Construction of Brownian motion on the Wiener measure space

メタデータ	言語: eng 出版者: 公開日: 2019-10-03 キーワード (Ja): キーワード (En): 作成者: 佐藤, 秀一 メールアドレス: 所属:
URL	http://hdl.handle.net/2297/00055659

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CONSTRUCTION OF BROWNIAN MOTION ON THE WIENER MEASURE SPACE

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Abstract. We give a self contained construction of the Wiener probability space.

1. Introduction

Let $W = W_d = C[0, \infty)$ be the collection of continuous $\mathbb{R}^d$-valued functions on the interval $\mathbb{R}_0 = [0, \infty)$. In this note we present a self contained construction of the Wiener probability space $(W, \mathcal{F}, \mu)$, where $\mathcal{F}$ is a sigma-algebra of subsets of W and μ is the Wiener measure. The mathematical theory of the Brownian motion is based on this probability space. We follow the methods of [4], but we intend to make our presentation more specific. (See also [2], [3], [6].)

2. Definition of the Wiener measure

Definition 2.1. We write $t = (t_1, \dots, t_n) \in (\mathbb{R}_0)^n_*$ if $t = (t_1, \dots, t_n) \in (\mathbb{R}_0)^n = \mathbb{R}_0 \times \cdots \times \mathbb{R}_0$ (n-fold product) and $0 < t_1 < t_2 < \cdots < t_n$. We also write $\mathbb{R}^d = \mathbb{R}_d$, when Cartesian products of $\mathbb{R}^d$ are considered. Define $\Phi_t : W \to \mathbb{R}^n_d$, $t \in (\mathbb{R}_0)^n_*$, by

$$\Phi_t(f) = (\phi_{t_1}(f), \dots, \phi_{t_n}(f)),$$

where $\phi_{t_j}(f) = f(t_j)$ and $\mathbb{R}^n_d = (\mathbb{R}_d)^n = \mathbb{R}_d \times \cdots \times \mathbb{R}_d$ (n-fold product). Set

$$\mathcal{G}_t = \left\{ \Phi_t^{-1}(B) : B \in \mathcal{B}(\mathbb{R}^n_d) \right\},$$

where $\Phi_t^{-1}(B) = \{f \in W : \Phi_t(f) \in B\}$ and $\mathcal{B}(\mathbb{R}^n_d)$ denotes the Borel class of $\mathbb{R}^n_d$.

Since $\mathcal{B}(\mathbb{R}^n_d)$ is a sigma-algebra, obviously we have the following result.

Lemma 2.2. *$\mathcal{G}_t$ is a sigma-algebra for every $t \in (\mathbb{R}_0)^n_*$.*

We observe the following result on $\mathcal{G}_t$.

Lemma 2.3. *Let $t \in (\mathbb{R}_0)^m_*$, $s \in (\mathbb{R}_0)^n_*$ with $m \le n$. Let $\tilde{t} = \{t_1, \dots, t_m\}, \tilde{s} = \{s_1, \dots, s_n\}$, which are sets of positive numbers, if $t = (t_1, \dots, t_m), s = (s_1, \dots, s_n)$. Suppose that $\tilde{t} \subset \tilde{s}$. Then $\mathcal{G}_t \subset \mathcal{G}_s$.*

Proof. Take $\sigma(s) = (s_{\sigma(1)}, s_{\sigma(2)}, \dots, s_{\sigma(n)})$ satisfying $s_{\sigma(1)} = t_1$, $s_{\sigma(2)} = t_2, \dots, s_{\sigma(m)} = t_m$ with some $\sigma \in \mathcal{S}_n$ (the permutation group). Suppose $A \in \mathcal{G}_t$. Then there exists $\Lambda \in \mathcal{B}(\mathbb{R}^m_d)$ such that $A = \Phi_t^{-1}(\Lambda)$. Let $\Lambda' = \Lambda \times \mathbb{R}^{n-m}_d$. Define

$$\sigma^{-1}(\Lambda') = \{(x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(n)}) : (x_1, \dots, x_n) \in \Lambda'\}.$$

Key Words and Phrases. Brownian motion, Wiener probability space.

The author is partly supported by Grant-in-Aid for Scientific Research (C) No. 16K05195, Japan Society for the Promotion of Science.

This note was essentially finalized in August 2016.

We see that $f \in \Phi_s^{-1}(\sigma^{-1}(\Lambda'))$ if and only if

$$(\phi_{s_1}(f), \dots, \phi_{s_n}(f)) \in \sigma^{-1}(\Lambda'),$$

which means that

$$(\phi_{s_{\sigma(1)}}(f), \dots, \phi_{s_{\sigma(n)}}(f)) \in \Lambda'.$$

The definition of Λ' implies that this is equivalent to

$$(\phi_{s_{\sigma(1)}}(f), \dots, \phi_{s_{\sigma(m)}}(f)) \in \Lambda.$$

This can be rewritten as

$$(\phi_{t_1}(f), \dots, \phi_{t_m}(f)) \in \Lambda,$$

which is equivalent to $f \in \Phi_t^{-1}(\Lambda)$. Thus $A = \Phi_t^{-1}(\Lambda) = \Phi_s^{-1}(\sigma^{-1}(\Lambda'))$, and hence $A \in \mathcal{G}_s$, since $\sigma^{-1}(\Lambda') \in \mathcal{B}(\mathbb{R}_d^n)$. □

Definition 2.4. Let $\mathcal{G}^{(n)} = \cup_{t \in (\mathbb{R}_0)_*^n} \mathcal{G}_t$ and $\mathcal{G} = \cup_{n=1}^{\infty} \mathcal{G}^{(n)}$.

Lemma 2.5. *$\mathcal{G}$ is an algebra of subsets of W.*

Proof. We easily see that $W \in \mathcal{G}$. Let $A \in \mathcal{G}$. Then there is $n \geq 1$ such that $A \in \mathcal{G}_t$ for some $t \in (\mathbb{R}_0)_*^n$. Since $\mathcal{G}_t$ is a sigma-algebra (Lemma 2.2), we have $A^c \in \mathcal{G}_t$ and hence $A^c \in \mathcal{G}$. Suppose that $A, B \in \mathcal{G}$. Lemma 2.3 implies that $A, B \in \mathcal{G}_t$ for some t. Thus $A \cup B \in \mathcal{G}_t$ by Lemma 2.2. Collecting results, we see that $\mathcal{G}$ is an algebra. □

Let $|x| = (\alpha_1^2 + \cdots + \alpha_d^2)^{1/2}$ be the Euclidean norm of $x = (\alpha_1, \dots, \alpha_d) \in \mathbb{R}^d$. Define

$$g(t, x, y) = \frac{1}{(\sqrt{2\pi t})^d} \exp\left(-\frac{|y - x|^2}{2t}\right), \quad x, y \in \mathbb{R}^d, \quad t > 0.$$

We have the following formulas:

Lemma 2.6.

$$\int_{\mathbb{R}^d} g(t, x, y)\, dy = 1,$$

$$\int_{\mathbb{R}^d} g(s, a, x) g(t, x, b)\, dx = g(s + t, a, b).$$

Definition 2.7. Define μ_t on $\mathcal{G}_t$, $t \in (\mathbb{R}_0)_*^n$, by

$$\mu_t(A) = \int_{\Lambda} g(t_1, 0, x_1) g(t_2 - t_1, x_1, x_2) \dots g(t_n - t_{n-1}, x_{n-1}, x_n)\, dx_1 \dots dx_n$$

with $A = \Phi_t^{-1}(\Lambda)$, $t = (t_1, \dots, t_n)$, $\Lambda \in \mathcal{B}(\mathbb{R}_d^n)$.

