, $\boldsymbol{K}_0$, and all rotations $\boldsymbol{Q}$ or

$$\boldsymbol{K}_0=\beta\,\boldsymbol{1} \quad (41)$$

with β a scalar.

The *initial anisotropy* is then defined, similarly to Eqs. (38) and (39), as

$$f(\boldsymbol{\sigma}_s, \boldsymbol{\sigma}_0, \boldsymbol{K}_0)\neq f(\boldsymbol{\sigma}_s, \boldsymbol{\sigma}_0, \boldsymbol{QK}_0\boldsymbol{Q}^T) \quad (42)$$

for some $\boldsymbol{\sigma}_s$, $\boldsymbol{\sigma}_0$, $\boldsymbol{K}_0$ or some rotations $\boldsymbol{Q}$ and

$$\boldsymbol{K}_0\neq\beta\,\boldsymbol{1} \quad (43)$$

The initial anisotropy is decomposed into three types depending on the function $\boldsymbol{k}$:

(1) $\boldsymbol{K}_0=\boldsymbol{k}(\boldsymbol{\sigma}_{s-})$ (44)
is called the *stress induced initial anisotropy*. (Eq. (35) and (43) require stress history $\boldsymbol{\sigma}_{s-}$ is not isotropic, $\boldsymbol{\sigma}_{s-}\neq p_{s-}\boldsymbol{1}$)

(2) $\boldsymbol{K}_0=\boldsymbol{k}(\boldsymbol{K})$ (45)
is called the *inherent initial anisotropy*.

(3) $\boldsymbol{K}_0=\boldsymbol{k}(\boldsymbol{\sigma}_{s-}, \boldsymbol{K})$ (46)
is called the *combined initial anisotropy*.

We note that the stress induced anisotropy in Eq. (44) is not the inherent anisotropy but the (inherent) isotropy.

Examples

Here we demonstrate several examples of constitutive equations of anisotropic material of soils and show how the principles derived in the previous parts are useful.

(1) *Dilatancy of clays*

Assume that the volumetric strain v_d by dilatancy from an initial state is given in the form

$$v_d=\delta\|\boldsymbol{K}_0p_0\boldsymbol{\sigma}_0^{-1}(\boldsymbol{\sigma}/p-\boldsymbol{\sigma}_0/p_0)\| \quad (47)$$

and

$$\boldsymbol{K}_0=\boldsymbol{k}(\boldsymbol{\sigma}_c, \boldsymbol{K}) \quad (48)$$

where δ is a scalar, $p=tr\boldsymbol{\sigma}/3$ is the *mean principal effective stress*, and $\boldsymbol{\sigma}_c$ is a certain dominant constant stress over $\boldsymbol{K}_0$, for instance, $\boldsymbol{\sigma}_c=\boldsymbol{\sigma}(s)$ constant for $0\leq s<t$. It is not difficult to show that Eq. (47) satisfies

the objectivity in Eq. (34) and $v_d=0$ when $\boldsymbol{\sigma}=\boldsymbol{\sigma}_0$. Eq. (48) also must satisfy Eq. (35). The general polynomial form is given in Smith(1971).

Let Eq. (48) satisfy Eqs. (38) and (39). Then the material assumed by Eq. (47) with Eq. (48) is an inherent anisotropy. If, furthermore, $\boldsymbol{K}_0 \neq \beta\mathbf{1}$, the material in Eq. (47) with Eq. (48) is an initial anisotropy since Eq. (47) satisfies Eq. (42). For the general $\boldsymbol{K}_0$, the material is a combined initial anisotropy by the definition Eq. (46). If $\boldsymbol{K}_0$ is independent of stress histories (There is such a possibility in the case of an isotropically consolidated clay.), the material in Eq. (47) with a form of $\boldsymbol{K}_0$ in Eq. (45) is an inherent initial anisotropy. Let $\boldsymbol{K}_0=\boldsymbol{K}$ simply and apply the pressure $\boldsymbol{\sigma}_0=p_0\mathbf{1}$ to the initial stress. Then Eq. (47) yields

$$v_d=\delta\|\boldsymbol{K\eta}\| \qquad (49)$$

where $\boldsymbol{\eta}=\boldsymbol{s}/p$ and $\boldsymbol{s}=\boldsymbol{\sigma}-p\mathbf{1}$. Eq. (49) gives a simple form of dilatancy equations of the inherent initial anisotropy in the case of isotropically pre-consolidated clays.

