A Numerical Simulation Study of Data-driven Pole Placement

DISSERTATION

A Numerical Simulation Study of Data-driven Pole Placement

Division of Electrical Engineering and Computer Science

Graduate School of Natural Science & Technology

Kanazawa University

Student Number: 1424042018

Pyone Ei Ei Shwe

Supervisor: Prof. Shigeru Yamamoto

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Abstract

This dissertation is concerned with numerical simulation studies on the state-feedback data-driven pole placement method. The data-driven pole placement method can precisely identify the state space model and pole placement gain simultaneously from a set of measurement data of the linear time-invariant system under certain conditions. In this study, solutions of several difficulties of the method for practical applications are investigated by numerical simulations.

First, the data-driven pole placement method is applied to a self-balancing robot which is a nonlinear system. By numerical simulations with nonlinear differential equation of the self-balancing robot, it is shown that the linearized model can be identified for the noisy case where the measurement noise exists together with noiseless cases. In particular, it is revealed that the suitable linearized model and pole placement gain can be identified by using the data sufficiently near the equilibrium.

Second, it is shown that the total least square and a prefilter are effective to the data-driven pole placement method when the measurement data is contaminated by noise. It is also shown that the random exciting signal is more suitable than the chirp exciting signal.

Finally, the data-driven pole placement method is extended to online tuning, real-time updating the closed loop system. Its capability is also investigated by numerical simulations of the self-balancing robot. It is shown that the method can update the state space model of the self-balancing robot and the pole placement gain for noisy measurement.

Contents

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Chapter 1

Introduction

1.1 Data-driven control

Data-based or data-driven control method involves designing a controller when the plant model is not available. Data-driven control doesn't base on the knowledge of a plant model and requires only the measured input-output data from the process to be controlled. It intends to determine the control input or the controller to obtain the desired closed loop performance in contrast to model-based control design which requires the knowledge of mathematical model of the system.

In the data-driven control framework, where no explicit mathematical plant model is used, a feedback controller must be derived that satisfies the prescribed closed-loop performance and fit to known experimental data. In contrast with traditional model-based controller designs, techniques such as controller identification [24] or a combination of a plant model and controller identification must be applied [25, 26].

Researches on data-driven approaches have extensively been proposed such as unfalsified control [6], virtual reference feedback tuning (VRFT) [19, 20], and fictitious reference iterative tuning (FRIT) [8, 21, 22, 23]. In VRFT and FRIT, the prescribed closed-loop performance should be given as a closed-loop reference transfer function. They are considered as transfer function approaches and they require the reference transfer function of the model. In [21], a FRIT method is shown which is able to identify a transfer function of the controlled system. The method is extended to a state feedback problem in [7]. Such FRIT methods can be applied to the data-driven pole placement problem by choosing a reference transfer function with the desired poles. However, it is not easy to specify the zeros of the reference transfer function from u to x because the zeros of the plant are unknown. In contrast, the data-driven pole placement method presented in [5] requires only a state space representation of the closed-loop system to specify the prescribed closed-loop performance, as shown in Section 1.2. This avoids the zero assignment issue that arises in the transfer function approach used in [5].

1.2 Data-driven pole placement

Pole placement, also called pole assignment or eigenvalue assignment, is a standard controller synthesis method in which the locations of the closed-loop poles can be determined by setting a controller gain. The eigenvalues of the system correspond to the pole locations and they affect the system response such as stability, convergence rate, disturbance rejection and noise immunity. For stability issue, the poles of the system should be inside the unit circle in the discrete time system or should be the left-half plane in the continuous time system. Pole placement method works on setting the desired pole location and then moving the poles of the system to these desired pole locations by using the feedback gain to specify the desired system response. For pole placement control design, all state variables are assumed to be measurable and available for feedback and, the system is assumed to be completely controllable. Various pole placement methods have broadly been developed.

In contrast to the standard pole placement approach that assumes the state-space model is known and given, a different pole placement approach that does not use such assumptions has recently been proposed. A salient feature of the approach is that from a pair of state and input measurement we can simultaneously obtain the state-space model and the pole placement gain. The basic principle of this approach is based on unfalsified control, which is also known as data-driven control.

Data-driven pole placement was proposed for the state feedback control of discrete-time linear systems. Various control methods for the pole placement problem have well-known for a long time. In state feedback pole placement problem, the state feedback gain must be determined for a given system such that the closed-loop poles coincide with the desired locations. This is also a well-known problem, and various pole placement methods have been extensively discussed in many works of literature [1, 2, 3, 17].

In standard pole placement methods, a state space model is assumed to be given by a system identification technique using data from past experiments. Whereas the traditional approach combines the identification of the state space model with the standard pole placement method, an alternative approach called "data-driven pole placement" has recently been proposed [5]. In this approach, the state space model and pole placement feedback gain are identified simultaneously from the set of state measurements and control input sequences. The method proposed in [5] is based on the data-driven control framework ([18] and references therein) such as unfalsified control [6], virtual reference feedback tuning (VRFT) [19, 20], or fictitious reference iterative tuning (FRIT) [8, 21, 22, 23].

Consider the discrete-time linear time-invariant system and a static state feedback

$$x(k+1) = Ax(k) + Bu(k), \tag{1.1}$$

$$u(k) = Fx(k) + v(k), \tag{1.2}$$

where $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times m}$, $x \in \mathbf{R}^n$ is the state vector, $u \in \mathbf{R}^m$ is the input vector, $v \in \mathbf{R}^m$ is the external input to the closed loop system, and $F \in \mathbf{R}^{m\times n}$ is the feedback gain.

The data-driven pole placement problem was formulated in [5] as follows.

Problem 1 *We assume that the order of the plant n is known, pair (A, B) is controllable but the exact value is unknown, and B is of full rank. Let* $\Lambda = \{p_1, \ldots, p_n\}$ *be a self-conjugate set of n complex numbers in the unit circle. Given the input and output measurement data sequence* $(x_0(k), u_0(k))$ *of (1.1), find a state feedback gain F from the observed data* $(x_0(k), u_0(k))$ *such that* $\{\lambda_i(A + BF)\} = \Lambda$.

In a conventional approach, this problem is solved in two steps: A and B are identified from $x_0(k), u_0(k)$, then F is derived using the standard pole placement algorithms. In contrast, the data-driven pole placement method solves the two steps simultaneously. To achieve this, the method uses the equivalency between the closed-loop system

$$x(k+1) = (A + BF)x(k) + Bv(k) \tag{1.3}$$

with the desired pole placement gain F and

$$x_{\mathrm{d}}(k+1) = A_{\mathrm{d}}x_{\mathrm{d}}(k) + B_{\mathrm{d}}v(k), \tag{1.4}$$

$$x_{\mathrm{d}}(k) = Tx(k), \tag{1.5}$$

where $(A_{\mathrm{d}}, B_{\mathrm{d}})$ with $\lambda_i(A_{\mathrm{d}}) = p_i$ is an appropriate controllable pair. This equivalency requires the nonsingular matrix T to exist. We remove v from (1.4) by using (1.2), to obtain

$$x_{\mathrm{d}}(k+1) = A_{\mathrm{d}}x_{\mathrm{d}}(k) + B_{\mathrm{d}}u(k) - B_{\mathrm{d}}Fx(k). \tag{1.6}$$

Then, using (1.5), we obtain

$$Tx(k+1) = A_{\mathrm{d}}Tx(k) + B_{\mathrm{d}}u(k) - B_{\mathrm{d}}Fx(k). \tag{1.7}$$

If $(x_0(k), u_0(k))$ $(k = i, \dots, i+N)$ satisfies (1.7),

$$TX_0P_1 = A_{\mathrm{d}}TX_0P_2 + B_{\mathrm{d}}U_0 - B_{\mathrm{d}}FX_0, \tag{1.8}$$

where

$$X_0 = \begin{bmatrix} x_0(i) & x_0(i+2) & \cdots & x_0(i+N) \end{bmatrix}, \tag{1.9}$$

$$U_0 = \begin{bmatrix} u_0(i) & u_0(i+1) & \cdots & u_0(i+N-1) \end{bmatrix}, \tag{1.10}$$

$$P_1 = \begin{bmatrix} 0_{1\times N} \\ I_N \end{bmatrix}, \quad P_2 = \begin{bmatrix} I_N \\ 0_{1\times N} \end{bmatrix}. \tag{1.11}$$

In [5], (1.8) is cast into

$$S_1 \begin{bmatrix} T \\ F \end{bmatrix} X_0P_1 + S_2 \begin{bmatrix} T \\ F \end{bmatrix} X_0P_2 = B_{\mathrm{d}}U_0 \tag{1.12}$$

$$S_1 = \begin{bmatrix} I_n & 0_{n\times m} \end{bmatrix}, \; S_2 = \begin{bmatrix} -A_{\mathrm{d}} & B_{\mathrm{d}} \end{bmatrix} \tag{1.13}$$

and

$$F = \begin{bmatrix} f_1 \\ \vdots \\ f_m \end{bmatrix} \in \mathbf{R}^{m\times n}, \quad T = \begin{bmatrix} t_1 \\ \vdots \\ t_n \end{bmatrix} \in \mathbf{R}^{n\times n}. \tag{1.14}$$

The system (1.4) can be interpreted as a reference model within VRFT (e.g., [19, 20]) and FRIT (e.g., [8, 21, 22, 23]). The idea of eliminating v in (1.6) is also based on FRIT. In [8, 22, 23], a similar state feedback control problem has been discussed within the FRIT framework. To apply these FRIT techniques to the data-driven pole placement problem, the desired transfer function must be specified from u to x, rather than x_{d}. When precise values for (A, B) are not available, it becomes impossible to specify the zeros of the desired transfer function.

To obtain the datasets (1.9) by applying state feedback (1.2) to the system (1.1), the initial feedback gain F should be based on (A, B). Hence, in Problem 1, the exact value of (A, B) is assumed to be unknown.

When applying the property of Kronecker product $\mathrm{vec}(MDN) = (N^\top \otimes M)\mathrm{vec}D$ (see for example Th.2.13 in [29]) to the transpose of (1.12) to solve (1.12) for F and T, a further linear equation is derived, as follows:

$$\mathcal{X}\eta = \mathcal{U}, \tag{1.15}$$

where

$$\eta = \begin{bmatrix} t_1 & \cdots & t_n & f_1 & \cdots & f_m \end{bmatrix}^\top \in \mathbf{R}^{(n+m)n} \tag{1.16}$$

$$\mathcal{X} = S_1 \otimes (X_0 P_1)^\top + S_2 \otimes (X_0 P_2)^\top \in \mathbf{R}^{nN\times(n+m)n}, \tag{1.17}$$

$$\mathcal{U} = \left(B_{\mathrm{d}} \otimes U_0^{\mathrm{T}}\right)(\mathrm{vec}\, I_m) \in \mathbf{R}^{nN}. \tag{1.18}$$

If T is nonsingular, the model coefficients can be obtained

$$A = T^{-1}A_{\mathrm{d}}T - T^{-1}B_{\mathrm{d}}F, \quad B = T^{-1}B_{\mathrm{d}}. \tag{1.19}$$

1.3 Motivations and objectives

It is known that data-driven pole placement method can be applied to linear time-invariant systems with measurable states. However, there were unclarified points such as the applicability to nonlinear systems, and/or noisy measurements. The data-driven pole placement method to handle noise remains an open issue, though in [5], the total least square (TLS) method [27] was claimed to be effective. To resolve this issue, we introduced a prefiltering technique that reduces the effect of measurement noise. More specifically, a finite impulse response (FIR) filter was used to prefilter the data, as this makes them easier to manipulate. In Section 3.3, we discuss the effect of applying this prefiltering technique, together with the least square (LS) and TLS methods, to a self-balancing robot model. We then report the results for the pole placement error and identification error when two different exciting signals were applied. Finally, we investigated the ability of the data-driven pole placement method.

1.4 Contributions of this dissertation

This dissertation is concerned with exploring several applications based on data-driven pole placement method. To see the effectiveness of the method and to implement this to practical applications, the contributions of this thesis are mainly discussed in Chapter 2 to 4. The results achieved in this dissertation can be summarized as follows.

- In [14], the data-driven pole placement method is applied to a 2D self-balancing robot which is a nonlinear system with a single input. It is evaluated that a more precise linearized model can be identified as the states approach to the equilibrium when a random exciting signal is used in the data-driven pole placement method. The ability of data-driven pole placement method is examined when there exist measurement noise. As measurement noise is an important issue in practical applications, it is considered and investigated for off-line tuning.

- In [15], results when applying LS and TLS data fitting methods and the random exciting signal are shown and TLS method gives better results than LS method. For dealing with measurement noise, a prefilter is designed. The results of applying TLS and prefilter with data-driven pole placement method show good performance. Then, the results while applying TLS and prefilter are investigated when a different exciting signal, chirp signal is applied other than random exciting signal. By comparing the results in numerical values and simulation results, applying the random exciting signal on data-driven pole placement method shows good response than applying the chirp exciting signal.

- Off-line tuning is extended to real-time updating (on-line tuning) in [16]. The convergence of the identification errors of plant model and feedback gain to small values can be seen while the data approaches equilibrium point in both noiseless and noisy cases. The data-driven pole placement method can stabilize the self-balancing robot in the real-time fashion.

1.5 Dissertation organization

This dissertation is organized as follows.

Chapter 2 explores the effect of data-driven pole placement problem when it is applied to a non-linear system. A self-balancing robot (inverted pendulum) is used which has non-linearity. By considering as the single input case to this while applying the random exciting signal, an effective region to linearized the model is explored and the plant model and feedback gain identifications are interrogated. Both noiseless and noisy conditions are considered to see the response of data-driven pole placement method. These results were discussed in [14].

Chapter 3 discusses for dealing with measurement noise more effectively. Two single-input cases to self-balancing robot are considered and a FIR prefilter is designed. Random exciting signal is applied with least square and total least square method for receiving the better suited data. Comparisons are made before applying and after applying the designed prefilter based on these settings. Then, another exciting signal, chirp signal, is applied to analyze the response of data-driven pole placement method. All the results are shown in [15].

Chapter 4 shows the real-time updating as an extension of the Chapter 2. It works as online tuning and it can update the plant model and pole placement feedback gain. The results under noiseless and noisy conditions are also explored in [16].

Chapter 5 summarizes the final conclusions for this dissertation.

The derivation of equation of motion for self-balancing robot is shown in A.

Chapter 2

An application to nonlinear system

2.1 Problem setup

To apply the data-driven pole placement method proposed in [5] for nonlinear systems, we consider a self-balancing robot which can be controlled as the inverted pendulum. We need to get the suitable measurement data sets for the proposed method other than the desired pole locations. So, we have to identify the linearized model. In this chapter, it is shown that we can theoretically derive an ideal linearized model of a self-balancing robot. The results show that a better-linearized model can be identified when the measurement data near the equilibrium point is available. That is, the accuracy of the linearized model can be improved when the measurement data exists in a small sphere centered the equilibrium point. That conclusion is derived under the noiseless conclusion. To support the results under a noiseless condition, we further investigate the ability of data-driven pole placement method in which measurement data are contaminated by noise. To this end, we discuss the identified model and how data-driven pole placement method works under noisy condition. As an extension, we consider two cases: 1) noiseless condition when there is no measurement noise and 2) noisy condition when there is measurement noise in and according to the uniform distribution.

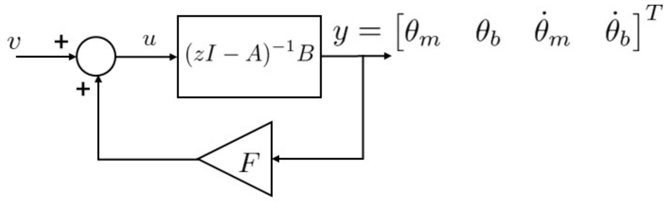

Figure 2.1: Self-balancing robot control system.

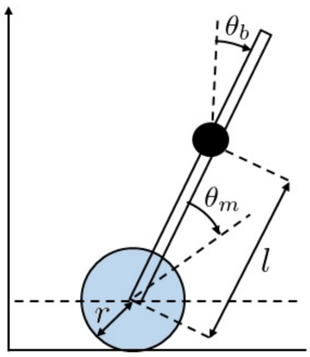

Figure 2.2: Coordinates of the 2D self-balancing robot.

2.2 Linearized model of self-balancing robot

We applied the data-driven pole placement method to control a 2D self-balancing robot [9, 28] (Fig. 2.1). We used $\theta_{\mathrm{m}}(t)$, $\theta_{\mathrm{b}}(t)$, $\dot{\theta}_{\mathrm{m}}(t)$, and $\dot{\theta}_{\mathrm{b}}(t)$ to construct the state vector of the self-balancing robot as

$$x = \begin{bmatrix} \theta_{\mathrm{m}} \\ \theta_{\mathrm{b}} \\ \dot{\theta}_{\mathrm{m}} \\ \dot{\theta}_{\mathrm{b}} \end{bmatrix}. \tag{2.1}$$

The goal of the control is to stabilize the self-balancing robot, that is $\lim_{t\to\infty} x(t) = 0$. State feedback was subsequently applied.

If the precise dynamics are known (A.233), an ideal model can be obtained by linearization as

$$\dot{x}(t) = A_c x(t) + B_c u(t), \tag{2.2}$$

where

$$A_c = \begin{bmatrix} 0_{2\times2} & I_{2\times2} \\ -J^{-1}K & -J^{-1}D \end{bmatrix},\ B_c = \begin{bmatrix} 0_{2\times1} \\ J^{-1}\zeta \end{bmatrix}, \tag{2.3}$$

$$J = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix},\ D = \begin{bmatrix} c & 0 \\ 0 & 0 \end{bmatrix}, K = M_{\mathrm{b}} l g \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}, \zeta = \begin{bmatrix} a \\ 0 \end{bmatrix}, \tag{2.4}$$

$$J_{11} = J'_{11}, \tag{2.5}$$

$$J_{12} = J'_{12} + M_{\mathrm{b}} l r, \tag{2.6}$$

$$J_{21} = J'_{21} + M_{\mathrm{b}} l r, \tag{2.7}$$

$$J_{22} = J'_{22} + 2M_{\mathrm{b}} l r. \tag{2.8}$$

Using parameters in [9],[28],

$$M_{\mathrm{w}} = 0.071 \text{ [kg]},\ M_{\mathrm{b}} = 0.5392 \text{ [kg]},\ J_{\mathrm{b}} = 2.16\times10^{-3} \text{ [kg}\cdot\text{m}^2\text{]},$$
$$J_{\mathrm{w}} = 8.63\times10^{-6} \text{ [kgm}^2\text{]},\ J_{\mathrm{m}} = 1.3\times10^{-7} \text{ [kg}\cdot\text{m}^2\text{]},$$
$$l = 0.1073 \text{ [m]},\ r = 0.0249 \text{ [m]},$$
$$c = 1\times10^{-4} \text{ [kg}\cdot\text{m}^2/\text{s]},\ g_{\mathrm{r}} = 30,\ a = 6.28\times10^{-7} \text{ [Nm/A]},$$

and discretizing A_c and B_c using a sampling period of 0.01s gives the ideal discrete-time model

$$A_0 = \begin{bmatrix} 1.0057 & 0 & 0.01 & 0 \\ -0.0207 & 1 & -0.0001 & 0.01 \\ 1.1388 & 0 & 1.0057 & 0.0007 \\ -4.1369 & 0 & -0.0207 & 0.9954 \end{bmatrix}, B_0 = \begin{bmatrix} -0.0023 \\ 0.0146 \\ -0.4577 \\ 2.9109 \end{bmatrix}. \tag{2.9}$$

For this ideal model, we computed an ideal feedback gain

$$F_0 = \begin{bmatrix} 32.2761 & 1.7605 & 4.3148 & 0.5209 \end{bmatrix}, \tag{2.10}$$

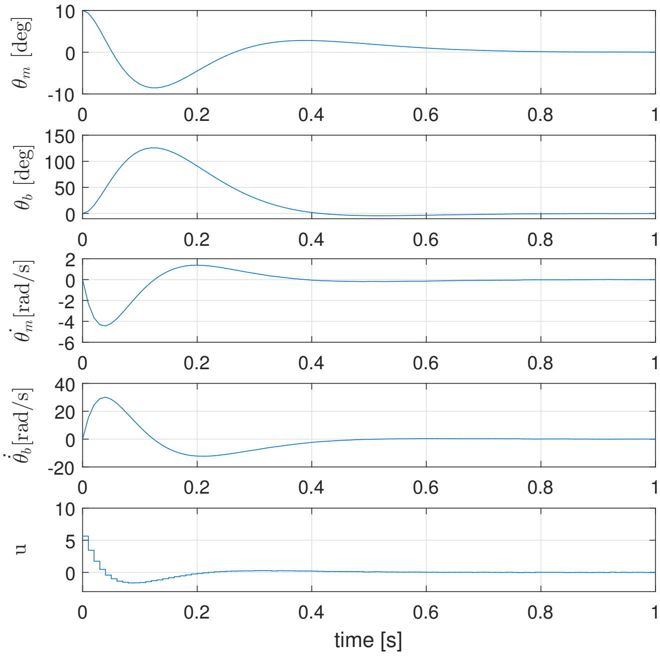

Figure 2.3: Simulation results with state feedback gain F_0.

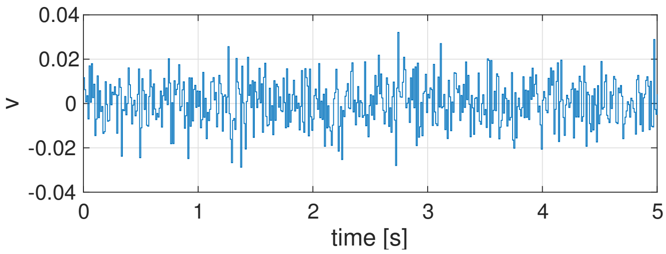

Figure 2.4: Exciting signal v used in Fig. 2.3.

to allow the standard pole placement method to be used such that

$$\lambda(A_0 + B_0 F_0) = 0.8, 0.9, 0.9 \pm 0.05j. \tag{2.11}$$

Fig. 2.3 shows the simulation results with the ideal state feedback gain F_0. In the simulation, we used a random sequence $v(k) \sim \mathcal{N}(0, 0.1^2)$, as shown in Fig. 2.4. Using this sampled data, we derived the pole placement gain F and the state-space model A and B to set i in (1.9)–(1.10). We used settings of $i = 10, 50, 90$, $N = 10$. These correspond to the time intervals $I_{10} = \{t \mid 0.1 \leq t \leq 0.2\}$, $I_{50} = \{t \mid 0.5 \leq t \leq 0.6\}$, and $I_{90} = \{t \mid 0.9 \leq t \leq 1.0\}$. As can be seen in Fig. 2.3, the first interval has a large $\sin\theta$ and the third interval has a small $\sin\theta$, making the latter the preferred interval for identifying the linearized model. We denote the obtained gain F and the identified A and B using the subscripts 10, 50, and 90. The obtained gains were as follows:

$$F_{10} = \begin{bmatrix} 50.8656 & 3.1287 & 7.5746 & 0.9511 \end{bmatrix}, \tag{2.12}$$

$$F_{50} = \begin{bmatrix} 32.2760 & 1.7722 & 4.3100 & 0.5209 \end{bmatrix}, \tag{2.13}$$

$$F_{90} = \begin{bmatrix} 32.2757 & 1.7605 & 4.3147 & 0.5209 \end{bmatrix}. \tag{2.14}$$

We further calculated the identification errors as

$$||A_{10} - A_0|| = 3028.6, \;\; ||B_{10} - B_0|| = 58.2747, \tag{2.15}$$

$$||A_{50} - A_0|| = 4.8651, \;\; ||B_{50} - B_0|| = 0.0168, \tag{2.16}$$

$$||A_{90} - A_0|| = 1.8343 \times 10^{-4}, \;\; ||B_{90} - B_0|| = 1.3032 \times 10^{-6}. \tag{2.17}$$

To evaluate the modeling errors, we defined

$$\Delta A := ||\tilde{A} - A||, \quad \Delta B := ||\tilde{B} - B||. \tag{2.18}$$

where $(\tilde{A}, \tilde{B})$ is the obtained model and (A, B) is the linearized model.

For pole location errors, we defined the accuracy measurement which has the maximum absolute difference between each eigenvalue of $(A + B\tilde{F})$ and the corresponding $p_j \in \Lambda_i$ as

$$\delta\lambda := max\{|\lambda_j(A + B\tilde{F}) - p_j|p_j \in \Lambda\}, \tag{2.19}$$

where

$$\Lambda = p_1, \cdots, p_n, \tag{2.20}$$

and the spectral radius ρ by taking the largest absolute value of eigenvalues of $(A + B\tilde{F})$ as

$$\rho(A + B\tilde{F}) = \underset{1 \leq j \leq n}{max} |\lambda_j(A + B\tilde{F})|. \tag{2.21}$$

and

2.3 Simulation results

For noiseless case, we firstly simulate the closed-loop response by using the ideal model; ideal feedback gain (2.10) and the random sequence of exciting signal for (1.2) that are assigned as uniform distribution $\mathrm{U}(-0.02, 0.02)$. The simulation result can be seen on Fig. 2.5.

As i of I_i becomes large, F_i converges to the ideal F_0 and (A_i, B_i) to the ideal (A_0, B_0). In Fig. 2.6, we can see the simulation results of identification errors of A and B, $\|F_i - F_0\|$ and $\max\{\|x(k)\| \mid k \in I_i\}$ with increasing values of i. According to these figures, the identified errors converge to small values when i becomes large and it is starting around $i = 90$. This suggests that the accuracy of the identified model increases when the measurement data used in the proposed method becomes concentrated in the vicinity of the equilibrium point.

Fig. 2.7 shows the pole location errors in noiseless case when '+' indicates the desired poles obtained by ideal feedback gain and '$\circ$' indicates those obtained by the derived feedback gain by data-driven pole placement method. For noisy condition, we set the exciting signal $v(k)$ as uniform distribution $\mathrm{U}(-0.05, 0.05)$ and the measurement noises as Gaussian distribution $\mathrm{N}(0, \sigma^2), \sigma = 0.4 \times 10^{-7}$ for θ_m and θ_b. Fig. 2.8 shows the initial response by measurement noise. Fig. 2.9 shows that the modeling errors and the pole location errors converges to small values. We can see that the eigenvalues by derived feedback gain shows fluctuations along and it may be because of the noises.

In Fig. 2.10, the pole location errors in the noisy case are shown while '+' represents the desired poles obtained by ideal feedback gain and '$\circ$' represents those obtained by the derived feedback gain by data-driven pole placement method.

2.4 Summary

We applied the data-driven pole placement method to stabilization of a self-balancing robot. The method can solve the pole placement problem (finding a state feedback gain such that the closed-loop poles are assigned at prescribed location) directly from the measurement data. At the same time, it can identify a state space model of the controlled system. Using numerical simulations of a self-balancing robot representing a nonlinear system, we demonstrated the accuracy of the method. We conclude that (i) to obtain a more precise linearized model, the measured data far from the equilibrium state must be discarded; (ii) as the state approaches equilibrium using state feedback, a more precise linearized model is obtained. Then, we examined the ability of the method when the measurement data are contaminated by noise. The obtained results by numerical simulations show degradation of identification of the state space model by noise. In addition, they show that the ideal feedback gain cannot be obtained in the noisy environment. Although stabilization can be achieved, to reduce the influence of noise is a key of its application to adaptive control [12].

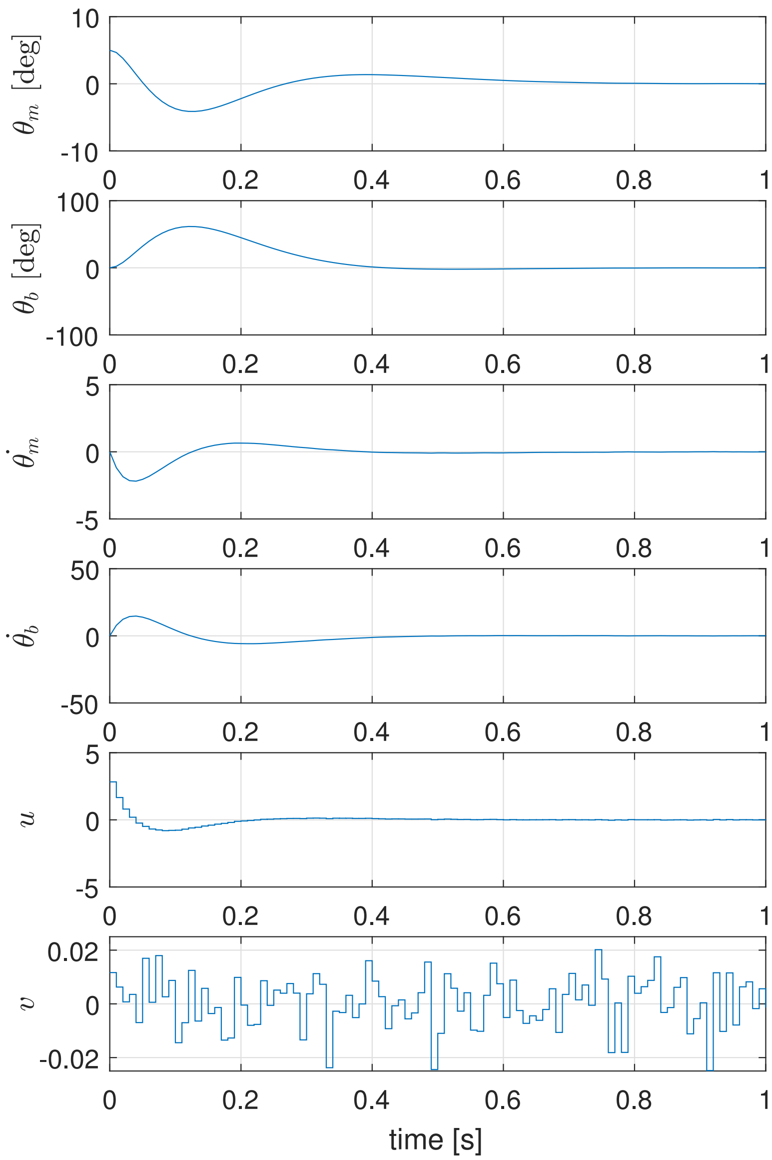

Figure 2.5: Response by ideal state feedback gain F_0 in noiseless case.

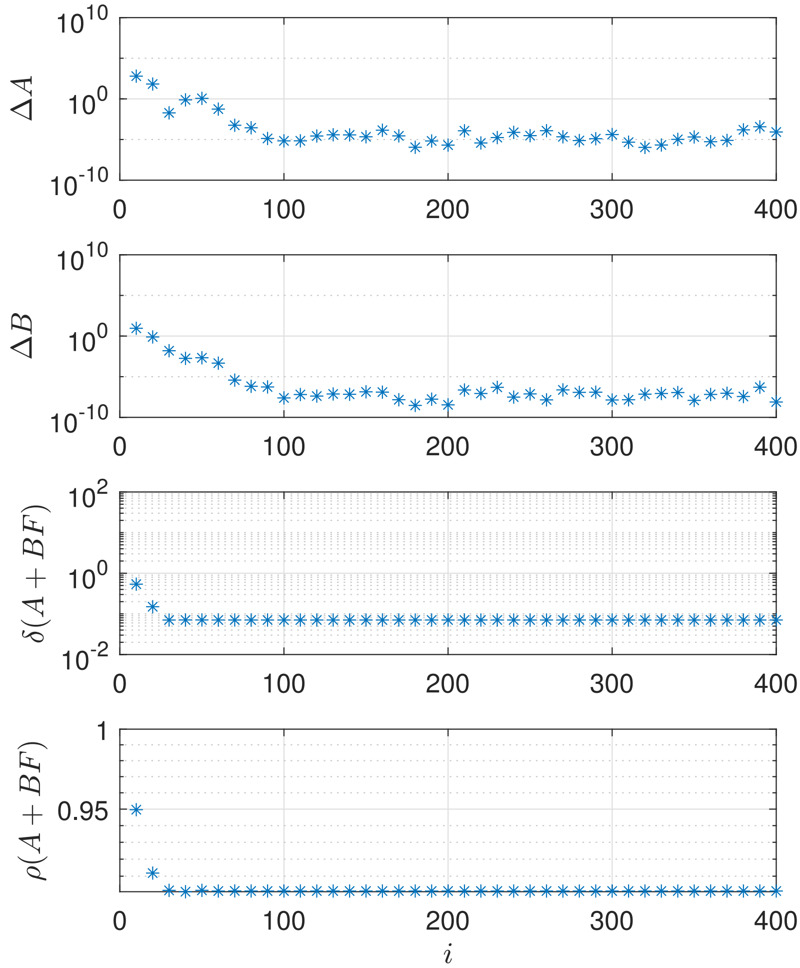

Figure 2.6: Identified results in noiseless case.