Lemma 2.8. *Let μ_t be as in Definition* 2.7*. Then, μ_t defines a probability measure on the sigma-algebra $\mathcal{G}_t$.*

Proof. If μ_t is well-defined on $\mathcal{G}_t$, it is easy to see that μ_t is a probability measure. Suppose that $A = \Phi_t^{-1}(\Lambda) = \Phi_t^{-1}(\Lambda')$ with $\Lambda, \Lambda' \in \mathcal{B}(\mathbb{R}_d^n)$. We show that $\Lambda = \Lambda'$. To see this, we notice that Φ_t is a surjection from W onto $\mathbb{R}_d^n$. Thus the relation

$$\Lambda \cap \Phi_t(W) = \Phi_t(A) = \Lambda' \cap \Phi_t(W)$$

implies $\Lambda = \Lambda'$. It follows that μ_t is well-defined on $\mathcal{G}_t$.

To show that μ_t is a probability measure, we have to prove

(1) $\mu_t(W) = 1$;

(2) if $A_k \in \mathcal{G}_t$, $k = 1, 2, \dots$, and $A_j \cap A_k = \emptyset$, $j \neq k$, then

$$\mu_t\left(\cup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu_t\left(A_k\right).$$

Obviously, we have part (1), since $W = \Phi_t^{-1}(\mathbb{R}_d^n)$. To prove part (2), let $A_k = \Phi_t^{-1}(\Lambda_k)$ with $\Lambda_k \in \mathcal{B}(\mathbb{R}_d^n)$. We note that

$$\Phi_t^{-1}(\Lambda_j \cap \Lambda_k) = A_j \cap A_k = \emptyset$$

if $j \neq k$. This implies that $\Lambda_j \cap \Lambda_k = \emptyset$, since Φ_t is a surjection. Thus the countable additivity of μ_t follows from the definition of μ_t with the countable additivity of Lebesgue integral.

□

We note that if $A = \phi_t^{-1}(\Lambda)$, $t > 0$,

$$\mu_t\{f \in W : f(t) \in \Lambda\} = \mu_t(A) = \int_\Lambda \frac{1}{(\sqrt{2\pi t})^d} \exp\left(-\frac{|x|^2}{2t}\right) dx. \tag{2.1}$$

Lemma 2.9. *Let $A \in \mathcal{G}$. Then we can define μ on $\mathcal{G}$ by $\mu(A) = \mu_t(A)$ with $t \in \cup_{n \geq 1}(\mathbb{R}_0)_*^n$ satisfying $A \in \mathcal{G}_t$.*

Proof. Suppose that $A = \Phi_s^{-1}(\Lambda) = \Phi_t^{-1}(\Lambda')$ with $s \in (\mathbb{R}_0)_*^m$, $t \in (\mathbb{R}_0)_*^n$ and $\Lambda \in \mathcal{B}(\mathbb{R}_d^m)$, $\Lambda' \in \mathcal{B}(\mathbb{R}_d^n)$. We show that

$$\int_\Lambda g(s_1, 0, x_1) g(s_2 - s_1, x_1, x_2) \dots g(s_m - s_{m-1}, x_{m-1}, x_m)\, dx_1 \dots dx_m$$
$$= \int_{\Lambda'} g(t_1, 0, x_1) g(t_2 - t_1, x_1, x_2) \dots g(t_n - t_{n-1}, x_{n-1}, x_n)\, dx_1 \dots dx_n.$$

If $m < n$, put $\Lambda^* = \Lambda \times \mathbb{R}_d^{n-m}$, $s^* = (s_1, \dots, s_m, s_m + 1, s_m + 2, \dots, s_m + n - m)$. Then $\Lambda^* \in \mathcal{B}(\mathbb{R}_d^n)$, $s^* \in (\mathbb{R}_0)_*^n$, $\Phi_s^{-1}(\Lambda) = \Phi_{s^*}^{-1}(\Lambda^*)$ and

$$\int_\Lambda g(s_1, 0, x_1) g(s_2 - s_1, x_1, x_2) \dots g(s_m - s_{m-1}, x_{m-1}, x_m)\, dx_1 \dots dx_m$$
$$= \int_{\Lambda^*} g(s_1^*, 0, x_1) g(s_2^* - s_1^*, x_1, x_2) \dots g(s_n^* - s_{n-1}^*, x_{n-1}, x_n)\, dx_1 \dots dx_n.$$

So, we may assume that $s, t \in (\mathbb{R}_0)_*^n$ and $\Lambda, \Lambda' \in \mathcal{B}(\mathbb{R}_d^n)$.

Let $\tilde{s} \cap \tilde{t} = \{s_{\sigma(1)}, \dots, s_{\sigma(k)}\} = \{t_{\tau(1)}, \dots, t_{\tau(k)}\}$, $\sigma(1) < \cdots < \sigma(k)$, $\tau(1) < \cdots < \tau(k)$ with $\sigma, \tau \in \mathcal{S}_n$. We show that $\sigma(\Lambda) = \tau(\Lambda') = \Gamma \times \mathbb{R}_d^{n-k}$ for some $\Gamma \in \mathcal{B}(\mathbb{R}_d^k)$. Let

$$\Gamma = \{(x_1, \dots, x_k) : (x_1, \dots, x_k, x_{k+1}, \dots, x_n) \in \sigma(\Lambda) \text{ for some } (x_{k+1}, \dots, x_n) \in \mathbb{R}_d^{n-k}\},$$

$$\Gamma' = \{(x_1, \dots, x_k) : (x_1, \dots, x_k, x_{k+1}, \dots, x_n) \in \tau(\Lambda') \text{ for some } (x_{k+1}, \dots, x_n) \in \mathbb{R}_d^{n-k}\}.$$

If $x = (x', x'') \in \mathbb{R}_d^k \times \mathbb{R}_d^{n-k}$, let $\pi_k(x) = x'$. Then $\Gamma = \pi_k(\sigma(\Lambda))$, $\Gamma' = \pi_k(\tau(\Lambda'))$. We can show $\Gamma = \Gamma'$ as follows. Let $(x_1, \dots, x_k) \in \Gamma$. Then $(x_1, \dots, x_k, x_{k+1}, \dots, x_n) \in \sigma(\Lambda)$ for some $(x_{k+1}, \dots, x_n) \in \mathbb{R}_d^{n-k}$. Thus

$$(x_1, \dots, x_k, x_{k+1}, \dots, x_n) = (y_{\sigma(1)}, \dots, y_{\sigma(k)}, y_{\sigma(k+1)}, \dots, y_{\sigma(n)})$$

with some $(y_1, \dots, y_k, y_{k+1}, \dots, y_n) \in \Lambda$. Since $\Phi_s(\Phi_s^{-1}(\Lambda)) = \Lambda$, there exists $f \in A$ such that

$$(f(s_1), \dots, f(s_k), f(s_{k+1}), \dots, f(s_n)) = (y_1, \dots, y_k, y_{k+1}, \dots, y_n).$$