If $\boldsymbol{K}_0$ is independent of a reference tensor $\boldsymbol{K}$ such as $\boldsymbol{K}_0=\boldsymbol{k}(\boldsymbol{\sigma}_c)$, Eq. (47) is a stress induced initial anisotropy as is defined in Eq. (44). The objectivity in Eq. (35) then requires $\boldsymbol{k}$ to be isotropic in $\boldsymbol{\sigma}_c$, but Eq. (43) to be $\boldsymbol{\sigma}_c \neq p_c\mathbf{1}$. That is, the stress induced initial anisotropy is always a result of the anisotropic consolidation before the initial state. Let

$$\boldsymbol{K}_0=\lambda\boldsymbol{\sigma}_c/p_c. \qquad (50)$$

Then Eq. (47) yields

$$v_d=\lambda\delta\|\boldsymbol{\sigma}_c\boldsymbol{\sigma}_0^{-1}p_0/p_c(\boldsymbol{\sigma}/p-\boldsymbol{\sigma}_0/p_0)\| \qquad (51)$$

Eq. (51) with (50) is a dilatancy equation of anisotropically overconsolidated clays if $p_c>p_0$. In particular, if $\boldsymbol{\sigma}_0=p_0\mathbf{1}$, Eq. (51) yields

$$v_d=\lambda\delta\|(\boldsymbol{\sigma}_c/p_c)\boldsymbol{\eta}\| \qquad (52)$$

Eq. (52) with (50) gives a simple form of dilatancy equations of the stress induced initial anisotropy in the case of initial isotropically consolidated clays. If $\boldsymbol{\sigma}_0=\boldsymbol{\sigma}_c$ in Eq. (51), we have

$$v_d=\lambda\delta\|\boldsymbol{\eta}-\boldsymbol{\eta}_0\| \qquad (53)$$

Eq. (53) under a condition such as Eq. (50) with $\boldsymbol{\sigma}_c=\boldsymbol{\sigma}_0$ is a stress induced dilatancy equation of normally consolidated clays employed by Sekiguchi and Ohta(1977).

If $\boldsymbol{K}_0=\beta\mathbf{1}$ in Eq. (47), the reduced form

$$v_d=\beta\delta\|p_0\boldsymbol{\sigma}_0^{-1}(\boldsymbol{\eta}-\boldsymbol{\eta}_0)\| \qquad (54)$$

is an initial isotropy by the definition in Eq. (41). In particular, if $\boldsymbol{\sigma}_0=p_0\mathbf{1}$, Eq. (54) yields the simplest initially isotropic material:

$$v_d=\beta\delta\|\boldsymbol{\eta}\|$$

We note that the form employed by Ohta and Hata(1971)

$$v_d=\delta(\|\boldsymbol{\eta}\|^2-\|\boldsymbol{\eta}_0\|^2)^{1/2} \qquad (55)$$

is not a stress induced anisotropy but an initial isotropy since Eq. (55) is independent of $\boldsymbol{K}_0$ if $\boldsymbol{\eta}_0$ is independent of $\boldsymbol{K}_0$ and even if $\boldsymbol{\eta}_0$ is related to $\boldsymbol{K}_0$ such as $\boldsymbol{\eta}_0$ $(\boldsymbol{K}_0)$, the objectivity requires

$$\|\boldsymbol{\eta}_0(\boldsymbol{QK}_0\boldsymbol{Q}^T)\|=\|\boldsymbol{Q\eta}_0(\boldsymbol{K}_0)\boldsymbol{Q}^T\|=\|\boldsymbol{\eta}_0(\boldsymbol{K}_0)\|$$

and Eq. (55) therefore satisfies Eq. (40). The material in Eq. (53) is also an initial isotropy if $\boldsymbol{\eta}_0$ is independent of $\boldsymbol{K}_0$. It is stress induced anisotropic only if $\boldsymbol{\eta}_0$ is related to $\boldsymbol{K}_0$ such as $\boldsymbol{\eta}_0=\boldsymbol{K}_0/\lambda-\mathbf{1}$ as Eq. (50) with $\boldsymbol{\sigma}_c=\boldsymbol{\sigma}_0$.

