CHARACTERIZATION OF H^1 SOBOLEV SPACES BY SQUARE FUNCTIONS OF MARCINKIEWICZ TYPE

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Abstract. We establish characterization of H^1 Sobolev spaces by certain square functions of Marcinkiewicz type. The square functions are related to the Lusin area integrals. Also, in the one dimensional case, the non-periodic version of the function of Marcinkiewicz is applied to characterize weighted H^1 Sobolev spaces.

1. Introduction

We recall that a function Φ belongs to $\mathcal{M}^\alpha$, $\alpha > 0$, if Φ is a bounded, compactly supported function on $\mathbb{R}^n$ satisfying $\int_{\mathbb{R}^n} \Phi(x)\, dx = 1$; if $\alpha \geq 1$, we further require that

$$\int_{\mathbb{R}^n} \Phi(x)x^\gamma\, dx = 0 \quad \text{for all } \gamma \text{ with } 1 \leq |\gamma| \leq [\alpha],$$

where γ is a multi-index, $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_n)$, $|\gamma| = \gamma_1 + \gamma_2 + \dots + \gamma_n$, $x^\gamma = x_1^{\gamma_1} \dots x_n^{\gamma_n}$ and $[\alpha]$ denotes the greatest integer not exceeding α. Let

$$G_\alpha(f)(x) = \left(\int_0^\infty |f(x) - \Phi_t * f(x)|^2 \frac{dt}{t^{1+2\alpha}} \right)^{1/2}, \quad \alpha > 0, \tag{1.1}$$

where $\Phi \in \mathcal{M}^\alpha$ and $\Phi_t(x) = t^{-n}\Phi(t^{-1}x)$.

Let L^p_w, $0 < p < \infty$, be the weighted Lebesgue space with the norm $\|f\|_{p,w}$ defined as $\|f\|_{p,w} = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x)\, dx\right)^{1/p}$. When $w = 1$ (the unweighted case), $\|f\|_{p,w}$ is written simply as $\|f\|_p$. Let $w \in A_p$, $1 < p < \infty$, where A_p denotes the weight class of Muckenhoupt (see [7]), and let $\alpha > 0$. The Sobolev space $W^{\alpha,p}_w(\mathbb{R}^n)$ is defined to be the collection of functions $f \in L^p_w(\mathbb{R}^n)$ which can be expressed as $f = J_\alpha(g)$ with $g \in L^p_w(\mathbb{R}^n)$; the norm is given by $\|f\|_{p,\alpha,w} = \|g\|_{p,w}$, where J_α is the Bessel potential operator defined as $J_\alpha(g) = K_\alpha * g$ with

$$\hat{K}_\alpha(\xi) = (1 + 4\pi^2|\xi|^2)^{-\alpha/2}.$$

The Fourier transform is defined as

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i\langle x,\xi\rangle}\, dx, \quad \langle x, \xi\rangle = \sum_{j=1}^n x_j\xi_j.$$

It was proved in [17] that G_α can characterize the weighted Sobolev spaces $W^{\alpha,p}_w$, $1 < p < \infty$.

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Theorem A. *Let $1 < p < \infty$, $0 < \alpha < n$ and $w \in A_p$, $\Phi \in \mathcal{M}^\alpha$. Let G_α be as in (1.1). Then $f \in W_w^{\alpha,p}(\mathbb{R}^n)$ if and only if $f \in L_w^p$ and $G_\alpha(f) \in L_w^p$; furthermore,*

$$\|f\|_{p,\alpha,w} \simeq \|f\|_{p,w} + \|G_\alpha(f)\|_{p,w},$$

that is,

$$c_1\|f\|_{p,\alpha,w} \le \|f\|_{p,w} + \|G_\alpha(f)\|_{p,w} \le c_2\|f\|_{p,\alpha,w}$$

for positive constants c_1, c_2 independent of f.

(See [17, Corollary 5.2].) A square function characterization of the Sobolev spaces of this type was established by [1]. It has been developed by [8, 15]. Special cases of Theorem A can be found in [15]. In [16] a square function characterization of $W_w^{\alpha,p}$ different from Theorem A was given when $\alpha = 2$. Also, an alternative proof of a result of [8] using a pointwise relation between a square function of Marcinkiewicz type and one arising from the Bochner-Riesz operators can be found in [18].

We now state an application of Theorem A. Let $B(x,t)$ be the ball centered at x with radius t: $B(x,t) = \{y \in \mathbb{R}^n : |x-y| < t\}$ and $B_0 = B(0,1)$. Put $\chi_0 = |B_0|^{-1}\chi_{B_0}$, where $|B_0|$ denotes the Lebesgue measure of B_0 and χ_{B_0} is the characteristic function of B_0. Then $\chi_0 \in \mathcal{M}^\alpha$ for $\alpha \in (0,2)$. If $\Phi = \chi_0$ in (1.1), $G_\alpha(f)$ can be expressed as

$$\left(\int_0^\infty \left|f(x) - \fint_{B(x,t)} f(y)\,dy\right|^2 \frac{dt}{t^{1+2\alpha}}\right)^{1/2},$$

where $\fint_{B(x,t)} f(y)\,dy = |B(x,t)|^{-1}\int_{B(x,t)} f(y)\,dy$. Theorem A applies to this square function for $0 < \alpha < \min(n,2)$, as shown in [1] in the unweighted case.