Therefore

$$(x_1, \dots, x_k, x_{k+1}, \dots, x_n) = (f(s_{\sigma(1)}), \dots, f(s_{\sigma(k)}), f(s_{\sigma(k+1)}), \dots, f(s_{\sigma(n)})).$$

Since $\Phi_s^{-1}(\Lambda) = \Phi_t^{-1}(\Lambda')$, we also have

$$(f(t_1), \dots, f(t_k), f(t_{k+1}), \dots, f(t_n)) \in \Lambda'.$$

Thus

$$(f(t_{\tau(1)}), \dots, f(t_{\tau(k)}), f(t_{\tau(k+1)}), \dots, f(t_{\tau(n)})) \in \tau(\Lambda').$$

Since

$$(x_1, \dots, x_k) = (f(s_{\sigma(1)}), \dots, f(s_{\sigma(k)})) = (f(t_{\tau(1)}), \dots, f(t_{\tau(k)})),$$

it follows that $(x_1, \dots, x_k) \in \Gamma'$. This proves $\Gamma \subset \Gamma'$. Similarly, we also have $\Gamma' \subset \Gamma$. Thus we have $\Gamma = \Gamma'$.

Next we show that $\sigma(\Lambda) = \Gamma \times \mathbb{R}_d^{n-k}$. Let $(x_1, \dots, x_k, x_{k+1}, \dots, x_n) \in \Gamma \times \mathbb{R}_d^{n-k}$. Then, since $(x_1, \dots, x_k) \in \Gamma' = \pi_k(\tau(\Lambda'))$ and Φ_t is surjective from $\Phi_t^{-1}(\Lambda')$ to Λ', we can find $f \in \Phi_t^{-1}(\Lambda')$ such that

$$(x_1, \dots, x_k) = \pi_k((f(t_{\tau(1)}), \dots, f(t_{\tau(k)}), f(t_{\tau(k+1)}), \dots, f(t_{\tau(n)})))$$

with

$$(f(t_1), \dots, f(t_k), f(t_{k+1}), \dots, f(t_n)) \in \Lambda'.$$

Since $s_{\sigma(k+1)}, \dots, s_{\sigma(n)} \notin \tilde{t}$, f can be chosen so that $f(s_{\sigma(k+1)}) = x_{k+1}, \dots, f(s_{\sigma(n)}) = x_n$ for any $x_{k+1}, \dots, x_n \in \mathbb{R}^d$. Since $f \in \Phi_s^{-1}(\Lambda)$ also and $s_{\sigma(j)} = t_{\tau(j)}$, $1 \le j \le k$,

$$(x_1, \dots, x_k, x_{k+1}, \dots, x_n) = (f(s_{\sigma(1)}), \dots, f(s_{\sigma(k)}), f(s_{\sigma(k+1)}), \dots, f(s_{\sigma(n)}))$$

with

$$(f(s_1), \dots, f(s_k), f(s_{k+1}), \dots, f(s_n)) \in \Lambda.$$

This implies that $\Gamma \times \mathbb{R}_d^{n-k} \subset \sigma(\Lambda)$. The reverse inclusion is obvious. Also we have $\tau(\Lambda') = \Gamma \times \mathbb{R}_d^{n-k}$.

If $\tilde{s} \cap \tilde{t} = \emptyset$, by the arguments above we have $\Lambda = \Lambda' = \mathbb{R}_d^n$ provided that $\Phi_s^{-1}(\Lambda) = \Phi_t^{-1}(\Lambda')$ ($\Lambda \neq \emptyset$, $\Lambda' \neq \emptyset$).

We note that

$$\int_\Lambda g(s_1, 0, x_1) g(s_2 - s_1, x_1, x_2) \dots g(s_n - s_{n-1}, x_{n-1}, x_n)\, dx_1 \dots dx_n$$
$$= \int_{\sigma(\Lambda)} g(s_1, 0, x_{\sigma^{-1}(1)}) g(s_2 - s_1, x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}) \dots g(s_n - s_{n-1}, x_{\sigma^{-1}(n-1)}, x_{\sigma^{-1}(n)})\, dx_1 \dots dx_n.$$

We show that

$$\int_{\sigma(\Lambda)} g(s_1, 0, x_{\sigma^{-1}(1)}) g(s_2 - s_1, x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}) \dots g(s_n - s_{n-1}, x_{\sigma^{-1}(n-1)}, x_{\sigma^{-1}(n)})\, dx_1 \dots dx_n$$
$$= \int_\Gamma g(s_{\sigma(1)}, 0, x_1) g(s_{\sigma(2)} - s_{\sigma(1)}, x_1, x_2) \dots g(s_{\sigma(k)} - s_{\sigma(k-1)}, x_{k-1}, x_k)\, dx_1 \dots dx_k. \tag{2.2}$$

To see this we first note that

$$\int_{\mathbb{R}_d^{\sigma(1)-1}} \prod_{i=0}^{\sigma(1)-1} g(s_{i+1} - s_i, x_{\sigma^{-1}(i)}, x_{\sigma^{-1}(i+1)})\, dx_{\sigma^{-1}(1)} \dots dx_{\sigma^{-1}(\sigma(1)-1)} = g(s_{\sigma(1)}, 0, x_1)$$

with $s_0 = 0$, $x_0 = 0$, $\sigma^{-1}(0) = 0$. We consider this integral when $\sigma(1) \geq 2$. Similarly,

$$\int_{\mathbb{R}_d^{\sigma(m+1)-\sigma(m)-1}} \prod_{i=\sigma(m)}^{\sigma(m+1)-1} g(s_{i+1}-s_i, x_{\sigma^{-1}(i)}, x_{\sigma^{-1}(i+1)})\, dx_{\sigma^{-1}(\sigma(m)+1)} \dots dx_{\sigma^{-1}(\sigma(m+1)-1)} = g(s_{\sigma(m+1)} - s_{\sigma(m)}, x_m, x_{m+1})$$

for $1 \leq m \leq k-1$, where this is considered when $\sigma(m+1) \geq \sigma(m)+2$, and

$$\int_{\mathbb{R}_d^{n-\sigma(k)}} \prod_{i=\sigma(k)}^{n-1} g(s_{i+1} - s_i, x_{\sigma^{-1}(i)}, x_{\sigma^{-1}(i+1)})\, dx_{\sigma^{-1}(\sigma(k)+1)} \dots dx_{\sigma^{-1}(n)} = 1.$$

This integral is considered when $\sigma(k) \leq n-1$. Let us denote by $F(x_1, \dots, x_n)$ the integrand of the left hand side of (2.2). Then the integral on the left hand side of (2.2) equals

$$\int_{\Gamma} \left(\int_{\mathbb{R}_d^{n-k}} F(x_1, \dots, x_k, x_{k+1}, \dots, x_n)\, dx_{k+1} \dots dx_n \right) dx_1 \dots, dx_k. \tag{2.3}$$

Collecting results above, we see that the inner integral is equal to

$$g(s_{\sigma(1)}, 0, x_1) g(s_{\sigma(2)} - s_{\sigma(1)}, x_1, x_2) \dots g(s_{\sigma(k)} - s_{\sigma(k-1)}, x_{k-1}, x_k).$$

Using this in (2.3), we get (2.2).