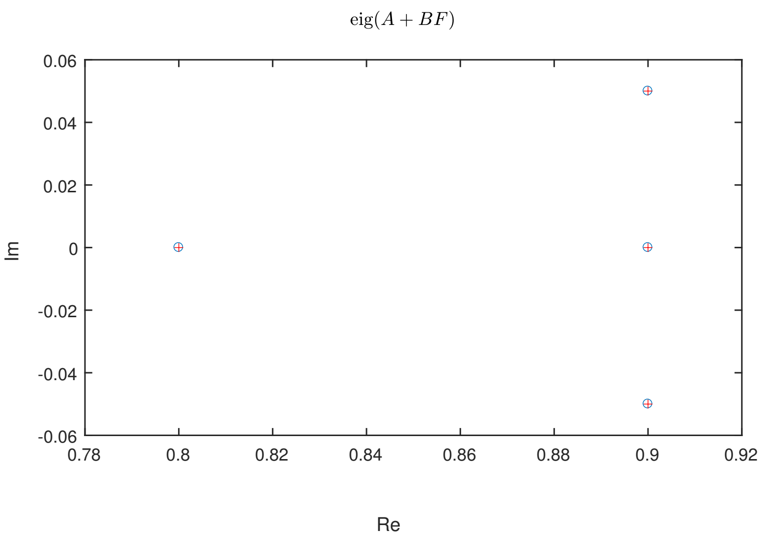

Figure 2.7: Pole locations in noiseless case ('+' indicates the desired poles obtained by ideal feedback gain, '○' those obtained by derived feedback gain).

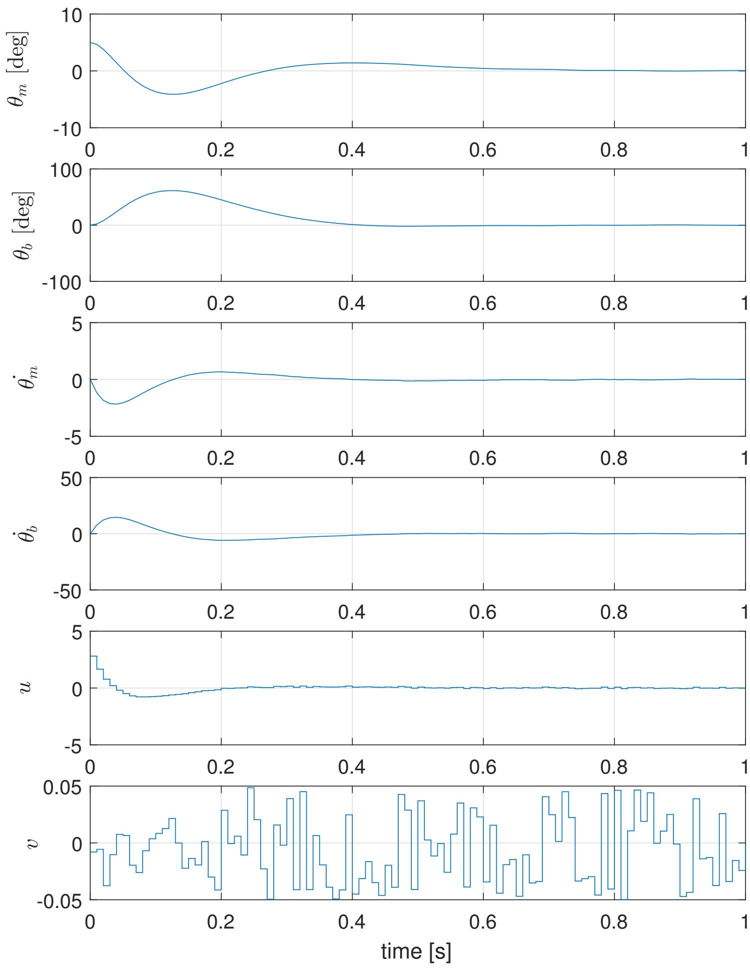

Figure 2.8: Response by ideal state feedback gain F_0 in noisy case.

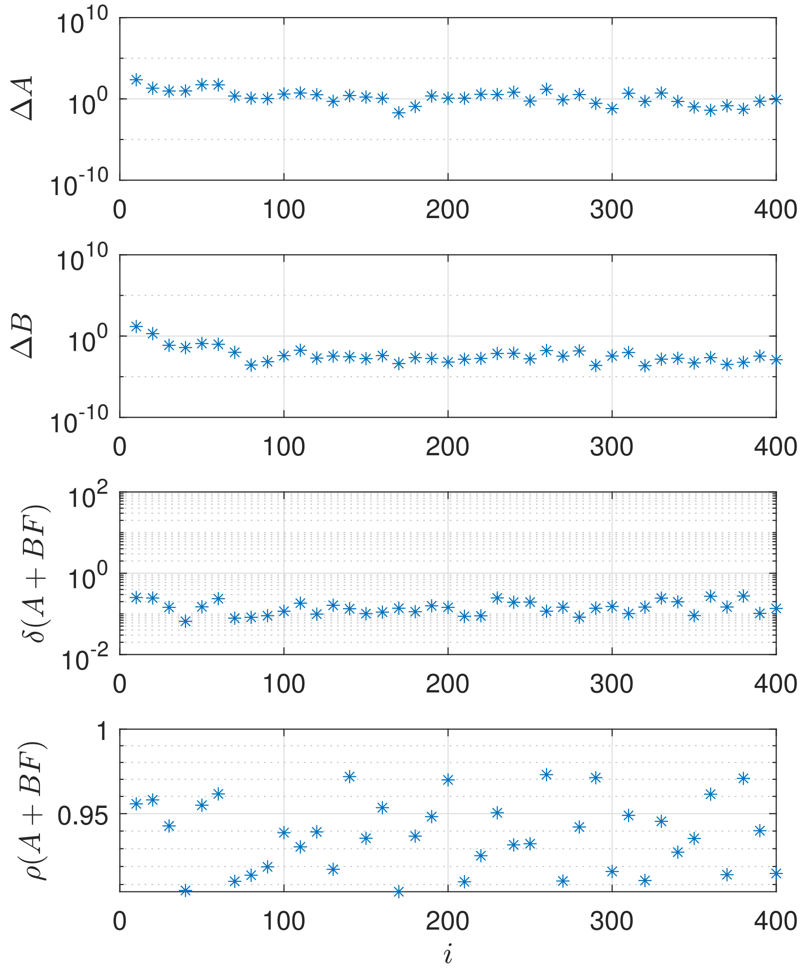

Figure 2.9: Identified results in noisy case.

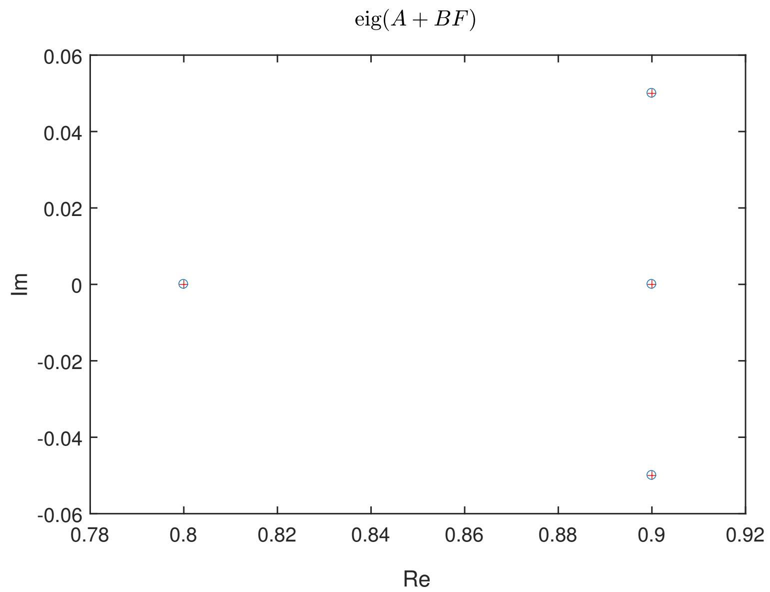

Figure 2.10: Pole locationsin noisy case ('+' indicates the desired poles obtained by ideal feedback gain, 'o' those obtained by derived feedback gain).

Chapter 3

An improvement for noisy measurement data

3.1 Problem setup

In previous chapters, we have shown the simulation results to see how noise takes effects on the performance of data-driven pole placement method. Although total least square (TLS) method was declared as an effective method in [5], we can see that dealing with noise in that method is still open. As every measurement of any physical quantity becomes uncertain because of it, we design FIR prefilter to deal with it effectively. Then, we apply the least square and total least square in order to get the best fit data together with the random exciting signal. We compare the results before and after applying the designed prefilter by numerical results and simulations. Then, to evaluate the response when we apply the different exciting signal, we also introduce the charp exciting signal and compare the results.

3.2 Prefiltering noisy measurement

When the measurement of x is contaminated by noise ε,

$$x_0(k) = x(k) + \varepsilon(k). \tag{3.1}$$

Then, (1.7) becomes

$$T(x_0(k+1) - \varepsilon(k+1)) = A_{\mathrm{d}}T(x_0(k) - T\varepsilon(k)) + B_{\mathrm{d}}u_0(k) \\ - B_{\mathrm{d}}F(x_0(k) - \varepsilon(k)). \tag{3.2}$$

Hence, if $(x_0(k), u_0(k))$ $(k = i, \ldots, i+N)$ satisfies the above equation,

$$T(X_0 - E)P_1 = A_{\mathrm{d}}T(X_0 - E)P_2 + B_{\mathrm{d}}U_0 - B_{\mathrm{d}}F(X_0 - E)P_2, \tag{3.3}$$

where

$$E = \begin{bmatrix} \varepsilon(i) & \varepsilon(i+1) & \cdots & \varepsilon(i+N) \end{bmatrix}. \tag{3.4}$$

Then, the resulting linear equation is given as

$$(\mathcal{X} + \Delta\mathcal{X})\eta = \mathcal{U} + \Delta\mathcal{U}, \tag{3.5}$$

where the effect of the noise is

$$\Delta\mathcal{X} = -S_1 \otimes (EP_1)^\top - S_2 \otimes (EP_2)^\top, \tag{3.6}$$

and $\Delta\mathcal{U}$ is the equation error. Following [5], we can solve $\eta \in \mathbf{R}^{(n+m)n}$ to (3.5) as a TLS problem [27], by minimizing the Frobenius norm $\left\| \begin{bmatrix} \Delta\mathcal{X} & \Delta\mathcal{U} \end{bmatrix} \right\|_F$. It is known that the TLS solution is given as

$$\eta = -\frac{1}{V_{22}} V_{12}, \tag{3.7}$$

based on the singular value decomposition

$$\begin{bmatrix} \mathcal{X} & \mathcal{U} \end{bmatrix} = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}^\top, \tag{3.8}$$

where these matrices are partitioned into blocks corresponding to $\mathcal{X}$ and $\mathcal{U}$.

Here, we assume that there exists $M > 0$ such that

$$\frac{1}{M} \sum_{j=1}^{M} \varepsilon(i+j) \approx 0, \tag{3.9}$$

for all i. This means that, when $N > M$,

$$EP_1\Phi \approx 0, \quad EP_2\Phi \approx 0, \tag{3.10}$$

for the matrix

$$\Phi = \frac{1}{M}\begin{bmatrix} 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 1 & & \ddots & 0 \\ 0 & \ddots & & 1 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \end{bmatrix} \in \mathbf{R}^{N\times(N-M+1)}, \tag{3.11}$$

where each column has M elements of 1. Therefore,

$$T\tilde{X}_0P_1 = A_\mathrm{d}T\tilde{X}_0P_2 + B_\mathrm{d}\tilde{U}_0 - B_\mathrm{d}F\tilde{X}_0P_2 \tag{3.12}$$

where

$$\tilde{X}_0 = X_0\Phi,\ \tilde{U}_0 = U_0\Phi. \tag{3.13}$$

This multiplication by Φ represents the prefiltering of signals via a Mth order FIR filter.

When the systems (1.1) and (1.4) are driven by the exciting signal, we have

$$(X_0 - E)P_1 = A(X_0 - E)P_2 + BU_0, \tag{3.14}$$

$$U_0 = F(X_0 - E)P_2 + V, \tag{3.15}$$

$$X_\mathrm{d} = T(X_0 - E)P_2, \tag{3.16}$$

$$X_\mathrm{d}P_1 = A_\mathrm{d}X_\mathrm{d}P_2 + B_\mathrm{d}V, \tag{3.17}$$

where

$$X_\mathrm{d} = \begin{bmatrix} x_\mathrm{d}(i) & x_\mathrm{d}(i+1) & \cdots & x_\mathrm{d}(i+N) \end{bmatrix}, \tag{3.18}$$

$$V = \begin{bmatrix} v(i) & x(i+1) & \cdots & v(i+N-1) \end{bmatrix}. \tag{3.19}$$

By applying Φ to these systems, we obtain

$$X_0P_1\Phi = AX_0P_2\Phi + BU_0\Phi, \tag{3.20}$$

$$U_0\Phi = FX_0P_2\Phi + V\Phi, \tag{3.21}$$

$$X_dP_2\Phi = TX_0P_2\Phi, \tag{3.22}$$

$$X_\mathrm{d}P_1\Phi = A_\mathrm{d}X_\mathrm{d}P_2\Phi + B_\mathrm{d}V\Phi. \tag{3.23}$$

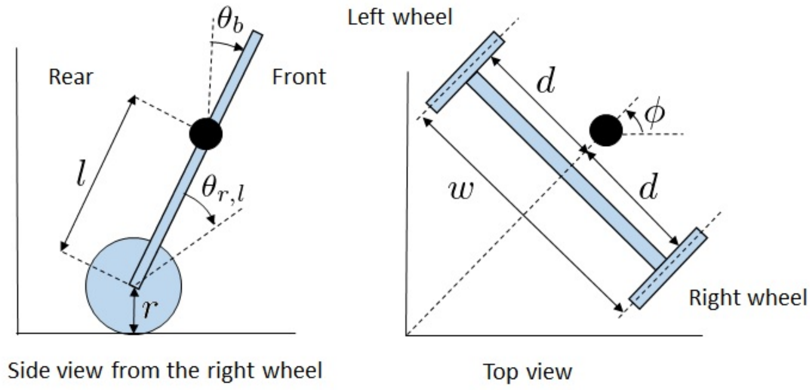

Figure 3.1: Coordinates of the self-balancing robot.

Here, if $V\Phi \approx 0$, (3.12) cannot be satisfied. Hence, for all i,

$$\frac{1}{M}\sum_{j=1}^{M} v(i+j) \neq 0 \tag{3.24}$$

must be satisfied.

3.3 Numerical example: self-balancing robot

We applied the data-driven pole placement method to the model of a 3D self-balancing robot [9, 28] shown in Fig. A.1.

3.3.1 Linear model and feedback gain

By defining the state vector

$$x_1(t) = \begin{bmatrix} x_a(t) \\ \dot{x}_a(t) \end{bmatrix} = \begin{bmatrix} \theta_{\mathrm{w}}(t) \\ \theta_{\mathrm{b}}(t) \\ \dot{\theta}_{\mathrm{w}}(t) \\ \dot{\theta}_{\mathrm{b}}(t) \end{bmatrix}, \quad x_2(t) = \begin{bmatrix} x_b(t) \\ \dot{x}_b(t) \end{bmatrix} = \begin{bmatrix} \phi(t) \\ \dot{\phi}(t) \end{bmatrix}, \tag{3.25}$$

the linear state space model can be derived as

$$\begin{cases} \dot{x}_1(t) = A_{c1}x_1(t) + B_{c1}u_1(t), \\ \dot{x}_2(t) = A_{c2}x_2(t) + B_{c2}u_2(t), \end{cases} \tag{3.26}$$

where

$$A_{c1} = \begin{bmatrix} 0_{2\times 2} & I_2 \\ -J_1^{-1}K_1 & -J_1^{-1}D_1 \end{bmatrix}, \quad B_{c1} = \begin{bmatrix} 0_{2\times 1} \\ b_{\mathrm{v}} J_1^{-1} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \end{bmatrix},$$

$$A_{c2} = \begin{bmatrix} 0 & 1 \\ 0 & -J_2^{-1}D_2 \end{bmatrix}, \quad B_{c2} = \begin{bmatrix} 0 \\ b_{\mathrm{v}} \frac{d}{r} J_2^{-1} \end{bmatrix}.$$

Then, the feedback can be independently designed as

$$u_1 = F_1 x_1 + v_1, \quad u_2 = F_2 x_2 + v_2. \tag{3.27}$$

Note that this can be more succinctly represented as

$$\dot{x}(t) = A_c x(t) + B_c u(t), \; u(t) = F x(t), \; x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \tag{3.28}$$

$$A_c = \begin{bmatrix} A_{c1} & 0_{4\times 2} \\ 0_{2\times 4} & A_{c2} \end{bmatrix}, \quad B_c = \begin{bmatrix} B_{c1} & 0_{4\times 1} \\ 0_{2\times 1} & B_{c2} \end{bmatrix}, F = \begin{bmatrix} F_1 & 0_{1\times 2} \\ 0_{1\times 4} & F_2 \end{bmatrix}. \tag{3.29}$$

When the parameters

$$\begin{aligned}
&g = 9.81 \; [\mathrm{m/s^2}], \; M_{\mathrm{b}} = 0.5 \; [\mathrm{kg}], \; M_{\mathrm{w}} = 0.07 \; [\mathrm{kg}], \; r = 0.025 \; [\mathrm{m}], \\
&J_{\mathrm{w}} = 8.75 \times 10^{-5} \; [\mathrm{kg \cdot m^2}], \; w = 2d = 0.12 \; [\mathrm{m}], \; l = 0.1073 \; [\mathrm{m}], \\
&J_{\mathrm{b}} = 6.7 \times 10^{-3} \; [\mathrm{kg \cdot m^2}], \; J_{\phi} = 6 \times 10^{-4} \; [\mathrm{kg \cdot m^2}], \\
&J_{\mathrm{m}} = 1.3 \times 10^{-4} \; [\mathrm{kg \cdot m^2}], \; R_{\mathrm{m}} = 0.035 \; [\Omega], \; K_{\mathrm{e}} = 0.02 \; [\mathrm{V \cdot s/rad}], \\
&K_{\mathrm{t}} = K_{\mathrm{e}} \; [\mathrm{N \cdot m/A}], \; g_{\mathrm{r}} = 30, \; d_{\mathrm{m}} = 0.0022, \; d_{\mathrm{w}} = 0,
\end{aligned}$$

are used and the sampling period is $h = 0.1$ s, the discrete-time model after discretizing (3.26) is

$$\begin{cases} x_1(k+1) = A_1 x_1(k) + B_1 u_1(k), \\ x_2(k+1) = A_2 x_2(k) + B_2 u_2(k), \end{cases} \tag{3.30}$$

where

$$A_1 = \begin{bmatrix} 1 & 0.1719 & 0.0226 & 0.0830 \\ 0 & 1.1722 & 0.0113 & 0.0944 \\ 0 & 3.5363 & 0.1388 & 1.0332 \\ 0 & 3.5299 & 0.1386 & 1.0336 \end{bmatrix}, \ B_1 = \begin{bmatrix} 0.0641 \\ -0.0094 \\ 0.7131 \\ -0.1148 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 1 & 0.0113 \\ 0 & 0.0001 \end{bmatrix}, \ B_2 = \begin{bmatrix} 0.0306 \\ 0.3450 \end{bmatrix}. \tag{3.31}$$

Here, we assume that the exact values of (3.31) are not available, but that uncertain values are available:

$$A_1 = \begin{bmatrix} 1 & 0.1897 & 0.0218 & 0.0844 \\ 0 & 1.1900 & 0.0115 & 0.0947 \\ 0 & 3.9115 & 0.1408 & 1.0489 \\ 0 & 3.9151 & 0.1407 & 1.0492 \end{bmatrix}, \ B_1 = \begin{bmatrix} 0.0648 \\ -0.0095 \\ 0.7115 \\ -0.1165 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 1 & 0.0103 \\ 0 & 0.0001 \end{bmatrix}, \ B_2 = \begin{bmatrix} 0.0310 \\ 0.3450 \end{bmatrix}. \tag{3.32}$$

The coefficients can be derived from J_1, J_2, with an assumed uncertainty of 10%. By applying linear quadratic optimal control theory to (3.32), the desired closed-loop pole locations can be chosen as

$$\lambda(A_1 + B_1F_1) \in \Lambda_1 = \{6.0355 \times 10^{-5}, \ 0.5253, \ 0.5745, \ 0.7630\}, \tag{3.33}$$

$$\lambda(A_2 + B_2F_2) \in \Lambda_2 = \{6.0426 \times 10^{-5}, \ 0.7835\}, \tag{3.34}$$

and the initial feedback gains needed to obtain datasets for the data-driven pole placement as

$$F_1 = \begin{bmatrix} 1.5216 & 124.181 & 2.3915 & 18.3089 \end{bmatrix}, \quad F_2 = \begin{bmatrix} -6.2764 & -0.0646 \end{bmatrix}. \tag{3.35}$$

3.3.2 Comparison of methods

Next, simulations were conducted and comparisons made of the results obtained when using different methods and exciting signals.

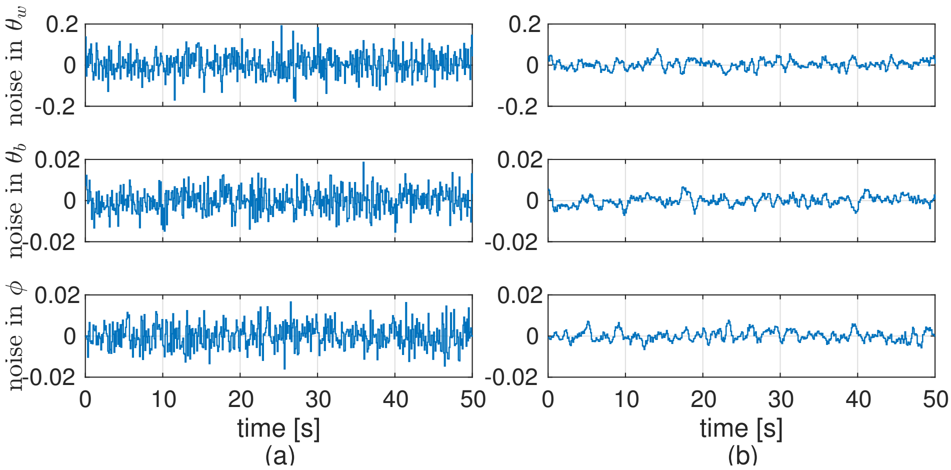

Figure 3.2: (a) Measurement noise (b) Prefiltered measurement noise.

Measurement noise was prepared with the Gaussian distribution $\mathcal{N}(0, \sigma^2)$, where $\sigma^2 = 1.0 \times 10^{-3}$, 1.0×10^{-4}, and 1.0×10^{-4} in θ_w, $\dot{\theta}_b$, and ϕ, respectively. This is shown in Fig. 3.2(a). We used the random exciting signal v shown in Fig. 3.3(a), with the uniform distribution $v_1(k) \sim \mathbf{U}(-0.5, 0.5)$ and $v_2(k) \sim \mathbf{U}(-0.1, 0.1)$, and the linear chirp signal $v(k)$ shown in Fig. 3.3(b). We set the order of the prefilter Φ (3.11) as $M = 6$. After prefiltering, the measurement noise in θ_w, $\dot{\theta}_b$, and ϕ was reduced, as shown in Fig. 3.2(b). Fig. 3.3(b) and Fig. 3.3(d) show the prefiltered exciting signals. It can be seen that the exciting signals v were not eliminated by the prefilter Φ, but that the high-frequency elements were reduced.

Fig. 3.4 shows a closed-loop response by state feedback (3.27), with initial gain (3.35), in the presence of measurement noise. Fig. 3.4(a) and Fig. 3.4(b) show the response to the random exciting signal and the chirp exciting signal, respectively. Of particular note is that the responses of θ_b, $\dot{\theta}_w$, and $\dot{\theta}_b$ seen in Fig. 3.4(b) showed the high-pass filter like gain characteristics of the transfer function from v to x.

For comparison, the dataset for the data-driven pole placement was chosen as $\{(x_0(k), u_0(k))\}_{k=50,\dots,450}$ where $i = 50$ and $N = 400$.

To evaluate the obtained pole placement gain $\tilde{F}$, we introduced an accuracy measurement that took the largest absolute difference in value between each eigenvalue of $A_i + B_i\tilde{F}_i$ and the corresponding $p_j \in \Lambda_i$

$$\delta\lambda(A_{di}) := \max\{|\lambda_j(A_i + B_i\tilde{F}_i) - p_j \mid p_j \in \Lambda_i\}. \tag{3.36}$$

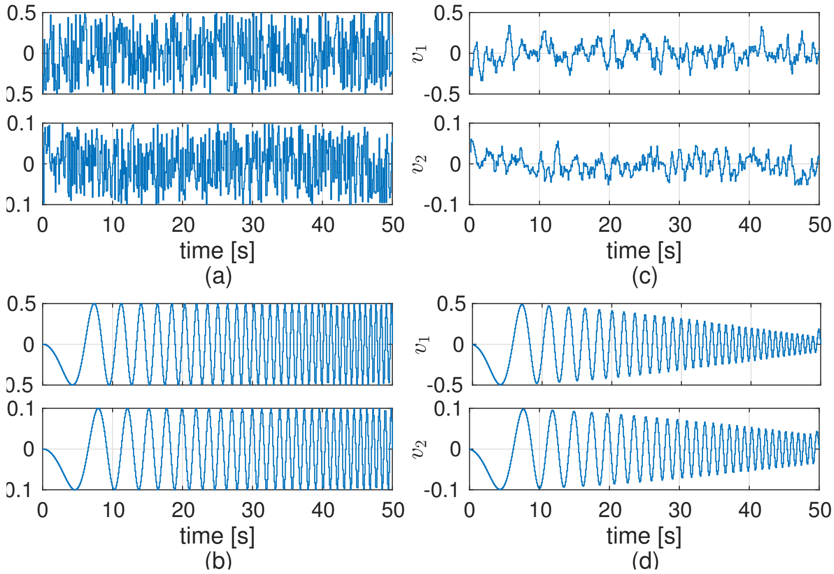

Figure 3.3: Exciting signal v (a) random (b) chirp (c) prefiltered random (d) prefiltered chirp.

To evaluate the obtained model $(\tilde{A}, \tilde{B})$, the following identification errors were used:

$$\Delta A_i := \|\tilde{A}_i - A_i\|, \quad \Delta B_i := \|\tilde{B}_i - B_i\|, \tag{3.37}$$

$$\delta\lambda(A_i) := \max\{|\lambda_j(A_i) - \lambda_j(\tilde{A}_i)|\}. \tag{3.38}$$

The eigenvalues λ_j were sorted by magnitude using the MATLAB command "sort". This further sorts the elements of equal magnitude by the phase angle on the interval $(-\pi, \pi]$. The impulse response $G(z) = (zI - \tilde{A})^{-1}\tilde{B}$ was used to evaluate the model obtained, as follows:

$$\Delta G_i := \sqrt{\sum_{k=0}^{10} \|x_{\tilde{A}_i,\tilde{B}_i}(k) - x_{A_i,B_i}(k)\|^2}, \tag{3.39}$$

where $x_{\tilde{A}_i,\tilde{B}_i}$ and x_{A_i,B_i} are the impulse responses of $G_i(z) := (zI - \tilde{A}_i)^{-1}\tilde{B}_i$ and $G(z) := (zI - A)^{-1}B$, respectively.

From the perspective of system control, smaller is better, particularly in the case of $\delta\lambda(A_{\mathrm{d}i})$, $\delta\lambda(A_i)$, and ΔG_i. The following key results were found:

1. **The initial model and feedback gain** were affected by uncertainty: The model errors and pole placement errors are shown in Table 3.1(initial).

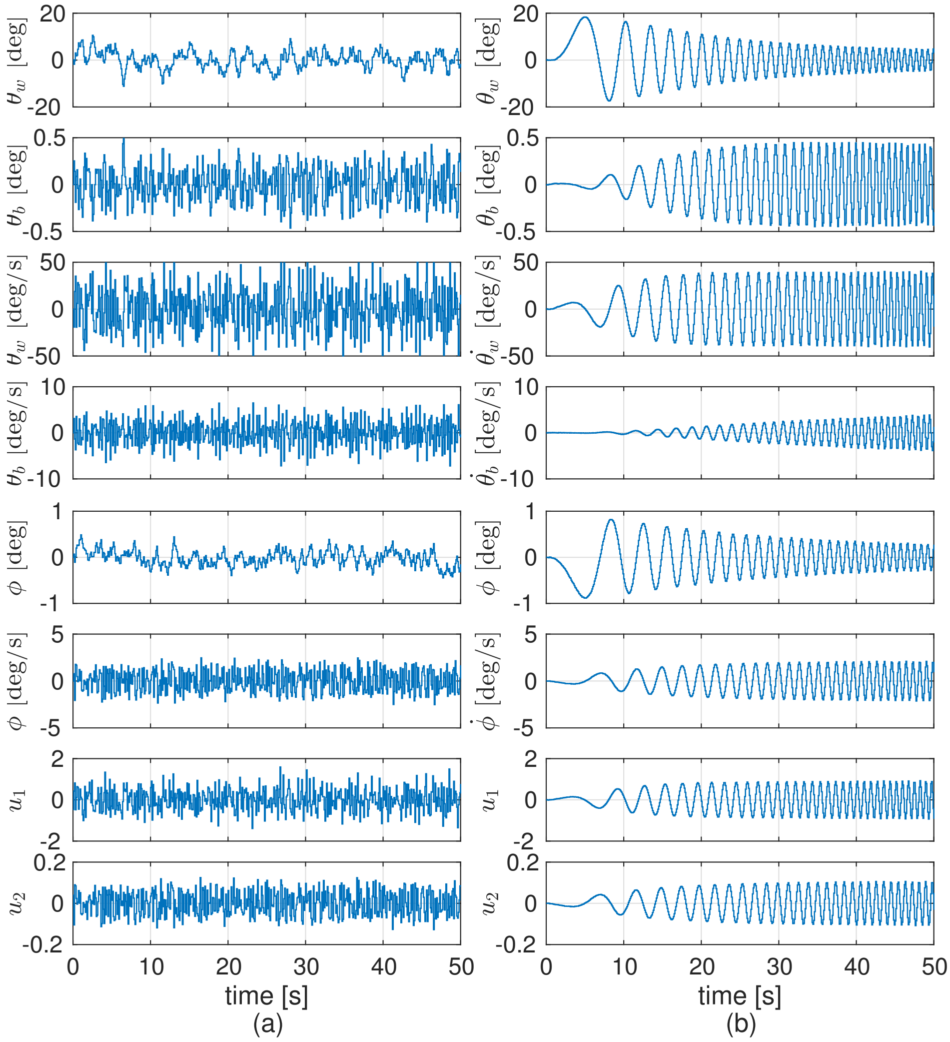

Figure 3.4: Closed-loop response by an initial state feedback via (a) random exciting signal v (b) chirp exciting signal v.

2. The results when using the LS method to solve linear equation (3.5) for **noiseless** data are shown in Table 3.1(a). All errors were reasonably small, confirming that the data-driven method performs well when the measurement data $(x_0(k), u_0(k))$ are noiseless.

3. The results when using the **LS method** to solve linear equation (3.5) for **noisy data** are shown in Table 3.1(b). All errors became larger when the noise was added, suggesting that LS analysis is inadequate when the measurement data are contaminated by noise.

4. The results when using the **TLS method** to solve linear equation (3.5) are

shown in Table 3.1(c). The errors were significantly smaller than those reported in [5], using the LS method.

5. The results when applying **prefiltering (PF)** and using the **TLS method** to solve linear equation (3.5) are shown in Table 3.1(d). The prefilter further reduced the errors, in particular, the pole placement error $\delta\lambda(A_{\mathrm{d}1})$ and the impulse response error ΔG_1.

6. The results when applying **PF** and using the **TLS method** to solve the linear equation (3.5), but with v as the **chirp signal**, are shown in Table 3.1(e). No significant improvement in error rates was found with respect to A_2 when using the chirp exciting signal. However, the errors with respect to A_1 became significantly worse than when a random exciting signal was used. This was assumed to be because A_1 has an unstable eigenvalue of 1.7838. We conclude that a random exciting signal is more appropriate than a chirp exciting signal when using data-driven methods.

We finally compared the pole locations obtained, as shown in Fig. 3.5. As can be seen, a better performance was achieved when using the random exciting signal.

