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メタデータ	言語: eng 出版者: 公開日: 2019-11-08 キーワード (Ja): キーワード (En): 作成者: メールアドレス: 所属:
URL	https://doi.org/10.24517/00056034

CHARACTERIZATION OF PARABOLIC HARDY SPACES BY LITTLEWOOD-PALEY FUNCTIONS

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Abstract. We consider Littlewood-Paley functions associated with non-isotropic dilations. We prove that they can be used to characterize the parabolic Hardy spaces of Calderón-Torchinsky.

1. Introduction

Let P be an $n \times n$ real matrix such that

$$\langle Px, x\rangle \geq \langle x, x\rangle \quad \text{for all } x = (x_1, \dots, x_n) \in \mathbb{R}^n,$$

where $\langle x, y\rangle = x_1y_1 + \cdots + x_ny_n$ is the inner product in $\mathbb{R}^n$. Let $\gamma = \text{trace } P$. Define a dilation group $\{A_t\}_{t>0}$ on $\mathbb{R}^n$ by $A_t = t^P = \exp((\log t)P)$. It is known that $|A_tx|$ is strictly increasing as a function of t on $\mathbb{R}_+ = (0, \infty)$ for $x \neq 0$, where $|x| = \langle x, x\rangle^{1/2}$. Define a norm function $\rho(x)$ to be the unique positive real number t such that $|A_{t^{-1}}x| = 1$ when $x \neq 0$ and $\rho(0) = 0$. Then $\rho(A_tx) = t\rho(x)$, $t > 0$, $x \in \mathbb{R}^n$, $\rho \in C^\infty(\mathbb{R}^n \setminus \{0\})$ and the following results are known (see [4, 6]):

(P.1) $\rho(x + y) \leq \rho(x) + \rho(y)$ for all $x, y \in \mathbb{R}^n$;
(P.2) $\rho(x) \leq 1$ if and only if $|x| \leq 1$;
(P.3) if $|x| \leq 1$, then $|x| \leq \rho(x)$;
(P.4) if $|x| \geq 1$, then $|x| \geq \rho(x)$;
(P.5) if $t \geq 1$, then $|A_tx| \geq t|x|$ for all $x \in \mathbb{R}^n$;
(P.6) if $0 < t \leq 1$, then $|A_tx| \leq t|x|$ for all $x \in \mathbb{R}^n$.

Similarly, we can consider a norm function $\rho^*(x)$ associated with the dilation group $\{A_t^*\}_{t>0}$, where A_t^* denotes the adjoint of A_t. We have properties analogous to those for $\rho(x)$, A_t above.

Let

$$g_\varphi(f)(x) = \left(\int_0^\infty |f * \varphi_t(x)|^2 \frac{dt}{t}\right)^{1/2} \tag{1.1}$$

be the Littlewood-Paley function on $\mathbb{R}^n$, where $\varphi_t(x) = t^{-\gamma}\varphi(A_t^{-1}x)$ and φ is a function in $L^1(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} \varphi(x)\, dx = 0. \tag{1.2}$$

Mapping properties of g_φ on $L^p(\mathbb{R}^n)$ for $p \in (1, \infty)$ can be found in [1, 16, 17] when $P = E$ (the identity matrix) and g_φ is defined by $\varphi_t(x) = t^{-n}\varphi(t^{-1}x)$ in (1.1).

2010 *Mathematics Subject Classification.* Primary 42B25; Secondary 42B30.
Key Words and Phrases. Littlewood-Paley functions, parabolic Hardy spaces.
The author is partly supported by Grant-in-Aid for Scientific Research (C) No. 25400130, Japan Society for the Promotion of Science.

We say that a tempered distribution f belongs to the parabolic H^p, $0 < p < \infty$, if $\|f\|_{H^p} = \|f^*\|_p < \infty$, where $f^*(x) = \sup_{t>0} |\Phi_t * f(x)|$ and $\|f^*\|_p = \|f^*\|_{L^p}$, with $\Phi \in \mathcal{S}(\mathbb{R}^n)$ satisfying $\int \Phi(x)\,dx = 1$, $\operatorname{supp}(\Phi) \subset \{|x| < 1\}$ (see [4, 5], [8]). We have denoted by $\mathcal{S}(\mathbb{R}^n)$ the Schwartz class of rapidly decreasing smooth functions on $\mathbb{R}^n$. It is known that H^p coincides with L^p when $1 < p < \infty$ and that a different choice of such Φ gives an equivalent norm.

Let $\varphi \in L^1(\mathbb{R}^n)$. We consider the non-degeneracy condition:

$$\sup_{t>0} |\hat{\varphi}(A_t^*\xi)| > 0 \quad \text{for all } \xi \neq 0, \tag{1.3}$$

where the Fourier transform is defined as

$$\hat{f}(\xi) = \mathcal{F}(f)(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \langle x, \xi \rangle}\,dx.$$

In this note we shall prove the following.

Theorem 1.1. *Let φ be a function in $\mathcal{S}(\mathbb{R}^n)$ satisfying (1.2) and (1.3). Let g_φ be as in (1.1). Suppose that $0 < p \le 1$. Then if $f \in H^p$, we have*

$$c_1\|f\|_{H^p} \le \|g_\varphi(f)\|_p \le c_2\|f\|_{H^p}$$

with some positive constants c_1, c_2 independent of f.

The second inequality of the conclusion can be shown by applying a theory of vector valued singular integrals. The first inequality is more difficult for us to prove, where we do not assume the condition that the Fourier transform of φ is supported on a compact set not containing the origin; if we have this condition, the result could be shown much more easily.

We recall some related results when $P = E$ and g_φ is defined by $\varphi_t(x) = t^{-n}\varphi(t^{-1}x)$ in (1.1). Then Theorem 1.1 is known (see [22] and also [12] for some background materials). Let $Q(x) = [(\partial/\partial t)P(x,t)]_{t=1}$, where

$$P(x,t) = c_n \frac{t}{(|x|^2 + t^2)^{(n+1)/2}}, \quad c_n = \frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}},$$

is the Poisson kernel associated with the upper half space $\mathbb{R}^n \times (0,\infty)$ (see [20, Chap. I]). We note that $\hat{Q}(\xi) = -2\pi|\xi|e^{-2\pi|\xi|}$. Then it is also known that

$$c_1\|f\|_{H^p} \le \|g_Q(f)\|_p \le c_2\|f\|_{H^p} \tag{1.4}$$

for $f \in H^p(\mathbb{R}^n)$, $0 < p < \infty$, with positive constants c_1, c_2 (see [8] and also [22]). In addition, we would like to mention that in [9, Chap. 7] we can find a relation between Hardy spaces and Littlewood-Paley functions associated with the heat kernels in the setting of homogeneous groups.

Uchiyama [22] gave a proof of the first inequality of (1.4) for $0 < p \le 1$ by methods of real analysis without the use of special properties of the Poisson kernel such as harmonicity, a semigroup property. Applying a similar argument, [22] also proved the first inequality of the conclusion of Theorem 1.1 (when $P = E$) for $0 < p \le 1$:

$$\|f\|_{H^p} \le c\|g_\varphi(f)\|_p. \tag{1.5}$$

Meanwhile, for a function F on $\mathbb{R}^n$ and positive real numbers N, R, if we define the Peetre maximal function $F^{**}_{N,R}$ by

$$F^{**}_{N,R}(x) = \sup_{y\in\mathbb{R}^n} \frac{|F(x-y)|}{(1+R|y|)^N}$$

(see [14]), then it is known that the maximal function $F^{**}_{N,R}$ can be used along with some well-known arguments to prove (1.5) for $0 < p \le 1$ when $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfies a non-degeneracy condition and the condition $\mathrm{supp}(\hat{\varphi}) \subset \{a_1 \le |\xi| \le a_2\}$ for some $a_1, a_2 > 0$.

In [18], (1.5) was proved for f in a dense subspace of $H^p(\mathbb{R}^n)$ and for φ in a class of functions including Q and a general $\varphi \in \mathcal{S}(\mathbb{R}^n)$, without the restriction on $\mathrm{supp}(\hat{\varphi})$ above, with (1.2) and (1.3), for $A_t^*\xi = t\xi$, by applying a vector valued inequality related to the Littlewood-Paley theory. The proof of the vector valued inequality is based on an application of the maximal function $F^{**}_{N,R}$. This application of $F^{**}_{N,R}$ in proving (1.5) was found by [18].