In the same way, we have

$$\begin{aligned} &\int_{\tau(\Lambda')} g(t_1, 0, x_{\tau^{-1}(1)}) g(t_2 - t_1, x_{\tau^{-1}(1)}, x_{\tau^{-1}(2)}) \\ &\qquad \dots g(t_n - t_{n-1}, x_{\tau^{-1}(n-1)}, x_{\tau^{-1}(n)})\, dx_1 \dots dx_n \\ &= \int_{\Gamma'} g(t_{\tau(1)}, 0, x_1) g(t_{\tau(2)} - t_{\tau(1)}, x_1, x_2) \dots g(t_{\tau(k)} - t_{\tau(k-1)}, x_{k-1}, x_k)\, dx_1 \dots dx_k. \end{aligned} \tag{2.4}$$

From (2.2) and (2.4), it follows that

$$\begin{aligned} &\int_{\sigma(\Lambda)} g(s_1, 0, x_{\sigma^{-1}(1)}) g(s_2 - s_1, x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}) \\ &\qquad \dots g(s_n - s_{n-1}, x_{\sigma^{-1}(n-1)}, x_{\sigma^{-1}(n)})\, dx_1 \dots dx_n \\ &\quad = \int_{\tau(\Lambda')} g(t_1, 0, x_{\tau^{-1}(1)}) g(t_2 - t_1, x_{\tau^{-1}(1)}, x_{\tau^{-1}(2)}) \\ &\qquad \dots g(t_n - t_{n-1}, x_{\tau^{-1}(n-1)}, x_{\tau^{-1}(n)})\, dx_1 \dots dx_n, \end{aligned}$$

since $\Gamma = \Gamma'$, $s_{\sigma(j)} = t_{\tau(j)}$ for $1 \leq j \leq k$, and hence

$$\begin{aligned} &\int_{\Lambda} g(s_1, 0, x_1) g(s_2 - s_1, x_1, x_2) \dots g(s_n - s_{n-1}, x_{n-1}, x_n)\, dx_1 \dots dx_n \\ &\quad = \int_{\Lambda'} g(t_1, 0, x_1) g(t_2 - t_1, x_1, x_2) \dots g(t_n - t_{n-1}, x_{n-1}, x_n)\, dx_1 \dots dx_n. \end{aligned}$$

This implies that μ is well-defined on $\mathcal{G}$. □

The set function μ will extend to the Wiener measure on the sigma-algebra generated by $\mathcal{G}$.

3. Countable additivity of the Wiener measure

We first prove countable additivity of μ on the algebra $\mathcal{G}$, which along with a result from the measure theory will imply what we want to show.

Proposition 3.1. *Let μ be as in Lemma 2.9. Then μ is countably additive on the algebra $\mathcal{G}$.*

It is easy to see that this follows from the next result.

Proposition 3.2. *Let μ be as in Lemma 2.9. Let $A_n \in \mathcal{G}$, $n = 1, 2, \ldots$. Suppose that $A_n \downarrow \emptyset$. Then $\lim_{n\to\infty} \mu(A_n) = 0$.*

To prove Proposition 3.2 we shall show the following.

Assertion 1. Suppose that $A_n \in \mathcal{G}$, $n = 1, 2, \ldots$, $A_n \downarrow$ and $\lim_{n\to\infty} \mu(A_n) \neq 0$. Then $\lim_{n\to\infty} A_n \neq \emptyset$.

To prove this we assume that $A_n = \Phi_{t_n}^{-1}(\Lambda_n)$ for some $t_n \in (\mathbb{R}_0)_*^{N_n}$ and $\Lambda_n \in \mathcal{B}(\mathbb{R}_d^{N_n})$, where $N_n = \operatorname{card} \tilde{t}_n$. We may also assume that

$$\tilde{t}_n = \left\{ t_1^{(n)}, t_2^{(n)}, \ldots, t_{a(n)}^{(n)}, \frac{1}{2^n}, \frac{2}{2^n}, \ldots, \frac{n2^n}{2^n} \right\},$$

where $\{a(n)\}$ is a strictly increasing sequence of positive integers, which can be seen by Lemma 2.3. Let

$$\tilde{t}_n^* = \left\{ t_1^{(n)}, t_2^{(n)}, \ldots, t_n^{(n)}, \frac{1}{2^n}, \frac{2}{2^n}, \ldots, \frac{n2^n}{2^n} \right\},$$

$$\tilde{t}_n^{**} = \left\{ \frac{1}{2^n}, \frac{2}{2^n}, \ldots, \frac{n2^n}{2^n} \right\},$$

which are subsets of $\tilde{t}_n$. We note that

$$\bigcup_{j\geq 1} \tilde{t}_j = \bigcup_{j\geq 1} \tilde{t}_j^* \tag{3.1}$$

and $M_n = \operatorname{card}(\tilde{t}_n^*) \leq 3n2^{n-1}$.

To prove the assertion we need the following.

Lemma 3.3. *For any $\epsilon > 0$, there exists $\eta > 1$ such that*

$$\sum_{n=1}^{\infty} \mu\{f \in W : \omega_n(f) > 2^{-n/3}\eta\} < \epsilon,$$

where

$$\omega_n(f) = \max\{|\phi_t(f) - \phi_s(f)| : 0 < t - s \leq 2^{-n}, s, t \in \tilde{t}_n^*\}.$$

In proving this we apply the following formula.