In this note we consider characterization of H^1 Sobolev spaces by square functions of Marcinkiewicz type. Let $\mathcal{S}(\mathbb{R}^n)$ be the Schwartz class of rapidly decreasing smooth functions. We choose $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfying $\int_{\mathbb{R}^n}\varphi\,dx = 1$, $\varphi \ge 0$ and $\mathrm{supp}(\varphi) \subset B(0,1)$. The function φ will be fixed in what follows. Let $H^p(\mathbb{R}^n)$, $0 < p < \infty$, be the Hardy space of tempered distributions on $\mathbb{R}^n$ such that $f^* \in L^p(\mathbb{R}^n)$, where $f^*(x) = \sup_{t>0}|\varphi_t * f(x)|$. The norm of f in $H^p(\mathbb{R}^n)$ is defined to be $\|f\|_{H^p} = \|f^*\|_p$. It is known that a different choice of such φ gives an equivalent norm. The space H^p coincides with L^p when $1 < p < \infty$ and H^1 is contained in L^1. We denote by $\mathcal{S}_0(\mathbb{R}^n)$ a dense subspace of $H^p(\mathbb{R}^n)$ consisting of those functions f in $\mathcal{S}(\mathbb{R}^n)$ such that $\hat{f} = 0$ outside a compact set not containing the origin (see [26, Chapters V and VII]).

Let $f \in L^1_{loc}$ and

$$\nu(f)(x) = \left(\int_0^\infty |f(x+t) + f(x-t) - 2f(x)|^2 \frac{dt}{t^3}\right)^{1/2}. \tag{1.2}$$

Put $\mu(f) = \nu(\mathcal{I}(f))$, where $\mathcal{I}(f)(x) = \int_0^x f(y)\,dy$. Then $\mu(f)$ is the function of Marcinkiewicz and we have

$$\|\mu(f)\|_p \simeq \|f\|_{H^p}$$

for $f \in \mathcal{S}_0(\mathbb{R})$ when $2/3 < p < \infty$ (see [14, Theorem 4]). It follows that

$$\|\nu(f)\|_p \simeq \|f'\|_{H^p} \tag{1.3}$$

if $f \in \mathcal{S}_0(\mathbb{R})$ when $2/3 < p < \infty$.

Let g_ψ be the Littlewood-Paley function on $\mathbb{R}^n$ defined as

$$g_\psi(f)(x) = \left(\int_0^\infty |f * \psi_t(x)|^2 \, \frac{dt}{t}\right)^{1/2},$$

with $\psi \in L^1(\mathbb{R}^n)$ satisfying $\int_{\mathbb{R}^n} \psi(x)\, dx = 0$. The Marcinkiewicz function μ was introduced by the author in 1938 (see [11]) in the setting of periodic functions. The non-periodic version $\mu(f)$ can be realized as a Littlewood-Paley function $g_\psi(f)$ on $\mathbb{R}$ with $\psi(x) = \chi_{[0,1]}(x) - \chi_{[-1,0]}(x)$.

We define the H^1 Sobolev space by

$$W^\alpha_{H^1}(\mathbb{R}^n) = \{f \in H^1(\mathbb{R}^n) : f = J_\alpha(h) \quad \text{for some } h \in H^1(\mathbb{R}^n)\}.$$

For $f \in W^\alpha_{H^1}$, define $\|f\|_{W^\alpha_{H^1}} = \|h\|_{H^1}$ with $f = J_\alpha(h)$. This is well-defined since J_α is an injection from H^1 to H^1. To see that J_α is injective on H^1, suppose that $J_\alpha(f) = J_\alpha(g)$ for $f, g \in H^1$. Then by part (2) of Lemma 2.5 below, we have

$$\begin{aligned}\int (K_\alpha * h)(y) f(y)\, dy &= \int (K_\alpha * f)(y) h(y)\, dy \\ &= \int (K_\alpha * g)(y) h(y)\, dy = \int (K_\alpha * h)(y) g(y)\, dy\end{aligned}$$

for all $h \in \mathcal{S}(\mathbb{R}^n)$, which implies that $f = g$, since J_α is a surjection from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$.

The estimates (1.3) indicates that the square function ν can characterize $W^1_{H^1}(\mathbb{R})$. Indeed, we have Theorem 1.2 below. In general dimensions, we can give a characterization of the space $W^\alpha_{H^1}(\mathbb{R}^n)$ in terms of Lusin area integral functions of Marcinkiewicz type. Let I_α be the Riesz potential operator defined by

$$\widehat{I_\alpha(f)}(\xi) = (2\pi|\xi|)^{-\alpha} \hat{f}(\xi), \quad f \in \mathcal{S}(\mathbb{R}^n).$$

If $L_\alpha(x) = \tau(\alpha)|x|^{\alpha-n}$, then $\widehat{L}_\alpha(\xi) = (2\pi|\xi|)^{-\alpha}$, $0 < \alpha < n$, where

$$\tau(\alpha) = \frac{\Gamma(n/2 - \alpha/2)}{\pi^{n/2} 2^\alpha \Gamma(\alpha/2)}$$

(see [23, Chapter V.1]). Let

$$\psi^{(\alpha)}(x) = L_\alpha(x) - L_\alpha * \Phi(x), \tag{1.4}$$

where $\Phi \in \mathcal{M}^\alpha$, $0 < \alpha < n$. We note that $\psi^{(\alpha)} \in L^1(\mathbb{R}^n)$ (see [17, p. 37]). Define

$$\begin{aligned}S_{\psi^{(\alpha)}}(f)(x) &= \left(\int_0^\infty \int_{B_0} |\psi_t^{(\alpha)} * f(x - tz)|^2 \, dz \, \frac{dt}{t}\right)^{1/2} \\ &= \left(\int_0^\infty \int_{B(x,t)} |\psi_t^{(\alpha)} * f(z)|^2 \, dz \, t^{-n} \, \frac{dt}{t}\right)^{1/2}.\end{aligned}$$