The purpose of this note is to generalize the methods of [18] to the case of the parabolic Hardy spaces and establish the characterization of the parabolic Hardy spaces in terms of Littlewood-Paley functions of (1.1) with $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfying conditions described in Theorem 1.1, where we do not require the condition that $\mathrm{supp}(\hat{\varphi})$ is a compact set not containing the origin. (See [19] for related results for the weighted Hardy spaces.)

In Section 2, we shall prove an analogue of a vector valued inequality in [18] for the general dilation group $\{A_t\}$, introducing a function space named as $\mathcal{B}$ which includes those φ considered in Theorem 1.1 (Theorem 2.2). In Section 3, we shall consider g_φ for $\varphi \in \mathcal{B}$ and prove (1.5) in the setting of the parabolic H^p for such φ and for f in a dense subspace of H^p as an application of Theorem 2.2 (Corollary 3.1). Theorem 2.2 will be stated more generally than needed for the proof of Corollary 3.1 as weighted vector valued inequalities.

Also, in Section 3, Theorem 1.1 will be derived from Corollary 3.1. Finally, discrete parameter versions of Theorems 1.1 and 2.2 will be stated (Theorems 3.6 and 3.7).

2. Weighted vector valued inequalities with non-isotropic dilations

To state our results, we introduce a set $\mathcal{B}$ of functions and recall the Muckenhoupt weight class.

Definition 2.1. Let φ be a function in $L^1(\mathbb{R}^n)$ satisfying (1.3). We assume that $\hat{\varphi} \in C^\infty(\mathbb{R}^n \setminus \{0\})$. Then, we say $\varphi \in \mathcal{B}$ if the following conditions are satisfied:

$$\varphi \in C^1(\mathbb{R}^n), \quad \partial_k\varphi \in L^1(\mathbb{R}^n), \quad 1 \le k \le n, \quad \text{where } \partial_k = \partial_{x_k} = \partial/\partial x_k; \tag{2.1}$$

$$|\hat{\varphi}(\xi)| \le C|\xi|^\epsilon \quad \text{for some } \epsilon > 0; \tag{2.2}$$

$$|\partial_\xi^\alpha \hat{\varphi}(\xi)| \le C_{\alpha,\tau}|\xi|^{-\tau} \quad \text{outside a neighborhood of the origin,} \tag{2.3}$$

for all α and $\tau > 0$ with a constant $C_{\alpha,\tau}$, where $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi-index, $\alpha_j \in \mathbb{Z}$, $\alpha_j \ge 0$, $|\alpha| = \alpha_1 + \dots + \alpha_n$ and $\partial_\xi^\alpha = \partial_{\xi_1}^{\alpha_1} \dots \partial_{\xi_n}^{\alpha_n}$.

If $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfies (1.2) and (1.3), then clearly $\varphi \in \mathcal{B}$.

We consider a ball in $\mathbb{R}^n$ with center x and radius t relative to ρ defined by

$$B(x,t) = \{y \in \mathbb{R}^n : \rho(x-y) < t\}.$$

We say that a weight function w belongs to the class A_p, $1 < p < \infty$, of Muckenhoupt if

$$[w]_{A_p} = \sup_B \left(|B|^{-1}\int_B w(x)\,dx\right)\left(|B|^{-1}\int_B w(x)^{-1/(p-1)}dx\right)^{p-1} < \infty,$$

where the supremum is taken over all balls B in $\mathbb{R}^n$ and $|B|$ denotes the Lebesgue measure of B. Let $A_\infty = \cup_{p>1} A_p$. Also, we define the class A_1 to be the family of weight functions w such that $M(w) \leq Cw$ almost everywhere. We denote by $[w]_{A_1}$ the infimum of all such C. Here, M is the Hardy-Littlewood maximal operator relative to ρ

$$M(f)(x) = \sup_{x\in B} |B|^{-1}\int_B |f(y)|\,dy,$$

where the supremum is taken over all balls B in $\mathbb{R}^n$ containing x. (See [2, 10].)

For a weight w, we denote by $\|f\|_{p,w}$ the weighted L^p norm

$$\left(\int_{\mathbb{R}^n} |f(x)|^p w(x)\,dx\right)^{1/p}.$$

Then we have the following result.

Theorem 2.2. *Let $\varphi \in \mathcal{B}$. Suppose that $0 < p, q < \infty$ and $w \in A_\infty$. Let $\psi \in \mathcal{S}(\mathbb{R}^n)$. Suppose that $\hat{\psi} = 0$ in a neighborhood of the origin. Then*

$$\left\|\left(\int_0^\infty |f*\psi_t|^q\,\frac{dt}{t}\right)^{1/q}\right\|_{p,w} \leq C\left\|\left(\int_0^\infty |f*\varphi_t|^q\,\frac{dt}{t}\right)^{1/q}\right\|_{p,w}$$

for $f \in \mathcal{S}(\mathbb{R}^n)$ with a positive constant C independent of f.

This can be generalized to more general ψ (see Theorem 2.9 of [18]). Function classes B^l_τ in $L^1(\mathbb{R}^n)$ are defined in Definition 2.7 of [18] for $\tau, l \geq 0$ with $l \in \mathbb{Z}$. If $\varphi \in B^l_\tau$, then $\hat{\varphi}$ satisfies a condition less restrictive than (2.3). Although Theorem 2.2 is formulated by using $\mathcal{B}$ for simplicity, it can be stated by using a suitable B^l_τ depending on p, q, w instead of using $\mathcal{B}$. A similar remark applies to Corollary 3.1 below.

Let

$$G^{**}_{N,R}(x) = \sup_{y\in\mathbb{R}^n} \frac{|G(x-y)|}{(1+R\rho(y))^N} \tag{2.4}$$

for a function G on $\mathbb{R}^n$ and positive real numbers N, R. To prove Theorem 2.2 we use the following.

Lemma 2.3. *Let $0 < q < \infty$, $N > 0$. Suppose that $\varphi \in \mathcal{B}$, $f \in \mathcal{S}(\mathbb{R}^n)$. Let $r = \gamma/N$. Then*

$$\int_0^\infty (f*\varphi_t)^{**}_{N,t^{-1}}(x)^q\,\frac{dt}{t} \leq C\int_0^\infty M(|f*\varphi_t|^r)(x)^{q/r}\,\frac{dt}{t}.$$

We apply the next result to show Lemma 2.3.

Lemma 2.4. *Let $N = \gamma/r$, $r > 0$ and let $\varphi \in L^1(\mathbb{R}^n)$, $f \in \mathcal{S}(\mathbb{R}^n)$. Suppose that φ satisfies* (2.1)*. Then*

$$(f * \varphi_t)^{**}_{N,t^{-1}}(x) \le C\delta^{-N} M(|f * \varphi_t|^r)(x)^{1/r} + C\delta |f * (\nabla\varphi)_t|^{**}_{N,t^{-1}}(x)$$

for all $\delta \in (0,1]$ with a constant C independent of δ and $t > 0$, where $\nabla\varphi = (\partial_1\varphi, \dots, \partial_n\varphi)$ and $f * (\nabla\varphi)_t = (f * (\partial_1\varphi)_t, \dots, f * (\partial_n\varphi)_t)$.

To prove Lemma 2.4, we use the following.