Lemma 3.4. *Let $0 < s < t < \infty$, $\Lambda \in \mathcal{B}(\mathbb{R}^d)$. Then*

$$\mu\left\{f \in W : \phi_t(f) - \phi_s(f) \in \Lambda\right\} = \int_\Lambda \frac{1}{(\sqrt{2\pi(t-s)})^d} \exp\left(-\frac{|x|^2}{2(t-s)}\right) dx.$$

Proof. Define a continuous function $h : \mathbb{R}_d^2 \to \mathbb{R}^d$ by $h(x, y) = y - x$. Let

$$\Phi_{(s,t)}(f) = (\phi_s(f), \phi_t(f)).$$

Then, we note that

$$\{f \in W : \phi_t(f) - \phi_s(f) \in \Lambda\} = \left\{f \in W : \Phi_{(s,t)}(f) \in h^{-1}(\Lambda)\right\}.$$

Therefore

$$\begin{aligned}
&\mu\left\{f \in W : \phi_t(f) - \phi_s(f) \in \Lambda\right\} = \mu_{(s,t)}(\Phi_{(s,t)}^{-1}(h^{-1}(\Lambda)))\\
&= \int_{h^{-1}(\Lambda)} g(s, 0, x_1) g(t-s, x_1, x_2)\, dx_1 dx_2\\
&= \int_{\mathbb{R}_d^2} \chi_\Lambda(x_2 - x_1) g(s, 0, x_1) g(t-s, x_1, x_2)\, dx_1 dx_2\\
&= \int_{\mathbb{R}^d} \left(\int_\Lambda g(s, 0, x_1) g(t-s, x_1, x_1 + x_2)\, dx_2 \right) dx_1\\
&= \int_{\mathbb{R}^d} g(s, 0, x_1)\, dx_1 \int_\Lambda g(t-s, 0, x_2)\, dx_2\\
&= \int_\Lambda g(t-s, 0, x_2)\, dx_2\\
&= \int_\Lambda \frac{1}{(\sqrt{2\pi(t-s)})^d} \exp\left(-\frac{|x_2|^2}{2(t-s)} \right) dx_2,
\end{aligned}$$

where χ_Λ denotes the characteristic function of Λ. □

Proof of Lemma 3.3. Note that

$$\text{(3.2)} \quad \{f \in W : \omega_n(f) > 2^{-n/3}\eta\} = \bigcup_{\substack{0 < t-s \leq 2^{-n},\\ s,t \in \tilde{t}_n^*}} \{f \in W : |\phi_t(f) - \phi_s(f)| > 2^{-n/3}\eta\}.$$

This in particular implies that the set on the left hand side is in $\mathcal{G}_{t_n} \subset \mathcal{G}$ (see the proof of (3.3) below). Since

$$\mu\{f \in W : |\phi_t(f) - \phi_s(f)| > c\} = \int_{|x| > c} \frac{1}{(\sqrt{2\pi(t-s)})^d} \exp\left(-\frac{|x|^2}{2(t-s)} \right) dx,$$

which follows from Lemma 3.4, by (3.2) we have

$$\begin{aligned}
\mu\{f \in W : \omega_n(f) > 2^{-n/3}\eta\} &\leq \sum_{\substack{0 < t-s \leq 2^{-n},\\ s,t \in \tilde{t}_n^*}} \int_{|x| > 2^{-n/3}\eta} \frac{1}{(\sqrt{2\pi(t-s)})^d} \exp\left(-\frac{|x|^2}{2(t-s)} \right) dx\\
&\leq \sum_{\substack{0 < t-s \leq 2^{-n},\\ s,t \in \tilde{t}_n^*}} C_d \sqrt{t-s}\, 2^{n/3} \eta^{-1} \exp\left(-\frac{\eta^2}{2(t-s) 2^{2n/3} d} \right),
\end{aligned}$$

where $C_d = \frac{d^{3/2}\sqrt{2}}{\sqrt{\pi}}$ and to get the last inequality we have used the estimate

$$\int_{|x|>c} \frac{1}{(\sqrt{2\pi t})^d} \exp\left(-\frac{|x|^2}{2t}\right) dx \\ \leq \sum_{i=1}^{d} \int_{|v_i|>c/\sqrt{d}} \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{v_i^2}{2t}\right) dv_i \leq d\frac{\sqrt{2td}}{\sqrt{\pi}c} \exp\left(-\frac{c^2}{2dt}\right),$$

with $c > 0$. Therefore,

$$\mu\{f \in W : \omega_n(f) > 2^{-n/3}\eta\} \leq \sum_{\substack{0<t-s\leq 2^{-n}, \\ s,t\in \tilde{t}_n^*}} \left(C_d 2^{-n/2} 2^{n/3} \eta^{-1}\right) \exp\left(-\frac{\eta^2}{2}\frac{2^{-2n/3}}{d2^{-n}}\right) \\ \leq M_n^2 C_d 2^{-n/6} \eta^{-1} \exp\left(-\eta^2 2^{n/3-1} d^{-1}\right) \\ \leq (9/4) C_d n^2 2^{2n-n/6} \eta^{-1} \exp\left(-\eta^2 2^{n/3-1} d^{-1}\right).$$

Thus

$$\sum_{n=1}^{\infty} \mu\{f \in W : \omega_n(f) > 2^{-n/3}\eta\} \leq \left((9/4) C_d \sum_{n=1}^{\infty} n^2 2^{11n/6} \exp\left(-2^{n/3-1} d^{-1}\right)\right) \eta^{-1},$$

and hence taking η large enough depending on ϵ, we get the conclusion. □

Choose $\epsilon_0 > 0$ so that $\lim \mu_{t_n}(A_n) > \epsilon_0$. Applying Lemma 3.3, take $\eta_0 > 1$ such that

$$\sum_{n=1}^{\infty} \mu_{t_n}\{f \in W : \omega_n(f) > 2^{-n/3}\eta_0\} < \epsilon_0/3.$$

Choose a compact set $\Lambda_n' \subset \Lambda_n$ such that

$$\mu_{t_n}(\Phi_{t_n}^{-1}(\Lambda_n \setminus \Lambda_n')) < \epsilon_0 3^{-n}$$

(see [5, p. 48, Theorem 2.18]). There exist a compact set Λ_n'' in $\mathbb{R}_d^{N_n}$ such that $\Lambda_n'' \subset \Lambda_n'$ and

$$\Phi_{t_n}^{-1}(\Lambda_n') \cap \{f \in W : \omega_n(f) \leq 2^{-n/3}\eta_0\} = \Phi_{t_n}^{-1}(\Lambda_n''). \tag{3.3}$$

This can be shown as follows.

Let $h : \mathbb{R}_d^2 \to \mathbb{R}^d$, $h(x,y) = y - x$ as above. Let $B(r) = \{x \in \mathbb{R}^d : |x| \le r\}$. Then

$$\begin{aligned}\{f \in W : \omega_n(f) \le 2^{-n/3}\eta_0\} &= \bigcup_{\substack{0<t-s\le 2^{-n},\\ s,t\in \tilde{t}_n^*}} \{f \in W : h(\phi_t(f), \phi_s(f)) \in B(2^{-n/3}\eta_0)\} \\ &= \bigcup_{\substack{0<t-s\le 2^{-n},\\ s,t\in \tilde{t}_n^*}} \{f \in W : (\phi_t(f), \phi_s(f)) \in h^{-1}(B(2^{-n/3}\eta_0))\} \\ &= \bigcup_{\substack{0<t-s\le 2^{-n},\\ s,t\in \tilde{t}_n^*}} \{f \in W : \Phi_{t_n}(f) \in \Lambda(n, \eta_0, s, t)\} \\ &= \Phi_{t_n}^{-1}\left(\bigcup_{\substack{0<t-s\le 2^{-n},\\ s,t\in \tilde{t}_n^*}} \Lambda(n, \eta_0, s, t) \right)\end{aligned}$$

for some $\Lambda(n, \eta_0, s, t) \in \mathcal{B}(\mathbb{R}_d^{N_n})$ which is closed and can be written as

$$\Lambda(n, \eta_0, s, t) = \sigma\left(h^{-1}(B(2^{-n/3}\eta_0)) \times \mathbb{R}_d^{N_n - 2}\right)$$

with some $\sigma = \sigma_{s,t} \in \mathcal{S}_{N_n}$. Thus we can take

$$\Lambda_n'' = \Lambda_n' \cap \left(\bigcup_{\substack{0<t-s\le 2^{-n},\\ s,t\in \tilde{t}_n^*}} \Lambda(n, \eta_0, s, t) \right).$$