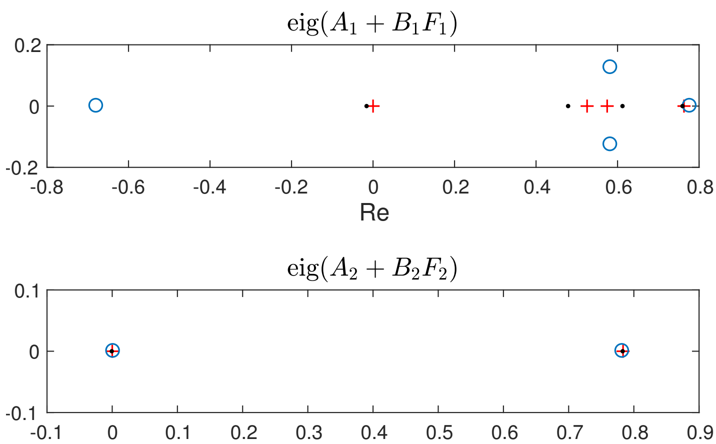

Figure 3.5: Comparison of pole locations ('+' indicates the desired poles, '·' those obtained by the random exciting signal and '∘' those obtained by the chirp exciting signal).

Table 3.1: Comparison of errors.

	(initial)	(a)	(b)	(c)	(d)	(e)
noise	-	noiseless	noisy	noisy	noisy	noisy
method	-	LS	LS	TLS	TLS+PF	TLS+PF
exciting sig.	-	Random	Random	Random	Random	Chirp
$\delta\lambda(A_{\mathrm{d1}})$	0.2426	0.0007	0.4597	0.1367	**0.0466**	1.2530
ΔA_1	0.5317	0.0016	36.295	1.6678	1.8763	17.246
ΔB_1	0.0025	0.0000	0.3400	0.0481	0.0415	0.2932
$\delta\lambda(A_1)$	0.0511	0.0000	0.3920	0.0194	**0.0177**	0.4695
ΔG_1	42.333	0.0082	629.67	44.718	**29.324**	106.04
$\delta\lambda(A_{\mathrm{d2}})$	0.0029	0.0000	0.0092	0.0024	**0.0007**	0.0017
ΔA_2	0.0001	0.0000	0.0288	0.0064	0.0005	0.0007
ΔB_2	0.0004	0.0000	0.0031	0.0002	0.0002	0.0002
$\delta\lambda(A_2)$	0.0001	0.0000	0.0090	0.0012	0.0004	0.0001
ΔG_2	0.0036	0.0002	0.0525	0.0073	0.0019	0.0019

3.4 Summary

In this study, we evaluated different approaches to reducing the effect of measurement noise in data-driven pole placement methods for deriving a state space model and pole placement state feedback. Using numerical simulations of a self-balancing robot, which is a nonlinear system, we demonstrated the important role that prefiltering can play in reducing the interference caused by noise. Again using numerical simulation, we compared the use of two exciting signals: a random signal and a chirp signal. The use of a random exciting signal was found to be more effective with our proposed method. Further developments are needed in the methods used to cope with noise. A method such as that used in [20] may be appropriate for use in practical applications where noise is present, and adaptive control based on real-time updating [16] is a future promising approach.

Chapter 4

Real-time data-driven pole placement

4.1 Problem setup

As we saw in Chapter 2, the simulation results based on off-line tuning on data-driven pole placement method are given. Based on that results, we investigate the ability of real-time update of the linearized state-space model to use data-driven pole placement in this chapter. To a way to adaptive control, we extend from off-line tuning to on-line tuning approach.

The measured data have to be sufficiently rich to confirm the non-singularity of vector T, and it is essential that for an index i,

$$\operatorname{rank}\begin{bmatrix} X_0 \\ U_0 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} x_0(i) & \cdots & x_0(i+n+m-1) \\ u_0(i) & \cdots & u_0(i+n+m-1) \end{bmatrix} = n+m. \tag{4.1}$$

Since the size of $\mathcal{X}$ is $nN \times (n+m)n$, $\mathcal{X}$ will be square when $N = n+m$. Therefore, we can calculate η uniquely when $\mathcal{X}$ is nonsingular. As an instance, for the single input case ($m = 1$),

$$F = \left(\begin{bmatrix} 0_{n\times n^2} & I_n \end{bmatrix} \mathcal{X}^{-1}\mathcal{U}\right)^{\top}. \tag{4.2}$$

When $(A_{\mathrm{d}}, B_{\mathrm{d}})$ is of the controllable canonical form and $N = n+m$,

$$|\mathcal{X}| \neq 0 \;\Leftrightarrow\; (4.1). \tag{4.3}$$

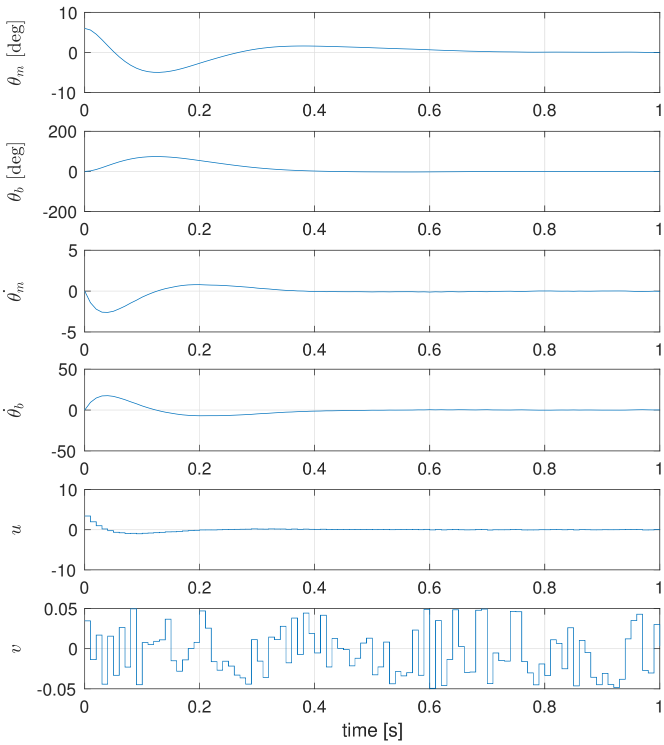

Figure 4.1: Responses of self-balancing robot by the ideal state feedback gain F_0 when v is applied and measurement noise is present.

4.2 Real-time update of the model and feedback gain

Data-driven pole placement is extended to real-time update of the model $(A(k), B(k))$ and the feedback gain $F(k)$ in each sampling time k according to the data-driven pole placement algorithm. Since the required rank condition (4.1) is not always ensured, $F(k)$ and $(A(k), B(k))$ are updated only when the rank condition (4.1) for $\mathcal{X}$ is satisfied. When $\mathcal{X}$ is singular, we set $F(k) := F(k-1)$ and $(A(k), B(k)) := (A(k-1), B(k-1))$. Algorithm 1 is the real-time update algorithm in MATLAB script.

4.3 Simulation results

We applied Algorithm 1 to the 2D self-balancing robot described in Appendix A. In all numerical simulations, we set $N = 5$ and used $v(k)$ according to the uniform distribution $\mathrm{U}(-0.05, 0.05)$ as the excitation signal. Moreover, we initially set $x(0) = \begin{bmatrix} 6\ [\mathrm{deg}] & 0 & 0 & 0 \end{bmatrix}^\top$, $F(0) = 0$, $A(0) = 0$, and $B(0) = 0$. Although we can design $F(0)$ if appropriate $A(0)$ and $B(0)$ are available, we assume here that there is no information on $A(0)$ and $B(0)$ to evaluate the ability of the proposed method to act as adaptive control.

For system identification, we applied the same equations (2.18), (2.19) and (2.21) as in Chapter 2.

We simulated two cases: 1) noise-free condition - measurement noise is absent, and 2) noisy condition - measurement noise is present in θ and ϕ according to the Gaussian distribution $\mathcal{N}(0, \sigma^2)$, $\sigma = 0.4 \times 10^{-6}$. In both cases, until $k = 4$, $\mathcal{X}$ is singular. Therefore, there is no feedback control because $F(k) = 0$, that is, $u(k) = v(k)$.

Fig. 4.1 shows the simulation result by using ideal F_0 in (2.10) when there is measurement noise. After applying real-time data-driven pole placement, the simulation results in the noise-free case are shown in Figs 4.2 and 4.3. As we see in Fig. 4.3, the identification errors ΔA, ΔB, and $\delta\lambda$ converge to a small value while the eigenvalues shows that average value is better than initial response. All the results in Fig. 4.3 works in real-time.

The simulation results of the noisy case are summarized in Figs. 4.4 and 4.5 and, we can make the same conclusion as in noiseless case. As we can see in Figs 4.2 and 4.4, the proposed real-time data-driven pole placement can stabilize the self-balancing robot even though it must wait for data to compute the data-driven pole placement method.

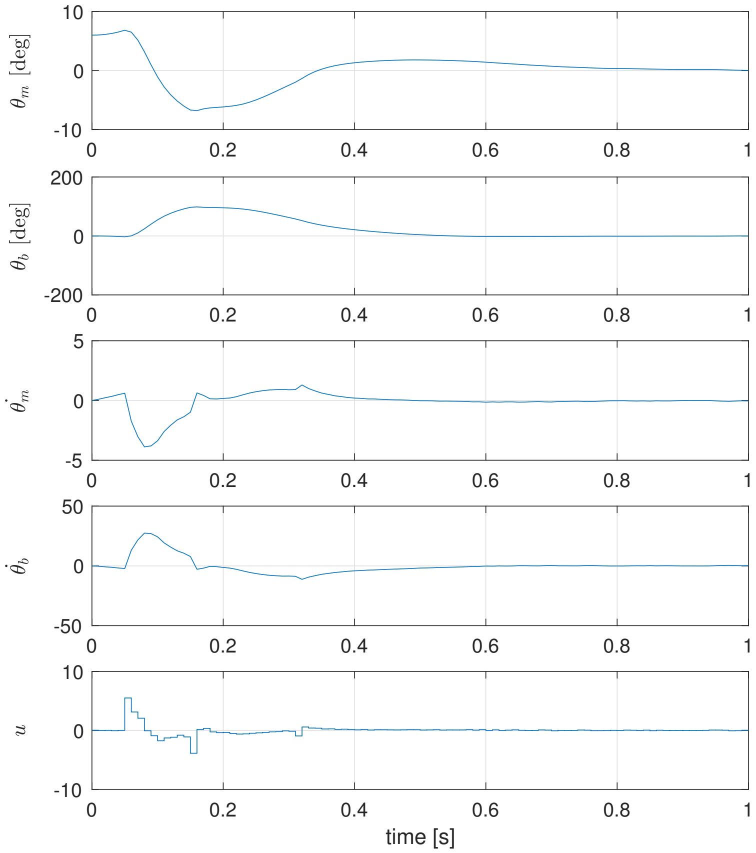

Figure 4.2: Responses of the real-time data-driven pole placement in the noise-free condition.

4.4 Summary

A data-driven pole placement method is extended to the real-time simultaneous update of a linearized state-space model and the pole placement gain. To demonstrate the effectiveness of the method, we applied it to a self-balancing robot, which is a nonlinear system, and we tested in noise-free and noisy conditions. Results of numerical simulations suggest that real-time data-driven pole placement is an adaptive control method that simultaneously gives us a linearized model and the pole placement gain.

Algorithm 1 Real-time data-driven pole placement.

```
Get:
  F(:,:,k-1), A(:,:,k-1), B(:,:,k-1),
  u(:,k-1) and x0(:,k);
X0p = x0(:,k-N+1:k);
X0 = x0(:,k-N:k-1);
U0 = u(:,k-N:k-1);
X = kron([eye(n) zeros(n,m)],X0p') + kron([-Ad Bd],X0');
U = kron(Bd,U0')*reshape(eye(m),m*m,1);
if rank(X) == n*N,
  eta = X\U;
  F = reshape(eta(n*n+1:end),n,m)';
  T = reshape(eta(1:n*n),n,n)';
  A = T\(Ad*T-Bd*F);
  B = T\Bd;
else
  F = F(:,:,k-1);
  A = A(:,:,k-1);
  B = B(:,:,k-1);
end
F(:,:,k) = F;
A(:,:,k) = A;
B(:,:,k) = B;
```

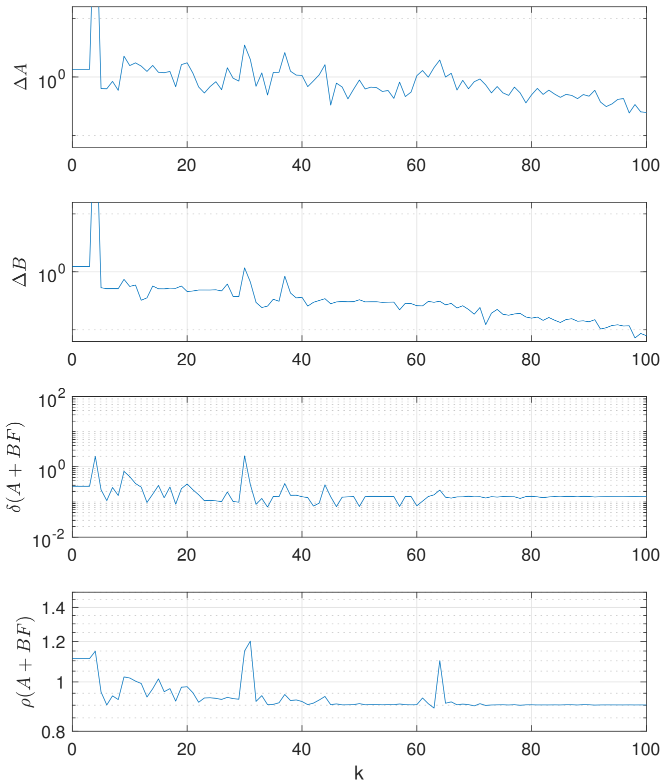

Figure 4.3: Modeling errors, difference between updated feedback gain and the ideal gain, and the norm of the state in the noise-free condition.

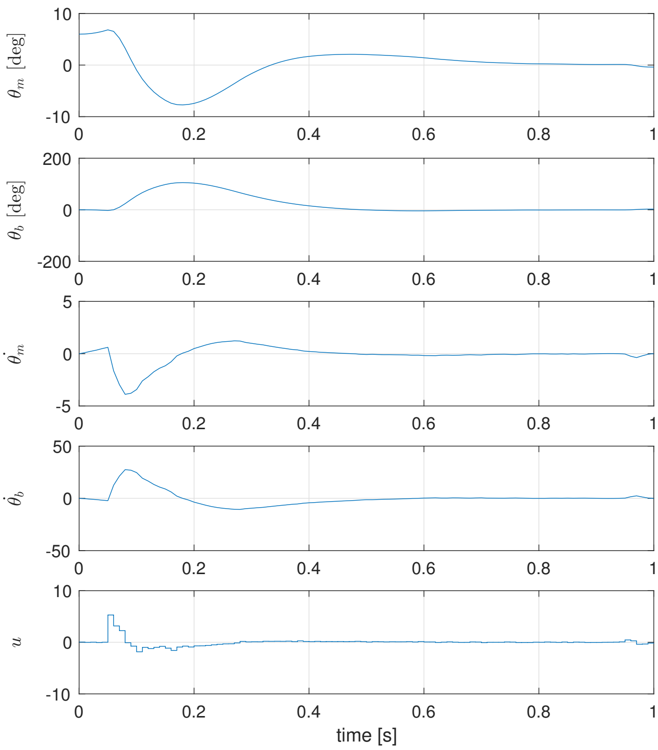

Figure 4.4: Responses by real-time data-driven pole placement in the noisy condition.

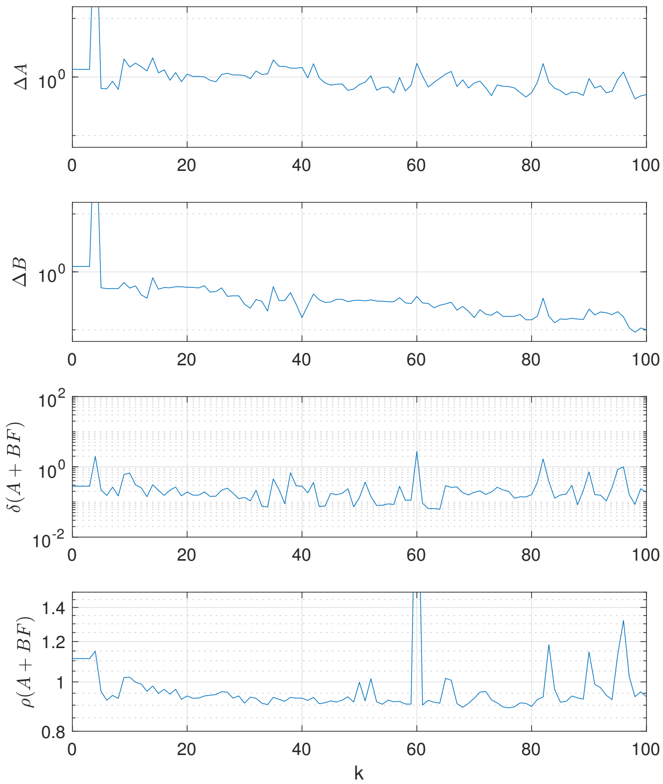

Figure 4.5: Modeling errors, difference between updated feedback gain and the ideal gain, and the norm of the state in the noisy condition.

Chapter 5

Conclusions

In this dissertation, we have studied the state feedback data-driven pole placement which can derive a linearized state-space model and pole placement gain. From Chapter 2 to 4, we have investigated many results based on it.

In Chapter 2, by applying the random exciting signal, we have examined the suitable region to identify the linearized model of self-balancing robot which is nonlinear system. We considered single input case to the robot. Then, we have identified the state space model and pole placement gain. Then, we have introduced the measurement noise to see the effects of data-driven pole placement method in noisy environment. We have compared the simulation results in noiseless and noisy conditions.

In Chapter 3, we have considered two single-input systems to the self-balancing robot. We have designed a FIR prefilter for forcibly dealing with measurement noise and we have studied the results when applying least square and total least square method. Then, we have examined the results when we introduced a different exciting signal, chirp signal to the system and we have compared the results by numerical values in the table. Finally, we have shown the comparison of random exciting signal and chirp exciting signal by numerical simulations.

In Chapter 4, we have extended the data-driven pole placement method to real time updating of a linearized state-space model and pole placement gain simultaneously. We have also considered both noiseless and noisy conditions for analyzing the simulation results. Then, we have studied that the method can stabilize the self-balancing robot.

Appendix A

Self-balancing robot and derivation of equation of motion

A.1 Self-balancing robot

The symbols are summarized in Table A.1.

The robot is equipped with right and left wheels driven by direct current (DC) motors whose voltages v_{r} and v_{l} can be controlled. We assume that the pitch angle θ_{b} and the pitch angular velocity $\dot{\theta}_{\mathrm{b}}$ of the body could be measured, as well as the angles θ_{wr} and θ_{wl} of the right and left wheels from the right and left motor angle $\psi_{\mathrm{r}}(t)$, $\psi_{\mathrm{l}}(t)$, and their angular velocities $\dot{\theta}_{\mathrm{wr}}$ and $\dot{\theta}_{\mathrm{wl}}$, respectively.

We assume that we use a clockwise rotation motor for the left wheel and a

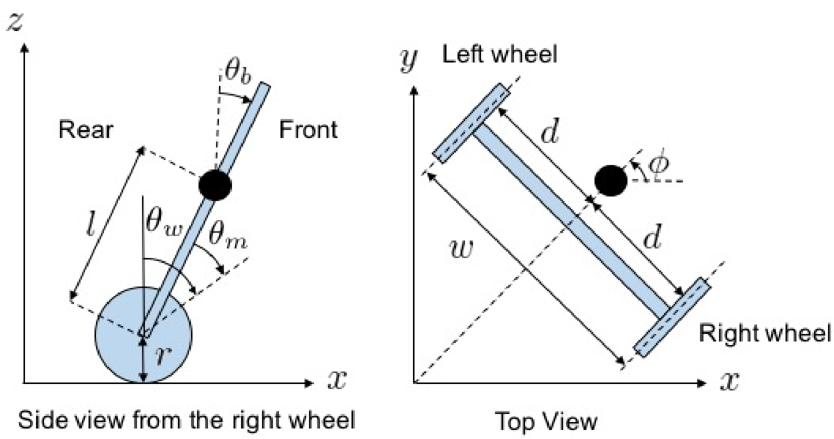

Figure A.1: Coordinates of the self-balancing robot.

Table A.1: Parameters of the self-balancing robot [9, 28].

g	acceleration due to gravity [m/s^2]
M_b	mass of body [kg]
M_w	mass of wheel [kg]
r	radius of wheel [m]
J_w	moment of inertia of wheel [kg·m^2]
w	vehicle width [m]
l	distance from wheel center to center of gravity of robot body [m]
J_b	moment of inertia of body (pitch) [kg·m^2]
J_ϕ	moment of inertia of body (yaw) [kg·m^2]
J_m	moment of inertia of DC motor [kg·m^2]
R_m	resistance of DC motor [Ω]
K_e	electromotive force constant of DC motor [V·s/rad]
$K_t = K_e$	torque constant of DC motor [N·m/A]
g_r	gear ratio
d_m	coefficient of friction between wheel and DC motor
d_w	coefficient of friction between wheel and floor
c	friction coefficient of axis [kg·m$^2/s$]
a	gain from current to torque of the tire [N·m/A]

counter-clockwise rotation motor of the left wheel. We define the positive rotation as a clockwise rotation for $\psi_r, \psi_l, \theta_{wr}, \theta_{wl}$, and θ_b in the side view from the right wheel. From $\psi_{r,l}(t)$ and the body pitch angle $\theta_b(t)$, we can calculate the wheel angle as

$$\theta_{wr} = \theta_{mr} + \theta_b, \tag{A.1}$$

$$\theta_{wl} = \theta_{ml} + \theta_b. \tag{A.2}$$

$$\theta_{mr} = \frac{1}{g_r}\psi_r, \tag{A.3}$$

$$\theta_{ml} = \frac{1}{g_r}\psi_l, \tag{A.4}$$

where g_r is the gear ratio.

We defined the mean values of the right and left wheel angles θ_{wr} and θ_{wl}, and the yaw angle of the body, as follows:

$$\theta_{\mathrm{w}} = \frac{1}{2}(\theta_{\mathrm{wr}} + \theta_{\mathrm{wl}}) = \frac{1}{2}(\theta_{\mathrm{mr}} + \theta_{\mathrm{ml}}) + \theta_{\mathrm{b}}, \tag{A.5}$$

$$\phi = \frac{r}{w}(\theta_{\mathrm{wr}} - \theta_{\mathrm{wl}}) = \frac{r}{w}(\theta_{\mathrm{mr}} - \theta_{\mathrm{ml}}), \tag{A.6}$$

where r is the radius of the wheel and $w = 2d$ is the distance between the two wheels.

Then,

$$\dot{\theta}_{\mathrm{w}} = \frac{1}{2}(\dot{\theta}_{\mathrm{wr}} + \dot{\theta}_{\mathrm{wl}}). \tag{A.7}$$

When the turning radius of the robot is $L(t)$, from the relation between the velocity and angular velocity,

$$r\dot{\theta}_{\mathrm{wr}}(t) = (L(t) + d)\dot{\phi}(t), \tag{A.8}$$

$$r\dot{\theta}_{\mathrm{wl}}(t) = (L(t) - d)\dot{\phi}(t). \tag{A.9}$$

By eliminating $L(t)$ from these,

$$\dot{\phi} = \frac{r}{2d}(\dot{\theta}_{\mathrm{wr}} - \dot{\theta}_{\mathrm{wl}}). \tag{A.10}$$

To estimate $\dot{\phi}(t)$, and $\theta_{\mathrm{b}}(t)$, we can use numerical integration and differentiation on $\phi(t)$, $\dot{\theta}_{\mathrm{w}}(t)$ and $\dot{\theta}_{\mathrm{b}}(t)$.

We define the coordinates of the middle point between the centers of the two wheels as

$$\begin{bmatrix} x_w \\ y_w \\ z_w \end{bmatrix} := \frac{1}{2}\left(\begin{bmatrix} x_r \\ y_r \\ z_r \end{bmatrix} + \begin{bmatrix} x_l \\ y_l \\ z_l \end{bmatrix}\right), \tag{A.11}$$

where the coordinates of the center of the right and left wheel are

$$\begin{bmatrix} x_r \\ y_r \\ z_r \end{bmatrix} = \begin{bmatrix} x_w + d\sin\phi \\ y_w - d\cos\phi \\ r \end{bmatrix}, \quad \begin{bmatrix} x_l \\ y_l \\ z_l \end{bmatrix} = \begin{bmatrix} x_w - d\sin\phi \\ y_w + d\cos\phi \\ r \end{bmatrix}. \tag{A.12}$$

Then, the coordinates of the center of gravity of the body is

$$\begin{bmatrix} x_b \\ y_b \\ z_b \end{bmatrix} = \begin{bmatrix} x_w + l\sin\theta_{\mathrm{b}}\cos\phi \\ y_w + l\sin\theta_{\mathrm{b}}\sin\phi \\ z_w + l\cos\theta_{\mathrm{b}} \end{bmatrix}. \tag{A.13}$$

From $\sqrt{\dot{x}_w^2 + \dot{y}_w^2} = r\dot{\theta}_\mathrm{w}$ and $\dot{z}_w = 0$,

$$\begin{bmatrix} \dot{x}_w \\ \dot{y}_w \\ \dot{z}_w \end{bmatrix} = \begin{bmatrix} r\dot{\theta}_\mathrm{w}\cos\phi \\ r\dot{\theta}_\mathrm{w}\sin\phi \\ 0 \end{bmatrix}, \tag{A.14}$$

$$\begin{bmatrix} \dot{x}_r \\ \dot{y}_r \\ \dot{z}_r \end{bmatrix} = \begin{bmatrix} \dot{x}_w + d\dot{\phi}\cos\phi \\ \dot{y}_w + d\dot{\phi}\sin\phi \\ 0 \end{bmatrix} = \begin{bmatrix} (r\dot{\theta}_\mathrm{w} + d\dot{\phi})\cos\phi \\ (r\dot{\theta}_\mathrm{w} + d\dot{\phi})\sin\phi \\ 0 \end{bmatrix}, \tag{A.15}$$

$$\begin{bmatrix} \dot{x}_l \\ \dot{y}_l \\ \dot{z}_l \end{bmatrix} = \begin{bmatrix} \dot{x}_w - d\dot{\phi}\cos\phi \\ \dot{y}_w - d\dot{\phi}\sin\phi \\ 0 \end{bmatrix} = \begin{bmatrix} (r\dot{\theta}_\mathrm{w} - d\dot{\phi})\cos\phi \\ (r\dot{\theta}_\mathrm{w} - d\dot{\phi})\sin\phi \\ 0 \end{bmatrix}, \tag{A.16}$$

$$\begin{aligned} \begin{bmatrix} \dot{x}_b \\ \dot{y}_b \\ \dot{z}_b \end{bmatrix} &= \begin{bmatrix} \dot{x}_w + l\dot{\theta}_\mathrm{b}\cos\theta_\mathrm{b}\cos\phi - l\dot{\phi}\sin\theta_\mathrm{b}\sin\phi \\ \dot{y}_w + l\dot{\theta}_\mathrm{b}\cos\theta_\mathrm{b}\sin\phi + l\dot{\phi}\sin\theta_\mathrm{b}\cos\phi \\ -l\dot{\theta}_\mathrm{b}\sin\theta_\mathrm{b} \end{bmatrix}, \\ &= \begin{bmatrix} (r\dot{\theta}_\mathrm{w} + l\dot{\theta}_\mathrm{b}\cos\theta_\mathrm{b})\cos\phi - l\dot{\phi}\sin\theta_\mathrm{b}\sin\phi \\ (r\dot{\theta}_\mathrm{w} + l\dot{\theta}_\mathrm{b}\cos\theta_\mathrm{b})\sin\phi + l\dot{\phi}\sin\theta_\mathrm{b}\cos\phi \\ -l\dot{\theta}_\mathrm{b}\sin\theta_\mathrm{b} \end{bmatrix}. \end{aligned} \tag{A.17}$$

A.2 Equation of motion in 3D (Part 1)

Kinetic energy of the right and left wheel are given by

$$\begin{aligned} T_r &= \frac{1}{2}M_\mathrm{w}\left(\dot{x}_r^2 + \dot{y}_r^2\right) + \frac{1}{2}J_\mathrm{w}\dot{\theta}_\mathrm{wr}^2 \\ T_l &= \frac{1}{2}M_\mathrm{w}\left(\dot{x}_l^2 + \dot{y}_l^2\right) + \frac{1}{2}J_\mathrm{w}\dot{\theta}_\mathrm{wl}^2. \end{aligned}$$

Kinetic energy of the right and left motors are given by

$$T_{mr} = \frac{1}{2}J_\mathrm{m}\dot{\psi}_\mathrm{r}^2 = \frac{1}{2}J_\mathrm{m}\left(g_\mathrm{r}\dot{\theta}_\mathrm{mr}\right)^2 \tag{A.18}$$

$$T_{ml} = \frac{1}{2}J_\mathrm{m}\dot{\psi}_\mathrm{l}^2 = \frac{1}{2}J_\mathrm{m}\left(g_\mathrm{r}\dot{\theta}_\mathrm{ml}\right)^2. \tag{A.19}$$

Here, we ignore mass of the motors. Kinetic energy of the body is given by

$$T_b = \frac{1}{2}M_\mathrm{b}(\dot{x}_b^2 + \dot{y}_b^2 + \dot{z}_b^2) + \frac{1}{2}J_\mathrm{b}\dot{\theta}_\mathrm{b}^2 + \frac{1}{2}J_\phi\dot{\phi}^2$$

On the other hand, potential energy is

$$V = M_{\mathrm{w}} g z_l + M_{\mathrm{w}} g z_r + M_{\mathrm{b}} g z_b = 2M_{\mathrm{w}} g r + M_{\mathrm{b}} g (r + l\cos\theta_{\mathrm{b}}). \tag{A.20}$$

Hence, the Lagrangian is given as

$$\begin{aligned} L &= T_r + T_l + T_{mr} + T_{ml} + T_b - V \\ &= \frac{1}{2} M_{\mathrm{w}} \left(\dot{x}_r^2 + \dot{y}_r^2 + \dot{x}_l^2 + \dot{y}_l^2\right) + \frac{1}{2} J_{\mathrm{w}} \left(\dot{\theta}_{\mathrm{wr}}^2 + \dot{\theta}_{\mathrm{wl}}^2\right) + \frac{1}{2} g_{\mathrm{r}}^2 J_{\mathrm{m}} \left(\dot{\theta}_{\mathrm{mr}}^2 + \dot{\theta}_{\mathrm{ml}}^2\right) \\ &\quad + \frac{1}{2} M_{\mathrm{b}} \left(\dot{x}_b^2 + \dot{y}_b^2 + \dot{z}_b^2\right) + \frac{1}{2} J_{\mathrm{b}} \dot{\theta}_{\mathrm{b}}^2 + \frac{1}{2} J_\phi \dot{\phi}^2 - 2M_{\mathrm{w}} g r - M_{\mathrm{b}} g \left(r + l\cos\theta_{\mathrm{b}}\right). \end{aligned}$$

Note that we can rewrite the terms as

$$\dot{x}_r^2 + \dot{y}_r^2 + \dot{x}_l^2 + \dot{y}_l^2 = \left(r\dot{\theta}_{\mathrm{w}} + d\dot{\phi}\right)^2 + \left(r\dot{\theta}_{\mathrm{w}} - d\dot{\phi}\right)^2 = 2\left(r^2\dot{\theta}_{\mathrm{w}}^2 + d^2\dot{\phi}^2\right), \tag{A.21}$$

$$\begin{aligned} \dot{x}_b^2 + \dot{y}_b^2 + \dot{z}_b^2 &= \left(r\dot{\theta}_{\mathrm{w}} + l\dot{\theta}_{\mathrm{b}}\cos\theta_{\mathrm{b}}\right)^2 + \left(l\dot{\phi}\sin\theta_{\mathrm{b}}\right)^2 + \left(l\dot{\theta}_{\mathrm{b}}\sin\theta_{\mathrm{b}}\right)^2 \\ &= r^2\dot{\theta}_{\mathrm{w}}^2 + 2lr\dot{\theta}_{\mathrm{w}}\dot{\theta}_{\mathrm{b}}\cos\theta_{\mathrm{b}} + l^2\dot{\theta}_{\mathrm{b}}^2 + l^2\dot{\phi}^2\sin^2\theta_{\mathrm{b}}. \end{aligned} \tag{A.22}$$