Then, by homogeneity of the Riesz potential,

$$\begin{aligned}S_{\psi^{(\alpha)}}(f)(x) &= \left(\int_0^\infty \int_{B_0} |I_\alpha f(x - tz) - \Phi_t * I_\alpha f(x - tz)|^2 \, dz \, t^{-2\alpha} \frac{dt}{t}\right)^{1/2} \\ &= \left(\int_0^\infty \int_{B(x,t)} |I_\alpha f(z) - \Phi_t * I_\alpha f(z)|^2 \, dz \, t^{-2\alpha - n} \frac{dt}{t}\right)^{1/2}.\end{aligned}$$

Also, let

$$U_\alpha(f)(x) = \left(\int_0^\infty \int_{B_0} |f(x-tz) - \Phi_t * f(x-tz)|^2\, dz\, t^{-2\alpha}\frac{dt}{t}\right)^{1/2} \tag{1.5}$$
$$= \left(\int_0^\infty \int_{B(x,t)} |f(z) - \Phi_t * f(z)|^2\, dz\, t^{-2\alpha-n}\frac{dt}{t}\right)^{1/2}.$$

Then U_α is available to characterize the Sobolev space $W^\alpha_{H^1}(\mathbb{R}^n)$.

Theorem 1.1. *Let U_α be as in (1.5). Suppose that $n/2 < \alpha < n$, $\Phi \in \mathcal{M}^\alpha$ and that there exists $\beta > 0$ such that*

$$|\hat{\Phi}(\xi)| \le C(1+|\xi|)^{-\beta}, \quad \alpha + \beta > n.$$

Then the following two statements are equivalent:

(1) $f \in W^\alpha_{H^1}(\mathbb{R}^n)$,
(2) $f \in H^1(\mathbb{R}^n)$ *and* $U_\alpha(f) \in L^1(\mathbb{R}^n)$.

Further, we have $\|f\|_{W^\alpha_{H^1}} \simeq \|f\|_{H^1} + \|U_\alpha(f)\|_1$.

We refer to [25] for a related previous work.

We note that

$$|\hat{\chi_0}(\xi)| \le C(1+|\xi|)^{-(n+1)/2}.$$

When $n = 1$, this can be seen by a direct computation. If $n \ge 2$, this follows by applications of Theorem 3.3 and Lemmas 3.11, 4.13 of [24, Chapter IV]. Further, $\chi_0 \in \mathcal{M}^\alpha$, $0 < \alpha < 2$. Thus Theorem 1.1 is available for $n = 1, 2, 3$ when $\Phi = \chi_0$. Also, if Φ is a bounded radial function with compact support, then $|\hat{\Phi}(\xi)| \le C(1+|\xi|)^{-(n-1)/2}$, which easily follows from a formula in [24, Theorem 3.3, Chapter IV].

If $\Phi = \chi_0$ in (1.5), then $U_\alpha(f)$ can be written, up to a constant factor, as

$$\left(\int_0^\infty \fint_{B(x,t)} \left| f(z) - \fint_{B(z,t)} f(y)\, dy\right|^2 dz\, t^{-2\alpha}\frac{dt}{t}\right)^{1/2}.$$

This may extend to the case of metric measure spaces, since the expression $\fint_{B(x,t)} f$ makes sense in general metric measure spaces. Thus, Theorem 1.1 suggests that we may define H^1 Sobolev spaces in metric measure spaces by considering $H^{1,\infty}$ on metric measure spaces as the definition of Hardy spaces, where $H^{1,\infty}$ denotes the atomic H^1 (see [10, Chap. 3] for $H^{1,\infty}$). The square function $U_\alpha(f)$ with $\Phi = \chi_0$ was used in [9] to characterize the classical Sobolev space $W^{\alpha,p}(\mathbb{R}^n)$ with $\alpha \in (0,2)$ and $p \in (1,\infty)$.

We can also consider weighted H^1 Sobolev spaces. Recall the weight class A_1 of Muckenhoupt consisting of weights w such that $M(w) \le Cw$ almost everywhere. We have denoted by M the Hardy-Littlewood maximal operator defined as

$$M(f)(x) = \sup_B |B|^{-1}\int_B |f(y)|\, dy,$$

where the supremum is taken over all balls B containing x. Let H^1_w, $w \in A_1$, be the subspace of L^1_w such that $f^* \in L^1_w$ with the norm $\|f\|_{H^1_w} = \|f^*\|_{1,w}$. Similarly, we can define $H^p_w(\mathbb{R}^n)$ for $p \in (0,1)$ with $w \in A_1$ to be the space of tempered distributions f such that $f^* \in L^p_w$. Then $\mathcal{S}_0(\mathbb{R}^n)$ is dense in $H^p_w(\mathbb{R}^n)$, $0 < p \le 1$ (see [26]).

Define

$$W^\alpha_{H^1_w}(\mathbb{R}^n) = \{f \in H^1_w(\mathbb{R}^n) : f = J_\alpha(h) \quad \text{for some } h \in H^1_w(\mathbb{R}^n)\}.$$

For $f \in W^\alpha_{H^1_w}$, let $\|f\|_{W^\alpha_{H^1_w}} = \|h\|_{H^1_w}$ with $f = J_\alpha(h)$, which is also well-defined since J_α is injective on H^1_w (see Lemma 2.5 (2) below and recall that $J_\alpha : \mathcal{S} \to \mathcal{S}$ is onto). Here we note that $J_\alpha(h) \in H^1_w$ whenever $h \in H^1_w$. To see this, we first observe that $J_\alpha(h)^* \le K_\alpha * h^*$, where we recall that K_α is the kernel of J_α and $K_\alpha > 0$ (see [23, Chapter V, identity (26)]). Also, we have $K_\alpha * w \le CM(w)$ by applying suitably [23, Chapter III, Theorem 2], since the least decreasing radial majorant of K_α is integrable (see [23, Chapter V, identities (29), (30)]). Thus

$$\|J_\alpha(h)^*\|_{1,w} \le C\|h^*\|_{1,M(w)} \le C\|h^*\|_{1,w} = C\|h\|_{H^1_w}.$$

In the one dimensional case, we have the following result.