Next, we show

$$\bigcap_{j=1}^{n} \Phi_{t_j}^{-1}(\Lambda_j'') \ne \emptyset, \quad n = 1, 2, \ldots. \tag{3.4}$$

To prove this we first observe that

$$\begin{aligned}A_n = \bigcap_{j=1}^{n} A_j = \bigcap_{j=1}^{n} \Phi_{t_j}^{-1}(\Lambda_j) &= \bigcap_{j=1}^{n} \left(\Phi_{t_j}^{-1}(\Lambda_j \setminus \Lambda_j') \cup \Phi_{t_j}^{-1}(\Lambda_j' \setminus \Lambda_j'') \cup \Phi_{t_j}^{-1}(\Lambda_j'') \right) \\ &\subset \left(\bigcup_{j=1}^{n} \Phi_{t_j}^{-1}(\Lambda_j \setminus \Lambda_j') \right) \cup \left(\bigcup_{j=1}^{n} \Phi_{t_j}^{-1}(\Lambda_j' \setminus \Lambda_j'') \right) \cup \left(\bigcap_{j=1}^{n} \Phi_{t_j}^{-1}(\Lambda_j'') \right).\end{aligned}$$

We next note that

$$\Phi_{t_j}^{-1}(\Lambda_j' \setminus \Lambda_j'') = \Phi_{t_j}^{-1}(\Lambda_j') \setminus \Phi_{t_j}^{-1}(\Lambda_j'') \subset \{f \in W : \omega_j(f) > 2^{-j/3}\eta_0\}.$$

Thus

$$\begin{aligned}\mu\left(\bigcap_{j=1}^{n} \Phi_{t_j}^{-1}(\Lambda_j'') \right) &\ge \mu_{t_n}(A_n) - \sum_{j=1}^{n} \mu_{t_j}\left(\Phi_{t_j}^{-1}\left(\Lambda_j \setminus \Lambda_j'\right) \right) - \sum_{j=1}^{n} \mu_{t_j}\left(\Phi_{t_j}^{-1}\left(\Lambda_j' \setminus \Lambda_j''\right) \right) \\ &> \epsilon_0 - \sum_{j=1}^{n} \epsilon_0 3^{-j} - \sum_{j=1}^{n} \mu_{t_j}\{f \in W : \omega_j(f) > 2^{-j/3}\eta_0\} \\ &> \epsilon_0 - \epsilon_0/2 - \epsilon_0/3 = \epsilon_0/6 > 0,\end{aligned}$$

from which we can deduce (3.4).

Let $f_n \in \cap_{j=1}^n \Phi_{t_j}^{-1}(\Lambda_j'')$. Since $\Phi_{t_1}(f_n) \in \Lambda_1''$, $n = 1, 2, \dots$, $\{\Phi_{t_1}(f_n) : n = 1, 2, \dots\}$ is a sequence in a compact set Λ_1'', and there is a convergent subsequence $\{\Phi_{t_1}(f_{n_j^{(1)}})\}_{j=1}^\infty$ in Λ_1''. Next, $\Phi_{t_2}(f_{n_j^{(1)}})$ is in Λ_2'' if $j \geq 2$. Thus $\{\Phi_{t_2}(f_{n_j^{(1)}})\}_{j=1}^\infty$ contains a convergent subsequence $\{\Phi_{t_2}(f_{n_j^{(2)}})\}_{j=1}^\infty$ in Λ_2''. We note that $\{\Phi_{t_1}(f_{n_j^{(2)}})\}_{j=1}^\infty$ is also convergent in Λ_1''. Continuing this, we have sequences $\{f_{n_j^{(k)}}\}_{j=1}^\infty$ for $k = 1, 2, \dots$ such that $\{\Phi_{t_k}(f_{n_j^{(k')}})\}_{j=1}^\infty$ is a convergent sequence in Λ_k'' if $k' \geq k$ and such that $\{f_{n_j^{(k')}}\}_{j=1}^\infty$ is a subsequence of $\{f_{n_j^{(k)}}\}_{j=1}^\infty$ if $k' \geq k$.

Consider $\{f_{n_k}\}$, $n_k = n_k^{(k)}$. Then $\{\Phi_{t_j}(f_{n_k})\}_{k \geq j}$ is a sequence in Λ_j'' and converges in Λ_j'' for each $j \geq 1$, since $\{\Phi_{t_j}(f_{n_l^{(j)}})\}_{l=1}^\infty$ converges in Λ_j'' and $\{f_{n_k^{(k)}}\}_{k \geq j}$ is a subsequence of $\{f_{n_l^{(j)}}\}_{l \geq 1}$. Thus, if $t \in \cup_{j \geq 1} \tilde{t}_j$, the limit $\lim_{k \to \infty} f_{n_k}(t)$ exists. Define $x(t)$ on $\cup_{j \geq 1} \tilde{t}_j$ by setting

$$x(t) = \lim_{k \to \infty} f_{n_k}(t), \quad t \in \cup_{j \geq 1} \tilde{t}_j.$$

Then,

$$(x(a_1), x(a_2), \dots, x(a_{N_j})) \in \Lambda_j'' \quad \text{if } t_j = (a_1, a_2, \dots, a_{N_j}). \tag{3.5}$$

Assertion 2. There exists $f \in W$ such that $f = x$ on $\cup_{j \geq 1} \tilde{t}_j$.

If we have this, $\Phi_{t_j}(f) \in \Lambda_j'' \subset \Lambda_j$ for all $j \geq 1$ by (3.5), and hence

$$f \in \bigcap_{j=1}^\infty \Phi_{t_j}^{-1}(\Lambda_j) = \bigcap_{j=1}^\infty A_j,$$

which will prove Assertion 1.

Proof of Assertion 2. If $f \in \Phi_{t_j}^{-1}(\Lambda_j'')$, then $\omega_j(f) \leq 2^{-j/3}\eta_0$. So, if $k \geq j$, we have

$$|\phi_t(f_{n_k}) - \phi_s(f_{n_k})| \leq 2^{-j/3}\eta_0$$

whenever $s, t \in \tilde{t}_j^*$ and $0 < t - s \leq 2^{-j}$. Thus, letting $k \to \infty$, we have

$$|x(t) - x(s)| \leq 2^{-j/3}\eta_0 \quad \text{if } s, t \in \tilde{t}_j^* \text{ and } 0 < t - s \leq 2^{-j}. \tag{3.6}$$

Since $\cup_{j \geq 1} \tilde{t}_j^{**}$ is dense in $\mathbb{R}_0$, by (3.6) it will be shown that Assertion 2 will follow from the next result.