(2) *Failure conditions*

Matsuoka, Nakai and Ishizaki(1980) proposed an interesting failure condition for anisotropic soils. Let their condition for isotropic soils be

$$f(\boldsymbol{\sigma})=\text{const}, \qquad (56)$$

in the concrete,

$$J_1J_2/J_3=\text{const}$$

where $J_1=tr\boldsymbol{\sigma}$, $J_2=\{tr\boldsymbol{\sigma}^2-(tr\boldsymbol{\sigma})^2\}/2$, $J_3=det\boldsymbol{\sigma}$.

In order to give anisotropic structures, they introduced an anisotropic parameter related to an undeformed state, that is, introduced a reference tensor $\boldsymbol{K}$ in our symbol and replaced $\boldsymbol{\sigma}$ by $\boldsymbol{K\sigma}$ in Eq. (56):

$$f(\boldsymbol{K\sigma})=\text{const} \qquad (57)$$

We note that it is not the isotropy but the objectivity that requires f in Eq. (56) to be isotropic, that is, to be a function of J_1, J_2 and J_3. If the function f is assumed as Eq. (57), the objectivity

$$f(\boldsymbol{QKQ}^T\boldsymbol{Q\sigma Q}^T)=f(\boldsymbol{Q}(\boldsymbol{K\sigma})\boldsymbol{Q}^T)=f(\boldsymbol{K\sigma}),$$

requires f to be isotropic in $\boldsymbol{K\sigma}$. Thus f is necessarily a function of $J_1^*=tr(\boldsymbol{K\sigma})$, $J_2^*=1/2\{tr(\boldsymbol{K\sigma})^2-(tr\boldsymbol{K\sigma})^2\}$, $J_3^*=det(\boldsymbol{K\sigma})$. If $\boldsymbol{K}$ in Eq. (57) is a reference tensor of an undeformed state, the failure condition in Eq. (57) is an expression of the inherent anisotropy.

(3) *Materials of differential type*

Constitutive equations in soil mechanics are often assumed to be elastic-plastic materials in the form of a rate independent differential type such as

$$\dot{\boldsymbol{e}}=\boldsymbol{H}(\boldsymbol{\sigma},\boldsymbol{\sigma}_0,\boldsymbol{K}_0)[\dot{\boldsymbol{\sigma}}] \qquad (58)$$

or

$$\dot{e}_{ij}=H_{ijkl}\dot{\sigma}_{kl}$$

with an initial condition $\boldsymbol{e}=0$ and $\boldsymbol{\sigma}=\boldsymbol{\sigma}_0$ at $t=t_0$. We note that if $\boldsymbol{\sigma}(\tau)$ is given for $t_0<\tau<t$,

$$\boldsymbol{e}=\int_{t_0}^{t}\boldsymbol{H}(\boldsymbol{\sigma},\boldsymbol{\sigma}_0,\boldsymbol{K}_0)[\dot{\boldsymbol{\sigma}}]d\tau$$

is a form of Eq. (30) or Eq. (32).

Thus all the principles stated in the previous section are similarly applied to the constitutive equation of differential type. However, noting that when we consider the objectivity in infinitesimal theories, the rotation $\boldsymbol{Q}$ is constant in time, (Note that $\boldsymbol{Q}$ is a function of time in finite theories.), we may easily apply the objectivity to Eq. (58) directly as

$$\boldsymbol{Q}\dot{\boldsymbol{e}}\boldsymbol{Q}^T=\boldsymbol{H}(\boldsymbol{Q\sigma Q}^T,\boldsymbol{Q\sigma}_0\boldsymbol{Q}^T,\boldsymbol{QK}_0\boldsymbol{Q}^T)[\boldsymbol{Q}\dot{\boldsymbol{\sigma}}\boldsymbol{Q}^T]$$

for all $\boldsymbol{\sigma}$, $\boldsymbol{\sigma}_0$, $\boldsymbol{\sigma}$, $\boldsymbol{K}_0$ and all rotations $\boldsymbol{Q}$. For simplicity, we assume that $\boldsymbol{H}$ is a function of $\boldsymbol{\sigma}$; then

$$\boldsymbol{QH}(\boldsymbol{\sigma})[\dot{\boldsymbol{\sigma}}]\boldsymbol{Q}^T=\boldsymbol{H}(\boldsymbol{Q\sigma Q}^T)[\boldsymbol{Q}\dot{\boldsymbol{\sigma}}\boldsymbol{Q}^T]$$