Theorem 1.2. *Let ν be as in* (1.2) *and $w \in A_1$. Then we have the equivalence of the following two statements:*

(1) $f \in W^1_{H^1_w}(\mathbb{R})$,
(2) $f \in H^1_w(\mathbb{R})$ *and* $\nu(f) \in L^1_w(\mathbb{R})$.

Also, we have $\|f\|_{W^1_{H^1_w}} \simeq \|f\|_{H^1_w} + \|\nu(f)\|_{1,w}$.

Remark 1.3. The square function $S_{\psi^{(\alpha)}}(f)$ can be treated in some respects similarly to

$$\mathcal{D}_\alpha(f)(x) = \left(\int_{\mathbb{R}^n} |I_\alpha(f)(x-y) - I_\alpha(f)(x)|^2 \frac{dy}{|y|^{n+2\alpha}}\right)^{1/2}.$$

(See [21], [3] and also [5], [22] for $\mathcal{D}_\alpha$.)

Remark 1.4. Let $0 < \alpha < n/2$ and $1 \le p < 2n/(2\alpha+n)$ (we note that $p < 2$). Then, $S_{\psi^{(\alpha)}}$ is not bounded on L^p, where $\psi^{(\alpha)}$ is as in (1.4). (See Section 5 for a proof, which is based on [3]; see also [21] for the weak type estimates at $p = 2n/(2\alpha+n)$.)

Remark 1.5. In Theorem 1.1, the condition $\alpha > n/2$ is optimal in the sense that if $0 < \alpha < n/2$, the estimate $\|U_\alpha(f)\|_1 \le C\|f\|_{W^\alpha_{H^1}}$ does not hold, where U_α is as in (1.5) with $\Phi \in \mathcal{M}^\alpha$. (A proof can be found in Section 5.)

Remark 1.6. The proof of Theorem A is based on the estimates $\|g_{\psi^{(\alpha)}}(f)\|_{p,w} \simeq \|f\|_{p,w}$, $w \in A_p$, $1 < p < \infty$, where $\psi^{(\alpha)}$ is as in (1.4). If $\|g_{\psi^{(\alpha)}}(f)\|_1 \simeq \|f\|_{H^1}$, then we would be able to characterize $W^\alpha_{H^1}$ by G_α.

We shall prove Theorem 1.1 in Section 2 and Theorem 1.2 in Section 4. The proof of Theorem 1.1 is based on the equivalence of $\|S_{\psi^{(\alpha)}}(f)\|_1$ and $\|f\|_{H^1}$, $f \in \mathcal{S}_0(\mathbb{R}^n)$ (Theorem 2.3). In proving Theorem 1.2, we shall show in Section 3 the estimates of the kind $\|f\|_{H^1_w} \le C\|g_\psi(f)\|_{1,w}$ for $f \in \mathcal{S}_0(\mathbb{R}^n)$ (see Theorem 3.2), which generalizes a result of [27] to the case of weighted Hardy spaces.

2. Proof of Theorem 1.1

We will show Theorem 1.1 by means of the equivalence $c\|f\|_{H^1} \le \|S_{\psi^{(\alpha)}}(f)\|_1 \le C\|f\|_{H^1}$ which will be established in Theorem 2.3. The first inequality is obtained via a duality argument based on Lemma 2.2 below, while the converse inequality will come from the application of a Calderón-Zygmund estimate for vector valued kernels in [7, Chapter V, Corollary 3.10], whose crucial step is checking the Hörmander condition in Lemma 2.1 below.

Lemma 2.1. *Let $\psi^{(\alpha)}$ be as in* (1.4) *with α and Φ as in Theorem* 1.1. *Then*

$$\int_{|x|>2|y|}\left[\iint_{B_0\times(0,\infty)}\left|\psi_t^{(\alpha)}(x-y-tz)-\psi_t^{(\alpha)}(x-tz)\right|^2\,dz\,\frac{dt}{t}\right]^{1/2}dx\le C,$$

with a constant C independent of $y\in\mathbb{R}^n$.

Proof. To prove the lemma, by a change of variables we may assume that $\operatorname{supp}(\Phi)\subset B_0$. This can be seen as follows. Suppose that $\operatorname{supp}(\Phi)\subset B(0,M)$ with $M>1$. Let $c=1/M$. Then by a change of variables, we see that

$$\begin{aligned}
&\int_{B_0\times(0,\infty)}\left|\psi_t^{(\alpha)}(x-y-tz)-\psi_t^{(\alpha)}(x-tz)\right|^2\,dz\,\frac{dt}{t}\\
&=c^{-n}\int_{B(0,c)\times(0,\infty)}\left|\psi_{ct}^{(\alpha)}(x-y-tz)-\psi_{ct}^{(\alpha)}(x-tz)\right|^2\,dz\,\frac{dt}{t}\\
&\le c^{-n}\int_{B_0\times(0,\infty)}\left|\psi_{ct}^{(\alpha)}(x-y-tz)-\psi_{ct}^{(\alpha)}(x-tz)\right|^2\,dz\,\frac{dt}{t}.
\end{aligned}$$

We note that

$$\psi_c^{(\alpha)}(x)=c^{-\alpha}(L_\alpha(x)-L_\alpha*\Phi_c(x))$$

and that $\operatorname{supp}(\Phi_c)\subset B_0$. This implies what we need.