Lemma 2.5. *Let $G \in C^1(\mathbb{R}^n)$ and $R > 0$, $N = \gamma/r$, $r > 0$. Then*

$$G^{**}_{N,1}(x) \le C\delta^{-N} M(|G|^r)(x)^{1/r} + C\delta |\nabla G|^{**}_{N,1}(x)$$

for all $\delta \in (0,1]$ with a constant C independent of δ, where $G^{**}_{N,R}$ is as in (2.4)*.*

Proof. Let $\fint_{B(x,t)} f(y)\,dy = |B(x,t)|^{-1} \int_{B(x,t)} f(y)\,dy$. Then, for $\delta \in (0,1], r > 0$ and $x, z \in \mathbb{R}^n$ we write

$$|G(x-z)| = \left(\fint_{B(x-z,\delta)} |G(y) + (G(x-z) - G(y))|^r\,dy\right)^{1/r}.$$

This is bounded by

$$C_r \left(\fint_{B(x-z,\delta)} |G(y)|^r\,dy\right)^{1/r} + C_r \left(\fint_{B(x-z,\delta)} |G(x-z) - G(y)|^r\,dy\right)^{1/r},$$

where $C_r = 1$ if $r \ge 1$ and $C_r = 2^{-1+1/r}$ if $0 < r < 1$. Thus we have

$$|G(x-z)| \le C_r \left(\fint_{B(x-z,\delta)} |G(y)|^r\,dy\right)^{1/r} + C_r \sup_{y:\rho(x-z-y)<\delta} |x-z-y||\nabla G(y)|.$$

If $|x-z-y| \le 1$, $|x-z-y| \le \rho(x-z-y)$ by (P.3). So, we see that

$$|G(x-z)| \le C_r \left(\fint_{B(x-z,\delta)} |G(y)|^r\,dy\right)^{1/r} + C_r \sup_{y:\rho(x-z-y)<\delta} \delta|\nabla G(y)|. \tag{2.5}$$

If $\rho(x-z-y) < \delta$, $\rho(x-y) < \delta + \rho(z)$. Therefore

$$\begin{aligned}|\nabla G(y)| &\le \frac{|\nabla G(x + (y-x))|}{(1+\rho(x-y))^N}(1+\delta+\rho(z))^N \\ &\le |\nabla G|^{**}_{N,1}(x)(1+\delta+\rho(z))^N \\ &\le 2^N |\nabla G|^{**}_{N,1}(x)(1+\rho(z))^N.\end{aligned}$$

Thus

$$\sup_{y:\rho(x-z-y)<\delta} \delta|\nabla G(y)| \le 2^N \delta |\nabla G|^{**}_{N,1}(x)(1+\rho(z))^N. \tag{2.6}$$

On the other hand,

$$\begin{aligned}\left(\fint_{B(x-z,\delta)} |G(y)|^r\,dy\right)^{1/r} &\le \left(\delta^{-\gamma}(\delta+\rho(z))^\gamma \fint_{B(x,\delta+\rho(z))} |G(y)|^r\,dy\right)^{1/r} \\ &\le \delta^{-\gamma/r}(\delta+\rho(z))^{\gamma/r} M(|G|^r)(x)^{1/r} \\ &\le \delta^{-\gamma/r}(1+\rho(z))^{\gamma/r} M(|G|^r)(x)^{1/r}.\end{aligned} \tag{2.7}$$

By (2.5), (2.6) and (2.7), we have

$$|G(x-z)| \le C_r\delta^{-\gamma/r}(1+\rho(z))^{\gamma/r}M(|G|^r)(x)^{1/r} + 2^N C_r\delta|\nabla G|^{**}_{N,1}(x)(1+\rho(z))^N.$$

Thus, if $N=\gamma/r$, we see that

$$\frac{|G(x-z)|}{(1+\rho(z))^N} \le C_r\delta^{-N}M(|G|^r)(x)^{1/r} + 2^N C_r\delta|\nabla G|^{**}_{N,1}(x).$$

Taking the supremum in z over $\mathbb{R}^n$, we get the desired estimate. □

Proof of Lemma 2.4. Let $(T_tf)(x)=f(A_tx)$. Then we note the following.

(T.1) $(T_tG^{**}_{N,R})(x) = (T_tG)^{**}_{N,tR}(x)$.
(T.2) $T_t(f*g)(x) = t^\gamma(T_tf)*(T_tg)(x)$.
(T.3) $T_t(M(f))(x) = M(T_tf)(x)$.

By (T.1) and (T.2) we have

$$T_t((f*\varphi_t)^{**}_{N,t^{-1}})(x) = (T_tf*\varphi)^{**}_{N,1}(x).$$

Using Lemma 2.5, we see that

$$(T_tf*\varphi)^{**}_{N,1}(x) \le C\delta^{-N}M(|T_tf*\varphi|^r)(x)^{1/r} + C\delta|T_tf*\nabla\varphi|^{**}_{N,1}(x). \tag{2.8}$$

Applying $T_{t^{-1}}$ to (2.8), we have

$$\begin{aligned}(f*\varphi_t)^{**}_{N,t^{-1}}(x) \\ \le C\delta^{-N}T_{t^{-1}}(M(|T_tf*\varphi|^r)(x)^{1/r}) + C\delta T_{t^{-1}}(|T_tf*\nabla\varphi|^{**}_{N,1})(x).\end{aligned} \tag{2.9}$$

From (T.2) and (T.3), it follows that

$$T_{t^{-1}}(M(|T_tf*\varphi|^r)(x)^{1/r}) = M(|f*\varphi_t|^r)(x)^{1/r}. \tag{2.10}$$

Also, (T.1) and (T.2) imply

$$T_{t^{-1}}(|T_tf*\nabla\varphi|^{**}_{N,1})(x) = |f*(\nabla\varphi)_t|^{**}_{N,t^{-1}}(x). \tag{2.11}$$

Using (2.10) and (2.11) in (2.9), we get the conclusion of Lemma 2.4.

□

To prove Lemma 2.3 we also need Lemma 2.7 below. To state it, here we have some preliminaries. We first establish a partition of unity on $\mathbb{R}^n\setminus\{0\}$ associated with $\varphi\in L^1(\mathbb{R}^n)$ satisfying (1.3).

Lemma 2.6. *Suppose that φ is a function in $L^1(\mathbb{R}^n)$ satisfying* (1.3). *We assume that $\hat\varphi\in C^\infty(\mathbb{R}^n\setminus\{0\})$. Then, there exist $b_0\in(0,1)$ and $r_1,r_2>0$, $r_1<r_2$, such that for any $b\in[b_0,1)$ there exists a function η with the following properties*:

(1) $\eta\in C^\infty(\mathbb{R}^n)$;
(2) $\hat\eta\in C^\infty(\mathbb{R}^n)$ *and* $\operatorname{supp}\hat\eta\subset\{r_1<\rho^*(\xi)<r_2\}$;
(3) $\sum_{j=-\infty}^{\infty}\hat\varphi(A^*_{b^j}\xi)\hat\eta(A^*_{b^j}\xi)=1 \quad$ *for* $\xi\in\mathbb{R}^n\setminus\{0\}$.

When P is the identity matrix, this is in [21, Chap. V]; see also [4].

Proof of Lemma 2.6. Let $S^{n-1}=\{\xi:|\xi|=1\}$. Since $\hat\varphi$ is continuous, by a compactness argument we can find a finite family $\{I_h\}_{h=1}^L$ of compact intervals in $(0,\infty)$ such that

$$\inf_{\xi\in S^{n-1}}\max_{1\le h\le L}\inf_{t\in I_h}|\hat\varphi(A^*_t\xi)|^2 \ge c$$

with a positive constant c.

We observe that there exists $b_0 \in (0,1)$ such that if $b \in [b_0, 1)$, $t > 0$ and $1 \le h \le L$, then we have $b^j t \in I_h$ for some $j \in \mathbb{Z}$ (the set of integers). This can be seen by taking $b_0 = \max_{1\le h\le L}(a_h/b_h)$, where $I_h = [a_h, b_h]$.