for all $\boldsymbol{\sigma}$, $\dot{\boldsymbol{\sigma}}$, and all rotations $\boldsymbol{Q}$. Then by a representation theorem by Wang(1970) or Smith(1971), we have

$$\begin{aligned}\boldsymbol{H}(\boldsymbol{\sigma})[\dot{\boldsymbol{\sigma}}]=&[\beta_1 tr\dot{\boldsymbol{\sigma}}+\beta_2 tr(\boldsymbol{\sigma}\dot{\boldsymbol{\sigma}})+\beta_3 tr(\boldsymbol{\sigma}^2\dot{\boldsymbol{\sigma}})]\boldsymbol{1}\\&+[\beta_4 tr\dot{\boldsymbol{\sigma}}+\beta_5 tr(\boldsymbol{\sigma}\dot{\boldsymbol{\sigma}})+\beta_6 tr(\boldsymbol{\sigma}^2\dot{\boldsymbol{\sigma}})]\boldsymbol{\sigma}\\&+[\beta_7 tr\dot{\boldsymbol{\sigma}}+\beta_8 tr(\boldsymbol{\sigma}\dot{\boldsymbol{\sigma}})+\beta_9(\boldsymbol{\sigma}^2\dot{\boldsymbol{\sigma}})]\boldsymbol{\sigma}^2\\&+\beta_{10}\dot{\boldsymbol{\sigma}}+\beta_{11}(\dot{\boldsymbol{\sigma}}\boldsymbol{\sigma}+\boldsymbol{\sigma}\dot{\boldsymbol{\sigma}})+\beta_{12}(\dot{\boldsymbol{\sigma}}\boldsymbol{\sigma}^2+\boldsymbol{\sigma}^2\dot{\boldsymbol{\sigma}})\end{aligned}$$

where $\beta_1, \cdots, \beta_{12}$ are functions of the principal invariants of $\boldsymbol{\sigma}$. We note that even if Eq. (58) is a form

$$\dot{\boldsymbol{e}}=\boldsymbol{H}(\boldsymbol{\sigma},\boldsymbol{\sigma}_0)[\dot{\boldsymbol{\sigma}}] \qquad (59)$$

with an initial condition $\boldsymbol{e}=0$ and $\boldsymbol{\sigma}=\boldsymbol{\sigma}_0$ at $t=t_0$, the material in Eq. (59) is an initially isotropy if $\boldsymbol{\sigma}_0$ is independent of $\boldsymbol{K}_0$. On the other hand, even the simplest material such as

$$\dot{\boldsymbol{e}}=\boldsymbol{H}(\boldsymbol{\sigma})[\dot{\boldsymbol{\sigma}}]$$

can be a stress induced initial anisotropy if the initial stress $\boldsymbol{\sigma}_0$ is related to $\boldsymbol{K}_0$.

CONCLUSIONS

In the first part we present brief discussions about the principles for constitutive equations in finite theories:

(1) Reference indifference
(2) Objectivity
(3) Material symmetry (anisotropy) and isotropy

Introducing reference vectors $\boldsymbol{b}$ and $\boldsymbol{a}=\boldsymbol{Fb}$, we then conclude that

$\boldsymbol{T}=\boldsymbol{f}(\boldsymbol{F},\ \boldsymbol{b})$ must be isotropic in $\boldsymbol{F}$ and $\boldsymbol{b}$ by (1) and (2)

$\boldsymbol{T}=\boldsymbol{g}(\boldsymbol{B},\ \boldsymbol{a})$ must be isotropic in $\boldsymbol{B}$ and $\boldsymbol{a}$ by (2)

$\boldsymbol{S}=\boldsymbol{h}(\boldsymbol{C},\ \boldsymbol{b})$ must be isotropic in $\boldsymbol{C}$ and $\boldsymbol{b}$ by (1).