Fix $x,y\in\mathbb{R}^n$ with $|x|>2|y|$. Let

$$I(x,y,t)=\int_{B_0}\left|\psi_t^{(\alpha)}(x-y-tz)-\psi_t^{(\alpha)}(x-tz)\right|^2\,dz.$$

We write $I(x,y,t)=I_1(x,y,t)+I_2(x,y,t)$, where

$$I_1(x,y,t)=\int_{z\in B_0,|x/t-z|<6}\left|\psi_t^{(\alpha)}(x-y-tz)-\psi_t^{(\alpha)}(x-tz)\right|^2\,dz,$$

$$I_2(x,y,t)=\int_{z\in B_0,|x/t-z|>6}\left|\psi_t^{(\alpha)}(x-y-tz)-\psi_t^{(\alpha)}(x-tz)\right|^2\,dz.$$

We first estimate $I_1(x,y,t)$. Decompose $I_1(x,y,t)=I_{1,1}(x,y,t)+I_{1,2}(x,y,t)$, where

$$I_{1,1}(x,y,t)=\int_{z\in B_0,2|y|/t<|x/t-z|<6}\left|\psi_t^{(\alpha)}(x-y-tz)-\psi_t^{(\alpha)}(x-tz)\right|^2\,dz,$$

$$I_{1,2}(x,y,t)=\int_{z\in B_0,|x/t-z|<6,|x/t-z|<2|y|/t}\left|\psi_t^{(\alpha)}(x-y-tz)-\psi_t^{(\alpha)}(x-tz)\right|^2\,dz.$$

Since $L_\alpha*\Phi$ is bounded and $n/2<\alpha<n$, we easily see that

$$I_{1,2}(x,y,t)\le Ct^{-2n}\int_{|z|<3|y|/t}|z|^{-2n+2\alpha}\,dz\le Ct^{-2n}\left(\frac{|y|}{t}\right)^{-n+2\alpha}.$$

If $|x/t-z|<6$ and $|z|<1$, then $t>|x|/7$. Thus

$$\int_0^\infty I_{1,2}(x,y,t)\,\frac{dt}{t} = \int_{|x|/7}^\infty I_{1,2}(x,y,t)\,\frac{dt}{t} \leq C\int_{|x|/7}^{\infty} t^{-2n}\left(\frac{|y|}{t}\right)^{-n+2\alpha}\frac{dt}{t} \leq C|y|^{-n+2\alpha}|x|^{-n-2\alpha}. \tag{2.1}$$

Let $I'_{1,1}(x,y,t)$ and $I''_{1,1}(x,y,t)$ be defined in the same manner as $I_{1,1}(x,y,t)$ with L_α and $L_\alpha*\Phi$ in place of $\psi^{(\alpha)}$, respectively. Then $I_{1,1}\leq 2I'_{1,1}+2I''_{1,1}$. By the mean value theorem we see that

$$I'_{1,1}(x,y,t)\leq Ct^{-2n}\left(\frac{|y|}{t}\right)^2\int_{2|y|/t<|x/t-z|<6}\left|\frac{x}{t}-z\right|^{2(-n-1+\alpha)}dz = Ct^{-2n}\left(\frac{|y|}{t}\right)^2\int_{2|y|/t<|z|<6}|z|^{2(-n-1+\alpha)}\,dz.$$

Thus, if $n/2<\alpha<n/2+1$,

$$I'_{1,1}(x,y,t)\leq Ct^{-2n}\left(\frac{|y|}{t}\right)^2(2|y|/t)^{-n-2+2\alpha}\leq Ct^{-2n}\left(\frac{|y|}{t}\right)^{-n+2\alpha}; \tag{2.2}$$

if $n/2+1<\alpha$,

$$I'_{1,1}(x,y,t)\leq Ct^{-2n}\left(\frac{|y|}{t}\right)^2; \tag{2.3}$$

and if $\alpha=n/2+1$,

$$I'_{1,1}(x,y,t)\leq Ct^{-2n}\left(\frac{|y|}{t}\right)^2\log_+\left(\frac{3t}{|y|}\right), \tag{2.4}$$

where $\log_+ s=\max(\log s,0)$. By (2.2), (2.3) and (2.4), we have

$$\int_0^\infty I'_{1,1}(x,y,t)\,\frac{dt}{t} = \int_{|x|/7}^\infty I'_{1,1}(x,y,t)\,\frac{dt}{t} \leq \begin{cases} C|y|^{-n+2\alpha}|x|^{-n-2\alpha} & \text{if } n/2<\alpha<n/2+1,\\ C|y|^2|x|^{-2n-2} & \text{if } \alpha>n/2+1,\\ C|y|^2|x|^{-2n-2}\log\left(\frac{|x|}{|y|}\right) & \text{if } \alpha=n/2+1. \end{cases} \tag{2.5}$$

We note that the case $\alpha\geq n/2+1$ may occur only when $n\geq 3$, if $n/2<\alpha<n$.