Consider an interval $[m, H]$ in $(0,\infty)$ such that $\cup_{h=1}^{L} I_h \subset [m, H]$ and choose $\theta \in C_0^\infty(\mathbb{R})$ such that $\theta = 1$ on $[m, H]$, $\operatorname{supp}\theta \subset [m/2, 2H]$, $\theta \ge 0$. Define

$$\Psi(\xi) = \sum_{j=-\infty}^{\infty} \theta(b^j\rho^*(\xi))|\hat{\varphi}(A^*_{b^j}\xi)|^2.$$

Then $\Psi(\xi) \ge c > 0$ for $\xi \ne 0$. Note that $\Psi(A^*_{b^k}\xi) = \Psi(\xi)$ for $k \in \mathbb{Z}$. Let

$$\hat{\eta}(\xi) = \theta(\rho^*(\xi))\overline{\hat{\varphi}(\xi)}\Psi(\xi)^{-1} \quad \text{for } \xi \ne 0$$

and $\hat{\eta}(0) = 0$, where $\overline{\hat{\varphi}(\xi)}$ denotes the complex conjugate. Then, η satisfies all the properties stated in the lemma. This completes the proof. □

Let $\varphi \in \mathcal{B}$. Then (2.1) implies that

$$\mathcal{F}(\partial_k\varphi)(\xi) = \Xi_k(\xi)\hat{\varphi}(\xi), \quad 1 \le k \le n, \tag{2.12}$$

where $\Xi_k(\xi) = 2\pi i\xi_k$. Let $b \in [b_0, 1)$ and η be as in Lemma 2.6. Define

$$\hat{\zeta}(\xi) = 1 - \sum_{j\ge 0} \hat{\varphi}(A^*_{b^j}\xi)\hat{\eta}(A^*_{b^j}\xi).$$

Then $\operatorname{supp}(\hat{\zeta}) \subset \{\rho^*(\xi) \le r_2\}$, $\hat{\zeta} = 1$ in $\{\rho^*(\xi) < r_1\}$ and by (2.12) we have

$$\begin{aligned}\mathcal{F}(\partial_k\varphi)(\xi) &= \sum_{j\ge 0} \mathcal{F}(\partial_k\varphi)(\xi)\hat{\varphi}(A^*_{b^j}\xi)\hat{\eta}(A^*_{b^j}\xi) + \hat{\zeta}(\xi)\mathcal{F}(\partial_k\varphi)(\xi) \\ &= \sum_{j\ge 0} \hat{\varphi}(A^*_{b^j}\xi)\mathcal{F}(\alpha^{(b^j)}_{(k)})(A^*_{b^j}\xi) + \hat{\varphi}(\xi)\mathcal{F}(\beta_{(k)})(\xi),\end{aligned}$$

where $\alpha^{(b^j)}_{(k)}(x) = (\partial_k\varphi)_{b^{-j}} * \eta(x)$ and $\mathcal{F}(\beta_{(k)})(\xi) = \hat{\zeta}(\xi)\Xi_k(\xi)$.

Thus we have

$$|F(\partial_k\varphi, f)(x,t)| \le \sum_{j\ge 0} |F(\alpha^{(b^j)}_{(k)} * \varphi, f)(x, b^j t)| + |F(\beta_{(k)} * \varphi, f)(x,t)| \tag{2.13}$$

for $f \in \mathcal{S}(\mathbb{R}^n)$. Here and in the sequel we also write $F(\psi, f)(x,t) = f * \psi_t(x)$ for appropriate functions ψ, f; further, we shall write $F(\psi, f)(x,t) = F_\psi(x,t)$ when f is fixed.

Let

$$C(\psi, t, L, x) = (1+\rho(x))^L \left| \int \hat{\psi}(A^*_{t^{-1}}\xi)\hat{\eta}(\xi)e^{2\pi i\langle x,\xi\rangle}\, d\xi \right|, \tag{2.14}$$

for $\psi \in L^1(\mathbb{R}^n)$ with $t > 0, L \ge 0$. Then

$$|\alpha^{(b^j)}_{(k)s}(x)| = C(\partial_k\varphi, b^j, L, A_s^{-1}x)s^{-\gamma}(1+\rho(x)/s)^{-L}$$

for $j \in \mathbb{Z}$. Similarly,

$$|\beta_{(k)s}(x)| = D(\Xi_k, L, A_s^{-1}x)s^{-\gamma}(1+\rho(x)/s)^{-L},$$

with

$$D(\Xi_k, L, x) = (1+\rho(x))^L \left| \int \hat{\zeta}(\xi)\Xi_k(\xi)e^{2\pi i\langle x,\xi\rangle}\, d\xi \right|. \tag{2.15}$$

Define

$$C(\psi, j, L) = \int_{\mathbb{R}^n} C(\psi, b^j, L, x)\,dx, \quad j \in \mathbb{Z}, \tag{2.16}$$

for an appropriate function ψ in $L^1(\mathbb{R}^n)$;

$$D(\Xi_k, L) = \int_{\mathbb{R}^n} D(\Xi_k, L, x)\,dx. \tag{2.17}$$

We have $D(\Xi_k, L), C(\partial_k\varphi, j, L) < \infty$ for all j, L, which are easily shown by a straightforward computation and can be seen from Lemma 2.9 below. We also write $C(\psi, j, L) = C_\varphi(\psi, j, L)$, $D(\Xi_k, L) = D_\varphi(\Xi_k, L)$ to indicate that these quantities are based on φ.

We also need the following result in proving Lemma 2.3.

Lemma 2.7. *Let $\varphi \in \mathcal{B}$, $b \in [b_0, 1)$, $N > 0$. Then we have*

$$F(\partial_k\varphi, f)(\cdot, t)^{**}_{N,t^{-1}}(x) \le C \sum_{j\ge 0} C_\varphi(\partial_k\varphi, j, N) b^{-jN} F(\varphi, f)(\cdot, b^j t)^{**}_{N,(b^j t)^{-1}}(x) + C D_\varphi(\Xi_k, N) F(\varphi, f)(\cdot, t)^{**}_{N,t^{-1}}(x)$$

for $1 \le k \le n$, where $f \in \mathcal{S}(\mathbb{R}^n)$.

Proof. By (2.13) we have

$$\begin{aligned} &|F_{\partial_k\varphi}(z,t)| \\ &\le C\sum_{j\ge 0}\int |F_\varphi(y, b^j t)| \left(1+\frac{\rho(z-y)}{b^j t}\right)^{-N} C(\partial_k\varphi, b^j, N, A^{-1}_{b^j t}(z-y))(b^j t)^{-\gamma}\,dy \\ &\qquad + C\int |F_\varphi(y,t)| \left(1+\frac{\rho(z-y)}{t}\right)^{-N} D(\Xi_k, N, A_{t^{-1}}(z-y))t^{-\gamma}\,dy. \end{aligned}$$

Multiplying both sides of the inequality by $(1+\rho(x-z)/t)^{-N}$ and noting that

$$\left(1+\frac{\rho(z-y)}{b^j t}\right)^{-N}\left(1+\frac{\rho(x-z)}{t}\right)^{-N} \le C_{A,N} b^{-Nj}\left(1+\frac{\rho(x-y)}{b^j t}\right)^{-N}$$

for any $x, y, z \in \mathbb{R}^n$ and $t > 0$ when $b^j \le A$, we have

$$\begin{aligned} &|F_{\partial_k\varphi}(z,t)|(1+\rho(x-z)/t)^{-N} \\ &\le C\sum_{j\ge 0} b^{-Nj}\int |F_\varphi(y, b^j t)| \left(1+\frac{\rho(x-y)}{b^j t}\right)^{-N} C(\partial_k\varphi, b^j, N, A^{-1}_{b^j t}(z-y))(b^j t)^{-\gamma}\,dy \\ &\quad + C\int |F_\varphi(y,t)|\left(1+\frac{\rho(x-y)}{t}\right)^{-N} D(\Xi_k, N, A_t^{-1}(z-y))t^{-\gamma}\,dy \\ &\le C\sum_{j\ge 0} b^{-Nj} F_\varphi(\cdot, b^j t)^{**}_{N,(b^j t)^{-1}}(x)\int C(\partial_k\varphi, b^j, N, A^{-1}_{b^j t}(z-y))(b^j t)^{-\gamma}\,dy \\ &\quad + CF_\varphi(\cdot, t)^{**}_{N,t^{-1}}(x)\int D(\Xi_k, N, A_t^{-1}(z-y))t^{-\gamma}\,dy \\ &\le C\sum_{j\ge 0} C_\varphi(\partial_k\varphi, j, N) b^{-Nj} F_\varphi(\cdot, b^j t)^{**}_{N,(b^j t)^{-1}}(x) + CD_\varphi(\Xi_k, N)F_\varphi(\cdot, t)^{**}_{N,t^{-1}}(x). \end{aligned}$$

Taking the supremum in z over $\mathbb{R}^n$, we reach the conclusion. □

Let $\varphi \in \mathcal{B}$. Then we have

$$\sup_{j\geq 0} C_\varphi(\nabla\varphi, j, L)b^{-\tau j} < \infty, \tag{2.18}$$

for all $L, \tau > 0$, where we write $C_\varphi(\nabla\varphi, j, L) = \sum_{k=1}^n C_\varphi(\partial_k\varphi, j, L)$, recalling $C_\varphi(\partial_k\varphi, j, L) = C(\partial_k\varphi, j, L)$ (see (2.14), (2.16)), and

$$D_\varphi(L) < \infty \tag{2.19}$$

for all $L > 0$, where $D_\varphi(L) = \sum_{k=1}^n D_\varphi(\Xi_k, L)$; recall $D_\varphi(\Xi_k, L) = D(\Xi_k, L)$ and see (2.15), (2.17). Let $\psi \in \mathcal{S}(\mathbb{R}^n)$. Then, we also note that

$$\sup_{j:b^j\leq r_2} C_\varphi(\psi, j, L)b^{-\tau j} < \infty \quad \text{for any } L, \tau > 0, \tag{2.20}$$

where r_2 is as in Lemma 2.6. These results, which will be used in what follows, are easily shown and can be found in Lemma 2.9 below.