Assertion 3. There exists $f \in W$ such that $f = x$ on $\cup_{j \geq 1} \tilde{t}_j^{**}$.

To prove this we show the following.

Lemma 3.5. *If* $s, t \in \cup_{j \geq 1} \tilde{t}_j^{**}$, $0 < s < t < j$ *and* $0 < t - s < 2^{-j}$, *then*

$$|x(t) - x(s)| \leq C 2^{-j/3}.$$

Obviously, this implies that x is uniformly continuous on every bounded subset of $\cup_{j \geq 1} \tilde{t}_j^{**}$.

Proof of Lemma 3.5. Take $j' \geq j$ such that $2^{-j'-1} \leq t - s < 2^{-j'}$. It suffices to prove the lemma with j' in place of j.

We can find $u_0 = q2^{-j'-1}$ with $1 \le q \le j'2^{j'+1}$ such that $s \le u_0 \le t$. We have $s, t \in \tilde{t}^{**}_{j'+n}$ for some n. We can take n as large as we wish. We have $s = k2^{-j'-n}$, $t = m2^{-j'-n}$ with $1 \le k \le j'2^{j'+n}$ and $1 \le m \le j'2^{j'+n}$. Then

$$u_0 - s = (q2^{n-1} - k)2^{-j'-n}.$$

Since $0 \le u_0 - s < 2^{-j'}$, we have $0 \le q2^{n-1} - k < 2^n$. Thus we can write

$$u_0 - s = (\sum_{p=0}^{n-1} \epsilon_p 2^p)2^{-j'-n} = \sum_{p=0}^{n-1} \epsilon_p 2^{-(j'+n-p)}$$

with $\epsilon_p = 0$ or 1. Put

$$s_l = s + \sum_{p=0}^{l-2} \epsilon_p 2^{-(j'+n-p)}$$

for $2 \le l \le n+1$ and $s_1 = s$. Then $s_{n+1} = u_0$.

By definition, $s_{n+1} \in \tilde{t}^{**}_{j'+1}$ and $s_1 \in \tilde{t}^{**}_{j'+n}$. Also, we observe that $s_l \in \tilde{t}^{**}_{j'+n-l+1}$ for $1 \le l \le n$. For $n \ge 2$, this can be shown as follows. First, since $s_n = u_0 - \epsilon_{n-1}2^{-(j'+1)}$, we have $s_n \in \tilde{t}^{**}_{j'+1}$. Next, suppose that $s_{l+1} \in \tilde{t}^{**}_{j'+n-l}$ for $1 \le l \le n-1$. Then

$$\begin{aligned} s_l &= s_{l+1} - \epsilon_{l-1}2^{-(j'+n-l+1)} = r2^{-(j'+n-l)} - \epsilon_{l-1}2^{-(j'+n-l+1)} \\ &= (2r - \epsilon_{l-1})2^{-(j'+n-l+1)}, \end{aligned}$$

where

$$0 < 2r - \epsilon_{l-1} \le 2r \le 2(j'+n-l)2^{j'+n-l} \le (j'+n-l+1)2^{j'+n-l+1}.$$

Therefore, we see that $s_l \in \tilde{t}^{**}_{j'+n-l+1}$.

We have $s_{l+1}, s_l \in \tilde{t}^{**}_{j'+n-l+1}$ for $1 \le l \le n$. Also,

$$0 \le s_{l+1} - s_l = \epsilon_{l-1}2^{-(j'+n-l+1)} \le 2^{-(j'+n-l+1)}$$

for $1 \le l \le n$. Therefore by (3.6)

$$|x(u_0) - x(s)| \le \sum_{l=1}^{n} |x(s_{l+1}) - x(s_l)| \le \eta_0 \sum_{l=1}^{n} 2^{-(j'+n-l+1)/3} \le c_0\eta_0 2^{-j'/3},$$

where $c_0 = \sum_{m=1}^{\infty} 2^{-m/3}$.

Similarly, $|x(t) - x(u_0)| \le c_0\eta_0 2^{-j'/3}$. To see this we write

$$t - u_0 = \sum_{v=0}^{n-1} \epsilon_v 2^{-(j'+1+v)}$$

with $\epsilon_v = 0$ or 1,

$$t_l = u_0 + \sum_{v=0}^{l-2} \epsilon_v 2^{-(j'+1+v)}$$

for $2 \le l \le n+1$ and $t_1 = u_0$. Then $t_{n+1} = t$.

We have $t_1 \in \tilde{t}^{**}_{j'+1}$ and $t_l \in \tilde{t}^{**}_{j'+l-1}$ for $2 \le l \le n+1$. To see this, we first note that $t_2 \in \tilde{t}^{**}_{j'+1}$, since $t_2 = u_0 + \epsilon_0 2^{-(j'+1)} = (q + \epsilon_0)2^{-(j'+1)}$ with $1 \le q \le j'2^{j'+1}$. Next, suppose that $t_l \in \tilde{t}^{**}_{j'+l-1}$ for $2 \le l \le n$. We write

$$t_{l+1} = t_l + \epsilon_{l-1}2^{-(j'+l)} = r2^{-(j'+l-1)} + \epsilon_{l-1}2^{-(j'+l)} = (2r + \epsilon_{l-1})2^{-(j'+l)}$$

with $0 < r \le (j'+l-1)2^{j'+l-1}$. We see that $0 < 2r + \epsilon_{l-1} \le (j'+l)2^{j'+l}$, which implies $t_{l+1} \in \tilde{t}^{**}_{j'+l}$. The desired result follows from this.

We have $t_l, t_{l+1} \in \tilde{t}^{**}_{j'+l}$ and $t_l \le t_{l+1} \le t_l + 2^{-(j'+l)}$ for $1 \le l \le n$. So, by applying (3.6) as above to $|x(t_{l+1}) - x(t_l)|$, we have the estimate as claimed.

Thus $|x(t) - x(s)| \le |x(t) - x(u_0)| + |x(u_0) - x(s)| \le 2c_0\eta_0 2^{-j'/3}$. This proves Lemma 3.5. □

Since$\cup_{j\ge1}\tilde{t}^{**}_j$ is dense in $\mathbb{R}_0$, by Lemma 3.5 and a well-known argument, we have Assertion 3.

Proof of Assertion 3. We write $x(t) = (x_1(t), x_2(t), \dots, x_d(t))$. For $t \in \mathbb{R}_0$, let

$$f_i(t) = \lim_{\epsilon\to0} \sup\left\{x_i(s) : |s-t| < \epsilon, s \in \cup_{j\ge1}\tilde{t}^{**}_j\right\}, \quad 1 \le i \le d,$$

and

$$f(t) = (f_1(t), f_2(t), \dots, f_d(t)).$$

Then, by Lemma 3.5 it is not difficult to see that f is continuous on $\mathbb{R}_0$ and equals x when restricted to $\cup_{j\ge1}\tilde{t}^{**}_j$. □

For any $t \in \cup_{j\ge1}\tilde{t}_j$, by (3.1) and (3.6), there exists a sequence $\{t_m\}$ in $\cup_{j\ge1}\tilde{t}^{**}_j$ such that $t_m \to t$ and $x(t_m) \to x(t)$. Thus $f(t) = \lim f(t_m) = \lim x(t_m) = x(t)$. This completes the proof of Assertion 2 and hence that of Assertion 1. □

This completes the proof of Proposition 3.2 and hence that of Proposition 3.1.