From (A.10) and (A.7),

$$\dot{\theta}_{\mathrm{wr}}^2 + \dot{\theta}_{\mathrm{wl}}^2 = \left(\dot{\theta}_{\mathrm{w}} - \frac{d}{r}\dot{\phi}\right)^2 + \left(\dot{\theta}_{\mathrm{w}} + \frac{d}{r}\dot{\phi}\right)^2 = 2\dot{\theta}_{\mathrm{w}}^2 + 2\frac{d^2}{r^2}\dot{\phi}^2. \tag{A.23}$$

$$\begin{aligned} \left(\dot{\theta}_{\mathrm{mr}}^2 + \dot{\theta}_{\mathrm{ml}}^2\right) &= \left(\dot{\theta}_{\mathrm{wr}} - \dot{\theta}_{\mathrm{b}}\right)^2 + \left(\dot{\theta}_{\mathrm{wl}} - \dot{\theta}_{\mathrm{b}}\right)^2 = \dot{\theta}_{\mathrm{wr}}^2 + \dot{\theta}_{\mathrm{wl}}^2 + 2\dot{\theta}_{\mathrm{b}}^2 - 2\dot{\theta}_{\mathrm{b}}\left(\dot{\theta}_{\mathrm{wr}} + \dot{\theta}_{\mathrm{wl}}\right) \\ &= 2\left(\dot{\theta}_{\mathrm{w}}^2 - 2\dot{\theta}_{\mathrm{w}}\dot{\theta}_{\mathrm{b}} + \dot{\theta}_{\mathrm{b}}^2 + \frac{d^2}{r^2}\dot{\phi}^2\right). \end{aligned} \tag{A.24}$$

Hence, for the generalized coordinate $(\theta_{\mathrm{w}}, \theta_{\mathrm{b}}, \phi)$, the Lagrangian is

$$\begin{aligned} & L(\dot{\theta}_{\mathrm{w}}, \dot{\theta}_{\mathrm{b}}, \dot{\phi}, \theta_{\mathrm{w}}, \theta_{\mathrm{b}}, \phi) \\ &= \left(\left(M_{\mathrm{w}} + \frac{M_{\mathrm{b}}}{2}\right)r^2 + J_{\mathrm{w}} + g_{\mathrm{r}}^2 J_{\mathrm{m}}\right)\dot{\theta}_{\mathrm{w}}^2 + \left(M_{\mathrm{b}} l r\cos\theta_{\mathrm{b}} - 2g_{\mathrm{r}}^2 J_{\mathrm{m}}\right)\dot{\theta}_{\mathrm{w}}\dot{\theta}_{\mathrm{b}} \\ &\quad + \left(\frac{M_{\mathrm{b}}}{2} l^2 + \frac{1}{2} J_{\mathrm{b}} + g_{\mathrm{r}}^2 J_{\mathrm{m}}\right)\dot{\theta}_{\mathrm{b}}^2 \\ &\quad + \left(M_{\mathrm{w}} d^2 + \frac{M_{\mathrm{b}}}{2} l^2\sin^2\theta_{\mathrm{b}} + \frac{1}{2} J_\phi + \frac{d^2}{r^2} J_{\mathrm{w}} + \frac{d^2}{r^2} g_{\mathrm{r}}^2 J_{\mathrm{m}}\right)\dot{\phi}^2 \\ &\quad - 2M_{\mathrm{w}} g r - M_{\mathrm{b}} g (r + l\cos\theta_{\mathrm{b}}) \\ &= \frac{1}{2} J_{11}\dot{\theta}_{\mathrm{w}}^2 + J_{12}(\theta_{\mathrm{b}})\dot{\theta}_{\mathrm{w}}\dot{\theta}_{\mathrm{b}} + \frac{1}{2} J_{22}\dot{\theta}_{\mathrm{b}}^2 + \frac{1}{2} J_2(\theta_{\mathrm{b}})\dot{\phi}^2 - 2M_{\mathrm{w}} g r - M_{\mathrm{b}} g (r + l\cos\theta_{\mathrm{b}}), \end{aligned}$$

where

$$J_{11} := (2M_\mathrm{w} + M_\mathrm{b})\, r^2 + 2J_\mathrm{w} + 2g_\mathrm{r}^2 J_\mathrm{m}, \tag{A.25}$$

$$J_{12}(\theta_\mathrm{b}) := M_\mathrm{b} l r \cos\theta_\mathrm{b} - 2g_\mathrm{r}^2 J_\mathrm{m} \tag{A.26}$$

$$J_{22} := M_\mathrm{b} l^2 + J_\mathrm{b} + 2g_\mathrm{r}^2 J_\mathrm{m}, \tag{A.27}$$

$$J_2(\theta_\mathrm{b}) := 2M_\mathrm{w} d^2 + M_\mathrm{b} l^2 \sin^2\theta_\mathrm{b} + J_\phi + 2\frac{d^2}{r^2}\left(J_\mathrm{w} + g_\mathrm{r}^2 J_\mathrm{m}\right). \tag{A.28}$$

A.2.1 Case 1: the generalized coordinate $(\theta_\mathrm{w}, \theta_\mathrm{b}, \phi)$

$$\begin{aligned}
\frac{\partial L}{\partial \theta_\mathrm{w}} &= 0,\\
\frac{\partial L}{\partial \theta_\mathrm{b}} &= -M_\mathrm{b} l r \dot\theta_\mathrm{w}\dot\theta_\mathrm{b}\sin\theta_\mathrm{b} + M_\mathrm{b} l \sin\theta_\mathrm{b}(g + l\dot\phi^2\cos\theta_\mathrm{b}),\\
\frac{\partial L}{\partial \phi} &= 0,\\
\frac{\partial L}{\partial \dot\theta_\mathrm{w}} &= J_{11}\dot\theta_\mathrm{w} + J_{12}(\theta_\mathrm{b})\dot\theta_\mathrm{b},\\
\frac{\partial L}{\partial \dot\theta_\mathrm{b}} &= J_{22}\dot\theta_\mathrm{b} + J_{12}(\theta_\mathrm{b})\dot\theta_\mathrm{w},\\
\frac{\partial L}{\partial \dot\phi} &= J_2(\theta_\mathrm{b})\dot\phi.
\end{aligned}$$

Furthermore,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot\theta_\mathrm{w}}\right) = J_{11}\ddot\theta_\mathrm{w} + J_{12}(\theta_\mathrm{b})\ddot\theta_\mathrm{b} - (M_\mathrm{b} l r \sin\theta_\mathrm{b})\dot\theta_\mathrm{b}^2, \tag{A.29}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot\theta_\mathrm{b}}\right) = J_{12}(\theta_\mathrm{b})\ddot\theta_\mathrm{w} + J_{22}\ddot\theta_\mathrm{b} - (M_\mathrm{b} l r \sin\theta_\mathrm{b})\dot\theta_\mathrm{w}\dot\theta_\mathrm{b}, \tag{A.30}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot\phi}\right) = J_2(\theta_\mathrm{b})\ddot\phi + (2M_\mathrm{b} l^2 \sin\theta_\mathrm{b}\cos\theta_\mathrm{b})\dot\theta_\mathrm{b}\dot\phi. \tag{A.31}$$

Hence, by the Lagrangian equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot\theta_\mathrm{w}}\right) - \frac{\partial L}{\partial \theta_\mathrm{w}} = J_{11}\ddot\theta_\mathrm{w} + J_{12}(\theta_\mathrm{b})\ddot\theta_\mathrm{b} - (M_\mathrm{b} l r \sin\theta_\mathrm{b})\dot\theta_\mathrm{b}^2 = F_1, \tag{A.32}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot\theta_\mathrm{b}}\right) - \frac{\partial L}{\partial \theta_\mathrm{b}} = J_{12}(\theta_\mathrm{b})\ddot\theta_\mathrm{w} + J_{22}\ddot\theta_\mathrm{b} - M_\mathrm{b} l \sin\theta_\mathrm{b}(g + l\dot\phi^2\cos\theta_\mathrm{b}) = F_2, \tag{A.33}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot\phi}\right) - \frac{\partial L}{\partial \phi} = J_2(\theta_\mathrm{b})\ddot\phi + (2M_\mathrm{b} l^2 \sin\theta_\mathrm{b}\cos\theta_\mathrm{b})\dot\theta_\mathrm{b}\dot\phi = F_3, \tag{A.34}$$

The external force are two torque τ_{r} and τ_{l} generated by two motors fixed on the body and affected as the torque

$$\tau_{\theta_{\mathrm{m}}} = \tau_{\mathrm{r}} + \tau_{\mathrm{l}} \tag{A.35}$$

$$\tau_{\theta_{\mathrm{b}}} = 0 \tag{A.36}$$

$$\tau_{\phi} = \frac{d}{r}(\tau_{\mathrm{r}} - \tau_{\mathrm{l}}). \tag{A.37}$$

Then, the generalized force for $(\theta_{\mathrm{w}}, \theta_{\mathrm{b}}, \phi)$ are

$$\begin{aligned}
F_1 &= \frac{\partial}{\partial \theta_{\mathrm{w}}}\left(\tau_{\theta_{\mathrm{m}}}\theta_{\mathrm{m}} + \tau_{\theta_{\mathrm{b}}}\theta_{\mathrm{b}} + \tau_{\phi}\phi\right) = \frac{\partial}{\partial \theta_{\mathrm{w}}}\left(\tau_{\theta_{\mathrm{m}}}(\theta_{\mathrm{w}} - \theta_{\mathrm{b}}) + \tau_{\theta_{\mathrm{b}}}\theta_{\mathrm{b}}\right) = \tau_{\theta_m} \\
F_2 &= \frac{\partial}{\partial \theta_{\mathrm{b}}}\left(\tau_{\theta_{\mathrm{m}}}\theta_{\mathrm{m}} + \tau_{\theta_{\mathrm{b}}}\theta_{\mathrm{b}} + \tau_{\phi}\phi\right) = \frac{\partial}{\partial \theta_{\mathrm{b}}}\left(\tau_{\theta_{\mathrm{m}}}(\theta_{\mathrm{w}} - \theta_{\mathrm{b}}) + \tau_{\theta_{\mathrm{b}}}\theta_{\mathrm{b}}\right) = -\tau_{\theta_{\mathrm{m}}} + \tau_{\theta_{\mathrm{b}}} = -\tau_{\theta_{\mathrm{m}}} \\
F_3 &= \frac{\partial}{\partial \phi}\left(\tau_{\theta_{\mathrm{m}}}\theta_{\mathrm{m}} + \tau_{\theta_{\mathrm{b}}}\theta_{\mathrm{b}} + \tau_{\phi}\phi\right) = \tau_{\phi}.
\end{aligned}$$

Hence, we can derive the equation of motion,

$$\begin{aligned}
&J_{11}\ddot{\theta}_{\mathrm{w}} + J_{12}(\theta_{\mathrm{b}})\ddot{\theta}_{\mathrm{b}} - (M_{\mathrm{b}} l r \sin\theta_{\mathrm{b}})\dot{\theta}_{\mathrm{b}}^2 = \tau_{\theta_m}, \\
&J_{12}(\theta_{\mathrm{b}})\ddot{\theta}_{\mathrm{w}} + J_{22}\ddot{\theta}_{\mathrm{b}} - M_{\mathrm{b}} l \sin\theta_{\mathrm{b}}(g + l\dot{\phi}^2\cos\theta_{\mathrm{b}}) = -\tau_{\theta_m}, \\
&J_2(\theta_{\mathrm{b}})\ddot{\phi} + (2M_{\mathrm{b}} l^2 \sin\theta_{\mathrm{b}}\cos\theta_{\mathrm{b}})\dot{\theta}_{\mathrm{b}}\dot{\phi} = \tau_{\phi}.
\end{aligned}$$

The electrical and torque characteristics of a DC motor can be represented by the equivalent circuit equation.

$$L_m \frac{\mathrm{d}}{\mathrm{d}t} i_{\mathrm{r}} + R_{\mathrm{m}} i_{\mathrm{r}} = v_{\mathrm{r}} - K_{\mathrm{e}}\dot{\psi}_{\mathrm{r}}, \tag{A.38}$$

$$L_m \frac{\mathrm{d}}{\mathrm{d}t} i_{\mathrm{l}} + R_{\mathrm{m}} i_{\mathrm{l}} = v_{\mathrm{l}} - K_{\mathrm{e}}\dot{\psi}_{\mathrm{l}}, \tag{A.39}$$

where v_{r} and v_{l} are the voltage applied to the motor's armature, i_{r} and i_{l} are the armature current, R_{m} is the electric resistance, L_m is the electric inductance, K_{e} is the back EMF constant. Here, we assume that the inductance L_m is negligibly small, can be approximated as

$$R_{\mathrm{m}} i_{\mathrm{r}} = v_{\mathrm{r}} - K_{\mathrm{e}}\dot{\psi}_{\mathrm{r}} \iff i_{\mathrm{r}} = \frac{v_{\mathrm{r}} - K_{\mathrm{e}}\dot{\psi}_{\mathrm{r}}}{R_{\mathrm{m}}}, \tag{A.40}$$

$$R_{\mathrm{m}} i_{\mathrm{l}} = v_{\mathrm{l}} - K_{\mathrm{e}}\dot{\psi}_{\mathrm{l}} \iff i_{\mathrm{l}} = \frac{v_{\mathrm{l}} - K_{\mathrm{e}}\dot{\psi}_{\mathrm{l}}}{R_{\mathrm{m}}}. \tag{A.41}$$

The motor torque is proportional to the armature current by a constant factor K_t.

$$\tau_\mathrm{mr} = K_\mathrm{t} i_\mathrm{r} = K_\mathrm{t} \frac{v_\mathrm{r} - K_\mathrm{e}\dot{\psi}_\mathrm{r}}{R_\mathrm{m}}, \tag{A.42}$$

$$\tau_\mathrm{ml} = K_\mathrm{t} i_\mathrm{l} = K_\mathrm{t} \frac{v_\mathrm{l} - K_\mathrm{e}\dot{\psi}_\mathrm{l}}{R_\mathrm{m}}. \tag{A.43}$$

In SI units, the motor torque and back EMF constants are equal, that is, $K_\mathrm{t} = K_\mathrm{e}$. Under the assumption that there are no electromagnetic losses, mechanical power is equal to the electrical power dissipated by the back EMF $e_r = K_\mathrm{e}\dot{\psi}_\mathrm{r}$ in the armature. That is,

$$\tau_\mathrm{mr}\dot{\psi}_\mathrm{r} = e_r i_\mathrm{r} \iff K_\mathrm{t} i_\mathrm{r}\dot{\psi}_\mathrm{r} = K_\mathrm{e}\dot{\psi}_\mathrm{r} i_\mathrm{r} \iff K_\mathrm{t} = K_\mathrm{e}. \tag{A.44}$$

Hence, the torque applied to the right wheel satisfies

$$\tau_\mathrm{r} = g_\mathrm{r}\tau_\mathrm{mr} - d_\mathrm{m}\dot{\psi}_\mathrm{r} - d_\mathrm{w}\dot{\theta}_\mathrm{wr} = g_\mathrm{r}\tau_\mathrm{mr} - g_\mathrm{r} d_\mathrm{m}\dot{\theta}_\mathrm{mr} - d_\mathrm{w}\dot{\theta}_\mathrm{wr} \tag{A.45}$$

where (A.3) is used. Hence,

$$\begin{aligned} \tau_\mathrm{r} &= g_\mathrm{r} K_\mathrm{t}\left(\frac{v_\mathrm{r} - K_\mathrm{e}\dot{\psi}_\mathrm{r}}{R_\mathrm{m}}\right) - g_\mathrm{r} d_\mathrm{m}\dot{\theta}_\mathrm{mr} - d_\mathrm{w}\dot{\theta}_\mathrm{wr} \\ &= \frac{g_\mathrm{r} K_\mathrm{t}}{R_\mathrm{m}} v_\mathrm{r} - \frac{g_\mathrm{r} K_\mathrm{t} K_\mathrm{e}}{R_\mathrm{m}}\dot{\psi}_\mathrm{r} - g_\mathrm{r} d_\mathrm{m}\dot{\theta}_\mathrm{mr} - d_\mathrm{w}\dot{\theta}_\mathrm{wr} \\ &= b_\mathrm{v} v_\mathrm{r} - d_\mathrm{b}\dot{\theta}_\mathrm{mr} - d_\mathrm{w}\dot{\theta}_\mathrm{wr}, \end{aligned} \tag{A.46}$$

where

$$b_\mathrm{v} := \frac{g_\mathrm{r} K_\mathrm{t}}{R_\mathrm{m}}, \tag{A.47}$$

$$d_\mathrm{b} := \frac{g_\mathrm{r}^2 K_\mathrm{t} K_\mathrm{e}}{R_\mathrm{m}} + g_\mathrm{r} d_\mathrm{m}. \tag{A.48}$$

In the same way,

$$\tau_\mathrm{l} = b_\mathrm{v} v_\mathrm{l} - d_\mathrm{b}\dot{\theta}_\mathrm{ml} - d_\mathrm{w}\dot{\theta}_\mathrm{wl}. \tag{A.49}$$

Hence,

$$\tau_{\theta_m} = \tau_r + \tau_l \tag{A.50}$$

$$= b_v(v_r + v_l) - d_b\dot{\theta}_{mr} - d_b\dot{\theta}_{ml} - d_w(\dot{\theta}_{wr} + \dot{\theta}_{wl}),$$

$$= b_v(v_r + v_l) - 2d_b\dot{\theta}_m - 2d_w\dot{\theta}_w, \tag{A.51}$$

$$\tau_\phi = \frac{d}{r}(\tau_r - \tau_l) \tag{A.52}$$

$$= \frac{d}{r}b_v(v_r - v_l) - \frac{d}{r}d_b(\dot{\theta}_{mr} - \dot{\theta}_{ml}) + \frac{d}{r}d_w(\dot{\theta}_{wr} - \dot{\theta}_{wl}) \tag{A.53}$$

$$= \frac{d}{r}b_v(v_r - v_l) - \frac{d}{r}d_b(\dot{\theta}_{wr} - \dot{\theta}_{wl}) + \frac{d}{r}d_w(\dot{\theta}_{wr} - \dot{\theta}_{wl}) \tag{A.54}$$

To decompose the motion by the change of variables, we define

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} v_r + v_l \\ v_r - v_l \end{bmatrix}. \tag{A.55}$$

Hence, (A.164) and (A.167) become

$$\tau_{\theta_m} = b_v(v_r + v_l) + 2d_b\dot{\theta}_b - 2\left(d_b + d_w\right)\dot{\theta}_w, \tag{A.56}$$

$$= 2b_v u_1 - 2(d_b + d_w)\dot{\theta}_w + 2d_b\dot{\theta}_b, \tag{A.57}$$

$$\tau_\phi = \frac{d}{r}b_v(v_r - v_l) - \frac{d}{r}(d_b + d_w)(\dot{\theta}_{wr} - \dot{\theta}_{wl}) \tag{A.58}$$

$$= \frac{2d}{r}b_v u_2 - 2\frac{d^2}{r^2}\left(d_b + d_w\right)\dot{\phi}. \tag{A.59}$$

From the above, the equation of motion is

$$J_{11}\ddot{\theta}_w + J_{12}(\theta_b)\ddot{\theta}_b - (M_b l r \sin\theta_b)\dot{\theta}_b^2 - 2d_b\dot{\theta}_b + 2\left(d_b + d_w\right)\dot{\theta}_w = 2b_v u_1, \tag{A.60}$$

$$J_{22}\ddot{\theta}_b + J_{12}(\theta_b)\ddot{\theta}_w - M_b l \sin\theta_b(g + l\dot{\phi}^2\cos\theta_b)$$

$$+ 2d_b\dot{\theta}_b - 2\left(d_b + d_w\right)\dot{\theta}_w = -2b_v u_1, \tag{A.61}$$

$$J_2(\theta_b)\ddot{\phi} + (2M_b l^2 \sin\theta_b\cos\theta_b)\dot{\theta}_b\dot{\phi} + 2\frac{d^2}{r^2}\left(d_b + d_w\right)\dot{\phi} = \frac{d}{r}2b_v u_2. \tag{A.62}$$

To summarize, the equation of motion can be derived as

$$\begin{cases} J_1(\theta_b(t))\begin{bmatrix} \ddot{\theta}_w(t) \\ \ddot{\theta}_b(t) \end{bmatrix} + D_1\begin{bmatrix} \dot{\theta}_w(t) \\ \dot{\theta}_b(t) \end{bmatrix} - M_b l \sin\theta_b(t)\begin{bmatrix} r\dot{\theta}_b^2(t) \\ g + l\cos\theta_b(t)\dot{\phi}^2(t) \end{bmatrix} = H_1 u(t), \\ J_2(\theta_b(t))\ddot{\phi}(t) + D_2\dot{\phi}(t) + \left(2M_b l^2 \sin\theta_b(t)\cos\theta_b(t)\right)\dot{\theta}_b(t)\dot{\phi}(t) = H_2 u(t), \end{cases} \tag{A.63}$$

where

$$J_1(\theta_b) = \begin{bmatrix} J_{11} & J_{12}(\theta_b) \\ J_{12}(\theta_b) & J_{22} \end{bmatrix},\ D_1 = 2\begin{bmatrix} d_b + d_w & -d_b \\ -d_b - d_w & d_b \end{bmatrix},\ H_1 = 2b_v\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \tag{A.64}$$

$$J_2(\theta_b) = J_2(\theta_b),\ D_2 = 2\frac{d^2}{r^2}(d_b + d_w),\ H_2 = 2b_v\frac{d}{r}\begin{bmatrix} 0 & 1 \end{bmatrix}. \tag{A.65}$$

We linearized the equations of motion (A.180) around equilibrium states $\theta_w = 0$, $\theta_b = 0$, $\phi = 0$, $\dot{\theta}_w = 0$, $\dot{\theta}_b = 0$, $\dot{\phi} = 0$, and $u = 0$. Then, under the assumption that $\sin\theta_b(t) \approx \theta_b(t)$, $\cos\theta_b(t) \approx 1$, $\sin^2\theta_b(t) \approx 0$, $\dot{\theta}_b^2(t) \approx 0$, $\dot{\phi}^2(t) \approx 0$, and $\sin\theta_b(t)\cos\theta_b(t)\dot{\theta}_b(t)\dot{\phi}(t) \approx 0$, the linearized equations of motion can be derived as

$$\begin{cases} J_1\ddot{x}_a(t) + D_1\dot{x}_a(t) + K_1x_a(t) = H_1u(t), \\ J_2\ddot{x}_b(t) + D_2\dot{x}_b(t) = H_2u(t), \end{cases} \tag{A.66}$$

where

$$x_a(t) := \begin{bmatrix} \theta_w(t) \\ \theta_b(t) \end{bmatrix},\ \ x_b(t) := \phi(t), \tag{A.67}$$

$$J_1 = J_1(0) = \begin{bmatrix} J_{11} & J_{12}(0) \\ J_{12}(0) & J_{22} \end{bmatrix} \tag{A.68}$$

$$J_2 = J_2(0) = J_2(0) = 2M_wd^2 + J_\phi + 2\frac{d^2}{r^2}\left(J_w + g_r^2J_m\right) \tag{A.69}$$

$$K_1 = M_blg\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \tag{A.70}$$

$$J_{12}(0) := M_blr - 2g_r^2J_m. \tag{A.71}$$

A.2.2 Case 2: the generalized coordinate $(\theta_\mathrm{m}, \theta_\mathrm{b}, \phi)$

For the generalized coordinate $(\theta_\mathrm{m}, \theta_\mathrm{b}, \phi)$, we obtain another Lagrangian

$$\begin{aligned}
&L(\dot\theta_\mathrm{m}, \dot\theta_\mathrm{b}, \dot\phi, \theta_\mathrm{m}, \theta_\mathrm{b}, \phi)\\
&= \frac{1}{2}J_{11}\left(\dot\theta_\mathrm{m} + \dot\theta_\mathrm{b}\right)^2 + J_{12}(\theta_\mathrm{b})\left(\dot\theta_\mathrm{m} + \dot\theta_\mathrm{b}\right)\dot\theta_\mathrm{b} + \frac{1}{2}J_{22}\dot\theta_\mathrm{b}^2 + \frac{1}{2}J_2(\theta_\mathrm{b})\dot\phi^2\\
&\quad - 2M_\mathrm{w}gr - M_\mathrm{b}g(r + l\cos\theta_\mathrm{b})\\
&= \frac{1}{2}J_{11}\dot\theta_\mathrm{m}^2 + J'_{12}(\theta_\mathrm{b})\dot\theta_\mathrm{m}\dot\theta_\mathrm{b} + \frac{1}{2}J'_{22}(\theta_\mathrm{b})\dot\theta_\mathrm{b}^2 + \frac{1}{2}J_2(\theta_\mathrm{b})\dot\phi^2\\
&\quad - 2M_\mathrm{w}gr - M_\mathrm{b}g(r + l\cos\theta_\mathrm{b})
\end{aligned}$$

where

$$J_{11} := (2M_\mathrm{w} + M_\mathrm{b})\,r^2 + 2J_\mathrm{w} + 2g_\mathrm{r}^2 J_\mathrm{m}, \tag{A.72}$$

$$J'_{12}(\theta_\mathrm{b}) := J_{11} + J_{12}(\theta_\mathrm{b}) = (2M_\mathrm{w} + M_\mathrm{b})\,r^2 + M_\mathrm{b}lr\cos\theta_\mathrm{b} + 2J_\mathrm{w}, \tag{A.73}$$

$$\begin{aligned}
J'_{22}(\theta_\mathrm{b}) &:= J_{11} + 2J_{12}(\theta_\mathrm{b}) + J_{22}(\theta_\mathrm{b})\\
&= (2M_\mathrm{w} + M_\mathrm{b})\,r^2 + M_\mathrm{b}l^2 + 2M_\mathrm{b}lr\cos\theta_\mathrm{b} + 2J_\mathrm{w} + J_\mathrm{b},
\end{aligned} \tag{A.74}$$

$$J_2(\theta_\mathrm{b}) := 2M_\mathrm{w}d^2 + M_\mathrm{b}l^2\sin^2\theta_\mathrm{b} + J_\phi + 2\frac{d^2}{r^2}\left(J_\mathrm{w} + g_\mathrm{r}^2 J_\mathrm{m}\right). \tag{A.75}$$

$$\begin{aligned}
\frac{\partial L}{\partial \theta_\mathrm{m}} &= 0,\\
\frac{\partial L}{\partial \theta_\mathrm{b}} &= -M_\mathrm{b}lr\dot\theta_\mathrm{w}\dot\theta_\mathrm{b}\sin\theta_\mathrm{b} + M_\mathrm{b}l\sin\theta_\mathrm{b}(g + l\dot\phi^2\cos\theta_\mathrm{b}),\\
\frac{\partial L}{\partial \phi} &= 0,\\
\frac{\partial L}{\partial \dot\theta_\mathrm{m}} &= J_{11}\dot\theta_\mathrm{m} + J'_{12}(\theta_\mathrm{b})\dot\theta_\mathrm{b},\\
\frac{\partial L}{\partial \dot\theta_\mathrm{b}} &= J'_{12}(\theta_\mathrm{b})\dot\theta_\mathrm{m} + J'_{22}\dot\theta_\mathrm{b},\\
\frac{\partial L}{\partial \dot\phi} &= J_2(\theta_\mathrm{b})\dot\phi.
\end{aligned}$$

Furthermore,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}_{\mathrm{m}}}\right) = J_{11}\ddot{\theta}_{\mathrm{m}} + J'_{12}(\theta_{\mathrm{b}})\ddot{\theta}_{\mathrm{b}} - (M_{\mathrm{b}} lr \sin\theta_{\mathrm{b}})\dot{\theta}_{\mathrm{b}}^2, \tag{A.76}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}_{\mathrm{b}}}\right) = J'_{12}(\theta_{\mathrm{b}})\ddot{\theta}_{\mathrm{m}} + J'_{22}(\theta_{\mathrm{b}})\ddot{\theta}_{\mathrm{b}} - (M_{\mathrm{b}} lr \sin\theta_{\mathrm{b}})\dot{\theta}_{\mathrm{w}}\dot{\theta}_{\mathrm{b}} - (2M_{\mathrm{b}} lr \sin\theta_{\mathrm{b}})\dot{\theta}_{\mathrm{b}}^2, \tag{A.77}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = J_2(\theta_{\mathrm{b}})\ddot{\phi} + (2M_{\mathrm{b}} l^2 \sin\theta_{\mathrm{b}}\cos\theta_{\mathrm{b}})\dot{\theta}_{\mathrm{b}}\dot{\phi}. \tag{A.78}$$

Hence, by the Lagrangian equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}_{\mathrm{m}}}\right) - \frac{\partial L}{\partial \theta_{\mathrm{m}}} = J_{11}\ddot{\theta}_{\mathrm{m}} + J'_{12}(\theta_{\mathrm{b}})\ddot{\theta}_{\mathrm{b}} - (M_{\mathrm{b}} lr \sin\theta_{\mathrm{b}})\dot{\theta}_{\mathrm{b}}^2 = F_1, \tag{A.79}$$

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}_{\mathrm{b}}}\right) - \frac{\partial L}{\partial \theta_{\mathrm{b}}} \\ &\quad = J'_{12}(\theta_{\mathrm{b}})\ddot{\theta}_{\mathrm{m}} + J'_{22}(\theta_{\mathrm{b}})\ddot{\theta}_{\mathrm{b}} - M_{\mathrm{b}} l \sin\theta_{\mathrm{b}}(g + l\dot{\phi}^2\cos\theta_{\mathrm{b}}) - (2M_{\mathrm{b}} lr \sin\theta_{\mathrm{b}})\dot{\theta}_{\mathrm{b}}^2 = F_2,\end{aligned} \tag{A.80}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = J_2(\theta_{\mathrm{b}})\ddot{\phi} + (2M_{\mathrm{b}} l^2 \sin\theta_{\mathrm{b}}\cos\theta_{\mathrm{b}})\dot{\theta}_{\mathrm{b}}\dot{\phi} = F_3, \tag{A.81}$$

The external force are two torque τ_{r} and τ_{l} generated by two motors fixed on the body and affected as the torque Then, the generalized force for $(\theta_{\mathrm{m}}, \theta_{\mathrm{b}}, \phi)$ are

$$F_1 = \tau_{\theta_m}, F_2 = 0, F_3 = \tau_{\phi}.$$

Hence, we can derive the equation of motion,

$$\begin{aligned}&J_{11}\ddot{\theta}_{\mathrm{m}} + J'_{12}(\theta_{\mathrm{b}})\ddot{\theta}_{\mathrm{b}} - (M_{\mathrm{b}} lr \sin\theta_{\mathrm{b}})\dot{\theta}_{\mathrm{b}}^2 = \tau_{\theta_m},\\ &J'_{12}(\theta_{\mathrm{b}})\ddot{\theta}_{\mathrm{m}} + J'_{22}(\theta_{\mathrm{b}})\ddot{\theta}_{\mathrm{b}} - M_{\mathrm{b}} l \sin\theta_{\mathrm{b}}(g + l\dot{\phi}^2\cos\theta_{\mathrm{b}}) - (2M_{\mathrm{b}} lr \sin\theta_{\mathrm{b}})\dot{\theta}_{\mathrm{b}}^2 = 0,\\ &J_2(\theta_{\mathrm{b}})\ddot{\phi} + (2M_{\mathrm{b}} l^2 \sin\theta_{\mathrm{b}}\cos\theta_{\mathrm{b}})\dot{\theta}_{\mathrm{b}}\dot{\phi} = \tau_{\phi}.\end{aligned}$$

Since

$$\begin{aligned}\tau_{\theta_{\mathrm{m}}} &= 2b_{\mathrm{v}}u_1 + 2d_{\mathrm{b}}\dot{\theta}_{\mathrm{b}} - 2\left(d_{\mathrm{b}} + d_{\mathrm{w}}\right)\dot{\theta}_{\mathrm{w}}\\ &= 2b_{\mathrm{v}}u_1 + 2d_{\mathrm{b}}\dot{\theta}_{\mathrm{b}} - 2\left(d_{\mathrm{b}} + d_{\mathrm{w}}\right)(\dot{\theta}_{\mathrm{m}} + \dot{\theta}_{\mathrm{b}}) = 2b_{\mathrm{v}}u_1 - 2d_{\mathrm{w}}\dot{\theta}_{\mathrm{b}} - 2\left(d_{\mathrm{b}} + d_{\mathrm{w}}\right)\dot{\theta}_{\mathrm{m}},\end{aligned} \tag{A.82}$$

$$\tau_{\phi} = \frac{2d}{r}b_{\mathrm{v}}u_2 - 2\frac{d^2}{r^2}\left(d_{\mathrm{b}} + d_{\mathrm{w}}\right)\dot{\phi}. \tag{A.83}$$