Proof of Lemma 2.3. Lemma 2.4 implies that

$$F(\varphi, f)(\cdot, t)^{**}_{N,t^{-1}}(x) \leq C\delta^{-N}M(|f*\varphi_t|^r)(x)^{1/r} + C\delta|f*(\nabla\varphi)_t|^{**}_{N,t^{-1}}(x), \tag{2.21}$$

where $r = \gamma/N$. Applying Lemma 2.7, we have

$$\begin{aligned} &|f*(\nabla\varphi)_t|^{**}_{N,t^{-1}}(x) \\ &\leq C\sum_{j\geq 0} C_\varphi(\nabla\varphi, j, N)b^{-jN}F(\varphi, f)(\cdot, b^jt)^{**}_{N,(b^jt)^{-1}}(x) + CD_\varphi(N)F(\varphi, f)(\cdot, t)^{**}_{N,t^{-1}}(x). \end{aligned}$$

Thus by (2.21) and Hölder's inequality when $q > 1$ we have

$$\begin{aligned} F(\varphi, f)(\cdot, t)^{**}_{N,t^{-1}}(x)^q &\leq C\delta^{-Nq}M(|f*\varphi_t|^r)(x)^{q/r} \\ &+ C\delta^q\sum_{j\geq 0} C_\varphi(\nabla\varphi, j, N)^q b^{-jNq}b^{-\tau c_q j}F(\varphi, f)(\cdot, b^jt)^{**}_{N,(b^jt)^{-1}}(x)^q \\ &\qquad + C\delta^q D_\varphi(N)^q F(\varphi, f)(\cdot, t)^{**}_{N,t^{-1}}(x)^q, \end{aligned} \tag{2.22}$$

where $\tau > 0$, $c_q = 1$ when $q > 1$ and $c_q = 0$ when $0 < q \leq 1$. Integrating both sides of the inequality (2.22) over $(0, \infty)$ with respect to the measure dt/t and applying termwise integration on the right hand side, we see that

$$\begin{aligned} \int_0^\infty F(\varphi, f)(\cdot, t)^{**}_{N,t^{-1}}(x)^q \,\frac{dt}{t} &\leq C\delta^{-Nq}\int_0^\infty M(|f*\varphi_t|^r)(x)^{q/r}(x)\,\frac{dt}{t} \\ + C\delta^q\left[\sum_{j\geq 0} C_\varphi(\nabla\varphi, j, N)^q b^{-jNq}b^{-\tau c_q j} + D_\varphi(N)^q\right] &\int_0^\infty F(\varphi, f)(\cdot, t)^{**}_{N,t^{-1}}(x)^q\,\frac{dt}{t}. \end{aligned} \tag{2.23}$$

By (2.18) the sum in j on the right hand side of (2.23) is finite. By (2.2) and (2.3) with $\alpha = 0$ we easily see that the last integral on the right hand side of (2.23) is finite with $f \in \mathcal{S}(\mathbb{R}^n)$. Thus, along with (2.19) we see that the second term on the right hand side of (2.23) is finite. So, choosing δ sufficiently small, we get the conclusion. $\square$

To prove Theorem 2.2, we also need the following version of the vector valued inequality for the Hardy-Littlewood maximal operator of Fefferman-Stein [7] with non-isotropic dilations and weights.

Lemma 2.8. *Let $1 < \mu, \nu < \infty$ and $w \in A_\nu$. Then we have*

$$\left\| \left(\int_0^\infty M(F^t)(x)^\mu \, \frac{dt}{t} \right)^{1/\mu} \right\|_{\nu,w} \le C \left(\int_{\mathbb{R}^n} \left(\int_0^\infty |F(x,t)|^\mu \, \frac{dt}{t} \right)^{\nu/\mu} w(x)\, dx \right)^{1/\nu}$$

for appropriate functions $F(x,t)$ on $\mathbb{R}^n \times (0,\infty)$ with $F^t(x) = F(x,t)$.

The case $P = E$ is stated in [18]. See [11, pp. 265–267] for a proof, where ℓ^μ-valued case is handled; the proof may be available also in the present situation (see [15] for related results).

Proof of Theorem 2.2. By a change of variables we may assume that $\hat{\psi} = 0$ on $\{|\xi| \le 1\} = \{\rho^*(\xi) \le 1\}$. We apply Lemma 2.6 for φ of Theorem 2.2 and use similar notation as above. Define

$$\hat{\zeta}(\xi) = 1 - \sum_{j: b^j \le r_2} \hat{\varphi}(A^*_{b^j}\xi)\hat{\eta}(A^*_{b^j}\xi).$$

Then $\operatorname{supp}(\hat{\zeta}) \subset \{\rho^*(\xi) \le 1\}$, $\hat{\zeta} = 1$ in $\{\rho^*(\xi) < r_1/r_2\}$. Since $\hat{\psi} = 0$ on $\{|\xi| \le 1\}$, we have

$$\begin{aligned} \hat{\psi}(\xi) &= \sum_{j: b^j \le r_2} \hat{\psi}(\xi)\hat{\varphi}(A^*_{b^j}\xi)\hat{\eta}(A^*_{b^j}\xi) \\ &= \sum_{j: b^j \le r_2} \hat{\varphi}(A^*_{b^j}\xi)\mathcal{F}(\alpha^{(b^j)})(A^*_{b^j}\xi), \end{aligned}$$

where $\alpha^{(b^j)}(x) = (\psi)_{b^{-j}} * \eta(x)$.

Thus by an easier version of arguments for the proof of Lemma 2.7 we see that

$$|F(\psi,f)(x,t)| \le C \sum_{j: b^j \le r_2} C_\varphi(\psi,j,N) F(\varphi,f)(\cdot, b^j t)^{**}_{N,(b^j t)^{-1}}(x)$$

for any $N > 0$, from which it follows that

$$|F(\psi,f)(x,t)|^q \le C \sum_{j: b^j \le r_2} C_\varphi(\psi,j,N)^q b^{-\tau c_q j} F(\varphi,f)(\cdot, b^j t)^{**}_{N,(b^j t)^{-1}}(x)^q,$$

where $\tau > 0$ and c_q is as in (2.22). Integrating with the measure dt/t over $(0,\infty)$, we have

$$\begin{aligned} &\int_0^\infty |F(\psi,f)(x,t)|^q \, \frac{dt}{t} \\ &\quad \le C \left[\sum_{j: b^j \le r_2} C_\varphi(\psi,j,N)^q b^{-\tau c_q j} \right] \int_0^\infty F(\varphi,f)(\cdot,t)^{**}_{N,t^{-1}}(x)^q \, \frac{dt}{t}. \end{aligned} \tag{2.24}$$

By (2.20) the series on the right hand side of (2.24) converges. Let $0 < p, q < \infty$ and $w \in A_\infty$. If N is sufficiently large so that $r = \gamma/N < q, p$ and $w \in A_{pN/\gamma}$, from

(2.24) and Lemma 2.3 it follows that

$$\left\| \left(\int_0^\infty |f * \psi_t|^q \frac{dt}{t} \right)^{1/q} \right\|_{p,w} \le C \left\| \left(\int_0^\infty M(|f * \varphi_t|^r)(x)^{q/r} \frac{dt}{t} \right)^{1/q} \right\|_{p,w} \tag{2.25}$$

$$= C \left\| \left(\int_0^\infty M(|f * \varphi_t|^r)(x)^{q/r} \frac{dt}{t} \right)^{r/q} \right\|_{p/r,w}^{1/r}$$

$$\le C \left\| \left(\int_0^\infty |f * \varphi_t(x)|^q \frac{dt}{t} \right)^{1/q} \right\|_{p,w},$$

where the last inequality follows form Lemma 2.8. This completes the proof of Theorem 2.2. □

To conclude this section, we give a proof of the following results mentioned above.