In the second part we note that, in infinitesimal theories, (1) is indistinguisheable from (2) but prefer calling them the objectivity. We then conclude that $\boldsymbol{\sigma}=\boldsymbol{f}(\boldsymbol{e},\ \boldsymbol{b})$ must be isotropic in $\boldsymbol{e}$ and $\boldsymbol{b}$ by the objectivity. We must note that even if the above functions are independent of reference vectors, they all must be isotropic functions. We then note that requiring them to be isotropic is *not the isotropy* in (3), but the principles in (1) or (2). More importantly, employing reference vectors, anisotropic materials can be represented by isotropic functions whose representations are well established in Wang(1970) or Smith (1971). Using the objectivity, we demonstrate a derivation of a classical anisotropic material. Furthermore we show that there may exist an elastic material in which an infinitesimal rotation $\boldsymbol{r}$ is included and conclude that if we assume that $\boldsymbol{\sigma}$ is a first order function of the strain and $\boldsymbol{r}$, only $\boldsymbol{r}$ is eliminated and we get a classical linear elastic material.

Introducing a reference tensor $\boldsymbol{K}$ in the last part, we define the initial anisotropy such as the inherent anisotropy and the stress induced anisotropy. Briefly concluding, the inherent anisotropy is that the initial reference tensor $\boldsymbol{K}_0$ is a tensor valued function of the undeformed reference tensor $\boldsymbol{K}$ and the stress induced anisotropy is that $\boldsymbol{K}_0$ is a tensor valued function of *only the stress histories* up to the initial time. Thus the stress induced anisotropy is not the inherent anisotropy but the (inherent) isotropy.

We must note that $\boldsymbol{K}_0$ does not often appear explicitly in the constitutive equations. If $\boldsymbol{\sigma}_0$ is related to $\boldsymbol{K}_0$, and $\boldsymbol{\sigma}_0$ is not included in the invariant forms in the constitutive equations, the material is a stress induced initial anisotropy.

We further note that if the constitutive equations are given in the differential type, not only $\boldsymbol{K}_0$ but also $\boldsymbol{\sigma}_0$ might not appear explicitly in the differential equations, since $\boldsymbol{\sigma}_0$ is often given as an initial condition. Then the material is a stress induced initial anisotropy or initial isotropy corresponding to whether $\boldsymbol{\sigma}_0$ is related to $\boldsymbol{K}_0$ or not.

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NOTATION

$\boldsymbol{a}$=director in the current state
$\boldsymbol{b}, \boldsymbol{b}^\alpha$=reference vectors
$\boldsymbol{B}$=left Cauchy-Green strain tensor
$\boldsymbol{C}$=right Cauchy-Green strain tensor
$\boldsymbol{e}$=infinitesimal strain tensor
$\dot{\boldsymbol{e}}$=rate of strain or strain increment
$\boldsymbol{E}$=elastic constant
$\boldsymbol{F}$=deformation gradient
$\boldsymbol{G}$=transformation gradient
$\boldsymbol{H}$=gradient of displacement
J_1, J_2, J_3=invariants of stress
$\boldsymbol{K}, \boldsymbol{K}_0$=reference tensor in the undeformed state and the initial stress state
p, p_0=mean principal effective stress in the current state and the initial stress state
$\boldsymbol{Q}$=orthogonal tensor corresponding to the rigid rotation
$\boldsymbol{r}$=infinitesimal rotation
$\boldsymbol{R}$=rigid rotation of the reference state in finite theories
$\boldsymbol{s}$=deviatoric stress tensor
$\boldsymbol{S}$=second Piola-Kirchhoff stress tensor
$\boldsymbol{T}$=Cauchy stress tensor
$\boldsymbol{u}$=displacement vector
$\boldsymbol{U}$=right strech tensor
v_d=volumetric strain by dilatancy
$\boldsymbol{V}$=left strech tensor
$\boldsymbol{x}, \bar{\boldsymbol{x}}$=position vectors of a particle in the current states
$\boldsymbol{X}, \bar{\boldsymbol{X}}$=position vectors of a particle in the reference state $\boldsymbol{\kappa}$ and $\bar{\boldsymbol{\kappa}}$
$\boldsymbol{\eta}=\boldsymbol{s}/p$, pressure normalized deviatoric stress tensor
$\boldsymbol{\kappa}, \bar{\boldsymbol{\kappa}}$=reference states
$\boldsymbol{\sigma}$=infinitesimal stress tensor
$\boldsymbol{\sigma}_s$=stress history from the initial stress state up to the current state
$\boldsymbol{\sigma}_{s-}$=stress history up to the initial stress state
$\boldsymbol{\sigma}_0$=initial stress
$\boldsymbol{\sigma}_c$=dominant constant stress over K_0
$\dot{\boldsymbol{\sigma}}$=rate of stress or stress increment

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