From the above, the equation of motion is

$$J_{11}\ddot{\theta}_{\mathrm{m}} + J'_{12}(\theta_{\mathrm{b}})\ddot{\theta}_{\mathrm{b}} + 2\left(d_{\mathrm{b}} + d_{\mathrm{w}}\right)\dot{\theta}_{\mathrm{m}} + 2d_{\mathrm{w}}\dot{\theta}_{\mathrm{b}} - (M_{\mathrm{b}}lr\sin\theta_{\mathrm{b}})\dot{\theta}_{\mathrm{b}}^{2} = 2b_{\mathrm{v}}u_{1}, \quad \text{(A.84)}$$

$$J'_{12}(\theta_{\mathrm{b}})\ddot{\theta}_{\mathrm{m}} + J'_{22}(\theta_{\mathrm{b}})\ddot{\theta}_{\mathrm{b}} - M_{\mathrm{b}}l\sin\theta_{\mathrm{b}}(g + l\dot{\phi}^{2}\cos\theta_{\mathrm{b}}) - (2M_{\mathrm{b}}lr\sin\theta_{\mathrm{b}})\dot{\theta}_{\mathrm{b}}^{2} = 0, \quad \text{(A.85)}$$

$$J_{2}(\theta_{\mathrm{b}})\ddot{\phi} + (2M_{\mathrm{b}}l^{2}\sin\theta_{\mathrm{b}}\cos\theta_{\mathrm{b}})\dot{\theta}_{\mathrm{b}}\dot{\phi} + 2\frac{d^{2}}{r^{2}}\left(d_{\mathrm{b}} + d_{\mathrm{w}}\right)\dot{\phi} = \frac{d}{r}2b_{\mathrm{v}}u_{2}. \quad \text{(A.86)}$$

By defining the control input as

$$u = \begin{bmatrix} u_{1} \\ u_{2} \end{bmatrix}, \quad \text{(A.87)}$$

the equation of motion for the self-balancing robot can be derived as

$$\begin{cases} J_{1}(\theta_{\mathrm{b}}(t))\begin{bmatrix} \ddot{\theta}_{\mathrm{m}}(t) \\ \ddot{\theta}_{\mathrm{b}}(t) \end{bmatrix} + D_{1}\begin{bmatrix} \dot{\theta}_{\mathrm{m}}(t) \\ \dot{\theta}_{\mathrm{b}}(t) \end{bmatrix} \\ \quad - M_{\mathrm{b}}l\sin\theta_{\mathrm{b}}(t)\begin{bmatrix} r\dot{\theta}_{\mathrm{b}}^{2}(t) \\ g + l\cos\theta_{\mathrm{b}}(t)\dot{\phi}^{2}(t) + 2r\dot{\theta}_{\mathrm{b}}^{2}(t) \end{bmatrix} = H_{1}u(t), \\ J_{2}(\theta_{\mathrm{b}}(t))\ddot{\phi}(t) + D_{2}\dot{\phi}(t) + \left(2M_{\mathrm{b}}l^{2}\sin\theta_{\mathrm{b}}(t)\cos\theta_{\mathrm{b}}(t)\right)\dot{\theta}_{\mathrm{b}}(t)\dot{\phi}(t) = H_{2}u(t), \end{cases} \quad \text{(A.88)}$$

where

$$J_{1}(\theta_{\mathrm{b}}) = \begin{bmatrix} J_{11} & J'_{12}(\theta_{\mathrm{b}}) \\ J'_{12}(\theta_{\mathrm{b}}) & J'_{22}(\theta_{\mathrm{b}}) \end{bmatrix},\ D_{1} = 2\begin{bmatrix} d_{\mathrm{b}} + d_{\mathrm{w}} & d_{\mathrm{w}} \\ 0 & 0 \end{bmatrix},\ H_{1} = 2b_{\mathrm{v}}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{(A.89)}$$

$$J_{2}(\theta_{\mathrm{b}}) = J_{2}(\theta_{\mathrm{b}}),\ D_{2} = 2\frac{d^{2}}{r^{2}}(d_{\mathrm{b}} + d_{\mathrm{w}}),\ H_{2} = 2b_{\mathrm{v}}\frac{d}{r}\begin{bmatrix} 0 & 1 \end{bmatrix}. \quad \text{(A.90)}$$

We linearized the equations of motion (A.88) around equilibrium states $\theta_{\mathrm{m}} = 0$, $\theta_{\mathrm{b}} = 0$, $\phi = 0$, $\dot{\theta}_{\mathrm{m}} = 0$, $\dot{\theta}_{\mathrm{b}} = 0$, $\dot{\phi} = 0$, and $u = 0$. Then, under the assumption that $\sin\theta_{\mathrm{b}}(t) \approx \theta_{\mathrm{b}}(t)$, $\cos\theta_{\mathrm{b}}(t) \approx 1$, $\sin^{2}\theta_{\mathrm{b}}(t) \approx 0$, $\dot{\theta}_{\mathrm{b}}^{2}(t) \approx 0$, $\dot{\phi}^{2}(t) \approx 0$, and $\sin\theta_{\mathrm{b}}(t)\cos\theta_{\mathrm{b}}(t)\dot{\theta}_{\mathrm{b}}(t)\dot{\phi}(t) \approx 0$, the linearized equations of motion can be derived as

$$\begin{cases} J_{1}\ddot{x}_{a}(t) + D_{1}\dot{x}_{a}(t) + K_{1}x_{a}(t) = H_{1}u(t), \\ J_{2}\ddot{x}_{b}(t) + D_{2}\dot{x}_{b}(t) = H_{2}u(t), \end{cases} \quad \text{(A.91)}$$

A.2.3 Case 3: the generalized coordinate $(\theta_{\mathrm{w}}, \theta_{\mathrm{b}}, \phi)$ in [9, 28]

$$T_{mr} = \frac{1}{2}J_{\mathrm{m}}\left(g_{\mathrm{r}}\dot{\theta}_{\mathrm{mr}} + \dot{\theta}_{\mathrm{b}}\right)^2 = \frac{1}{2}J_{\mathrm{m}}\left(g_{\mathrm{r}}(\dot{\theta}_{\mathrm{wr}} - \dot{\theta}_{\mathrm{b}}) + \dot{\theta}_{\mathrm{b}}\right)^2 \tag{A.92}$$

$$T_{ml} = \frac{1}{2}J_{\mathrm{m}}\left(g_{\mathrm{r}}\dot{\theta}_{\mathrm{ml}} + \dot{\theta}_{\mathrm{b}}\right)^2 = \frac{1}{2}J_{\mathrm{m}}\left(g_{\mathrm{r}}(\dot{\theta}_{\mathrm{wl}} - \dot{\theta}_{\mathrm{b}}) + \dot{\theta}_{\mathrm{b}}\right)^2. \tag{A.93}$$

Then,

$$\begin{aligned} T_{mr} + T_{ml} &= \frac{1}{2}J_{\mathrm{m}}\left(g_{\mathrm{r}}^2\dot{\theta}_{\mathrm{wr}}^2 + 2g_{\mathrm{r}}(1-g_{\mathrm{r}})\dot{\theta}_{\mathrm{wr}}\dot{\theta}_{\mathrm{b}} + (1-g_{\mathrm{r}})^2\dot{\theta}_{\mathrm{b}}^2\right) \\ &\quad + \frac{1}{2}J_{\mathrm{m}}\left(g_{\mathrm{r}}^2\dot{\theta}_{\mathrm{wl}}^2 + 2g_{\mathrm{r}}(1-g_{\mathrm{r}})\dot{\theta}_{\mathrm{wl}}\dot{\theta}_{\mathrm{b}} + (1-g_{\mathrm{r}})^2\dot{\theta}_{\mathrm{b}}^2\right) && \text{(A.94)} \\ &= \frac{1}{2}J_{\mathrm{m}}\left(g_{\mathrm{r}}^2\left(\dot{\theta}_{\mathrm{wr}}^2 + \dot{\theta}_{\mathrm{wl}}^2\right) + 2g_{\mathrm{r}}(1-g_{\mathrm{r}})\left(\dot{\theta}_{\mathrm{wl}} + \dot{\theta}_{\mathrm{wr}}\right)\dot{\theta}_{\mathrm{b}} + 2(1-g_{\mathrm{r}})^2\dot{\theta}_{\mathrm{b}}^2\right) && \text{(A.95)} \\ &= \frac{1}{2}J_{\mathrm{m}}\left(2g_{\mathrm{r}}^2\left(\dot{\theta}_{\mathrm{w}}^2 + \frac{d^2}{r^2}\dot{\phi}^2\right) + 4g_{\mathrm{r}}(1-g_{\mathrm{r}})\dot{\theta}_{\mathrm{w}}\dot{\theta}_{\mathrm{b}} + 2(1-g_{\mathrm{r}})^2\dot{\theta}_{\mathrm{b}}^2\right). && \text{(A.96)} \end{aligned}$$

The Lagrangian is given as

$$
\begin{aligned}
&L(\dot{\theta}_{\mathrm{w}}, \dot{\theta}_{\mathrm{b}}, \dot{\phi}, \theta_{\mathrm{w}}, \theta_{\mathrm{b}}, \phi) \\
&= T_r + T_l + T_{mr} + T_{ml} + T_b - V \\
&= \frac{1}{2} M_{\mathrm{w}} \left(\dot{x}_r^2 + \dot{y}_r^2 + \dot{x}_l^2 + \dot{y}_l^2 \right) + \frac{1}{2} J_{\mathrm{w}} \left(\dot{\theta}_{\mathrm{wr}}^2 + \dot{\theta}_{\mathrm{wl}}^2 \right) \\
&\quad + J_{\mathrm{m}} \left(g_{\mathrm{r}}^2 \left(\dot{\theta}_{\mathrm{w}}^2 + \frac{d^2}{r^2} \dot{\phi}^2 \right) + 2 g_{\mathrm{r}} (1 - g_{\mathrm{r}}) \dot{\theta}_{\mathrm{w}} \dot{\theta}_{\mathrm{b}} + (1 - g_{\mathrm{r}})^2 \dot{\theta}_{\mathrm{b}}^2 \right) \\
&\quad + \frac{1}{2} M_{\mathrm{b}} \left(\dot{x}_b^2 + \dot{y}_b^2 + \dot{z}_b^2 \right) + \frac{1}{2} J_{\mathrm{b}} \dot{\theta}_{\mathrm{b}}^2 + \frac{1}{2} J_{\phi} \dot{\phi}^2 - 2 M_{\mathrm{w}} g r - M_{\mathrm{b}} g \left(r + l \cos \theta_{\mathrm{b}} \right) \\
&= M_{\mathrm{w}} \left(r^2 \dot{\theta}_{\mathrm{w}}^2 + d^2 \dot{\phi}^2 \right) + J_{\mathrm{w}} \left(\dot{\theta}_{\mathrm{w}}^2 + \frac{d^2}{r^2} \dot{\phi}^2 \right) \\
&\quad + g_{\mathrm{r}}^2 J_{\mathrm{m}} \left(\dot{\theta}_{\mathrm{w}}^2 + \frac{d^2}{r^2} \dot{\phi}^2 \right) + 2 J_{\mathrm{m}} g_{\mathrm{r}} (1 - g_{\mathrm{r}}) \dot{\theta}_{\mathrm{w}} \dot{\theta}_{\mathrm{b}} + J_{\mathrm{m}} (1 - g_{\mathrm{r}})^2 \dot{\theta}_{\mathrm{b}}^2 \\
&\quad + \frac{1}{2} M_{\mathrm{b}} \left(r^2 \dot{\theta}_{\mathrm{w}}^2 + 2 l r \dot{\theta}_{\mathrm{w}} \dot{\theta}_{\mathrm{b}} \cos \theta_{\mathrm{b}} + l^2 \dot{\theta}_{\mathrm{b}}^2 + l^2 \dot{\phi}^2 \sin^2 \theta_{\mathrm{b}} \right) \\
&\quad + \frac{1}{2} J_{\mathrm{b}} \dot{\theta}_{\mathrm{b}}^2 + \frac{1}{2} J_{\phi} \dot{\phi}^2 - 2 M_{\mathrm{w}} g r - M_{\mathrm{b}} g \left(r + l \cos \theta_{\mathrm{b}} \right) \\
&= \left(\left(M_{\mathrm{w}} + \frac{M_{\mathrm{b}}}{2} \right) r^2 + J_{\mathrm{w}} + g_{\mathrm{r}}^2 J_{\mathrm{m}} \right) \dot{\theta}_{\mathrm{w}}^2 + \left(M_{\mathrm{b}} l r \cos \theta_{\mathrm{b}} + 2 g_{\mathrm{r}} (1 - g_{\mathrm{r}}) J_{\mathrm{m}} \right) \dot{\theta}_{\mathrm{w}} \dot{\theta}_{\mathrm{b}} \\
&\quad + \left(\frac{M_{\mathrm{b}}}{2} l^2 + \frac{1}{2} J_{\mathrm{b}} + (1 - g_{\mathrm{r}})^2 J_{\mathrm{m}} \right) \dot{\theta}_{\mathrm{b}}^2 \\
&\quad + \left(\frac{1}{2} J_{\phi} + \frac{d^2}{r^2} J_{\mathrm{w}} + \frac{d^2}{r^2} J_{\mathrm{m}} + M_{\mathrm{w}} d^2 + \frac{M_{\mathrm{b}}}{2} l^2 \sin^2 \theta_{\mathrm{b}} \right) \dot{\phi}^2 \\
&\quad - 2 M_{\mathrm{w}} g r - M_{\mathrm{b}} g (r + l \cos \theta_{\mathrm{b}}) \\
&= \frac{1}{2} J_{11} \dot{\theta}_{\mathrm{w}}^2 + J_{12}(\theta_{\mathrm{b}}) \dot{\theta}_{\mathrm{w}} \dot{\theta}_{\mathrm{b}} + \frac{1}{2} J_{22} \dot{\theta}_{\mathrm{b}}^2 + \frac{1}{2} J_2(\theta_{\mathrm{b}}) \dot{\phi}^2 - 2 M_{\mathrm{w}} g r - M_{\mathrm{b}} g (r + l \cos \theta_{\mathrm{b}}),
\end{aligned}
$$

where

$$J_{11} := \left(2 M_{\mathrm{w}} + M_{\mathrm{b}} \right) r^2 + 2 J_{\mathrm{w}} + 2 g_{\mathrm{r}}^2 J_{\mathrm{m}}, \tag{A.97}$$

$$J_{12}(\theta_{\mathrm{b}}) := M_{\mathrm{b}} l r \cos \theta_{\mathrm{b}} + 2 g_{\mathrm{r}} (1 - g_{\mathrm{r}}) J_{\mathrm{m}}, \tag{A.98}$$

$$J_{22} := M_{\mathrm{b}} l^2 + J_{\mathrm{b}} + 2 (1 - g_{\mathrm{r}})^2 J_{\mathrm{m}}, \tag{A.99}$$

$$J_2(\theta_{\mathrm{b}}) := 2 M_{\mathrm{w}} d^2 + M_{\mathrm{b}} l^2 \sin^2 \theta_{\mathrm{b}} + J_{\phi} + 2 \frac{d^2}{r^2} \left(J_{\mathrm{w}} + g_{\mathrm{r}}^2 J_{\mathrm{m}} \right). \tag{A.100}$$

$$\frac{\partial L}{\partial \theta_{\mathrm{w}}} = 0,$$
$$\frac{\partial L}{\partial \theta_{\mathrm{b}}} = -(M_{\mathrm{b}} lr \sin\theta_{\mathrm{b}})\dot{\theta}_{\mathrm{w}}\dot{\theta}_{\mathrm{b}} + M_{\mathrm{b}} l \sin\theta_{\mathrm{b}}(g + l\dot{\phi}^2 \cos\theta_{\mathrm{b}}),$$
$$\frac{\partial L}{\partial \phi} = 0,$$
$$\frac{\partial L}{\partial \dot{\theta}_{\mathrm{w}}} = J_{11}\dot{\theta}_{\mathrm{w}} + J_{12}(\theta_{\mathrm{b}})\dot{\theta}_{\mathrm{b}},$$
$$\frac{\partial L}{\partial \dot{\theta}_{\mathrm{b}}} = J_{12}(\theta_{\mathrm{b}})\dot{\theta}_{\mathrm{w}} + J_{22}\dot{\theta}_{\mathrm{b}},$$
$$\frac{\partial L}{\partial \dot{\phi}} = J_2(\theta_{\mathrm{b}})\dot{\phi}.$$

Furthermore,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}_{\mathrm{w}}}\right) = J_{11}\ddot{\theta}_{\mathrm{w}} + J_{12}(\theta_{\mathrm{b}})\ddot{\theta}_{\mathrm{b}} - (M_{\mathrm{b}} lr \sin\theta_{\mathrm{b}})\dot{\theta}_{\mathrm{b}}^2, \tag{A.101}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}_{\mathrm{b}}}\right) = J_{22}\ddot{\theta}_{\mathrm{b}} + J_{12}(\theta_{\mathrm{b}})\ddot{\theta}_{\mathrm{w}} - (M_{\mathrm{b}} lr \sin\theta_{\mathrm{b}})\dot{\theta}_{\mathrm{w}}\dot{\theta}_{\mathrm{b}}, \tag{A.102}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = J_2(\theta_{\mathrm{b}})\ddot{\phi} + (2M_{\mathrm{b}} l^2 \sin\theta_{\mathrm{b}} \cos\theta_{\mathrm{b}})\dot{\theta}_{\mathrm{b}}\dot{\phi}. \tag{A.103}$$

Hence, by the Lagrangian equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}_{\mathrm{w}}}\right) - \frac{\partial L}{\partial \theta_{\mathrm{w}}} = J_{11}\ddot{\theta}_{\mathrm{w}} + J_{12}(\theta_{\mathrm{b}})\ddot{\theta}_{\mathrm{b}} - (M_{\mathrm{b}} lr \sin\theta_{\mathrm{b}})\dot{\theta}_{\mathrm{b}}^2 = F_1, \tag{A.104}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}_{\mathrm{b}}}\right) - \frac{\partial L}{\partial \theta_{\mathrm{b}}} = J_{12}(\theta_{\mathrm{b}})\ddot{\theta}_{\mathrm{w}} + J_{22}\ddot{\theta}_{\mathrm{b}} - M_{\mathrm{b}} l \sin\theta_{\mathrm{b}}(g + l\dot{\phi}^2 \cos\theta_{\mathrm{b}}) = F_2, \tag{A.105}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = J_2(\theta_{\mathrm{b}})\ddot{\phi} + (2M_{\mathrm{b}} l^2 \sin\theta_{\mathrm{b}} \cos\theta_{\mathrm{b}})\dot{\theta}_{\mathrm{b}}\dot{\phi} = F_3, \tag{A.106}$$

The external force are two torque τ_{r} and τ_{l} generated by two motors fixed on the body and affected as the torque

$$\tau_{\theta_{\mathrm{m}}} = \tau_{\mathrm{r}} + \tau_{\mathrm{l}} \tag{A.107}$$
$$\tau_{\theta_{\mathrm{b}}} = 0 \tag{A.108}$$
$$\tau_{\phi} = \frac{d}{r}(\tau_{\mathrm{r}} - \tau_{\mathrm{l}}). \tag{A.109}$$

Then, the generalized force for $(\theta_\mathrm{w}, \theta_\mathrm{b}, \phi)$ are

$$
\begin{aligned}
F_1 &= \frac{\partial}{\partial \theta_\mathrm{w}} \left(\tau_{\theta_\mathrm{m}} \theta_\mathrm{m} + \tau_{\theta_\mathrm{b}} \theta_\mathrm{b} + \tau_\phi \phi\right) = \frac{\partial}{\partial \theta_\mathrm{w}} \left(\tau_{\theta_\mathrm{m}} (\theta_\mathrm{w} - \theta_\mathrm{b}) + \tau_{\theta_\mathrm{b}} \theta_\mathrm{b}\right) = \tau_{\theta_m} \\
F_2 &= \frac{\partial}{\partial \theta_\mathrm{b}} \left(\tau_{\theta_\mathrm{m}} \theta_\mathrm{m} + \tau_{\theta_\mathrm{b}} \theta_\mathrm{b} + \tau_\phi \phi\right) = \frac{\partial}{\partial \theta_\mathrm{b}} \left(\tau_{\theta_\mathrm{m}} (\theta_\mathrm{w} - \theta_\mathrm{b}) + \tau_{\theta_\mathrm{b}} \theta_\mathrm{b}\right) = -\tau_{\theta_\mathrm{m}} + \tau_{\theta_\mathrm{b}} = -\tau_{\theta_\mathrm{m}} \\
F_3 &= \frac{\partial}{\partial \phi} \left(\tau_{\theta_\mathrm{m}} \theta_\mathrm{m} + \tau_{\theta_\mathrm{b}} \theta_\mathrm{b} + \tau_\phi \phi\right) = \tau_\phi.
\end{aligned}
$$

Hence, we can derive the equation of motion,

$$
\begin{aligned}
& J_{11} \ddot{\theta}_\mathrm{w} + J_{12}(\theta_\mathrm{b}) \ddot{\theta}_\mathrm{b} - (M_\mathrm{b} l r \sin \theta_\mathrm{b}) \dot{\theta}_\mathrm{b}^2 = \tau_{\theta_m}, \\
& J_{12}(\theta_\mathrm{b}) \ddot{\theta}_\mathrm{w} + J_{22} \ddot{\theta}_\mathrm{b} - M_\mathrm{b} l \sin \theta_\mathrm{b} (g + l \dot{\phi}^2 \cos \theta_\mathrm{b}) = -\tau_{\theta_m}, \\
& J_2(\theta_\mathrm{b}) \ddot{\phi} + (2 M_\mathrm{b} l^2 \sin \theta_\mathrm{b} \cos \theta_\mathrm{b}) \dot{\theta}_\mathrm{b} \dot{\phi} = \tau_\phi.
\end{aligned}
$$

A.2.4 Case 4: the generalized coordinate $(\theta_\mathrm{m}, \theta_\mathrm{b}, \phi)$ in [9, 28]

The Lagrangian is given as

$$
\begin{aligned}
& L(\dot{\theta}_\mathrm{m}, \dot{\theta}_\mathrm{b}, \dot{\phi}, \theta_\mathrm{m}, \theta_\mathrm{b}, \phi) \\
&= \frac{1}{2} J_{11} \left(\dot{\theta}_\mathrm{m} + \dot{\theta}_\mathrm{b}\right)^2 + J_{12}(\theta_\mathrm{b}) \left(\dot{\theta}_\mathrm{m} + \dot{\theta}_\mathrm{b}\right) \dot{\theta}_\mathrm{b} + \frac{1}{2} J_{22}(\theta_\mathrm{b}) \dot{\theta}_\mathrm{b}^2 + \frac{1}{2} J_2(\theta_\mathrm{b}) \dot{\phi}^2 \\
&\quad - 2 M_\mathrm{w} g r - M_\mathrm{b} g (r + l \cos \theta_\mathrm{b}) \\
&= \frac{1}{2} J_{11} \dot{\theta}_\mathrm{m}^2 + J'_{12}(\theta_\mathrm{b}) \dot{\theta}_\mathrm{m} \dot{\theta}_\mathrm{b} + \frac{1}{2} J'_{22}(\theta_\mathrm{b}) \dot{\theta}_\mathrm{b}^2 + \frac{1}{2} J_2(\theta_\mathrm{b}) \dot{\phi}^2 - 2 M_\mathrm{w} g r - M_\mathrm{b} g (r + l \cos \theta_\mathrm{b}).
\end{aligned}
$$

where

$$
J_{11} := (2 M_\mathrm{w} + M_\mathrm{b}) r^2 + 2 \left(J_\mathrm{w} + g_\mathrm{r}^2 J_\mathrm{m}\right), \tag{A.110}
$$

$$
J'_{12}(\theta_\mathrm{b}) := J_{11} + J_{12}(\theta_\mathrm{b}) = (2 M_\mathrm{w} + M_\mathrm{b}) r^2 + M_\mathrm{b} l r \cos \theta_\mathrm{b} + 2 J_\mathrm{w} + 2 g_\mathrm{r} J_\mathrm{m} \tag{A.111}
$$

$$
\begin{aligned}
J'_{22}(\theta_\mathrm{b}) &:= J_{11} + 2 J_{12}(\theta_\mathrm{b}) + J_{22}(\theta_\mathrm{b}) \\
&= (2 M_\mathrm{w} + M_\mathrm{b}) r^2 + M_\mathrm{b} l^2 + 2 M_\mathrm{b} l r \cos \theta_\mathrm{b} + 2 J_\mathrm{w} + 2 J_\mathrm{m} + J_\mathrm{b}
\end{aligned} \tag{A.112}
$$

$$
J_2(\theta_\mathrm{b}) := 2 M_\mathrm{w} d^2 + M_\mathrm{b} l^2 \sin^2 \theta_\mathrm{b} + J_\phi + 2 \frac{d^2}{r^2} \left(J_\mathrm{w} + g_\mathrm{r}^2 J_\mathrm{m}\right). \tag{A.113}
$$

$$
\begin{aligned}
\frac{\partial L}{\partial \theta_{\mathrm{m}}} &= 0, \\
\frac{\partial L}{\partial \theta_{\mathrm{b}}} &= -M_{\mathrm{b}} l r \dot{\theta}_{\mathrm{w}} \dot{\theta}_{\mathrm{b}} \sin\theta_{\mathrm{b}} + M_{\mathrm{b}} l \sin\theta_{\mathrm{b}} (g + l\dot{\phi}^2 \cos\theta_{\mathrm{b}}), \\
\frac{\partial L}{\partial \phi} &= 0, \\
\frac{\partial L}{\partial \dot{\theta}_{\mathrm{m}}} &= J_{11}\dot{\theta}_{\mathrm{m}} + J'_{12}(\theta_{\mathrm{b}})\dot{\theta}_{\mathrm{b}}, \\
\frac{\partial L}{\partial \dot{\theta}_{\mathrm{b}}} &= J'_{12}(\theta_{\mathrm{b}})\dot{\theta}_{\mathrm{m}} + J'_{22}\dot{\theta}_{\mathrm{b}}, \\
\frac{\partial L}{\partial \dot{\phi}} &= J_2(\theta_{\mathrm{b}})\dot{\phi}.
\end{aligned}
$$

Furthermore,

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}_{\mathrm{m}}}\right) = J_{11}\ddot{\theta}_{\mathrm{m}} + J'_{12}(\theta_{\mathrm{b}})\ddot{\theta}_{\mathrm{b}} - (M_{\mathrm{b}} l r \sin\theta_{\mathrm{b}})\dot{\theta}_{\mathrm{b}}^2, \tag{A.114}
$$

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}_{\mathrm{b}}}\right) = J'_{12}(\theta_{\mathrm{b}})\ddot{\theta}_{\mathrm{m}} + J'_{22}(\theta_{\mathrm{b}})\ddot{\theta}_{\mathrm{b}} - (M_{\mathrm{b}} l r \sin\theta_{\mathrm{b}})\dot{\theta}_{\mathrm{w}}\dot{\theta}_{\mathrm{b}} - (2M_{\mathrm{b}} l r \sin\theta_{\mathrm{b}})\dot{\theta}_{\mathrm{b}}^2, \tag{A.115}
$$

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = J_2(\theta_{\mathrm{b}})\ddot{\phi} + (2M_{\mathrm{b}} l^2 \sin\theta_{\mathrm{b}} \cos\theta_{\mathrm{b}})\dot{\theta}_{\mathrm{b}}\dot{\phi}. \tag{A.116}
$$

Hence, by the Lagrangian equation

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}_{\mathrm{m}}}\right) - \frac{\partial L}{\partial \theta_{\mathrm{m}}} = J_{11}\ddot{\theta}_{\mathrm{m}} + J'_{12}(\theta_{\mathrm{b}})\ddot{\theta}_{\mathrm{b}} - (M_{\mathrm{b}} l r \sin\theta_{\mathrm{b}})\dot{\theta}_{\mathrm{b}}^2 = F_1, \tag{A.117}
$$

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}_{\mathrm{b}}}\right) - \frac{\partial L}{\partial \theta_{\mathrm{b}}} \\
&\quad = J'_{12}(\theta_{\mathrm{b}})\ddot{\theta}_{\mathrm{m}} + J'_{22}(\theta_{\mathrm{b}})\ddot{\theta}_{\mathrm{b}} - M_{\mathrm{b}} l \sin\theta_{\mathrm{b}}(g + l\dot{\phi}^2\cos\theta_{\mathrm{b}}) - (2M_{\mathrm{b}} l r \sin\theta_{\mathrm{b}})\dot{\theta}_{\mathrm{b}}^2 = F_2,
\end{aligned} \tag{A.118}
$$

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = J_2(\theta_{\mathrm{b}})\ddot{\phi} + (2M_{\mathrm{b}} l^2 \sin\theta_{\mathrm{b}} \cos\theta_{\mathrm{b}})\dot{\theta}_{\mathrm{b}}\dot{\phi} = F_3, \tag{A.119}
$$

The external force are two torque τ_{r} and τ_{l} generated by two motors fixed on the body and affected as the torque Then, the generalized force for $(\theta_{\mathrm{m}}, \theta_{\mathrm{b}}, \phi)$ are

$$
F_1 = \tau_{\theta_m}, F_2 = 0, F_3 = \tau_{\phi}.
$$

Hence, we can derive the equation of motion,

$$J_{11}\ddot{\theta}_{\rm m} + J'_{12}(\theta_{\rm b})\ddot{\theta}_{\rm b} - (M_{\rm b}lr\sin\theta_{\rm b})\dot{\theta}_{\rm b}^2 = \tau_{\theta_m},$$
$$J'_{12}(\theta_{\rm b})\ddot{\theta}_{\rm m} + J'_{22}(\theta_{\rm b})\ddot{\theta}_{\rm b} - M_{\rm b}l\sin\theta_{\rm b}(g + l\dot{\phi}^2\cos\theta_{\rm b}) - (2M_{\rm b}lr\sin\theta_{\rm b})\dot{\theta}_{\rm b}^2 = 0,$$
$$J_2(\theta_{\rm b})\ddot{\phi} + (2M_{\rm b}l^2\sin\theta_{\rm b}\cos\theta_{\rm b})\dot{\theta}_{\rm b}\dot{\phi} = \tau_{\phi}.$$

Since

$$\begin{aligned}\tau_{\theta_{\rm m}} &= 2b_{\rm v}u_1 + 2d_{\rm b}\dot{\theta}_{\rm b} - 2\left(d_{\rm b} + d_{\rm w}\right)\dot{\theta}_{\rm w} = 2b_{\rm v}u_1 + 2d_{\rm b}\dot{\theta}_{\rm b} - 2\left(d_{\rm b} + d_{\rm w}\right)(\dot{\theta}_{\rm m} + \dot{\theta}_{\rm b})\\ &= 2b_{\rm v}u_1 - 2d_{\rm w}\dot{\theta}_{\rm b} - 2\left(d_{\rm b} + d_{\rm w}\right)\dot{\theta}_{\rm m}, \qquad \text{(A.120)}\end{aligned}$$

$$\tau_{\phi} = \frac{2d}{r}b_{\rm v}u_2 - 2\frac{d^2}{r^2}\left(d_{\rm b} + d_{\rm w}\right)\dot{\phi}. \qquad \text{(A.121)}$$

From the above, the equation of motion is

$$J_{11}\ddot{\theta}_{\rm m} + J'_{12}(\theta_{\rm b})\ddot{\theta}_{\rm b} + 2\left(d_{\rm b} + d_{\rm w}\right)\dot{\theta}_{\rm m} + 2d_{\rm w}\dot{\theta}_{\rm b} - (M_{\rm b}lr\sin\theta_{\rm b})\dot{\theta}_{\rm b}^2 = 2b_{\rm v}u_1, \qquad \text{(A.122)}$$

$$J'_{12}(\theta_{\rm b})\ddot{\theta}_{\rm m} + J'_{22}(\theta_{\rm b})\ddot{\theta}_{\rm b} - M_{\rm b}l\sin\theta_{\rm b}(g + l\dot{\phi}^2\cos\theta_{\rm b}) - (2M_{\rm b}lr\sin\theta_{\rm b})\dot{\theta}_{\rm b}^2 = 0, \qquad \text{(A.123)}$$

$$J_2(\theta_{\rm b})\ddot{\phi} + (2M_{\rm b}l^2\sin\theta_{\rm b}\cos\theta_{\rm b})\dot{\theta}_{\rm b}\dot{\phi} + 2\frac{d^2}{r^2}\left(d_{\rm b} + d_{\rm w}\right)\dot{\phi} = \frac{d}{r}2b_{\rm v}u_2. \qquad \text{(A.124)}$$