Lemma 2.9. *Let $\varphi \in \mathcal{B}$. Suppose that $\psi \in L^1$, $\hat{\psi} \in C^\infty(\mathbb{R}^n \setminus \{0\})$ and we have the estimates* (2.3) *with $\hat{\psi}$ in place of $\hat{\varphi}$ for all multi-indices α and all $\tau > 0$. Let $L, J \ge 0$. Then*

1. $\sup_{j: b^j \le J} C_\varphi(\psi, j, L) b^{-j\tau} < \infty$ *for any $\tau > 0$, where $C_\varphi(\psi, j, L) = C(\psi, j, L)$ is as in* (2.16).
2. $D_\varphi(\Xi_k, L) < \infty$, $1 \le k \le n$, *where $\Xi_k(\xi) = 2\pi i \xi_k$ as above and $D_\varphi(\Xi_k, L) = D(\Xi_k, L)$ is as in* (2.17).

Proof. To prove part (1), since $1 + \rho(x) \le c(1 + |x|)$ by (P.4), we have

$$(1 + |x|)^{[n/2]+1} C(\psi, t, L, x)$$
$$\le C \left| \int \hat{\psi}(A^*_{t^{-1}}\xi)\hat{\eta}(\xi) e^{2\pi i \langle x, \xi \rangle} \, d\xi \right| + C \sup_{|\alpha| = L + [n/2]+1} \left| \int \partial_\xi^\alpha \left[\hat{\psi}(A^*_{t^{-1}}\xi)\hat{\eta}(\xi) \right] e^{2\pi i \langle x, \xi \rangle} \, d\xi \right|,$$

where $C(\psi, t, L, x)$ is as in (2.14) and $[a]$ denotes the largest integer not exceeding a. We recall that $\hat{\eta} \in C^\infty(\mathbb{R}^n)$ with support in $\{r_1 < \rho^*(\xi) < r_2\}$, which is in Lemma 2.6. It is known that $\|A^*_{t^{-1}}\| \le t^{-\kappa}$ for $t \in (0, 1]$ with some $\kappa \ge 1$. Thus (2.3) for $\hat{\psi}$ implies

$$\left| \partial_\xi^\alpha \left[\hat{\psi}(A^*_{t^{-1}}\xi)\hat{\eta}(\xi) \right] \right| \le C_{\alpha, M} t^\tau, \quad 0 < t \le M,$$

for any $M > 0$, if $|\alpha| = L + [n/2] + 1$ or $\alpha = 0$. Thus

$$C(\psi, t, L, x) \le C(1 + |x|)^{-[n/2]-1} G(x)$$

with some $G \in L^2$ such that $\|G\|_2 \le Ct^\tau$, and hence, the Schwarz inequality implies

$$\int_{\mathbb{R}^n} C(\psi, t, L, x) \, dx \le Ct^\tau, \tag{2.26}$$

since $[n/2] + 1 > n/2$. From (2.26) with $t = b^j$ we obtain the conclusion of part (1).

Similarly, we can see that

$$\int_{\mathbb{R}^n} D(\Xi_k, L, x) \, dx < \infty,$$

where $D(\Xi_k, L, x)$ is as in (2.15). This completes the proof of part (2). □

3. Littlewood-Paley functions and parabolic Hardy spaces

As an application of Theorem 2.2 we have the following.

Corollary 3.1. *Let $0 < p \le 1$. Suppose that $\varphi \in \mathcal{B}$. Then we have*

$$\|f\|_{H^p} \le C_p \|g_\varphi(f)\|_p$$

for $f \in H^p(\mathbb{R}^n) \cap \mathcal{S}(\mathbb{R}^n)$ with a positive constant C_p independent of f.

Let $\mathcal{H}$ be the Hilbert space of functions $u(t)$ on $\mathbb{R}_+$ such that $\|u\|_{\mathcal{H}} = \left(\int_0^\infty |u(t)|^2\, dt/t\right)^{1/2} < \infty$. Let $L^q_{\mathcal{H}}(\mathbb{R}^n)$ be the Lebesgue space of functions $h(y,t)$ with the norm

$$\|h\|_{q,\mathcal{H}} = \left(\int_{\mathbb{R}^n} \|h^y\|^q_{\mathcal{H}}\, dy\right)^{1/q},$$

where $h^y(t) = h(y,t)$.

Let $0 < p \le 1$. We consider the parabolic Hardy space of functions on $\mathbb{R}^n$ with values in $\mathcal{H}$, which is denoted by $H^p_{\mathcal{H}}(\mathbb{R}^n)$. Choose $\Phi \in \mathcal{S}(\mathbb{R}^n)$ as in the definition of H^p in Section 1. Let $h \in L^2_{\mathcal{H}}(\mathbb{R}^n)$. We say $h \in H^p_{\mathcal{H}}(\mathbb{R}^n)$ if $\|h\|_{H^p_{\mathcal{H}}} = \|h^*\|_{L^p} < \infty$ with

$$h^*(x) = \sup_{s>0} \left(\int_0^\infty |\Phi_s * h^t(x)|^2 \frac{dt}{t}\right)^{1/2},$$

where we write $h^t(x) = h(x,t)$.

To prove Corollary 3.1 we need the following.

Lemma 3.2. *Let $\hat\psi$ be a function of $\mathcal{S}(\mathbb{R}^n)$ with support in $\{1 \le \rho^*(\xi) \le 2\}$. Suppose that*

$$\int_0^\infty |\hat\psi(A_t^*\xi)|^2 \frac{dt}{t} = 1 \quad \text{for all } \xi \ne 0.$$

Let $F(y,t) = f * \psi_t(y)$ with $f \in H^p(\mathbb{R}^n) \cap \mathcal{S}(\mathbb{R}^n)$, $0 < p \le 1$. Then $F \in H^p_{\mathcal{H}}(\mathbb{R}^n)$ and

$$\|f\|_{H^p} \le C\|F\|_{H^p_{\mathcal{H}}}.$$

In what follows, we write

$$E^\epsilon_\psi(h)(x) = \int_0^\infty \int_{\mathbb{R}^n} \psi_t(x-y) h_{(\epsilon)}(y,t)\, dy\, \frac{dt}{t},$$

where $h \in L^2_{\mathcal{H}}$ and $h_{(\epsilon)}(y,t) = h(y,t)\chi_{(\epsilon,\epsilon^{-1})}(t)$, $0 < \epsilon < 1$, and ψ is an appropriate function. Here χ_S denotes the characteristic function of a set S.

We apply the following result in proving Lemma 3.2.

Lemma 3.3. *Suppose that $\psi \in \mathcal{S}(\mathbb{R}^n)$ and $\operatorname{supp} \hat\psi \subset \{1 \le \rho^*(\xi) \le 2\}$. Then*

$$\sup_{\epsilon \in (0,1)} \|E^\epsilon_\psi(h)\|_{H^p} \le C\|h\|_{H^p_{\mathcal{H}}}, \quad 0 < p \le 1.$$

Let a be a (p,∞) atom in $H^p_{\mathcal{H}}(\mathbb{R}^n)$:

(i) $\left(\int_0^\infty |a(x,t)|^2\, dt/t\right)^{1/2} \le |B|^{-1/p}$, where B is a ball in $\mathbb{R}^n$ with respect to ρ;
(ii) $\operatorname{supp}(a(\cdot,t)) \subset B$ for all $t > 0$, where B is as in (i);
(iii) $\int_{\mathbb{R}^n} a(x,t)x^\alpha\, dx = 0$ for all $t > 0$ and α such that $|\alpha| \le [\gamma(1/p-1)]$, where $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi-index with $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$.

To prove Lemma 3.3 we use the following atomic decomposition.