Let $\mathcal{F} = \mathcal{F}_d$ be the sigma-algebra generated by $\mathcal{G}$. Then, by Proposition 3.1, μ uniquely extends to a measure on $\mathcal{F}$, which is again denoted by μ (see [1, p. 30, Theorem (1.14)]). We also write $\mu = \mu_d$. The measure space $(W_d, \mathcal{F}_d, \mu_d)$ is called the Wiener probability space.

Remark 3.6. We have

$$\mu\{f \in W : f(0) = 0\} = 1. \tag{3.7}$$

This can be shown as follows. By (2.1) we see that

$$\mu\{f \in W : f(t) \in B(r)\} = \int_{B(r)} \frac{1}{(\sqrt{2\pi t})^d} \exp\left(-\frac{|x|^2}{2t}\right) dx$$

for all $t, r > 0$ (recall that $B(r) = \{x \in \mathbb{R}^d : |x| \le r\}$). Letting $t \to 0$, we have

$$\mu\{f \in W : f(0) \in B(r)\} = 1$$

for all $r > 0$. Thus (3.7) follows by taking the limit as $r \to 0$.

4. Wiener measure of nowhere differentiable functions

Let D be the subset of W_1 consisting of the functions f for which there exists at least one point in $\mathbb{R}_0$ at which they are differentiable.

Theorem 4.1. *We have $\mu(D) = 0$, where $\mu = \mu_1$.*

To prove this we need the following.

Theorem 4.2. *Let* $0 < t_1 < t_2 < \cdots < t_n$ *and* $\Lambda_1, \Lambda_2, \ldots, \Lambda_n \in \mathcal{B}(\mathbb{R}^d)$. *Then*

$$\begin{aligned}\mu\left\{f \in W : \phi_{t_1}(f) \in \Lambda_1, \phi_{t_2}(f) - \phi_{t_1}(f) \in \Lambda_2, \ldots, \phi_{t_n}(f) - \phi_{t_{n-1}}(f) \in \Lambda_n\right\} \\ = \mu\left\{f \in W : \phi_{t_1}(f) \in \Lambda_1\right\} \mu\left\{f \in W : \phi_{t_2}(f) - \phi_{t_1}(f) \in \Lambda_2\right\} \ldots \\ \times \mu\left\{f \in W : \phi_{t_n}(f) - \phi_{t_{n-1}}(f) \in \Lambda_n\right\}.\end{aligned}$$

Proof. Define a continuous function $\varphi : \mathbb{R}_d^n \to \mathbb{R}_d^n$ by

$$\varphi(x_1, x_2, \ldots, x_n) = (x_1, x_2 - x_1, x_3 - x_2, \ldots, x_n - x_{n-1}).$$

Then

$$\begin{aligned}\left\{f \in W : \phi_{t_1}(f) \in \Lambda_1, \phi_{t_2}(f) - \phi_{t_1}(f) \in \Lambda_2, \ldots, \phi_{t_n}(f) - \phi_{t_{n-1}}(f) \in \Lambda_n\right\} \\ = \Phi_t^{-1}\left(\varphi^{-1}\left(\Lambda_1 \times \Lambda_2 \times \cdots \times \Lambda_n\right)\right),\end{aligned}$$

where $t = (t_1, \ldots, t_n)$. Thus

$$\begin{aligned}&\mu\left\{f \in W : \phi_{t_1}(f) \in \Lambda_1, \phi_{t_2}(f) - \phi_{t_1}(f) \in \Lambda_2, \ldots, \phi_{t_n}(f) - \phi_{t_{n-1}}(f) \in \Lambda_n\right\} \\ &= \mu_t\left(\Phi_t^{-1}\left(\varphi^{-1}\left(\Lambda_1 \times \Lambda_2 \times \cdots \times \Lambda_n\right)\right)\right) \\ &= \int_{\varphi^{-1}(\Lambda_1 \times \Lambda_2 \times \cdots \times \Lambda_n)} g(t_1, 0, x_1) g(t_2 - t_1, x_1, x_2) \ldots g(t_n - t_{n-1}, x_{n-1}, x_n)\, dx_1 \ldots dx_n \\ &= \int_{\Lambda_1 \times \Lambda_2 \times \cdots \times \Lambda_n} g(t_1, 0, x_1) g(t_2 - t_1, 0, x_2) \ldots g(t_n - t_{n-1}, 0, x_n)\, dx_1 \ldots dx_n \\ &= \int_{\Lambda_1} g(t_1, 0, x_1)\, dx_1 \int_{\Lambda_2} g(t_2 - t_1, 0, x_2)\, dx_2 \cdots \int_{\Lambda_n} g(t_n - t_{n-1}, 0, x_n)\, dx_n,\end{aligned}$$

which will imply the conclusion if we recall Lemma 3.4 and the formula (2.1). □

Proof of Theorem 4.1. Let D_N be the subset of D consisting of the functions f for which there exists at least one point in $[0, N]$ at which they are differentiable. It suffices to prove $\mu(D_N) = 0$ for $N = 1, 2, \ldots$, since $D_N \uparrow D$. Define

$$E_{k,n} = \bigcup_{j=1}^{nN} \bigcap_{i=j}^{j+2} \{f \in W : |\phi_{(i-1)/n}(f) - \phi_{i/n}(f)| < k/n\}$$

for $k = 1, 2, \ldots; n = 1, 2, \ldots$. Then

$$D_N \subset \bigcup_{k \geq 1} \left(\liminf_{n \to \infty} E_{k,n}\right).$$

Thus to prove the claim, it suffices to show that $\lim_{n\to\infty} \mu(E_{k,n}) = 0$ for every k.

By Theorem 4.2 and Lemma 3.4 with $d = 1$, we see that

$$\begin{aligned}&\mu\left(\bigcap_{i=j}^{j+2} \{f \in W : |\phi_{(i-1)/n}(f) - \phi_{i/n}(f)| < k/n\}\right) \\ &= \prod_{i=j}^{j+2} \mu\{f \in W : |\phi_{(i-1)/n}(f) - \phi_{i/n}(f)| < k/n\} \\ &= \left(2 \int_0^{k/n} \frac{1}{\sqrt{2\pi/n}} \exp\left(-\frac{x^2}{2/n}\right)\, dx\right)^3 \\ &\leq (2k/\sqrt{2\pi})^3 n^{-3/2}.\end{aligned}$$

Thus
$$\mu(E_{k,n}) \leq nN(2k/\sqrt{2\pi})^3 n^{-3/2},$$
from which we get the result as claimed. □

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