By defining the control input as

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \qquad \text{(A.125)}$$

the equation of motion for the self-balancing robot can be derived as

$$\begin{cases} J_1(\theta_{\rm b}(t))\begin{bmatrix}\ddot{\theta}_{\rm m}(t)\\ \ddot{\theta}_{\rm b}(t)\end{bmatrix} + D_1\begin{bmatrix}\dot{\theta}_{\rm m}(t)\\ \dot{\theta}_{\rm b}(t)\end{bmatrix} \\ \quad - M_{\rm b}l\sin\theta_{\rm b}(t)\begin{bmatrix} r\dot{\theta}_{\rm b}^2(t) \\ g + l\cos\theta_{\rm b}(t)\dot{\phi}^2(t) + 2r\dot{\theta}_{\rm b}^2(t)\end{bmatrix} = H_1u(t), \\ J_2(\theta_{\rm b}(t))\ddot{\phi}(t) + D_2\dot{\phi}(t) + \left(2M_{\rm b}l^2\sin\theta_{\rm b}(t)\cos\theta_{\rm b}(t)\right)\dot{\theta}_{\rm b}(t)\dot{\phi}(t) = H_2u(t), \end{cases} \qquad \text{(A.126)}$$

where

$$J_1(\theta_\mathrm{b}) = \begin{bmatrix} J_{11} & J'_{12}(\theta_\mathrm{b}) \\ J'_{12}(\theta_\mathrm{b}) & J'_{22}(\theta_\mathrm{b}) \end{bmatrix},\ D_1 = 2\begin{bmatrix} d_\mathrm{b} + d_\mathrm{w} & d_\mathrm{w} \\ 0 & 0 \end{bmatrix},\ H_1 = 2b_\mathrm{v}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \tag{A.127}$$

$$J_2(\theta_\mathrm{b}) = J_2(\theta_\mathrm{b}),\ D_2 = 2\frac{d^2}{r^2}(d_\mathrm{b} + d_\mathrm{w}),\ H_2 = 2b_\mathrm{v}\frac{d}{r}\begin{bmatrix} 0 & 1 \end{bmatrix}. \tag{A.128}$$

We linearized the equations of motion (A.126) around equilibrium states $\theta_\mathrm{m} = 0$, $\theta_\mathrm{b} = 0$, $\phi = 0$, $\dot{\theta}_\mathrm{m} = 0$, $\dot{\theta}_\mathrm{b} = 0$, $\dot{\phi} = 0$, and $u = 0$. Then, under the assumption that $\sin\theta_\mathrm{b}(t) \approx \theta_\mathrm{b}(t)$, $\cos\theta_\mathrm{b}(t) \approx 1$, $\sin^2\theta_\mathrm{b}(t) \approx 0$, $\dot{\theta}_\mathrm{b}^2(t) \approx 0$, $\dot{\phi}^2(t) \approx 0$, and $\sin\theta_\mathrm{b}(t)\cos\theta_\mathrm{b}(t)\dot{\theta}_\mathrm{b}(t)\dot{\phi}(t) \approx 0$, the linearized equations of motion can be derived as

$$\begin{cases} J_1\ddot{x}_a(t) + D_1\dot{x}_a(t) + K_1 x_a(t) = H_1 u(t), \\ J_2\ddot{x}_b(t) + D_2\dot{x}_b(t) = H_2 u(t), \end{cases} \tag{A.129}$$

where

$$x_a(t) := \begin{bmatrix} \theta_\mathrm{m}(t) \\ \theta_\mathrm{b}(t) \end{bmatrix},\quad x_b(t) := \phi(t), \tag{A.130}$$

$$J_1 = J_1(0),\quad J_2 = J_2(0),\quad K_1 = M_\mathrm{b} lg\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}. \tag{A.131}$$

A.3 Equation of motion in 3D (Part 2)

Kinetic energy of the right and left wheel are given by

$$T_r = \frac{1}{2}M_\mathrm{w}\left(\dot{x}_r^2 + \dot{y}_r^2\right) + \frac{1}{2}J_\mathrm{w}\dot{\theta}_\mathrm{wr}^2$$
$$T_l = \frac{1}{2}M_\mathrm{w}\left(\dot{x}_l^2 + \dot{y}_l^2\right) + \frac{1}{2}J_\mathrm{w}\dot{\theta}_\mathrm{wl}^2.$$

Here, we ignore mass of the motors and kinetic energy T_{mr} and T_{ml} of the right and left motors. Kinetic energy of the body is given by

$$T_b = \frac{1}{2}M_\mathrm{b}(\dot{x}_b^2 + \dot{y}_b^2 + \dot{z}_b^2) + \frac{1}{2}J_\mathrm{b}\dot{\theta}_\mathrm{b}^2 + \frac{1}{2}J_\phi\dot{\phi}^2$$

On the other hand, potential energy is

$$V = M_\mathrm{w} g z_l + M_\mathrm{w} g z_r + M_\mathrm{b} g z_b = 2M_\mathrm{w} g r + M_\mathrm{b} g(r + l\cos\theta_\mathrm{b}). \tag{A.132}$$

Hence, the Lagrangian is given as

$$
\begin{aligned}
L &= T_r + T_l + T_b - V \\
&= \frac{1}{2}M_{\mathrm{w}}\left(\dot{x}_r^2 + \dot{y}_r^2 + \dot{x}_l^2 + \dot{y}_l^2\right) + \frac{1}{2}J_{\mathrm{w}}\left(\dot{\theta}_{\mathrm{wr}}^2 + \dot{\theta}_{\mathrm{wl}}^2\right) \\
&\quad + \frac{1}{2}M_{\mathrm{b}}\left(\dot{x}_b^2 + \dot{y}_b^2 + \dot{z}_b^2\right) + \frac{1}{2}J_{\mathrm{b}}\dot{\theta}_{\mathrm{b}}^2 + \frac{1}{2}J_{\phi}\dot{\phi}^2 - 2M_{\mathrm{w}}gr - M_{\mathrm{b}}g\left(r + l\cos\theta_{\mathrm{b}}\right).
\end{aligned}
$$

Note that we can rewrite the terms as

$$
\dot{x}_r^2 + \dot{y}_r^2 + \dot{x}_l^2 + \dot{y}_l^2 = \left(r\dot{\theta}_{\mathrm{w}} + d\dot{\phi}\right)^2 + \left(r\dot{\theta}_{\mathrm{w}} - d\dot{\phi}\right)^2 = 2\left(r^2\dot{\theta}_{\mathrm{w}}^2 + d^2\dot{\phi}^2\right), \tag{A.133}
$$

$$
\begin{aligned}
\dot{x}_b^2 + \dot{y}_b^2 + \dot{z}_b^2 &= \left(r\dot{\theta}_{\mathrm{w}} + l\dot{\theta}_{\mathrm{b}}\cos\theta_{\mathrm{b}}\right)^2 + \left(l\dot{\phi}\sin\theta_{\mathrm{b}}\right)^2 + \left(l\dot{\theta}_{\mathrm{b}}\sin\theta_{\mathrm{b}}\right)^2 \\
&= r^2\dot{\theta}_{\mathrm{w}}^2 + 2lr\dot{\theta}_{\mathrm{w}}\dot{\theta}_{\mathrm{b}}\cos\theta_{\mathrm{b}} + l^2\dot{\theta}_{\mathrm{b}}^2 + l^2\dot{\phi}^2\sin^2\theta_{\mathrm{b}}.
\end{aligned} \tag{A.134}
$$

From (A.10) and (A.7),

$$
\dot{\theta}_{\mathrm{wr}}^2 + \dot{\theta}_{\mathrm{wl}}^2 = \left(\dot{\theta}_{\mathrm{w}} - \frac{d}{r}\dot{\phi}\right)^2 + \left(\dot{\theta}_{\mathrm{w}} + \frac{d}{r}\dot{\phi}\right)^2 = 2\dot{\theta}_{\mathrm{w}}^2 + 2\frac{d^2}{r^2}\dot{\phi}^2. \tag{A.135}
$$

Hence, for the generalized coordinate $(\theta_{\mathrm{w}}, \theta_{\mathrm{b}}, \phi)$, the Lagrangian is

$$
\begin{aligned}
&L(\dot{\theta}_{\mathrm{w}}, \dot{\theta}_{\mathrm{b}}, \dot{\phi}, \theta_{\mathrm{w}}, \theta_{\mathrm{b}}, \phi) \\
&= \left(\left(M_{\mathrm{w}} + \frac{M_{\mathrm{b}}}{2}\right)r^2 + J_{\mathrm{w}}\right)\dot{\theta}_{\mathrm{w}}^2 + \left(M_{\mathrm{b}}lr\cos\theta_{\mathrm{b}}\right)\dot{\theta}_{\mathrm{w}}\dot{\theta}_{\mathrm{b}} + \left(\frac{M_{\mathrm{b}}}{2}l^2 + \frac{1}{2}J_{\mathrm{b}}\right)\dot{\theta}_{\mathrm{b}}^2 \\
&\quad + \left(M_{\mathrm{w}}d^2 + \frac{M_{\mathrm{b}}}{2}l^2\sin^2\theta_{\mathrm{b}} + \frac{1}{2}J_{\phi} + \frac{d^2}{r^2}J_{\mathrm{w}}\right)\dot{\phi}^2 - 2M_{\mathrm{w}}gr - M_{\mathrm{b}}g(r + l\cos\theta_{\mathrm{b}}) \\
&= \frac{1}{2}\bar{J}_{11}\dot{\theta}_{\mathrm{w}}^2 + \bar{J}_{12}(\theta_{\mathrm{b}})\dot{\theta}_{\mathrm{w}}\dot{\theta}_{\mathrm{b}} + \frac{1}{2}\bar{J}_{22}\dot{\theta}_{\mathrm{b}}^2 + \frac{1}{2}\bar{J}_2(\theta_{\mathrm{b}})\dot{\phi}^2 - 2M_{\mathrm{w}}gr - M_{\mathrm{b}}g(r + l\cos\theta_{\mathrm{b}}),
\end{aligned}
$$

where

$$
\bar{J}_{11} := \left(2M_{\mathrm{w}} + M_{\mathrm{b}}\right)r^2 + 2J_{\mathrm{w}}, \tag{A.136}
$$

$$
\bar{J}_{12}(\theta_{\mathrm{b}}) := M_{\mathrm{b}}lr\cos\theta_{\mathrm{b}} \tag{A.137}
$$

$$
\bar{J}_{22} := M_{\mathrm{b}}l^2 + J_{\mathrm{b}}, \tag{A.138}
$$

$$
\bar{J}_2(\theta_{\mathrm{b}}) := 2M_{\mathrm{w}}d^2 + M_{\mathrm{b}}l^2\sin^2\theta_{\mathrm{b}} + J_{\phi} + 2\frac{d^2}{r^2}J_{\mathrm{w}}. \tag{A.139}
$$

A.3.1 Case 1: the generalized coordinate $(\theta_{\mathrm{w}}, \theta_{\mathrm{b}}, \phi)$

$$\frac{\partial L}{\partial \theta_{\mathrm{w}}} = 0,$$
$$\frac{\partial L}{\partial \theta_{\mathrm{b}}} = -M_{\mathrm{b}} l r \dot{\theta}_{\mathrm{w}} \dot{\theta}_{\mathrm{b}} \sin\theta_{\mathrm{b}} + M_{\mathrm{b}} l \sin\theta_{\mathrm{b}} (g + l\dot{\phi}^2 \cos\theta_{\mathrm{b}}),$$
$$\frac{\partial L}{\partial \phi} = 0,$$
$$\frac{\partial L}{\partial \dot{\theta}_{\mathrm{w}}} = \bar{J}_{11} \dot{\theta}_{\mathrm{w}} + \bar{J}_{12}(\theta_{\mathrm{b}}) \dot{\theta}_{\mathrm{b}},$$
$$\frac{\partial L}{\partial \dot{\theta}_{\mathrm{b}}} = \bar{J}_{22} \dot{\theta}_{\mathrm{b}} + \bar{J}_{12}(\theta_{\mathrm{b}}) \dot{\theta}_{\mathrm{w}},$$
$$\frac{\partial L}{\partial \dot{\phi}} = \bar{J}_{2}(\theta_{\mathrm{b}}) \dot{\phi}.$$

Furthermore,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}_{\mathrm{w}}}\right) = \bar{J}_{11} \ddot{\theta}_{\mathrm{w}} + \bar{J}_{12}(\theta_{\mathrm{b}}) \ddot{\theta}_{\mathrm{b}} - (M_{\mathrm{b}} l r \sin\theta_{\mathrm{b}}) \dot{\theta}_{\mathrm{b}}^2, \tag{A.140}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}_{\mathrm{b}}}\right) = \bar{J}_{12}(\theta_{\mathrm{b}}) \ddot{\theta}_{\mathrm{w}} + \bar{J}_{22} \ddot{\theta}_{\mathrm{b}} - (M_{\mathrm{b}} l r \sin\theta_{\mathrm{b}}) \dot{\theta}_{\mathrm{w}} \dot{\theta}_{\mathrm{b}}, \tag{A.141}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = \bar{J}_{2}(\theta_{\mathrm{b}}) \ddot{\phi} + (2M_{\mathrm{b}} l^2 \sin\theta_{\mathrm{b}} \cos\theta_{\mathrm{b}}) \dot{\theta}_{\mathrm{b}} \dot{\phi}. \tag{A.142}$$

Hence, by the Lagrangian equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}_{\mathrm{w}}}\right) - \frac{\partial L}{\partial \theta_{\mathrm{w}}} = \bar{J}_{11} \ddot{\theta}_{\mathrm{w}} + \bar{J}_{12}(\theta_{\mathrm{b}}) \ddot{\theta}_{\mathrm{b}} - (M_{\mathrm{b}} l r \sin\theta_{\mathrm{b}}) \dot{\theta}_{\mathrm{b}}^2 = F_1, \tag{A.143}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}_{\mathrm{b}}}\right) - \frac{\partial L}{\partial \theta_{\mathrm{b}}} = \bar{J}_{12}(\theta_{\mathrm{b}}) \ddot{\theta}_{\mathrm{w}} + \bar{J}_{22} \ddot{\theta}_{\mathrm{b}} - M_{\mathrm{b}} l \sin\theta_{\mathrm{b}} (g + l\dot{\phi}^2 \cos\theta_{\mathrm{b}}) = F_2, \tag{A.144}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = \bar{J}_{2}(\theta_{\mathrm{b}}) \ddot{\phi} + (2M_{\mathrm{b}} l^2 \sin\theta_{\mathrm{b}} \cos\theta_{\mathrm{b}}) \dot{\theta}_{\mathrm{b}} \dot{\phi} = F_3, \tag{A.145}$$

The external force are two torque τ_{r} and τ_{l} generated by two motors fixed on the body and affected as the torque

$$\tau_{\theta_{\mathrm{m}}} = \tau_{\mathrm{r}} + \tau_{\mathrm{l}} \tag{A.146}$$
$$\tau_{\theta_{\mathrm{b}}} = 0 \tag{A.147}$$
$$\tau_{\phi} = \frac{d}{r}(\tau_{\mathrm{r}} - \tau_{\mathrm{l}}). \tag{A.148}$$

Then, the generalized force for $(\theta_\mathrm{w}, \theta_\mathrm{b}, \phi)$ are

$$F_1 = \frac{\partial}{\partial \theta_\mathrm{w}} \left(\tau_{\theta_\mathrm{m}}\theta_\mathrm{m} + \tau_{\theta_\mathrm{b}}\theta_\mathrm{b} + \tau_\phi \phi\right) = \frac{\partial}{\partial \theta_\mathrm{w}} \left(\tau_{\theta_\mathrm{m}}(\theta_\mathrm{w} - \theta_\mathrm{b}) + \tau_{\theta_\mathrm{b}}\theta_\mathrm{b}\right) = \tau_{\theta_m}$$
$$F_2 = \frac{\partial}{\partial \theta_\mathrm{b}} \left(\tau_{\theta_\mathrm{m}}\theta_\mathrm{m} + \tau_{\theta_\mathrm{b}}\theta_\mathrm{b} + \tau_\phi \phi\right) = \frac{\partial}{\partial \theta_\mathrm{b}} \left(\tau_{\theta_\mathrm{m}}(\theta_\mathrm{w} - \theta_\mathrm{b}) + \tau_{\theta_\mathrm{b}}\theta_\mathrm{b}\right) = -\tau_{\theta_\mathrm{m}} + \tau_{\theta_\mathrm{b}} = -\tau_{\theta_\mathrm{m}}$$
$$F_3 = \frac{\partial}{\partial \phi} \left(\tau_{\theta_\mathrm{m}}\theta_\mathrm{m} + \tau_{\theta_\mathrm{b}}\theta_\mathrm{b} + \tau_\phi \phi\right) = \tau_\phi.$$

Hence, we can derive the equation of motion,

$$J_{11}\ddot{\theta}_\mathrm{w} + J_{12}(\theta_\mathrm{b})\ddot{\theta}_\mathrm{b} - (M_\mathrm{b} l r \sin\theta_\mathrm{b})\dot{\theta}_\mathrm{b}^2 = \tau_{\theta_m},$$
$$J_{12}(\theta_\mathrm{b})\ddot{\theta}_\mathrm{w} + J_{22}\ddot{\theta}_\mathrm{b} - M_\mathrm{b} l \sin\theta_\mathrm{b}(g + l\dot{\phi}^2\cos\theta_\mathrm{b}) = -\tau_{\theta_m},$$
$$J_2(\theta_\mathrm{b})\ddot{\phi} + (2M_\mathrm{b} l^2 \sin\theta_\mathrm{b}\cos\theta_\mathrm{b})\dot{\theta}_\mathrm{b}\dot{\phi} = \tau_\phi.$$

The electrical and torque characteristics of a DC motor can be represented by the equivalent circuit equation.

$$L_m \frac{\mathrm{d}}{\mathrm{d}t} i_\mathrm{r} + R_\mathrm{m} i_\mathrm{r} = v_\mathrm{r} - K_\mathrm{e}\dot{\psi}_\mathrm{r}, \tag{A.149}$$
$$L_m \frac{\mathrm{d}}{\mathrm{d}t} i_\mathrm{l} + R_\mathrm{m} i_\mathrm{l} = v_\mathrm{l} - K_\mathrm{e}\dot{\psi}_\mathrm{l}, \tag{A.150}$$

where v_r and v_l are the voltage applied to the motor's armature, i_r and i_l are the armature current, R_m is the electric resistance, L_m is the electric inductance, K_e is the back EMF constant. Here, we assume that the inductance L_m is negligibly small, can be approximated as

$$R_\mathrm{m} i_\mathrm{r} = v_\mathrm{r} - K_\mathrm{e}\dot{\psi}_\mathrm{r} \iff i_\mathrm{r} = \frac{v_\mathrm{r} - K_\mathrm{e}\dot{\psi}_\mathrm{r}}{R_\mathrm{m}}, \tag{A.151}$$
$$R_\mathrm{m} i_\mathrm{l} = v_\mathrm{l} - K_\mathrm{e}\dot{\psi}_\mathrm{l} \iff i_\mathrm{l} = \frac{v_\mathrm{l} - K_\mathrm{e}\dot{\psi}_\mathrm{l}}{R_\mathrm{m}}. \tag{A.152}$$

The motor torque is proportional to the armature current by a constant factor K_t.

$$\tau_\mathrm{mr} = K_\mathrm{t} i_\mathrm{r} = K_\mathrm{t}\frac{v_\mathrm{r} - K_\mathrm{e}\dot{\psi}_\mathrm{r}}{R_\mathrm{m}}, \tag{A.153}$$
$$\tau_\mathrm{ml} = K_\mathrm{t} i_\mathrm{l} = K_\mathrm{t}\frac{v_\mathrm{l} - K_\mathrm{e}\dot{\psi}_\mathrm{l}}{R_\mathrm{m}}. \tag{A.154}$$

In SI units, the motor torque and back EMF constants are equal, that is, $K_\mathrm{t} = K_\mathrm{e}$. Under the assumption that there are no electromagnetic losses, mechanical power is equal to the electrical power dissipated by the back EMF $e_r = K_\mathrm{e}\dot{\psi}_\mathrm{r}$ in the armature. That is,

$$\tau_\mathrm{mr}\dot{\psi}_\mathrm{r} = e_r i_\mathrm{r} \iff K_\mathrm{t} i_\mathrm{r}\dot{\psi}_\mathrm{r} = K_\mathrm{e}\dot{\psi}_\mathrm{r} i_\mathrm{r} \iff K_\mathrm{t} = K_\mathrm{e}. \tag{A.155}$$

Hence, the torque applied to the right wheel satisfies

$$J_\mathrm{m}\frac{\mathrm{d}}{\mathrm{d}t}\dot{\psi}_\mathrm{r} = \tau_\mathrm{mr} - g_\mathrm{r}^{-1}\tau_\mathrm{r} - g_\mathrm{r}^{-1}d_\mathrm{m}\dot{\theta}_\mathrm{mr} - g_\mathrm{r}^{-1}d_\mathrm{w}\dot{\theta}_\mathrm{wr} \tag{A.156}$$

$$\tau_\mathrm{r} = g_\mathrm{r}\tau_\mathrm{mr} - g_\mathrm{r}J_\mathrm{m}\frac{\mathrm{d}}{\mathrm{d}t}\dot{\psi}_\mathrm{r} - d_\mathrm{m}\dot{\theta}_\mathrm{mr} - d_\mathrm{w}\dot{\theta}_\mathrm{wr} \tag{A.157}$$

$$= g_\mathrm{r}\tau_\mathrm{mr} - g_\mathrm{r}^2 J_\mathrm{m}\frac{\mathrm{d}}{\mathrm{d}t}\dot{\theta}_\mathrm{mr} - d_\mathrm{m}\dot{\theta}_\mathrm{mr} - d_\mathrm{w}\dot{\theta}_\mathrm{wr}, \tag{A.158}$$

where (A.3) is used. Hence,

$$\begin{aligned}\tau_\mathrm{r} &= g_\mathrm{r}K_\mathrm{t}\left(\frac{v_\mathrm{r} - K_\mathrm{e}\dot{\psi}_\mathrm{r}}{R_\mathrm{m}}\right) - g_\mathrm{r}^2 J_\mathrm{m}\frac{\mathrm{d}}{\mathrm{d}t}\dot{\theta}_\mathrm{mr} - d_\mathrm{m}\dot{\theta}_\mathrm{mr} - d_\mathrm{w}\dot{\theta}_\mathrm{wr}\\ &= \frac{g_\mathrm{r}K_\mathrm{t}}{R_\mathrm{m}}v_\mathrm{r} - \frac{g_\mathrm{r}^2 K_\mathrm{t}K_\mathrm{e}}{R_\mathrm{m}}\dot{\theta}_\mathrm{mr} - g_\mathrm{r}^2 J_\mathrm{m}\frac{\mathrm{d}}{\mathrm{d}t}\dot{\theta}_\mathrm{mr} - d_\mathrm{m}\dot{\theta}_\mathrm{mr} - d_\mathrm{w}\dot{\theta}_\mathrm{wr}\\ &= b_\mathrm{v}v_\mathrm{r} - g_\mathrm{r}^2 J_\mathrm{m}\frac{\mathrm{d}}{\mathrm{d}t}\dot{\theta}_\mathrm{mr} - d_\mathrm{b}\dot{\theta}_\mathrm{mr} - d_\mathrm{w}\dot{\theta}_\mathrm{wr},\end{aligned} \tag{A.159}$$

where

$$b_\mathrm{v} := \frac{g_\mathrm{r}K_\mathrm{t}}{R_\mathrm{m}}, \tag{A.160}$$

$$d_\mathrm{b} := \frac{g_\mathrm{r}^2 K_\mathrm{t}K_\mathrm{e}}{R_\mathrm{m}} + d_\mathrm{m}. \tag{A.161}$$

In the same way,

$$\tau_\mathrm{l} = b_\mathrm{v}v_\mathrm{l} - g_\mathrm{r}^2 J_\mathrm{m}\frac{\mathrm{d}}{\mathrm{d}t}\dot{\theta}_\mathrm{ml} - d_\mathrm{b}\dot{\theta}_\mathrm{ml} - d_\mathrm{w}\dot{\theta}_\mathrm{wl}. \tag{A.162}$$

Hence,

$$\tau_{\theta_{\mathrm{m}}} = \tau_{\mathrm{r}} + \tau_{\mathrm{l}} \tag{A.163}$$

$$= b_{\mathrm{v}}(v_{\mathrm{r}} + v_{\mathrm{l}}) - g_{\mathrm{r}}^2 J_{\mathrm{m}} \left(\frac{\mathrm{d}}{\mathrm{d}t}\dot{\theta}_{\mathrm{mr}} + \frac{\mathrm{d}}{\mathrm{d}t}\dot{\theta}_{\mathrm{ml}} \right) - d_{\mathrm{b}} \left(\dot{\theta}_{\mathrm{mr}} + \dot{\theta}_{\mathrm{ml}} \right) - d_{\mathrm{w}} \left(\dot{\theta}_{\mathrm{wr}} + \dot{\theta}_{\mathrm{wl}} \right),$$

$$= b_{\mathrm{v}}(v_{\mathrm{r}} + v_{\mathrm{l}}) - 2g_{\mathrm{r}}^2 J_{\mathrm{m}} \frac{\mathrm{d}}{\mathrm{d}t}\dot{\theta}_{\mathrm{m}} - 2d_{\mathrm{b}}\dot{\theta}_{\mathrm{m}} - 2d_{\mathrm{w}}\dot{\theta}_{\mathrm{w}}, \tag{A.164}$$

$$\tau_{\phi} = \frac{d}{r}(\tau_{\mathrm{r}} - \tau_{\mathrm{l}}) \tag{A.165}$$

$$= \frac{d}{r} b_{\mathrm{v}}(v_{\mathrm{r}} - v_{\mathrm{l}}) - g_{\mathrm{r}}^2 J_{\mathrm{m}} \frac{d}{r} \left(\frac{\mathrm{d}}{\mathrm{d}t}\dot{\theta}_{\mathrm{mr}} - \frac{\mathrm{d}}{\mathrm{d}t}\dot{\theta}_{\mathrm{ml}} \right) - \frac{d}{r} d_{\mathrm{b}} \left(\dot{\theta}_{\mathrm{mr}} - \dot{\theta}_{\mathrm{ml}} \right) + \frac{d}{r} d_{\mathrm{w}} \left(\dot{\theta}_{\mathrm{wr}} - \dot{\theta}_{\mathrm{wl}} \right) \tag{A.166}$$

$$= \frac{d}{r} b_{\mathrm{v}}(v_{\mathrm{r}} - v_{\mathrm{l}}) - g_{\mathrm{r}}^2 J_{\mathrm{m}} \frac{d}{r} \left(\frac{\mathrm{d}}{\mathrm{d}t}\dot{\theta}_{\mathrm{wr}} - \frac{\mathrm{d}}{\mathrm{d}t}\dot{\theta}_{\mathrm{wl}} \right) - \frac{d}{r} d_{\mathrm{b}} \left(\dot{\theta}_{\mathrm{wr}} - \dot{\theta}_{\mathrm{wl}} \right) + \frac{d}{r} d_{\mathrm{w}} \left(\dot{\theta}_{\mathrm{wr}} - \dot{\theta}_{\mathrm{wl}} \right) \tag{A.167}$$

To decompose the motion by the change of variables, we define

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} v_{\mathrm{r}} + v_{\mathrm{l}} \\ v_{\mathrm{r}} - v_{\mathrm{l}} \end{bmatrix}. \tag{A.168}$$

Hence, (A.164) and (A.167) become

$$\tau_{\theta_{\mathrm{m}}} = b_{\mathrm{v}}(v_{\mathrm{r}} + v_{\mathrm{l}}) - 2g_{\mathrm{r}}^2 J_{\mathrm{m}} \frac{\mathrm{d}}{\mathrm{d}t}\dot{\theta}_{\mathrm{w}} + 2g_{\mathrm{r}}^2 J_{\mathrm{m}} \frac{\mathrm{d}}{\mathrm{d}t}\dot{\theta}_{\mathrm{b}} + 2d_{\mathrm{b}}\dot{\theta}_{\mathrm{b}} - 2(d_{\mathrm{b}} + d_{\mathrm{w}})\dot{\theta}_{\mathrm{w}}, \tag{A.169}$$

$$= 2b_{\mathrm{v}}u_1 - 2g_{\mathrm{r}}^2 J_{\mathrm{m}} \frac{\mathrm{d}}{\mathrm{d}t}\dot{\theta}_{\mathrm{w}} + 2g_{\mathrm{r}}^2 J_{\mathrm{m}} \frac{\mathrm{d}}{\mathrm{d}t}\dot{\theta}_{\mathrm{b}} - 2(d_{\mathrm{b}} + d_{\mathrm{w}})\dot{\theta}_{\mathrm{w}} + 2d_{\mathrm{b}}\dot{\theta}_{\mathrm{b}}, \tag{A.170}$$

$$\tau_{\phi} = \frac{d}{r} b_{\mathrm{v}}(v_{\mathrm{r}} - v_{\mathrm{l}}) - g_{\mathrm{r}}^2 J_{\mathrm{m}} \frac{d}{r} \left(\frac{\mathrm{d}}{\mathrm{d}t}\dot{\theta}_{\mathrm{wr}} - \frac{\mathrm{d}}{\mathrm{d}t}\dot{\theta}_{\mathrm{wl}} \right) - \frac{d}{r}(d_{\mathrm{b}} + d_{\mathrm{w}})(\dot{\theta}_{\mathrm{wr}} - \dot{\theta}_{\mathrm{wl}}) \tag{A.171}$$

$$= \frac{2d}{r} b_{\mathrm{v}}u_2 - 2g_{\mathrm{r}}^2 J_{\mathrm{m}} \frac{d^2}{r^2} \frac{\mathrm{d}}{\mathrm{d}t}\dot{\phi} - 2\frac{d^2}{r^2}(d_{\mathrm{b}} + d_{\mathrm{w}})\dot{\phi}. \tag{A.172}$$

From the above, the equation of motion is

$$J_{11}\ddot{\theta}_{\mathrm{w}} + J_{12}(\theta_{\mathrm{b}})\ddot{\theta}_{\mathrm{b}} - (M_{\mathrm{b}} l r \sin\theta_{\mathrm{b}})\dot{\theta}_{\mathrm{b}}^2 - 2d_{\mathrm{b}}\dot{\theta}_{\mathrm{b}} + 2(d_{\mathrm{b}} + d_{\mathrm{w}})\dot{\theta}_{\mathrm{w}} = 2b_{\mathrm{v}}u_1, \tag{A.173}$$

$$J_{22}\ddot{\theta}_{\mathrm{b}} + J_{12}(\theta_{\mathrm{b}})\ddot{\theta}_{\mathrm{w}} - M_{\mathrm{b}} l \sin\theta_{\mathrm{b}}(g + l\dot{\phi}^2\cos\theta_{\mathrm{b}})$$
$$+ 2d_{\mathrm{b}}\dot{\theta}_{\mathrm{b}} - 2(d_{\mathrm{b}} + d_{\mathrm{w}})\dot{\theta}_{\mathrm{w}} = -2b_{\mathrm{v}}u_1, \tag{A.174}$$

$$J_2(\theta_{\mathrm{b}})\ddot{\phi} + (2M_{\mathrm{b}} l^2 \sin\theta_{\mathrm{b}}\cos\theta_{\mathrm{b}})\dot{\theta}_{\mathrm{b}}\dot{\phi} + 2\frac{d^2}{r^2}(d_{\mathrm{b}} + d_{\mathrm{w}})\dot{\phi} = \frac{d}{r}2b_{\mathrm{v}}u_2, \tag{A.175}$$

where

$$J_{11} := \bar{J}_{11} + 2g_{\mathrm{r}}^2 J_{\mathrm{m}} \tag{A.176}$$

$$J_{12}(\theta_{\mathrm{b}}) := \bar{J}_{12}(\theta_{\mathrm{b}}) - 2g_{\mathrm{r}}^2 J_{\mathrm{m}} \tag{A.177}$$

$$\bar{J}_{22} := J_{22} + 2g_{\mathrm{r}}^2 J_{\mathrm{m}} \tag{A.178}$$

$$J_2(\theta_{\mathrm{b}}) := \bar{J}_2(\theta_{\mathrm{b}}) + 2\frac{d^2}{r^2} g_{\mathrm{r}}^2 J_{\mathrm{m}} \tag{A.179}$$