To estimate $I''_{1,1}$, we note that

$$|L_\alpha*\Phi(x-y)-L_\alpha*\Phi(x)| = \left|\int_{\mathbb{R}^n}|2\pi\xi|^{-\alpha}\hat{\Phi}(\xi)e^{2\pi i\langle x,\xi\rangle}\left(e^{-2\pi i\langle y,\xi\rangle}-1\right)d\xi\right| \leq C\int_{\mathbb{R}^n}|\xi|^{-\alpha}(1+|\xi|)^{-\beta}|y|^\epsilon|\xi|^\epsilon\,d\xi \leq C|y|^\epsilon,$$

where $0 < \epsilon < \min(\alpha + \beta - n, 1)$. Thus

$$I''_{1,1}(x, y, t) \le Ct^{-2n} \left(\frac{|y|}{t}\right)^{2\epsilon},$$

and hence

$$\int_0^\infty I''_{1,1}(x, y, t)\, \frac{dt}{t} = \int_{|x|/7}^\infty I''_{1,1}(x, y, t)\, \frac{dt}{t} \le C|y|^{2\epsilon}|x|^{-2n-2\epsilon}. \tag{2.6}$$

We recall that it is assumed that $|x| > 2|y|$. To estimate $I_2(x, y, t)$, we observe that if $|x/t - z| > 6$ and $|z| < 1$, then $|x|/t \ge 5$ and $|x/t - y/t - z| \ge |x|/(2t) - |z| > 3/2$. So, further if $|x/t - z| < c_0|x|/t$ with $0 < c_0 < 2/3$, we would have $|z| \ge (1 - c_0)|x|/t > 1$. Thus

(2.7)
$$I_2(x, y, t) = \int_{z \in B_0, |x/t - z| > 6, |x/t - z| \ge c_0|x|/t} \left|\psi_t^{(\alpha)}(x - y - tz) - \psi_t^{(\alpha)}(x - tz)\right|^2 dz.$$

Take $c_0 \in (1/2, 2/3)$. Since $\Phi \in \mathcal{M}^\alpha$, by Taylor's formula, we see that

$$\psi_t^{(\alpha)}(x - y - tz) = t^{-n} \int \left[L_\alpha\left(\frac{x}{t} - \frac{y}{t} - z\right) - L_\alpha\left(\frac{x}{t} - \frac{y}{t} - z - w\right)\right] \Phi(w)\, dw$$
$$= -t^{-n} \sum_{|\gamma| = [\alpha]+1} \frac{[\alpha] + 1}{\gamma!} \int \int_0^1 (1 - s)^{[\alpha]} (\partial^\gamma L_\alpha)\left(\frac{x}{t} - \frac{y}{t} - z - sw\right) ds\, (-w)^\gamma \Phi(w)\, dw$$

and

$$\psi_t^{(\alpha)}(x - y - tz) - \psi_t^{(\alpha)}(x - tz)$$
$$= -t^{-n} \sum_{|\gamma| = [\alpha]+1} \frac{[\alpha] + 1}{\gamma!} \int \int_0^1 (1 - s)^{[\alpha]} N_\gamma(x, y, z, w, t, s)\, ds\, (-w)^\gamma \Phi(w)\, dw,$$

where

$$\begin{aligned} N_\gamma(x, y, z, w, t, s) &= (\partial^\gamma L_\alpha)\left(\frac{x}{t} - \frac{y}{t} - z - sw\right) - (\partial^\gamma L_\alpha)\left(\frac{x}{t} - z - sw\right) \\ &= \sum_{|\beta| = 1} \frac{-y^\beta}{t} \int_0^1 (\partial^{\gamma+\beta} L_\alpha)\left(\frac{x}{t} - \frac{uy}{t} - z - sw\right) du. \end{aligned}$$

Here $\partial^\gamma = \partial_1^{\gamma_1} \dots \partial_n^{\gamma_n}$, $\partial_j = \partial/\partial x_j$, $1 \le j \le n$, and $\gamma! = \gamma_1! \dots \gamma_n!$.

We note that $|x/t - uy/t - z - sw| \ge c|x/t - z|$ for some $c > 0$ if $|x/t - z| > 6$, $2|y| < |x|$, $z, w \in B_0$, $u, s \in [0, 1]$ and $|x/t - z| \ge c_0|x|/t$. This can be seen as follows. First, as in the case $u = 1$ above we have $|x/t - uy/t - z| \ge 3/2$. Thus

$$\left|\frac{x}{t} - \frac{uy}{t} - z - sw\right| \ge \frac{1}{3}\left|\frac{x}{t} - \frac{uy}{t} - z\right| \ge \frac{1}{3}\left(1 - \frac{1}{2c_0}\right)\left|\frac{x}{t} - z\right|,$$

as claimed. Using these results in (2.7), we have

$$\begin{aligned} I_2(x,y,t) &\le C \int_{z\in B_0,|x/t-z|>6,|x/t-z|\ge c_0|x|/t} t^{-2n}\left(\frac{|y|}{t}\right)^2 \left|\frac{x}{t}-z\right|^{2(-n+\alpha-[\alpha]-2)}\,dz \\ &\le Ct^{-2n}\left(\frac{|y|}{t}\right)^2\left(\frac{|x|}{t}\right)^{2(-n+\alpha-[\alpha]-2)} \\ &\le Ct^{-2\alpha+2[\alpha]+2}|y|^2|x|^{2(-n+\alpha-[\alpha]-2)} \end{aligned}$$

and hence

$$\int_0^\infty I_2(x,y,t)\,\frac{dt}{t} = \int_0^{|x|/5} I_2(x,y,t)\,\frac{dt}{t} \le C|y|^2|x|^{-2n-2}. \tag{2.8}$$

Since

$$\begin{aligned} \int_{B_0\times(0,\infty)} \left|\psi_t^{(\alpha)}(x-y-tz)-\psi_t^{(\alpha)}(x-tz)\right|^2 dz\,\frac{dt}{t} &\le C\int_0^\infty I_{1,2}(x,y,t)\,\frac{dt}{t} \\ + C\int_0^\infty I'_{1,1}(x,y,t)\,\frac{dt}{t} + C\int_0^\infty I''_{1,1}(x,y,t)\,\frac{dt}{t} &+ C\int_0^\infty I_2(x,y,t)\,\frac{dt}{t}, \end{aligned}$$

using (2.1), (2.5), (2.6), (2.8) and recalling that $\alpha > n/2$, we can get the conclusion. □

Also, we need the next result for the proof of Theorem 2.3.