Lemma 3.4. *Let $h \in L^2_{\mathcal{H}}(\mathbb{R}^n)$. Suppose that $h \in H^p_{\mathcal{H}}(\mathbb{R}^n)$. Then we can find a sequence $\{a_k\}$ of (p,∞) atoms in $H^p_{\mathcal{H}}(\mathbb{R}^n)$ and a sequence $\{\lambda_k\}$ of positive numbers such that $h = \sum_{k=1}^\infty \lambda_k a_k$ in $H^p_{\mathcal{H}}(\mathbb{R}^n)$ and in $L^2_{\mathcal{H}}(\mathbb{R}^n)$ and such that $\sum_{k=1}^\infty \lambda_k^p \le C\|h\|^p_{H^p_{\mathcal{H}}}$ with a constant C independent of h.*

We can find in [13] a proof of the atomic decomposition for $H^p(\mathbb{R}^n)$ (see also [3], [10] and [21]). The vector valued case can be treated similarly.

Proof of Lemma 3.3. Let a be a (p,∞) atom in $H^p_{\mathcal{H}}(\mathbb{R}^n)$ supported on the ball B of the definition of the atom. We choose a C^∞ function Φ on $\mathbb{R}^n$ supported on $\{|x|<1\}$ with $\int \Phi(x)\,dx = 1$ as above. We prove

$$\int_{\mathbb{R}^n} \sup_{s>0} \left|\Phi_s * E^\epsilon_\psi(a)(x)\right|^p \, dx \le C. \tag{3.1}$$

To show this, by applying translation and dilation arguments, we may assume that $B = B(0,1)$. Let $\widetilde{B} = B(0,2)$. Then $2\rho(y) \le \rho(x)$ if $y \in B$ and $x \in \mathbb{R}^n \setminus \widetilde{B}$. Let $\Psi_{s,t} = \Phi_s * \psi_t$, $s,t>0$. We note that $\Phi_s * \psi_t = (\Phi_{s/t} * \psi)_t$ and $\Phi_u * \psi$, $u>0$, belongs to a bounded subset of the topological vector space $\mathcal{S}(\mathbb{R}^n)$, which can easily be seen by noting that $\mathcal{F}(\Phi_u * \psi)(\xi) = \hat{\Phi}(A^*_u\xi)\hat{\psi}(\xi)$ and recalling that $\hat{\psi}(\xi)$ is supported on $\{1 \le \rho^*(\xi) \le 2\}$.

Let $P_x(y)$ be the Taylor polynomial in y of order $M = [\gamma(1/p-1)]$ at 0 for $\Phi_{s/t} * \psi(x-y)$. Then, if $\rho(x) \ge 2\rho(y)$, we have

$$|\Phi_{s/t} * \psi(x-y) - P_x(y)| \le C|y|^{M+1}(1+\rho(x))^{-L},$$

where L is a sufficiently large positive number, which will be specified below, and the constant C is independent of s,t,x,y. This implies

$$|\Psi_{s,t}(x-y) - t^{-\gamma}P_{A_{t^{-1}}x}(A_{t^{-1}}y)| \le Ct^{-\gamma}|A_{t^{-1}}y|^{M+1}(1+\rho(x)/t)^{-L}.$$

Thus, using the properties of an atom and the Schwarz inequality, for $x \in \mathbb{R}^n \setminus \widetilde{B}$ we see that

(3.2)

$$\begin{aligned}
&\left|\Phi_s * E^\epsilon_\psi(a)(x)\right| = \left|\iint \left(\Psi_{s,t}(x-y) - t^{-\gamma}P_{A_{t^{-1}}x}(A_{t^{-1}}y)\right) a_{(\epsilon)}(y,t)\,dy\,\frac{dt}{t}\right| \\
&\le \int_B \left(\int_0^\infty \left|\Psi_{s,t}(x-y) - t^{-\gamma}P_{A_{t^{-1}}x}(A_{t^{-1}}y)\right|^2 \frac{dt}{t}\right)^{1/2} \left(\int_0^\infty |a(y,t)|^2\,\frac{dt}{t}\right)^{1/2} dy \\
&\le C|B|^{-1/p}\int_B \left(\int_0^\infty \left(t^{-\gamma}|A_{t^{-1}}y|^{M+1}(1+\rho(x)/t)^{-L}\right)^2 \frac{dt}{t}\right)^{1/2} dy.
\end{aligned}$$

Now we show that

$$I(x,y) := \int_0^\infty \left(t^{-\gamma}|A_{t^{-1}}y|^{M+1}(1+\rho(x)/t)^{-L}\right)^2 \frac{dt}{t} \le C\rho(x)^{-2(\gamma+M+1)} \tag{3.3}$$

for $y \in B$, $x \in \mathbb{R}^n \setminus \widetilde{B}$, if L is sufficiently large. We first see that

$$I(x,y) = \rho(x)^{-2\gamma}\int_0^\infty \left(t^{-\gamma}|A_{(\rho(x)t)^{-1}}y|^{M+1}(1+t^{-1})^{-L}\right)^2 \frac{dt}{t}. \tag{3.4}$$

By (P.6) we have

$$\begin{aligned}(3.5)\qquad &\int_{\rho(x)^{-1}}^{\infty} \left(t^{-\gamma}|A_{(\rho(x)t)^{-1}}y|^{M+1}(1+t^{-1})^{-L}\right)^2 \frac{dt}{t}\\ &\le C|y|^{2(M+1)}\rho(x)^{-2(M+1)}\int_{\rho(x)^{-1}}^{\infty} t^{-2(\gamma+M+1)}(1+t^{-1})^{-2L}\frac{dt}{t}\\ &\le C|y|^{2(M+1)}\rho(x)^{-2(M+1)}\end{aligned}$$

if $L > \gamma + M + 1$. If $s \ge 1$, $|A_s y| \le Cs^{\kappa}|y|$ for some $\kappa \ge 1$. Thus

$$\begin{aligned}(3.6)\qquad &\int_0^{\rho(x)^{-1}} \left(t^{-\gamma}|A_{(\rho(x)t)^{-1}}y|^{M+1}(1+t^{-1})^{-L}\right)^2 \frac{dt}{t}\\ &\le C|y|^{2(M+1)}\rho(x)^{-2\kappa(M+1)}\int_0^{\rho(x)^{-1}} t^{-2(\gamma+\kappa(M+1))}(1+t^{-1})^{-2L}\frac{dt}{t}\\ &\le C|y|^{2(M+1)}\rho(x)^{-2\kappa(M+1)},\end{aligned}$$

if $L > \gamma + \kappa(M+1)$. By (3.4), (3.5) and (3.6) we obtain (3.3). From (3.2) and (3.3) we have

$$\left|\Phi_s * E_\psi^\epsilon(a)(x)\right| \le C\rho(x)^{-(\gamma+M+1)}$$

for $x \in \mathbb{R}^n \setminus \widetilde{B}$.

Since $p > \gamma/(\gamma+M+1)$, it follows that

$$(3.7)\qquad \int_{\mathbb{R}^n\setminus\widetilde{B}} \sup_{s>0}\left|\Phi_s * E_\psi^\epsilon(a)(x)\right|^p dx \le C\int_{\mathbb{R}^n\setminus\widetilde{B}} \rho(x)^{-p(\gamma+M+1)}\, dx \le C$$

(see [6]).

Using $\int_0^\infty |\hat{\psi}(A_t^*\xi)|^2\, dt/t \le C$, by duality we can easily see that

$$\sup_{\epsilon\in(0,1)} \|E_\psi^\epsilon(h)\|_2 \le C\|h\|_{L^2_{\mathcal{H}}}, \quad h \in L^2_{\mathcal{H}}(\mathbb{R}^n).$$

So, by Hölder's inequality and the properties (i), (ii) of a, we get

$$\begin{aligned}(3.8)\qquad \int_{\widetilde{B}} \sup_{s>0}\left|\Phi_s * E_\psi^\epsilon(a)(x)\right|^p dx &\le C\left(\int_{\widetilde{B}} |M(E_\psi^\epsilon(a))(x)|^2\, dx\right)^{p/2}\\ &\le C\left(\int_B\int_0^\infty |a(y,t)|^2\frac{dt}{t}\,dy\right)^{p/2}\\ &\le C.\end{aligned}$$

Combining (3.7) and (3.8), we have (3.1). By Lemma 3.4 and (3.1) we can prove

$$\int_{\mathbb{R}^n} \sup_{s>0}\left|\Phi_s * E_\psi^\epsilon(h)(x)\right|^p dx \le C\|h\|^p_{H^p_{\mathcal{H}}}.$$

This completes the proof. □

Proof of Lemma 3.2. By using the atomic decomposition for $H^p(\mathbb{R}^n)$, we can prove the fact that $F \in H^p_{\mathcal{H}}(\mathbb{R}^n)$ similarly to the proof of Lemma 3.3 (see [22, Lemma 3.6]).