To summarize, the equation of motion can be derived as

$$\begin{cases} J_1(\theta_{\mathrm{b}}(t)) \begin{bmatrix} \ddot{\theta}_{\mathrm{w}}(t) \\ \ddot{\theta}_{\mathrm{b}}(t) \end{bmatrix} + D_1 \begin{bmatrix} \dot{\theta}_{\mathrm{w}}(t) \\ \dot{\theta}_{\mathrm{b}}(t) \end{bmatrix} - M_{\mathrm{b}} l \sin\theta_{\mathrm{b}}(t) \begin{bmatrix} r\dot{\theta}_{\mathrm{b}}^2(t) \\ g + l\cos\theta_{\mathrm{b}}(t)\dot{\phi}^2(t) \end{bmatrix} = H_1 u(t), \\ J_2(\theta_{\mathrm{b}}(t))\ddot{\phi}(t) + D_2\dot{\phi}(t) + \left(2M_{\mathrm{b}} l^2 \sin\theta_{\mathrm{b}}(t)\cos\theta_{\mathrm{b}}(t)\right)\dot{\theta}_{\mathrm{b}}(t)\dot{\phi}(t) = H_2 u(t), \end{cases} \tag{A.180}$$

where

$$J_1(\theta_{\mathrm{b}}) = \begin{bmatrix} J_{11} & J_{12}(\theta_{\mathrm{b}}) \\ J_{12}(\theta_{\mathrm{b}}) & J_{22} \end{bmatrix},\ D_1 = 2\begin{bmatrix} d_{\mathrm{b}} + d_{\mathrm{w}} & -d_{\mathrm{b}} \\ -d_{\mathrm{b}} - d_{\mathrm{w}} & d_{\mathrm{b}} \end{bmatrix},\ H_1 = 2b_{\mathrm{v}}\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \tag{A.181}$$

$$J_2(\theta_{\mathrm{b}}) = J_2(\theta_{\mathrm{b}}),\ D_2 = 2\frac{d^2}{r^2}(d_{\mathrm{b}} + d_{\mathrm{w}}),\ H_2 = 2b_{\mathrm{v}}\frac{d}{r}\begin{bmatrix} 0 & 1 \end{bmatrix}. \tag{A.182}$$

We linearized the equations of motion (A.180) around equilibrium states $\theta_{\mathrm{w}} = 0$, $\theta_{\mathrm{b}} = 0$, $\phi = 0$, $\dot{\theta}_{\mathrm{w}} = 0$, $\dot{\theta}_{\mathrm{b}} = 0$, $\dot{\phi} = 0$, and $u = 0$. Then, under the assumption that $\sin\theta_{\mathrm{b}}(t) \approx \theta_{\mathrm{b}}(t)$, $\cos\theta_{\mathrm{b}}(t) \approx 1$, $\sin^2\theta_{\mathrm{b}}(t) \approx 0$, $\dot{\theta}_{\mathrm{b}}^2(t) \approx 0$, $\dot{\phi}^2(t) \approx 0$, and $\sin\theta_{\mathrm{b}}(t)\cos\theta_{\mathrm{b}}(t)\dot{\theta}_{\mathrm{b}}(t)\dot{\phi}(t) \approx 0$, the linearized equations of motion can be derived as

$$\begin{cases} J_1\ddot{x}_a(t) + D_1\dot{x}_a(t) + K_1 x_a(t) = H_1 u(t), \\ J_2\ddot{x}_b(t) + D_2\dot{x}_b(t) = H_2 u(t), \end{cases} \tag{A.183}$$

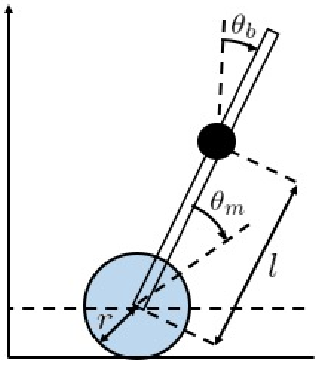

Figure A.2: Coordinates of the 2D self-balancing robot.

where

$$x_a(t) := \begin{bmatrix} \theta_{\mathrm{w}}(t) \\ \theta_{\mathrm{b}}(t) \end{bmatrix}, \quad x_b(t) := \phi(t), \tag{A.184}$$

$$J_1 = J_1(0) = \begin{bmatrix} J_{11} & J_{12}(0) \\ J_{12}(0) & J_{22} \end{bmatrix} \tag{A.185}$$

$$J_2 = J_2(0) = J_2(0) = 2M_{\mathrm{w}}d^2 + J_\phi + 2\frac{d^2}{r^2}\left(J_{\mathrm{w}} + g_{\mathrm{r}}^2 J_{\mathrm{m}}\right) \tag{A.186}$$

$$K_1 = M_{\mathrm{b}}lg \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \tag{A.187}$$

$$J_{12}(0) := M_{\mathrm{b}}lr - 2g_{\mathrm{r}}^2 J_{\mathrm{m}}. \tag{A.188}$$

A.4 Equation of motion in 2D

The motion of the self-balancing robot is restricted in 2D to make $\ddot{\phi}(t) = 0, \dot{\phi}(t) = 0$, and $v_{\mathrm{r}} = v_{\mathrm{l}}$. Then, $\psi_{\mathrm{r}}, \psi_{\mathrm{l}}, \theta_{\mathrm{wr}}, \theta_{\mathrm{wl}}$, and θ_{b} in the side view from the right wheel. From

$\psi_{\mathrm{r,l}}(t)$ and the body pitch angle $\theta_{\mathrm{b}}(t)$, we can calculate the wheel angle as

$$\theta_{\mathrm{wr}}(t) = \theta_{\mathrm{m}}(t) + \theta_{\mathrm{b}}(t),\ \theta_{\mathrm{m}}(t) = \frac{1}{g_{\mathrm{r}}}\psi_{\mathrm{r}}(t), \tag{A.189}$$

$$\theta_{\mathrm{wl}}(t) = \theta_{\mathrm{wr}}(t),\ \psi_{\mathrm{r}}(t) = \psi_{\mathrm{l}}(t) \tag{A.190}$$

$$\theta_{\mathrm{w}} = \theta_{\mathrm{wr}}, \tag{A.191}$$

$$\phi = \frac{r}{w}(\theta_{\mathrm{wr}} - \theta_{\mathrm{wl}}) = 0, \tag{A.192}$$

$$\dot{\theta}_{\mathrm{w}} = \frac{1}{2}(\dot{\theta}_{\mathrm{wr}} + \dot{\theta}_{\mathrm{wl}}) = \dot{\theta}_{\mathrm{wr}}, \tag{A.193}$$

$$\dot{\phi} = \frac{r}{2d}(\dot{\theta}_{\mathrm{wr}} - \dot{\theta}_{\mathrm{wl}}) = 0. \tag{A.194}$$

The coordinates are assumed to be

$$y_w = 0,\ y_r = 0,\ y_b = 0,\ \dot{y}_w = 0,\ \dot{y}_r = 0,\ \dot{y}_b = 0, \tag{A.195}$$

$$\begin{bmatrix} x_w \\ z_w \end{bmatrix} = \begin{bmatrix} x_r \\ z_r \end{bmatrix},\ \begin{bmatrix} x_r \\ z_r \end{bmatrix} = \begin{bmatrix} x_w \\ r \end{bmatrix},\ \begin{bmatrix} x_b \\ z_b \end{bmatrix} = \begin{bmatrix} x_w + l\sin\theta_{\mathrm{b}} \\ z_w + l\cos\theta_{\mathrm{b}} \end{bmatrix}. \tag{A.196}$$

From $\dot{x}_w = r\dot{\theta}_{\mathrm{w}}$ and $\dot{z}_w = 0$,

$$\begin{bmatrix} \dot{x}_w \\ \dot{z}_w \end{bmatrix} = \begin{bmatrix} \dot{x}_r \\ \dot{z}_r \end{bmatrix} = \begin{bmatrix} r\dot{\theta}_{\mathrm{w}} \\ 0 \end{bmatrix}, \tag{A.197}$$

$$\begin{bmatrix} \dot{x}_b \\ \dot{z}_b \end{bmatrix} = \begin{bmatrix} \dot{x}_w + l\dot{\theta}_{\mathrm{b}}\cos\theta_{\mathrm{b}} \\ -l\dot{\theta}_{\mathrm{b}}\sin\theta_{\mathrm{b}} \end{bmatrix} = \begin{bmatrix} r\dot{\theta}_{\mathrm{w}} + l\dot{\theta}_{\mathrm{b}}\cos\theta_{\mathrm{b}} \\ -l\dot{\theta}_{\mathrm{b}}\sin\theta_{\mathrm{b}} \end{bmatrix}. \tag{A.198}$$

Kinetic energy of the right and left wheel are given by

$$T_r = \frac{1}{2}M_{\mathrm{w}}\dot{x}_r^2 + \frac{1}{2}J_{\mathrm{w}}\dot{\theta}_{\mathrm{wr}}^2$$
$$T_l = \frac{1}{2}M_{\mathrm{w}}\dot{x}_l^2 + \frac{1}{2}J_{\mathrm{w}}\dot{\theta}_{\mathrm{wl}}^2.$$

Kinetic energy of the right and left motors are given by

$$T_{mr} = \frac{1}{2}J_{\mathrm{m}}\dot{\psi}_{\mathrm{r}}^2 = \frac{1}{2}J_{\mathrm{m}}\left(g_{\mathrm{r}}\dot{\theta}_{\mathrm{mr}}\right)^2,$$
$$T_{ml} = \frac{1}{2}J_{\mathrm{m}}\dot{\psi}_{\mathrm{l}}^2 = \frac{1}{2}J_{\mathrm{m}}\left(g_{\mathrm{r}}\dot{\theta}_{\mathrm{ml}}\right)^2.$$

Here, we ignore mass of the motors. Kinetic energy of the body is given by

$$T_b = \frac{1}{2}M_{\mathrm{b}}(\dot{x}_b^2 + \dot{z}_b^2) + \frac{1}{2}J_{\mathrm{b}}\dot{\theta}_{\mathrm{b}}^2$$

On the other hand, potential energy is

$$V = M_{\mathrm{w}} g z_l + M_{\mathrm{w}} g z_r + M_{\mathrm{b}} g z_b = 2M_{\mathrm{w}} g r + M_{\mathrm{b}} g(r + l\cos\theta_{\mathrm{b}}). \tag{A.199}$$

Hence, the Lagrangian is given as

$$\begin{aligned}
L &= T_r + T_l + T_{mr} + T_{ml} + T_b - V \\
&= \frac{1}{2}M_{\mathrm{w}}(\dot{x}_r^2 + \dot{x}_l^2) + \frac{1}{2}J_{\mathrm{w}}(\dot{\theta}_{\mathrm{wr}}^2 + \dot{\theta}_{\mathrm{wl}}^2) + \frac{1}{2}g_{\mathrm{r}}^2 J_{\mathrm{m}}(\dot{\theta}_{\mathrm{mr}}^2 + \dot{\theta}_{\mathrm{ml}}^2) \\
&\quad + \frac{1}{2}M_{\mathrm{b}}(\dot{x}_b^2 + \dot{z}_b^2) + \frac{1}{2}J_{\mathrm{b}}\dot{\theta}_{\mathrm{b}}^2 - 2M_{\mathrm{w}} g r - M_{\mathrm{b}} g(r + l\cos\theta_{\mathrm{b}}) \\
&= M_{\mathrm{w}}\dot{x}_r^2 + J_{\mathrm{w}}\dot{\theta}_{\mathrm{w}}^2 + g_{\mathrm{r}}^2 J_{\mathrm{m}}\dot{\theta}_{\mathrm{m}}^2 + \frac{1}{2}M_{\mathrm{b}}(\dot{x}_b^2 + \dot{z}_b^2) + \frac{1}{2}J_{\mathrm{b}}\dot{\theta}_{\mathrm{b}}^2 - 2M_{\mathrm{w}} g r - M_{\mathrm{b}} g(r + l\cos\theta_{\mathrm{b}}).
\end{aligned}$$

Note that we can rewrite the terms as

$$\begin{aligned}
&\dot{x}_l^2 = \dot{x}_r^2 = \left(r\dot{\theta}_{\mathrm{w}}\right)^2 = r^2\dot{\theta}_{\mathrm{w}}^2, \\
&\dot{x}_b^2 + \dot{z}_b^2 = \left(r\dot{\theta}_{\mathrm{w}} + l\dot{\theta}_{\mathrm{b}}\cos\theta_{\mathrm{b}}\right)^2 + \left(l\dot{\theta}_{\mathrm{b}}\sin\theta_{\mathrm{b}}\right)^2 = r^2\dot{\theta}_{\mathrm{w}}^2 + 2lr\dot{\theta}_{\mathrm{w}}\dot{\theta}_{\mathrm{b}}\cos\theta_{\mathrm{b}} + l^2\dot{\theta}_{\mathrm{b}}^2
\end{aligned}$$

Hence, for the generalized coordinate $(\theta_{\mathrm{w}}, \theta_{\mathrm{b}})$, the Lagrangian is

$$\begin{aligned}
&L(\dot{\theta}_{\mathrm{w}}, \dot{\theta}_{\mathrm{b}}, \theta_{\mathrm{w}}, \theta_{\mathrm{b}}) \\
&= M_{\mathrm{w}} r^2\dot{\theta}_{\mathrm{w}}^2 + J_{\mathrm{w}}\dot{\theta}_{\mathrm{w}}^2 + g_{\mathrm{r}}^2 J_{\mathrm{m}}\left(\dot{\theta}_{\mathrm{w}} - \dot{\theta}_{\mathrm{b}}\right)^2 + \frac{1}{2}M_{\mathrm{b}}\left(r^2\dot{\theta}_{\mathrm{w}}^2 + 2lr\dot{\theta}_{\mathrm{w}}\dot{\theta}_{\mathrm{b}}\cos\theta_{\mathrm{b}} + l^2\dot{\theta}_{\mathrm{b}}^2\right) \\
&\quad + \frac{1}{2}J_{\mathrm{b}}\dot{\theta}_{\mathrm{b}}^2 - 2M_{\mathrm{w}} g r - M_{\mathrm{b}} g(r + l\cos\theta_{\mathrm{b}}) \\
&= \left(\left(M_{\mathrm{w}} + \frac{M_{\mathrm{b}}}{2}\right)r^2 + J_{\mathrm{w}} + g_{\mathrm{r}}^2 J_{\mathrm{m}}\right)\dot{\theta}_{\mathrm{w}}^2 + \left(M_{\mathrm{b}} lr\cos\theta_{\mathrm{b}} - 2g_{\mathrm{r}}^2 J_{\mathrm{m}}\right)\dot{\theta}_{\mathrm{w}}\dot{\theta}_{\mathrm{b}} \\
&\quad + \left(\frac{M_{\mathrm{b}}}{2}l^2 + \frac{1}{2}J_{\mathrm{b}} + g_{\mathrm{r}}^2 J_{\mathrm{m}}\right)\dot{\theta}_{\mathrm{b}}^2 - 2M_{\mathrm{w}} g r - M_{\mathrm{b}} g(r + l\cos\theta_{\mathrm{b}}) \\
&= \frac{1}{2}J_{11}\dot{\theta}_{\mathrm{w}}^2 + J_{12}(\theta_{\mathrm{b}})\dot{\theta}_{\mathrm{w}}\dot{\theta}_{\mathrm{b}} + \frac{1}{2}J_{22}\dot{\theta}_{\mathrm{b}}^2 - 2M_{\mathrm{w}} g r - M_{\mathrm{b}} g(r + l\cos\theta_{\mathrm{b}}),
\end{aligned}$$

where

$$J_{11} := (2M_{\mathrm{w}} + M_{\mathrm{b}})\, r^2 + 2\left(J_{\mathrm{w}} + g_{\mathrm{r}}^2 J_{\mathrm{m}}\right), \tag{A.200}$$

$$J_{12}(\theta_{\mathrm{b}}) := M_{\mathrm{b}} lr\cos\theta_{\mathrm{b}} - 2g_{\mathrm{r}}^2 J_{\mathrm{m}}, \tag{A.201}$$

$$J_{22} := M_{\mathrm{b}} l^2 + J_{\mathrm{b}} + 2g_{\mathrm{r}}^2 J_{\mathrm{m}}. \tag{A.202}$$

A.4.1 Case 1: the generalized coordinate $(\theta_{\mathrm{w}}, \theta_{\mathrm{b}})$

$$
\begin{aligned}
\frac{\partial L}{\partial \theta_{\mathrm{w}}} &= 0, \\
\frac{\partial L}{\partial \theta_{\mathrm{b}}} &= -M_{\mathrm{b}} l r \dot{\theta}_{\mathrm{w}} \dot{\theta}_{\mathrm{b}} \sin\theta_{\mathrm{b}} + M_{\mathrm{b}} l g \sin\theta_{\mathrm{b}}, \\
\frac{\partial L}{\partial \dot{\theta}_{\mathrm{w}}} &= J_{11} \dot{\theta}_{\mathrm{w}} + J_{12}(\theta_{\mathrm{b}}) \dot{\theta}_{\mathrm{b}}, \\
\frac{\partial L}{\partial \dot{\theta}_{\mathrm{b}}} &= J_{12}(\theta_{\mathrm{b}}) \dot{\theta}_{\mathrm{w}} + J_{22} \dot{\theta}_{\mathrm{b}},
\end{aligned}
$$

Furthermore,

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}_{\mathrm{w}}}\right) = J_{11} \ddot{\theta}_{\mathrm{w}} + J_{12}(\theta_{\mathrm{b}}) \ddot{\theta}_{\mathrm{b}} - (M_{\mathrm{b}} l r \sin\theta_{\mathrm{b}}) \dot{\theta}_{\mathrm{b}}^2, \tag{A.203}
$$

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}_{\mathrm{b}}}\right) = J_{12}(\theta_{\mathrm{b}}) \ddot{\theta}_{\mathrm{w}} + J_{22} \ddot{\theta}_{\mathrm{b}} - (M_{\mathrm{b}} l r \sin\theta_{\mathrm{b}}) \dot{\theta}_{\mathrm{w}} \dot{\theta}_{\mathrm{b}}. \tag{A.204}
$$

Hence, by the Lagrangian equation

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}_{\mathrm{w}}}\right) - \frac{\partial L}{\partial \theta_{\mathrm{w}}} = \tau_{\theta_m} \tag{A.205}
$$

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}_{\mathrm{b}}}\right) - \frac{\partial L}{\partial \theta_{\mathrm{b}}} = -\tau_{\theta_m}, \tag{A.206}
$$

we can derive the equation of motion,

$$
J_{11} \ddot{\theta}_{\mathrm{w}} + J_{12}(\theta_{\mathrm{b}}) \ddot{\theta}_{\mathrm{b}} - (M_{\mathrm{b}} l r \sin\theta_{\mathrm{b}}) \dot{\theta}_{\mathrm{b}}^2 = \tau_{\theta_m}, \tag{A.207}
$$

$$
J_{12}(\theta_{\mathrm{b}}) \ddot{\theta}_{\mathrm{w}} + J_{22} \ddot{\theta}_{\mathrm{b}} - M_{\mathrm{b}} l g \sin\theta_{\mathrm{b}} = -\tau_{\theta_m}, \tag{A.208}
$$

where τ_{θ_w} and $\tau_{\theta_{\mathrm{b}}}$ are generalized force (torque).

Under these restrictions,

$$
\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} v_{\mathrm{r}} \\ 0 \end{bmatrix}, \tag{A.209}
$$

then, (A.175) yields a trivial equation. Furthermore, we define

$$
u = v_r, \tag{A.210}
$$

then (A.220) and (A.221) become

$$
J_{11} \ddot{\theta}_{\mathrm{w}} + J_{12}(\theta_{\mathrm{b}}) \ddot{\theta}_{\mathrm{b}} - 2 d_{\mathrm{b}} \dot{\theta}_{\mathrm{b}} + 2\left(d_{\mathrm{b}} + g_{\mathrm{r}} d_{\mathrm{w}}\right) \dot{\theta}_{\mathrm{w}} - (M_{\mathrm{b}} l r \sin\theta_{\mathrm{b}}) \dot{\theta}_{\mathrm{b}}^2 = 2 b_{\mathrm{v}} u, \tag{A.211}
$$

$$
J_{12}(\theta_{\mathrm{b}}) \ddot{\theta}_{\mathrm{w}} + J_{22} \ddot{\theta}_{\mathrm{b}} - M_{\mathrm{b}} l g \sin\theta_{\mathrm{b}} = 0. \tag{A.212}
$$

A.4.2 Case 2: the generalized coordinate $(\theta_\mathrm{m}, \theta_\mathrm{b})$

For the generalized coordinate $(\theta_\mathrm{m}, \theta_\mathrm{b})$, we obtain another Lagrangian

$$\begin{aligned}
&L(\dot\theta_\mathrm{m}, \dot\theta_\mathrm{b}, \theta_\mathrm{m}, \theta_\mathrm{b}) \\
&= \frac{1}{2}J_{11}\left(\dot\theta_\mathrm{m} + \dot\theta_\mathrm{b}\right)^2 + J_{12}(\theta_\mathrm{b})\left(\dot\theta_\mathrm{m} + \dot\theta_\mathrm{b}\right)\dot\theta_\mathrm{b} + \frac{1}{2}J_{22}\dot\theta_\mathrm{b}^2 - 2M_\mathrm{w}gr - M_\mathrm{b}g(r + l\cos\theta_\mathrm{b}) \\
&= \frac{1}{2}J_{11}\dot\theta_\mathrm{m}^2 + J'_{12}(\theta_\mathrm{b})\dot\theta_\mathrm{m}\dot\theta_\mathrm{b} + \frac{1}{2}J'_{22}(\theta_\mathrm{b})\dot\theta_\mathrm{b}^2 - 2M_\mathrm{w}gr - M_\mathrm{b}g(r + l\cos\theta_\mathrm{b})
\end{aligned}$$

where

$$J_{11} := (2M_\mathrm{w} + M_\mathrm{b})\, r^2 + 2\left(J_\mathrm{w} + g_\mathrm{r}^2 J_\mathrm{m}\right), \tag{A.213}$$

$$J'_{12}(\theta_\mathrm{b}) := J_{11} + J_{12}(\theta_\mathrm{b}) = (2M_\mathrm{w} + M_\mathrm{b})\, r^2 + M_\mathrm{b}lr\cos\theta_\mathrm{b} + 2J_\mathrm{w}, \tag{A.214}$$

$$\begin{aligned}
J'_{22}(\theta_\mathrm{b}) &:= J_{11} + 2J_{12}(\theta_\mathrm{b}) + J_{22} \\
&= (2M_\mathrm{w} + M_\mathrm{b})\, r^2 + M_\mathrm{b}l^2 + 2M_\mathrm{b}lr\cos\theta_\mathrm{b} + J_\mathrm{b} + 2J_\mathrm{w}.
\end{aligned} \tag{A.215}$$

Then,

$$\begin{aligned}
\frac{\partial L}{\partial \theta_\mathrm{m}} &= 0, \\
\frac{\partial L}{\partial \theta_\mathrm{b}} &= -M_\mathrm{b}lr\dot\theta_\mathrm{m}\dot\theta_\mathrm{b}\sin\theta_\mathrm{b} - M_\mathrm{b}lr\sin\theta_\mathrm{b}\dot\theta_\mathrm{b}^2 + M_\mathrm{b}lg\sin\theta_\mathrm{b}, \\
\frac{\partial L}{\partial \dot\theta_\mathrm{m}} &= J_{11}\dot\theta_\mathrm{m} + J'_{12}(\theta_\mathrm{b})\dot\theta_\mathrm{b}, \\
\frac{\partial L}{\partial \dot\theta_\mathrm{b}} &= J'_{12}(\theta_\mathrm{b})\dot\theta_\mathrm{w} + J'_{22}(\theta_\mathrm{b})\dot\theta_\mathrm{b},
\end{aligned}$$

Furthermore,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot\theta_\mathrm{m}}\right) = J_{11}\ddot\theta_\mathrm{m} + J'_{12}(\theta_\mathrm{b})\ddot\theta_\mathrm{b} - (M_\mathrm{b}lr\sin\theta_\mathrm{b})\dot\theta_\mathrm{b}^2, \tag{A.216}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot\theta_\mathrm{b}}\right) = J'_{12}(\theta_\mathrm{b})\ddot\theta_\mathrm{m} + J'_{22}(\theta_\mathrm{b})\ddot\theta_\mathrm{b} - (M_\mathrm{b}lr\sin\theta_\mathrm{b})\dot\theta_\mathrm{m}\dot\theta_\mathrm{b} - 2(M_\mathrm{b}lr\sin\theta_\mathrm{b})\dot\theta_\mathrm{b}^2. \tag{A.217}$$

Hence, by the Lagrangian equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot\theta_\mathrm{m}}\right) - \frac{\partial L}{\partial \theta_\mathrm{m}} = \tau_{\theta_m} \tag{A.218}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot\theta_\mathrm{b}}\right) - \frac{\partial L}{\partial \theta_\mathrm{b}} = 0, \tag{A.219}$$

we can derive the equation of motion,

$$J_{11}\ddot{\theta}_{\mathrm{m}} + J'_{12}(\theta_{\mathrm{b}})\ddot{\theta}_{\mathrm{b}} - (M_{\mathrm{b}}lr\sin\theta_{\mathrm{b}})\dot{\theta}_{\mathrm{b}}^2 = \tau_{\theta_m}, \tag{A.220}$$

$$J'_{12}(\theta_{\mathrm{b}})\ddot{\theta}_{\mathrm{m}} + J'_{22}(\theta_{\mathrm{b}})\ddot{\theta}_{\mathrm{b}} - (M_{\mathrm{b}}lr\sin\theta_{\mathrm{b}})\dot{\theta}_{\mathrm{b}}^2 - M_{\mathrm{b}}lg\sin\theta_{\mathrm{b}} = 0, \tag{A.221}$$

where τ_{θ_w} and $\tau_{\theta_{\mathrm{b}}}$ are generalized force (torque).

$$\tau_{\theta_m} = 2b_{\mathrm{v}}u_1 + 2d_{\mathrm{b}}\dot{\theta}_{\mathrm{b}} - 2\,(d_{\mathrm{b}} + d_{\mathrm{w}})\,(\dot{\theta}_{\mathrm{m}} + \dot{\theta}_{\mathrm{b}}), \tag{A.222}$$

$$= 2b_{\mathrm{v}}u_1 - 2\,(d_{\mathrm{b}} + d_{\mathrm{w}})\,\dot{\theta}_{\mathrm{m}} - 2d_{\mathrm{w}}\dot{\theta}_{\mathrm{b}}, \tag{A.223}$$

$$J_{11}\ddot{\theta}_{\mathrm{m}} + J'_{12}(\theta_{\mathrm{b}})\ddot{\theta}_{\mathrm{b}} + 2\,(d_{\mathrm{b}} + d_{\mathrm{w}})\,\dot{\theta}_{\mathrm{m}} + 2d_{\mathrm{w}}\dot{\theta}_{\mathrm{b}} - (M_{\mathrm{b}}lr\sin\theta_{\mathrm{b}})\dot{\theta}_{\mathrm{b}}^2 = 2b_{\mathrm{v}}u, \tag{A.224}$$

$$J'_{12}(\theta_{\mathrm{b}})\ddot{\theta}_{\mathrm{m}} + J'_{22}(\theta_{\mathrm{b}})\ddot{\theta}_{\mathrm{b}} - (M_{\mathrm{b}}lr\sin\theta_{\mathrm{b}})\dot{\theta}_{\mathrm{b}}^2 - M_{\mathrm{b}}lg\sin\theta_{\mathrm{b}} = 0. \tag{A.225}$$

$$\begin{bmatrix} J_{11} & J'_{12}(\theta_{\mathrm{b}}) \\ J'_{12}(\theta_{\mathrm{b}}) & J'_{22}(\theta_{\mathrm{b}}) \end{bmatrix}\begin{bmatrix} \ddot{\theta}_{\mathrm{m}}(t) \\ \ddot{\theta}_{\mathrm{b}}(t) \end{bmatrix} + \begin{bmatrix} 2d_{\mathrm{b}} + 2d_{\mathrm{w}} & 2d_{\mathrm{w}} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{\theta}_{\mathrm{m}}(t) \\ \dot{\theta}_{\mathrm{b}}(t) \end{bmatrix}$$
$$- M_{\mathrm{b}}lr\sin\theta_{\mathrm{b}}(t)\begin{bmatrix} 1 \\ 1 \end{bmatrix}\dot{\theta}_{\mathrm{b}}^2(t) - M_{\mathrm{b}}gl\sin\theta_{\mathrm{b}}(t)\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2b_{\mathrm{v}} \\ 0 \end{bmatrix}u. \tag{A.226}$$

By changing the constants as

$$J_{\mathrm{w}} \leftarrow 2J_{\mathrm{w}}, \tag{A.227}$$

$$J_{\mathrm{m}} \leftarrow 2J_{\mathrm{m}}, \tag{A.228}$$

$$M_{\mathrm{w}} \leftarrow 2M_{\mathrm{w}}, \tag{A.229}$$

$$d_{\mathrm{b}} \leftarrow 2d_{\mathrm{b}}, \tag{A.230}$$

$$d_{\mathrm{w}} \leftarrow 2d_{\mathrm{w}}, \tag{A.231}$$

$$b_{\mathrm{v}} \leftarrow 2b_{\mathrm{v}}, \tag{A.232}$$

We linearized the equations of motion (A.226) around equilibrium states $\theta_{\mathrm{w}} = 0$, $\theta_{\mathrm{b}} = 0$, $\dot{\theta}_{\mathrm{w}} = 0$, $\dot{\theta}_{\mathrm{b}} = 0$, and $u = 0$. Then, under the assumption that $\sin\theta_{\mathrm{b}} \approx \theta_{\mathrm{b}}$, $\cos\theta_{\mathrm{b}} \approx 1$, $\sin^2\theta_{\mathrm{b}} \approx 0$, and $\theta_{\mathrm{b}}^2 \approx 0$, the linearized equations of motion can be derived as

$$J\begin{bmatrix} \ddot{\theta}_{\mathrm{w}} \\ \ddot{\theta}_{\mathrm{b}} \end{bmatrix} + D\begin{bmatrix} \dot{\theta}_{\mathrm{w}} \\ \dot{\theta}_{\mathrm{b}} \end{bmatrix} - K\begin{bmatrix} \theta_{\mathrm{w}} \\ \theta_{\mathrm{b}} \end{bmatrix} = \zeta u(t), \tag{A.233}$$

where

$$J = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} := \begin{bmatrix} J'_{11} & J'_{12} + M_\mathrm{b} lr \\ J'_{21} + M_\mathrm{b} lr & J'_{22} + 2M_\mathrm{b} lr \end{bmatrix}, \tag{A.234}$$

$$D = \begin{bmatrix} c & 0 \\ 0 & 0 \end{bmatrix},\ K = M_\mathrm{b} lg \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix},\ \zeta = \begin{bmatrix} a \\ 0 \end{bmatrix}, \tag{A.235}$$

The state space model can be obtained as

$$\dot{x}(t) = A_c x(t) + B_c u(t), \tag{A.236}$$

where

$$A_c = \begin{bmatrix} 0_{2\times 2} & I_{2\times 2} \\ -J^{-1}K & -J^{-1}D \end{bmatrix},\ B_c = \begin{bmatrix} 0_{2\times 1} \\ J^{-1}\zeta \end{bmatrix}. \tag{A.237}$$

A.4.3 Case 3: the generalized coordinate $(\theta_\mathrm{w}, \theta_\mathrm{b})$ in [9, 28],

$$T_{mr} = \frac{1}{2} J_\mathrm{m} \left(g_\mathrm{r} \dot{\theta}_\mathrm{mr} + \dot{\theta}_\mathrm{b} \right)^2, \tag{A.238}$$

$$T_{ml} = \frac{1}{2} J_\mathrm{m} \left(g_\mathrm{r} \dot{\theta}_\mathrm{ml} + \dot{\theta}_\mathrm{b} \right)^2. \tag{A.239}$$

Then,

$$T_{mr} + T_{ml} = J_\mathrm{m} \left(g_\mathrm{r}^2 \dot{\theta}_\mathrm{ml}^2 + 2 g_\mathrm{r} \dot{\theta}_\mathrm{ml} \dot{\theta}_\mathrm{b} + \dot{\theta}_\mathrm{b}^2 \right) = J_\mathrm{m} \left(g_\mathrm{r}^2 \dot{\theta}_\mathrm{m}^2 + 2 g_\mathrm{r} \dot{\theta}_\mathrm{m} \dot{\theta}_\mathrm{b} + \dot{\theta}_\mathrm{b}^2 \right) \tag{A.240}$$

$$= J_\mathrm{m} \left(g_\mathrm{r}^2 (\dot{\theta}_\mathrm{w} - \dot{\theta}_\mathrm{b})^2 + 2 g_\mathrm{r} (\dot{\theta}_\mathrm{w} - \dot{\theta}_\mathrm{b}) \dot{\theta}_\mathrm{b} + \dot{\theta}_\mathrm{b}^2 \right) \tag{A.241}$$