Lemma 2.2. *Let $\psi^{(\alpha)}$ be as in* (1.4). *Then, if $f \in \mathcal{S}_0(\mathbb{R}^n)$ and $g \in \mathrm{BMO}(\mathbb{R}^n)$, we have*

$$\left|\int_{\mathbb{R}^n} f(x)g(x)\,dx\right| \le C\|g\|_{\mathrm{BMO}} \int_{\mathbb{R}^n} S_{\psi^{(\alpha)}}(f)(x)\,dx.$$

This can be shown by the methods of part (a) of Remarks on pp. 148–149 of [6].

Proof of Lemma 2.2. There exists $\eta \in \mathcal{S}_0(\mathbb{R}^n)$ such that

$$\int_0^\infty \widehat{\psi^{(\alpha)}}(t\xi)\hat{\eta}(-t\xi)\,\frac{dt}{t} = 1 \quad \text{for all } \xi \ne 0.$$

We can find such η since

$$\widehat{\psi^{(\alpha)}}(\xi) = (2\pi|\xi|)^{-\alpha}(1-\hat{\Phi}(\xi)) \tag{2.9}$$

satisfies a non-degeneracy condition $\sup_{t>0}|\widehat{\psi^{(\alpha)}}(t\xi)| > 0$ for $\xi \ne 0$. (See [2, Lemma 4.1] and its proof.) Since $g \in \mathrm{BMO}$, we have the Carleson measure estimate (see [6, p. 145])

$$\sup_{y\in\mathbb{R}^n,h>0} h^{-n}\int_0^h\int_{B(y,h)} |g*\eta_t(x)|^2\,dx\,\frac{dt}{t} \le C\|g\|_{\mathrm{BMO}}^2. \tag{2.10}$$

Let

$$S_\eta^{(h)}(g)(x) = \left(\int_0^h\int_{\mathbb{R}^n}\chi_{[0,1]}\left(\frac{|x-y|}{t}\right)|g*\eta_t(y)|^2\,dy\,t^{-n}\,\frac{dt}{t}\right)^{1/2}.$$

Then (2.10) implies that

$$\sup_{z\in\mathbb{R}^n,h>0}|B(z,h)|^{-1}\int_{B(z,h)} S_\eta^{(h)}(g)(x)^2\,dx \le C_0^2\|g\|_{\mathrm{BMO}}^2. \tag{2.11}$$

This can be seen as follows.

$$\begin{aligned}\int_{B(z,h)} S_\eta^{(h)}(g)(x)^2\,dx &= \int_0^h \int_{\mathbb{R}^n} |B(z,h)\cap B(y,t)||g*\eta_t(y)|^2\,dy\,t^{-n}\,\frac{dt}{t}\\ &\le C\int_0^h\int_{B(z,2h)} |g*\eta_t(y)|^2\,dy\,\frac{dt}{t}\\ &\le C\|g\|_{\mathrm{BMO}}^2 h^n.\end{aligned}$$

Let

$$h(x) = \sup\{h : S_\eta^{(h)}(g)(x) \le 2^{1/2}C_0\|g\|_{\mathrm{BMO}}\}.$$

Then

$$|\{x\in B(z,h_0) : h(x)\ge h_0\}| \ge |B(z,h_0)|/2 \tag{2.12}$$

for all $z\in\mathbb{R}^n$ and $h_0>0$. To see this, by (2.11) we observe that

$$\begin{aligned}C_0^2\|g\|_{\mathrm{BMO}}^2|B(z,h_0)| &\ge \int\limits_{B(z,h_0)\setminus\{x:h(x)\ge h_0\}} S_\eta^{(h_0)}(g)(x)^2\,dx\\ &\ge 2C_0^2\|g\|_{\mathrm{BMO}}^2\left(|B(z,h_0)| - |\{x\in B(z,h_0) : h(x)\ge h_0\}|\right),\end{aligned}$$

from which (2.12) follows.

Now we can show

$$\int_0^\infty\int_{\mathbb{R}^n} |f*\psi_t^{(\alpha)}(y)||g*\eta_t(y)|\,dy\,\frac{dt}{t} \le C\|g\|_{\mathrm{BMO}}\int_{\mathbb{R}^n} S_{\psi^{(\alpha)}}(f)(x)\,dx. \tag{2.13}$$

To see this, we first note that by (2.12) the left hand side is bounded up to a constant factor by

$$\begin{aligned}&\int_0^\infty\int_{\mathbb{R}^n} |\{x\in B(y,t) : h(x)\ge t\}|t^{-n}|f*\psi_t^{(\alpha)}(y)||g*\eta_t(y)|\,dy\,\frac{dt}{t}\\ &\quad = \int_0^\infty\int_{\mathbb{R}^n}\left(\int\limits_{h(x)\ge t}\chi_{[0,1]}\left(\frac{|x-y|}{t}\right)dx\right)|f*\psi_t^{(\alpha)}(y)||g*\eta_t(y)|\,dy\,t^{-n}\,\frac{dt}{t}.\end{aligned}$$

Fubini's theorem implies that this quantity is equal to

$$\int_{\mathbb{R}^n}\left(\int_0^{h(x)}\int_{\mathbb{R}^n}\chi_{[0,1]}\left(\frac{|x-y|}{t}\right)|f*\psi_t^{(\alpha)}(y)||g*\eta_t(y)|\,dy\,t^{-n}\,\frac{dt}{t}\right)dx.$$

Via Schwarz's inequality, this is bounded by

$$\int_{\mathbb{R}^n} S_{\psi^{(\alpha)}}(f)(x)S_\eta^{(h(x))}(g)(x)\,dx,$$

from which (2.13) follows, since $S_\eta^{(h(x))}(g)(x)\le 2^{1/2}C_0\|g\|_{\mathrm{BMO}}$.