We note that

$$E^\epsilon_{\bar{\tilde{\psi}}}(F)(x) = \int_\epsilon^{\epsilon^{-1}}\int_{\mathbb{R}^n} \psi_t * f(y)\bar{\psi}_t(y-x)\,dy\,\frac{dt}{t} = \int_{\mathbb{R}^n} \Psi^{(\epsilon)}(x-z)f(z)\,dz,$$

where $\widetilde{g}(x) = g(-x)$ and

$$\Psi^{(\epsilon)}(x) = \int_{\epsilon}^{\epsilon^{-1}} \int_{\mathbb{R}^n} \psi_t(x+y)\bar{\psi}_t(y)\, dy\, \frac{dt}{t}.$$

We have

$$\widehat{\Psi^{(\epsilon)}}(\xi) = \int_{\epsilon}^{\epsilon^{-1}} \hat{\psi}(A_t^*\xi)\hat{\bar{\psi}}(-A_t^*\xi)\, \frac{dt}{t} = \int_{\epsilon}^{\epsilon^{-1}} |\hat{\psi}(A_t^*\xi)|^2\, \frac{dt}{t} \to 1$$

as $\epsilon \to 0$ for $\xi \neq 0$. This and Lemma 3.3 imply

$$\|f\|_{H^p} \leq C \liminf_{\epsilon\to 0} \|E^{\epsilon}_{\tilde{\bar{\psi}}}(F)\|_{H^p} \leq C\|F\|_{H^p_{\mathcal{H}}}.$$

□

We also need the following result to prove Corollary 3.1.

Lemma 3.5. *Let* $\eta \in \mathcal{S}(\mathbb{R}^n)$ *satisfy* $\operatorname{supp}(\hat{\eta}) \subset \{1/2 \leq \rho^*(\xi) \leq 4\}$ *and* $\hat{\eta}(\xi) = 1$ *on* $\{1 \leq \rho^*(\xi) \leq 2\}$. *Let* ψ *be as in Lemma* 3.2. *Suppose that* $\Phi \in \mathcal{S}(\mathbb{R}^n)$ *satisfies* $\int_{\mathbb{R}^n} \Phi(x)\, dx = 1$ *and* $\operatorname{supp}(\Phi) \subset B(0,1)$. *Then for* $p, q > 0$ *and* $f \in \mathcal{S}(\mathbb{R}^n)$ *we have*

$$\left\| \left(\int_0^\infty \sup_{s>0} |\Phi_s * \psi_t * f|^q\, \frac{dt}{t} \right)^{1/q} \right\|_p \leq C \left\| \left(\int_0^\infty |\eta_t * f|^q\, \frac{dt}{t} \right)^{1/q} \right\|_p.$$

Proof. Since $\hat{\Phi}(A_s^*\xi)\hat{\psi}(A_t^*\xi) = \hat{\Phi}(A_s^*\xi)\hat{\psi}(A_t^*\xi)\hat{\eta}(A_t^*\xi)$, we have

$$\begin{aligned} |\Phi_s * \psi_t * f(x)| &\leq (f*\eta_t)^{**}_{N,t^{-1}}(x) \int_{\mathbb{R}^n} |\Phi_s * \psi_t(w)|(1 + t^{-1}\rho(w))^N\, dw \\ &= (f*\eta_t)^{**}_{N,t^{-1}}(x) \int_{\mathbb{R}^n} |\Phi_{s/t} * \psi(w)|(1+\rho(w))^N\, dw \\ &\leq C_N (f*\eta_t)^{**}_{N,t^{-1}}(x) \end{aligned}$$

for any $N > 0$, where C_N is independent of s, t. This follows from the observation that $\Phi_{s/t} * \psi$, $s, t > 0$, belongs to a bounded subset of the topological vector space $\mathcal{S}(\mathbb{R}^n)$, as in the proof of Lemma 3.3. Thus

$$\left(\int_0^\infty \sup_{s>0} |\Phi_s * \psi_t * f(x)|^q\, \frac{dt}{t} \right)^{1/q} \leq C \left(\int_0^\infty |(f*\eta_t)^{**}_{N,t^{-1}}(x)|^q\, \frac{dt}{t} \right)^{1/q}. \tag{3.9}$$

By (3.9) and Lemma 2.3 with η in place of φ, we have

$$\left(\int_0^\infty \sup_{s>0} |\Phi_s * \psi_t * f(x)|^q\, \frac{dt}{t} \right)^{1/q} \leq C \left(\int_0^\infty M(|f*\eta_t|^r)(x)(x)^{q/r}\, \frac{dt}{t} \right)^{1/q},$$

where $N = \gamma/r$. This and Lemma 2.8 prove Lemma 3.5 as in (2.25). □

Proof of Corollary 3.1. Let η be as in Lemma 3.5. Applying Lemma 3.2 and Lemma 3.5 with $q = 2$ and $p \in (0,1]$, we see that

$$\|f\|_{H^p} \leq C \left\|g_\eta(f)\right\|_p, \quad f \in H^p(\mathbb{R}^n) \cap \mathcal{S}(\mathbb{R}^n),$$

which combined with Theorem 2.2 with $q = 2$, $p \in (0,1]$, $w = 1$ and with η in place of ψ proves Corollary 3.1.

□

Proof of Theorem 1.1. Let φ be as in Theorem 1.1. Let $0 < p \le 1$. The inequality

$$\|g_\varphi(f)\|_p \le C\|f\|_{H^p}$$

can be proved similarly to the proof of the statement $F \in H^p_{\mathcal{H}}(\mathbb{R}^n)$ in Lemma 3.2 for $f \in H^p(\mathbb{R}^n) \cap \mathcal{S}(\mathbb{R}^n)$ by using the atomic decomposition for $H^p(\mathbb{R}^n)$. This and Corollary 3.1 imply

$$c_1\|f\|_{H^p} \le \|g_\varphi(f)\|_p \le c_2\|f\|_{H^p}$$

for $f \in H^p(\mathbb{R}^n) \cap \mathcal{S}(\mathbb{R}^n)$, from which the conclusion of Theorem 1.1 follows by arguments similar to the one in [22, pp. 149–150], since $H^p(\mathbb{R}^n) \cap \mathcal{S}(\mathbb{R}^n)$ is dense in $H^p(\mathbb{R}^n)$ (see [5]). □

It is not difficult to see that we have discrete parameter versions of Theorems 1.1 and 2.2. To conclude this note we remark the following results.

Theorem 3.6. *Let φ be as in Theorem 1.1 and $0 < p \le 1$. Let a positive number b be as in Lemma 2.6. Then, there exist positive constants c_1, c_2 such that*

$$c_1\|f\|_{H^p} \le \left\| \left(\sum_{j=-\infty}^{\infty} |f * \varphi_{b^j}|^2 \right)^{1/2} \right\|_p \le c_2\|f\|_{H^p}$$

for $f \in H^p(\mathbb{R}^n)$.

Theorem 3.7. *Let $0 < p, q < \infty$ and $w \in A_\infty$. Suppose that φ and ψ fulfill the hypotheses of Theorem 2.2. Then we have, for $f \in \mathcal{S}(\mathbb{R}^n)$,*

$$\left\| \left(\sum_{j=-\infty}^{\infty} |f * \psi_{b^j}|^q \right)^{1/q} \right\|_{p,w} \le C \left\| \left(\sum_{j=-\infty}^{\infty} |f * \varphi_{b^j}|^q \right)^{1/q} \right\|_{p,w},$$

where b is as in Lemma 2.6.

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