The Lagrangian is given as

$$
\begin{aligned}
&L(\dot{\theta}_{\mathrm{w}}, \dot{\theta}_{\mathrm{b}}, \theta_{\mathrm{w}}, \theta_{\mathrm{b}}) \\
&= T_r + T_l + T_{mr} + T_{ml} + T_b - V \\
&= \frac{1}{2} M_{\mathrm{w}} \left(\dot{x}_r^2 + \dot{x}_l^2 \right) + \frac{1}{2} J_{\mathrm{w}} \left(\dot{\theta}_{\mathrm{wr}}^2 + \dot{\theta}_{\mathrm{wl}}^2 \right) + J_{\mathrm{m}} \left(g_{\mathrm{r}}^2 (\dot{\theta}_{\mathrm{w}} - \dot{\theta}_{\mathrm{b}})^2 + 2 g_{\mathrm{r}} (\dot{\theta}_{\mathrm{w}} - \dot{\theta}_{\mathrm{b}}) \dot{\theta}_{\mathrm{b}} + \dot{\theta}_{\mathrm{b}}^2 \right) \\
&\quad + \frac{1}{2} M_{\mathrm{b}} \left(\dot{x}_b^2 + \dot{z}_b^2 \right) + \frac{1}{2} J_{\mathrm{b}} \dot{\theta}_{\mathrm{b}}^2 - 2 M_{\mathrm{w}} g r - M_{\mathrm{b}} g \left(r + l \cos \theta_{\mathrm{b}} \right) \\
&= M_{\mathrm{w}} \dot{x}_r^2 + J_{\mathrm{w}} \dot{\theta}_{\mathrm{w}}^2 + J_{\mathrm{m}} \left(g_{\mathrm{r}}^2 \dot{\theta}_{\mathrm{w}}^2 - 2 g_{\mathrm{r}}^2 \dot{\theta}_{\mathrm{w}} \dot{\theta}_{\mathrm{b}} + 2 g_{\mathrm{r}} \dot{\theta}_{\mathrm{w}} \dot{\theta}_{\mathrm{b}} + g_{\mathrm{r}}^2 \dot{\theta}_{\mathrm{b}}^2 - 2 g_{\mathrm{r}} \dot{\theta}_{\mathrm{b}}^2 + \dot{\theta}_{\mathrm{b}}^2 \right) \\
&\quad + \frac{1}{2} M_{\mathrm{b}} \left(r^2 \dot{\theta}_{\mathrm{w}}^2 + 2 l r \dot{\theta}_{\mathrm{w}} \dot{\theta}_{\mathrm{b}} \cos \theta_{\mathrm{b}} + l^2 \dot{\theta}_{\mathrm{b}}^2 \right) \\
&\quad + \frac{1}{2} J_{\mathrm{b}} \dot{\theta}_{\mathrm{b}}^2 - 2 M_{\mathrm{w}} g r - M_{\mathrm{b}} g \left(r + l \cos \theta_{\mathrm{b}} \right) \\
&= M_{\mathrm{w}} r^2 \dot{\theta}_{\mathrm{w}}^2 + \left(J_{\mathrm{w}} + g_{\mathrm{r}}^2 J_{\mathrm{m}} \right) \dot{\theta}_{\mathrm{w}}^2 - 2 J_{\mathrm{m}} g_{\mathrm{r}} \left(g_{\mathrm{r}} - 1 \right) \dot{\theta}_{\mathrm{w}} \dot{\theta}_{\mathrm{b}} + J_{\mathrm{m}} \left(g_{\mathrm{r}} - 1 \right)^2 \dot{\theta}_{\mathrm{b}}^2 \\
&\quad + \frac{1}{2} M_{\mathrm{b}} \left(r^2 \dot{\theta}_{\mathrm{w}}^2 + 2 l r \cos \theta_{\mathrm{b}} \dot{\theta}_{\mathrm{w}} \dot{\theta}_{\mathrm{b}} + l^2 \dot{\theta}_{\mathrm{b}}^2 \right) \\
&\quad + \frac{1}{2} J_{\mathrm{b}} \dot{\theta}_{\mathrm{b}}^2 - 2 M_{\mathrm{w}} g r - M_{\mathrm{b}} g \left(r + l \cos \theta_{\mathrm{b}} \right) \\
&= \left(\left(M_{\mathrm{w}} + \frac{M_{\mathrm{b}}}{2} \right) r^2 + J_{\mathrm{w}} + g_{\mathrm{r}}^2 J_{\mathrm{m}} \right) \dot{\theta}_{\mathrm{w}}^2 + \left(M_{\mathrm{b}} l r \cos \theta_{\mathrm{b}} - 2 g_{\mathrm{r}} (g_{\mathrm{r}} - 1) J_{\mathrm{m}} \right) \dot{\theta}_{\mathrm{w}} \dot{\theta}_{\mathrm{b}} \\
&\quad + \left(\frac{M_{\mathrm{b}}}{2} l^2 + \frac{1}{2} J_{\mathrm{b}} + \left(g_{\mathrm{r}} - 1 \right)^2 J_{\mathrm{m}} \right) \dot{\theta}_{\mathrm{b}}^2 - 2 M_{\mathrm{w}} g r - M_{\mathrm{b}} g (r + l \cos \theta_{\mathrm{b}}) \\
&= \frac{1}{2} J_{11} \dot{\theta}_{\mathrm{w}}^2 + J_{12}(\theta_{\mathrm{b}}) \dot{\theta}_{\mathrm{w}} \dot{\theta}_{\mathrm{b}} + \frac{1}{2} J_{22} \dot{\theta}_{\mathrm{b}}^2 - 2 M_{\mathrm{w}} g r - M_{\mathrm{b}} g (r + l \cos \theta_{\mathrm{b}}).
\end{aligned}
$$

where

$$J_{11} := \left(2 M_{\mathrm{w}} + M_{\mathrm{b}} \right) r^2 + 2 \left(J_{\mathrm{w}} + g_{\mathrm{r}}^2 J_{\mathrm{m}} \right), \tag{A.242}$$

$$J_{12}(\theta_{\mathrm{b}}) := M_{\mathrm{b}} l r \cos \theta_{\mathrm{b}} - 2 g_{\mathrm{r}} (g_{\mathrm{r}} - 1) J_{\mathrm{m}}, \tag{A.243}$$

$$J_{22} := M_{\mathrm{b}} l^2 + J_{\mathrm{b}} + 2 \left(g_{\mathrm{r}} - 1 \right)^2 J_{\mathrm{m}}, \tag{A.244}$$

$$
\begin{aligned}
&\frac{\partial L}{\partial \theta_{\mathrm{w}}} = 0, \\
&\frac{\partial L}{\partial \theta_{\mathrm{b}}} = -(M_{\mathrm{b}} l r \sin \theta_{\mathrm{b}}) \dot{\theta}_{\mathrm{w}} \dot{\theta}_{\mathrm{b}} + M_{\mathrm{b}} l g \sin \theta_{\mathrm{b}}, \\
&\frac{\partial L}{\partial \dot{\theta}_{\mathrm{w}}} = J_{11} \dot{\theta}_{\mathrm{w}} + J_{12}(\theta_{\mathrm{b}}) \dot{\theta}_{\mathrm{b}}, \\
&\frac{\partial L}{\partial \dot{\theta}_{\mathrm{b}}} = J_{12}(\theta_{\mathrm{b}}) \dot{\theta}_{\mathrm{w}} + J_{22} \dot{\theta}_{\mathrm{b}},
\end{aligned}
$$

Furthermore,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}_{\mathrm{w}}}\right) = J_{11}\ddot{\theta}_{\mathrm{w}} + J_{12}(\theta_{\mathrm{b}})\ddot{\theta}_{\mathrm{b}} - (M_{\mathrm{b}}lr\sin\theta_{\mathrm{b}})\dot{\theta}_{\mathrm{b}}^2, \tag{A.245}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}_{\mathrm{b}}}\right) = J_{12}(\theta_{\mathrm{b}})\ddot{\theta}_{\mathrm{w}} + J_{22}\ddot{\theta}_{\mathrm{b}} - (M_{\mathrm{b}}lr\sin\theta_{\mathrm{b}})\dot{\theta}_{\mathrm{w}}\dot{\theta}_{\mathrm{b}}, \tag{A.246}$$

Hence, by the Lagrangian equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}_{\mathrm{w}}}\right) - \frac{\partial L}{\partial \theta_{\mathrm{w}}} = J_{11}\ddot{\theta}_{\mathrm{w}} + J_{12}(\theta_{\mathrm{b}})\ddot{\theta}_{\mathrm{b}} - (M_{\mathrm{b}}lr\sin\theta_{\mathrm{b}})\dot{\theta}_{\mathrm{b}}^2 = F_1, \tag{A.247}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}_{\mathrm{b}}}\right) - \frac{\partial L}{\partial \theta_{\mathrm{b}}} = J_{12}(\theta_{\mathrm{b}})\ddot{\theta}_{\mathrm{w}} + J_{22}\ddot{\theta}_{\mathrm{b}} - M_{\mathrm{b}}lg\sin\theta_{\mathrm{b}} = F_2, \tag{A.248}$$

The external force are two torque τ_{r} and τ_{l} generated by two motors fixed on the body and affected as the torque

$$\tau_{\theta_{\mathrm{m}}} = \tau_{\mathrm{r}} + \tau_{\mathrm{l}} \tag{A.249}$$

$$\tau_{\theta_{\mathrm{b}}} = 0. \tag{A.250}$$

Then, the generalized force for $(\theta_{\mathrm{w}}, \theta_{\mathrm{b}}, \phi)$ are

$$F_1 = \frac{\partial}{\partial \theta_{\mathrm{w}}}\left(\tau_{\theta_{\mathrm{m}}}\theta_{\mathrm{m}} + \tau_{\theta_{\mathrm{b}}}\theta_{\mathrm{b}} + \tau_{\phi}\phi\right) = \frac{\partial}{\partial \theta_{\mathrm{w}}}\left(\tau_{\theta_{\mathrm{m}}}(\theta_{\mathrm{w}} - \theta_{\mathrm{b}}) + \tau_{\theta_{\mathrm{b}}}\theta_{\mathrm{b}}\right) = \tau_{\theta_m}$$

$$F_2 = \frac{\partial}{\partial \theta_{\mathrm{b}}}\left(\tau_{\theta_{\mathrm{m}}}\theta_{\mathrm{m}} + \tau_{\theta_{\mathrm{b}}}\theta_{\mathrm{b}} + \tau_{\phi}\phi\right) = \frac{\partial}{\partial \theta_{\mathrm{b}}}\left(\tau_{\theta_{\mathrm{m}}}(\theta_{\mathrm{w}} - \theta_{\mathrm{b}}) + \tau_{\theta_{\mathrm{b}}}\theta_{\mathrm{b}}\right) = -\tau_{\theta_{\mathrm{m}}} + \tau_{\theta_{\mathrm{b}}} = -\tau_{\theta_{\mathrm{m}}}.$$

Hence, we can derive the equation of motion,

$$J_{11}\ddot{\theta}_{\mathrm{w}} + J_{12}(\theta_{\mathrm{b}})\ddot{\theta}_{\mathrm{b}} - (M_{\mathrm{b}}lr\sin\theta_{\mathrm{b}})\dot{\theta}_{\mathrm{b}}^2 = \tau_{\theta_m},$$

$$J_{12}(\theta_{\mathrm{b}})\ddot{\theta}_{\mathrm{w}} + J_{22}\ddot{\theta}_{\mathrm{b}} - M_{\mathrm{b}}lg\sin\theta_{\mathrm{b}} = -\tau_{\theta_m}$$

A.4.4 Case 4: the generalized coordinate $(\theta_{\mathrm{m}}, \theta_{\mathrm{b}})$ in [9, 28]

The Lagrangian is given as

$$\begin{aligned} L &= \frac{1}{2}J_{11}\left(\dot{\theta}_{\mathrm{m}} + \dot{\theta}_{\mathrm{b}}\right)^2 + J_{12}(\theta_{\mathrm{b}})\left(\dot{\theta}_{\mathrm{m}} + \dot{\theta}_{\mathrm{b}}\right)\dot{\theta}_{\mathrm{b}} + \frac{1}{2}J_{22}\dot{\theta}_{\mathrm{b}}^2 \\ &\quad - 2M_{\mathrm{w}}gr - M_{\mathrm{b}}g(r + l\cos\theta_{\mathrm{b}}) \\ &= \frac{1}{2}J_{11}\dot{\theta}_{\mathrm{m}}^2 + J'_{12}(\theta_{\mathrm{b}})\dot{\theta}_{\mathrm{m}}\dot{\theta}_{\mathrm{b}} + \frac{1}{2}J'_{22}(\theta_{\mathrm{b}})\dot{\theta}_{\mathrm{b}}^2 - 2M_{\mathrm{w}}gr - M_{\mathrm{b}}g(r + l\cos\theta_{\mathrm{b}}). \end{aligned}$$

where

$$J_{11} := (2M_{\mathrm{w}} + M_{\mathrm{b}})\, r^2 + 2\left(J_{\mathrm{w}} + g_{\mathrm{r}}^2 J_{\mathrm{m}}\right), \tag{A.251}$$

$$J'_{12}(\theta_{\mathrm{b}}) := J_{11} + J_{12}(\theta_{\mathrm{b}}) = (2M_{\mathrm{w}} + M_{\mathrm{b}})\, r^2 + M_{\mathrm{b}} lr \cos\theta_{\mathrm{b}} + 2J_{\mathrm{w}} + 2g_{\mathrm{r}} J_{\mathrm{m}}, \tag{A.252}$$

$$\begin{aligned} J'_{22}(\theta_{\mathrm{b}}) &:= J_{11} + 2J_{12}(\theta_{\mathrm{b}}) + J_{22} \\ &= (2M_{\mathrm{w}} + M_{\mathrm{b}})\, r^2 + M_{\mathrm{b}} l^2 + 2M_{\mathrm{b}} lr \cos\theta_{\mathrm{b}} + 2J_{\mathrm{w}} + J_{\mathrm{b}} + 2J_{\mathrm{m}}. \end{aligned} \tag{A.253}$$

$$\begin{aligned} \frac{\partial L}{\partial \theta_{\mathrm{m}}} &= 0, \\ \frac{\partial L}{\partial \theta_{\mathrm{b}}} &= -M_{\mathrm{b}} lr \left(\dot\theta_{\mathrm{m}} + \dot\theta_{\mathrm{b}}\right) \dot\theta_{\mathrm{b}} \sin\theta_{\mathrm{b}} + M_{\mathrm{b}} lg \sin\theta_{\mathrm{b}}, \\ \frac{\partial L}{\partial \dot\theta_{\mathrm{m}}} &= \left((2M_{\mathrm{w}} + M_{\mathrm{b}})\, r^2 + 2\left(J_{\mathrm{w}} + g_{\mathrm{r}}^2 J_{\mathrm{m}}\right)\right) \dot\theta_{\mathrm{m}} \\ &\quad + \left((2M_{\mathrm{w}} + M_{\mathrm{b}})\, r^2 + 2J_{\mathrm{w}} + 2g_{\mathrm{r}} J_{\mathrm{m}} + M_{\mathrm{b}} lr \cos\theta_{\mathrm{b}}\right) \dot\theta_{\mathrm{b}} \\ &= J_{11}\dot\theta_{\mathrm{m}} + J'_{12}(\theta_{\mathrm{b}})\dot\theta_{\mathrm{b}}, \\ \frac{\partial L}{\partial \dot\theta_{\mathrm{b}}} &= \left((2M_{\mathrm{w}} + M_{\mathrm{b}})\, r^2 + 2J_{\mathrm{w}}\right) \dot\theta_{\mathrm{m}} + \left((2M_{\mathrm{w}} + M_{\mathrm{b}})\, r^2 + 2J_{\mathrm{w}}\right) \dot\theta_{\mathrm{b}} \\ &\quad + (M_{\mathrm{b}} lr \cos\theta_{\mathrm{b}} + 2g_{\mathrm{r}} J_{\mathrm{m}})\, \dot\theta_{\mathrm{m}} + \left(M_{\mathrm{b}} l^2 + J_{\mathrm{b}} + 2J_{\mathrm{m}} + 2M_{\mathrm{b}} lr \cos\theta_{\mathrm{b}}\right) \dot\theta_{\mathrm{b}} \\ &= J'_{12}(\theta_{\mathrm{b}})\dot\theta_{\mathrm{m}} + J'_{22}\dot\theta_{\mathrm{b}}, \end{aligned}$$

Furthermore,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot\theta_{\mathrm{m}}}\right) = J_{11}\ddot\theta_{\mathrm{m}} + J'_{12}(\theta_{\mathrm{b}})\ddot\theta_{\mathrm{b}} - (M_{\mathrm{b}} lr \sin\theta_{\mathrm{b}})\dot\theta_{\mathrm{b}}^2, \tag{A.254}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot\theta_{\mathrm{b}}}\right) = J'_{12}(\theta_{\mathrm{b}})\ddot\theta_{\mathrm{m}} + J'_{22}(\theta_{\mathrm{b}})\ddot\theta_{\mathrm{b}} - (M_{\mathrm{b}} lr \sin\theta_{\mathrm{b}})\dot\theta_{\mathrm{m}}\dot\theta_{\mathrm{b}} - (2M_{\mathrm{b}} lr \sin\theta_{\mathrm{b}})\dot\theta_{\mathrm{b}}^2. \tag{A.255}$$

Hence, by the Lagrangian equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot\theta_{\mathrm{m}}}\right) - \frac{\partial L}{\partial \theta_{\mathrm{m}}} = J_{11}\ddot\theta_{\mathrm{m}} + J'_{12}(\theta_{\mathrm{b}})\ddot\theta_{\mathrm{b}} - (M_{\mathrm{b}} lr \sin\theta_{\mathrm{b}})\dot\theta_{\mathrm{b}}^2 = F_1, \tag{A.256}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot\theta_{\mathrm{b}}}\right) - \frac{\partial L}{\partial \theta_{\mathrm{b}}} = J'_{12}(\theta_{\mathrm{b}})\ddot\theta_{\mathrm{m}} + J'_{22}(\theta_{\mathrm{b}})\ddot\theta_{\mathrm{b}} + (M_{\mathrm{b}} lr \sin\theta_{\mathrm{b}})\dot\theta_{\mathrm{b}}^2 - M_{\mathrm{b}} lg \sin\theta_{\mathrm{b}} = F_2. \tag{A.257}$$

The external force are two torque τ_{r} and τ_{l} generated by two motors fixed on the body and affected as the torque Then, the generalized force for $(\theta_{\mathrm{m}}, \theta_{\mathrm{b}})$ are

$$F_1 = \tau_{\theta_m}, F_2 = 0.$$

Hence, we can derive the equation of motion,

$$J_{11}\ddot{\theta}_{\mathrm{m}} + J'_{12}(\theta_{\mathrm{b}})\ddot{\theta}_{\mathrm{b}} - (M_{\mathrm{b}}lr\sin\theta_{\mathrm{b}})\dot{\theta}_{\mathrm{b}}^2 = \tau_{\theta_m},$$
$$J'_{12}(\theta_{\mathrm{b}})\ddot{\theta}_{\mathrm{m}} + J'_{22}(\theta_{\mathrm{b}})\ddot{\theta}_{\mathrm{b}} + (M_{\mathrm{b}}lr\sin\theta_{\mathrm{b}})\dot{\theta}_{\mathrm{b}}^2 - M_{\mathrm{b}}lg\sin\theta_{\mathrm{b}} = 0.$$

Since

$$\begin{aligned}\tau_{\theta_{\mathrm{m}}} &= 2b_{\mathrm{v}}u_1 + 2d_{\mathrm{b}}\dot{\theta}_{\mathrm{b}} - 2\left(d_{\mathrm{b}} + d_{\mathrm{w}}\right)\dot{\theta}_{\mathrm{w}} \\ &= 2b_{\mathrm{v}}u_1 + 2d_{\mathrm{b}}\dot{\theta}_{\mathrm{b}} - 2\left(d_{\mathrm{b}} + d_{\mathrm{w}}\right)(\dot{\theta}_{\mathrm{m}} + \dot{\theta}_{\mathrm{b}}) = 2b_{\mathrm{v}}u_1 - 2d_{\mathrm{w}}\dot{\theta}_{\mathrm{b}} - 2\left(d_{\mathrm{b}} + d_{\mathrm{w}}\right)\dot{\theta}_{\mathrm{m}},\end{aligned} \tag{A.258}$$

From the above, the equation of motion is

$$J_{11}\ddot{\theta}_{\mathrm{m}} + J'_{12}(\theta_{\mathrm{b}})\ddot{\theta}_{\mathrm{b}} + 2\left(d_{\mathrm{b}} + d_{\mathrm{w}}\right)\dot{\theta}_{\mathrm{m}} + 2d_{\mathrm{w}}\dot{\theta}_{\mathrm{b}} - (M_{\mathrm{b}}lr\sin\theta_{\mathrm{b}})\dot{\theta}_{\mathrm{b}}^2 = 2b_{\mathrm{v}}u_1, \tag{A.259}$$

$$J'_{12}(\theta_{\mathrm{b}})\ddot{\theta}_{\mathrm{m}} + J'_{22}(\theta_{\mathrm{b}})\ddot{\theta}_{\mathrm{b}} + (M_{\mathrm{b}}lr\sin\theta_{\mathrm{b}})\dot{\theta}_{\mathrm{b}}^2 - M_{\mathrm{b}}lg\sin\theta_{\mathrm{b}} = 0. \tag{A.260}$$

By defining the control input as

$$u = u_1 = v_{\mathrm{r}},\ u_2 = 0, \tag{A.261}$$

the equation of motion for the self-balancing robot can be derived as

$$J_1(\theta_{\mathrm{b}}(t))\begin{bmatrix}\ddot{\theta}_{\mathrm{m}}(t)\\ \ddot{\theta}_{\mathrm{b}}(t)\end{bmatrix} + D_1\begin{bmatrix}\dot{\theta}_{\mathrm{m}}(t)\\ \dot{\theta}_{\mathrm{b}}(t)\end{bmatrix} - M_{\mathrm{b}}l\sin\theta_{\mathrm{b}}(t)\begin{bmatrix}r\dot{\theta}_{\mathrm{b}}^2(t)\\ g - r\dot{\theta}_{\mathrm{b}}^2(t)\end{bmatrix} = H_1u(t), \tag{A.262}$$

where

$$J(\theta_{\mathrm{b}}) = \begin{bmatrix}J_{11} & J'_{12}(\theta_{\mathrm{b}})\\ J'_{12}(\theta_{\mathrm{b}}) & J'_{22}(\theta_{\mathrm{b}})\end{bmatrix},\ D = 2\begin{bmatrix}d_{\mathrm{b}} + d_{\mathrm{w}} & d_{\mathrm{w}}\\ 0 & 0\end{bmatrix},\ H = 2b_{\mathrm{v}}\begin{bmatrix}1\\ 0\end{bmatrix}. \tag{A.263}$$

We linearized the equations of motion (A.262) around equilibrium states $\theta_{\mathrm{m}} = 0$, $\theta_{\mathrm{b}} = 0$, $\phi = 0$, $\dot{\theta}_{\mathrm{m}} = 0$, $\dot{\theta}_{\mathrm{b}} = 0$, $\dot{\phi} = 0$, and $u = 0$. Then, under the assumption that $\sin\theta_{\mathrm{b}}(t) \approx \theta_{\mathrm{b}}(t)$, $\cos\theta_{\mathrm{b}}(t) \approx 1$, $\sin^2\theta_{\mathrm{b}}(t) \approx 0$, $\dot{\theta}_{\mathrm{b}}^2(t) \approx 0$, $\dot{\phi}^2(t) \approx 0$, and $\sin\theta_{\mathrm{b}}(t)\cos\theta_{\mathrm{b}}(t)\dot{\theta}_{\mathrm{b}}(t)\dot{\phi}(t) \approx 0$, the linearized equations of motion can be derived as

$$J\ddot{x}_a(t) + D\dot{x}_a(t) + Kx_a(t) = Hu(t) \tag{A.264}$$

where

$$x_a(t) := \begin{bmatrix} \theta_{\mathrm{m}}(t) \\ \theta_{\mathrm{b}}(t) \end{bmatrix}, \tag{A.265}$$

$$J = J(0) = \begin{bmatrix} J_{11} & J'_{12}(0) \\ J'_{12}(0) & J'_{22}(0) \end{bmatrix} \tag{A.266}$$

$$K = M_{\mathrm{b}} l g \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}. \tag{A.267}$$

$$J_{11} := 2J_{\mathrm{w}} + (2M_{\mathrm{w}} + M_{\mathrm{b}})\, r^2 + 2g_{\mathrm{r}}^2 J_{\mathrm{m}}, \tag{A.268}$$

$$J'_{12}(0) := 2J_{\mathrm{w}} + (2M_{\mathrm{w}} + M_{\mathrm{b}})\, r^2 + 2g_{\mathrm{r}} J_{\mathrm{m}} + M_{\mathrm{b}} l r \cos\theta_{\mathrm{b}}, \tag{A.269}$$

$$J'_{22}(0) := 2J_{\mathrm{w}} + (2M_{\mathrm{w}} + M_{\mathrm{b}})\, r^2 + 2J_{\mathrm{m}} + J_{\mathrm{b}} + M_{\mathrm{b}} l^2 + 2M_{\mathrm{b}} l r \cos\theta_{\mathrm{b}}. \tag{A.270}$$

Then, the linearized model can be obtained as

$$\dot{x}(t) = A_c x(t) + B_c u(t), \tag{A.271}$$

where

$$A_c = \begin{bmatrix} 0_{2\times 2} & I_{2\times 2} \\ -J^{-1}K & -J^{-1}D \end{bmatrix}, \; B_c = \begin{bmatrix} 0_{2\times 1} \\ J^{-1}H \end{bmatrix}. \tag{A.272}$$

Publications

Journal paper

1. P.E.E. Shwe and S. Yamamoto: An Improvement on Data-Driven Pole Placement for State Feedback Control and Model Identification, *Journal of Intelligent Control and Automation* (ISSN: 2153-0661)Vol. 8, No. 3, August 2017

Conference papers

1. P.E.E. Shwe and S. Yamamoto: An Application of Data-Driven Pole Placement; Simultaneously Deriving Linearized State-Space Model and Pole Placement Gain, *Proceedings of the 7th International Conference on Science and Engineering (ICSE2016)*, pp. 136–139, December 10th -12th , 2016, Yangon, Myanmar.

2. P.E.E. Shwe and S. Yamamoto: Real-time Simultaneously Updating a Linearized State-Space Model and Pole Placement Gain, *Proceedings of SICE 2016*, pp. 196–201, September 2016, Tsukuba, Japan.

Bibliography

[1] W.M. Wonham, “On pole assignment in multi-input controllable linear systems,” *IEEE Transactions on Automatic Control*, vol. 12, no. 6, pp. 660-665, 1967.

[2] J. E. Ackermann, “On the synthesis of linear control systems with specified characteristics,” *Automatica*, vol. 13, no. 1, pp. 89-94, 1977.

[3] H. Hikita, S. Koyama, and R. Miura, “The redundancy of feedback gain matrix and the derivation of low feedback gain matrix in pole assignment,” *The Society of Instrument and Control Engineers*, vol. 11, no. 5, pp. 556-560, 1975. (in Japanese)

[4] S. Yamamoto, Y. Okano, and O. Kaneko, “A pole placement method by using state measurements,” *Proc. of the 59th Annual Conference of the Institute of Systems, Control and Information Engineers*, 2015. (in Japanese)

[5] S. Yamamoto, Y. Okano, and O. Kaneko, “A data-driven pole placement method simultaneously identifying a state-space model,” *Trans. of the Institute of Systems, Control and Information Engineers*, vol. 29, no. 4, pp. 275-284, 2016. (in Japanese)

[6] M. G. Safonov and T.-C. Tsao, “The unfalsified control concept and learning,” *IEEE Transactions on Automatic Control*, vol. 42, no. 6, pp. 843-847, 1997.

[7] O. Kaneko, F. Sawakawa, and S. Yamamoto, “Data-driven update of state feedback gains,” *The Society of Instrument and Control Engineers*, vol. 49, no. 6, pp. 632-638, 2013. (in Japanese)

[8] Y. Matsui, S. Akamatsu, T. Kimura, K. Nakano, and K. Sakurama, “An application of fictitious reference iterative tuning to state feedback,” *The Institute of Electrical Engineers of Japan C*, vol. 132, no. 6, pp. 851-859, 2012. (in Japanese)

[9] ZMP Inc., Stabilization and Control of stable running of the Wheeled Inverted Pendulum — Development of Educational Wheeled Inverted Robot e-nuvo WHEEL ver.1.0, http://www.zmp.co.jp/e-nuvo/ .

[10] T. Nomura, Y. Kitsuka, H. Suemitsu, and T. Matsuo, “Adaptive backstepping control for a two-wheeled autonomous robot,” *ICCAS-SICE*, pp. 4687-4692, 2009.

[11] J. Y. Jaung,“On parametric pole-placement for a class of linear systems,” *Journal of Marine Science and Technology*, vol. 5, no. 1, pp. 61-12, 1997.

[12] P. Li,“Pole placement via state feedback,” *Lecture note, Univ. of Minnesota*, 21 April 2008.

[13] A. A. Wahab, R. Mamat, and S.S. Shamsudin, “The effectiveness of pole placement method in control system design for an autonomous helicopter model in hovering flight,” *International Journal of Integrated Engineering*, vol. 1, no. 3, pp. 33-46, 2001.

[14] P. E. E. Shwe and S. Yamamoto, “Data-driven method to simultaneously obtain a linearized state-space model and pole placement gain,” *Proc. of the 3rd Multi-Symposium on Control Systems*, 3B3-2, 2016.

[15] P. E. E. Shwe and S. Yamamoto, “An Improvement on Data-Driven Pole Placement for State Feedback Control and Model Identification,” *Journal of Intelligent Control and Automation* , vol. 8, no. 3, pp. 139-153, 2017.

[16] P. E. E. Shwe and S. Yamamoto, “Real-time simultaneously updating a linearized state-space model and pole placement gain,” *Proc. of SICE 2016*, pp. 196-201, 2016.

[17] H. Kimura, “Pole Assignment by Gain Output Feedback,” *IEEE Trans. on Automatic Control*, vol. 20, no. 4, pp. 509-516, 1975.

[18] Z. S. Hou and Z. Wang, “From Model-based Control to Data-driven Control: Survey, Classification and Perspective,” *Information Sciences*, vol. 235, pp. 3-35, 2013.

[19] M. C. Campi, A. Lecchini and S. M. Savaresi, “Virtual Reference Feedback Tuning: A Direct Method for the Design of Feedback Controllers,” *Automatica*, vol. 30,n0. 8, pp. 1337-1346, 2002.

[20] A. Sala and A. Esparza, “Extensions to “ Virtual Reference Feedback Tuning: A Direct Method for the Design of Feedback Controllers,” *Automatica*, vol. 41,n0. 8, pp. 1473-1476, 2005.

[21] S. Souma, O. Kaneko and T. Fujii, “A New Method of a Controller Parameter Tuning Based on Input-output Data, Fictitious Reference Iterative Tuning,” *In Proceedings of the 2nd IFAC Workshop on Adaptation and Learning in Control and Signal Processing*, pp. 789-794, 2004.

[22] Y. Matsui, S. Akamatsu, T. Kimura, K. Nakano and K. Sakurama, “Fictitious Reference Iterative Tuning for State Feedback Control of Inverted Pendulum with Inertia Rotor,” *SICE Annual Conference 2011*, pp. 1087-1092, 2011.

[23] O. Kaneko, “The Canonical Controller Approach to Data-driven Update of State Feedback Gain,” *In Proceedings of the 10th Asian Control Conference 2015 (ASCC 2015)*, pp. 2980-2985, 2015.

[24] K. V. Heusden, A. Karimi and T. Soderstrom, “Extensions to ”On Identification Methods for Direct Data driven Controller Tuning,” *International Journal of Adaptive Control and Signal Processing*, vol. 25, no. 5, pp. 448-465, 2011.

[25] O. Kaneko, M. Miyachi and T. Fujii, “Extensions to ”Simultaneous Updating of a Model and a Controller Based on the Data-driven Fictitious Controller,” *In Decision and Control, 2008. CDC 2008. 47th IEEE Conference on*, pp. 1358-1363, 2008.

[26] O. Kaneko, M. Miyachi and T. Fujii, "Extensions to " Simultaneous Updating of Model and Controller Based on Fictitious Reference Iterative Tuning," *SICE Journal of Control, Measurement, and System Integration*, vol. 4,no. 1, pp. 63-70, 2011.

[27] I. Markovsky and S. V. Huffel, "Overview of Total Least-squares Methods," *Signal Processing*, pp. 2283-2302.

[28] T. Nomura, Y. Kitsuka, H. Suemitsu and T. Matsuo, "Adaptive Backstepping Control for a Two-wheeled Autonomous Robot," *In Proceedings of ICCAS-SICE*, pp. 4687-4692, 2009.

[29] J. Brewer, "Kronecker Products and Matrix Calculus in System Theory," *IEEE Transactions on Circuits and Systems*, vol. 25,no. 9, pp. 772–781, 1978.