Finally we prove

$$\int_{\mathbb{R}^n} f(x)g(x)\,dx = \int_0^\infty\int_{\mathbb{R}^n} f*\psi_t^{(\alpha)}(y)g*\eta_t(y)\,dy\,\frac{dt}{t}, \tag{2.14}$$

for $f \in \mathcal{S}_0(\mathbb{R}^n)$ and $g \in BMO(\mathbb{R}^n)$, assuming that $S_{\psi^{(\alpha)}}(f) \in L^1(\mathbb{R}^n)$. Combining (2.13) and (2.14), we get the conclusion of the lemma. To see (2.14), we first note that $\int |g(z)|(1+|z|)^{-n-1}\,dz < \infty$, since $g \in \mathrm{BMO}(\mathbb{R}^n)$, and hence

$$\int |g(z)|(1+|y-z|)^{-n-1}\,dz \le C(1+|y|)^{n+1}.$$

From this we have

$$\int_{\mathbb{R}^n} f * \psi_t^{(\alpha)}(y) g * \eta_t(y)\,dy = \lim_{m\to\infty} \int_{\mathbb{R}^n} f * \psi_t^{(\alpha)}(y) g_{(m)} * \eta_t(y)\,dy$$

for each $t > 0$, where $g_{(m)}(x) = g(x)\chi_{[0,m]}(|x|)$, $m = 1, 2, \dots$, which can be seen by the dominated convergence theorem, since $f*\psi_t^{(\alpha)} \in \mathcal{S}$, $|g_{(m)}*\eta_t(y)| \le C_t(1+|y|)^{n+1}$ and $g_{(m)} * \eta_t \to g * \eta_t$ pointwise. We notice that $g_{(m)} \in L^1$ and

$$I_m(t) := \int_{\mathbb{R}^n} f * \psi_t^{(\alpha)}(y) g_{(m)} * \eta_t(y)\,dy = \int_{\mathbb{R}^n} \hat{f}(\xi)\widehat{\psi^{(\alpha)}}(t\xi)\widehat{g_{(m)}}(-\xi)\hat{\eta}(-t\xi)\,d\xi.$$

Using this and the fact $f, \eta \in \mathcal{S}_0$, we can see that there exists $\epsilon \in (0,1)$ such that $I_m(t) = 0$ if $t \notin (\epsilon, \epsilon^{-1})$ and

$$1 = \int_0^\infty \widehat{\psi^{(\alpha)}}(t\xi)\hat{\eta}(-t\xi)\,\frac{dt}{t} = \int_\epsilon^{\epsilon^{-1}} \widehat{\psi^{(\alpha)}}(t\xi)\hat{\eta}(-t\xi)\,\frac{dt}{t} \quad \text{for } \xi \in \operatorname{supp}(\hat{f}).$$

Also, we note that

$$\sup_{m\ge 1, t \in [\epsilon, \epsilon^{-1}]} \int_{\mathbb{R}^n} \left| f * \psi_t^{(\alpha)}(y) g_{(m)} * \eta_t(y) \right| dy \le C.$$

Applying these results, we have

$$\begin{aligned}
\int_0^\infty \int_{\mathbb{R}^n} f * \psi_t^{(\alpha)}(y) g * \eta_t(y)\,dy\,\frac{dt}{t} &= \int_0^\infty \left(\lim_{m\to\infty} \int_{\mathbb{R}^n} f * \psi_t^{(\alpha)}(y) g_{(m)} * \eta_t(y)\,dy \right) \frac{dt}{t} \\
&= \int_\epsilon^{\epsilon^{-1}} \left(\lim_{m\to\infty} \int_{\mathbb{R}^n} f * \psi_t^{(\alpha)}(y) g_{(m)} * \eta_t(y)\,dy \right) \frac{dt}{t} \\
&= \lim_{m\to\infty} \int_\epsilon^{\epsilon^{-1}} \left(\int_{\mathbb{R}^n} f * \psi_t^{(\alpha)}(y) g_{(m)} * \eta_t(y)\,dy \right) \frac{dt}{t} \\
&= \lim_{m\to\infty} \int_\epsilon^{\epsilon^{-1}} \left(\int_{\mathbb{R}^n} \hat{f}(\xi)\widehat{\psi^{(\alpha)}}(t\xi)\widehat{g_{(m)}}(-\xi)\hat{\eta}(-t\xi)\,d\xi \right) \frac{dt}{t} \\
&= \lim_{m\to\infty} \int_{\mathbb{R}^n} \hat{f}(\xi)\widehat{g_{(m)}}(-\xi) \left(\int_\epsilon^{\epsilon^{-1}} \widehat{\psi^{(\alpha)}}(t\xi)\hat{\eta}(-t\xi)\,\frac{dt}{t} \right) d\xi \\
&= \lim_{m\to\infty} \int_{\mathbb{R}^n} \hat{f}(\xi)\widehat{g_{(m)}}(-\xi)\,d\xi = \lim_{m\to\infty} \int_{\mathbb{R}^n} f(x) g_{(m)}(x)\,dx \\
&= \int_{\mathbb{R}^n} f(x) g(x)\,dx.
\end{aligned}$$

This completes the proof of (2.14) and hence that of the lemma. □

Now we can prove the following result, which will be applied in the proof of Theorem 1.1.

Theorem 2.3. *Suppose that* $\psi^{(\alpha)}$ *is as in* (1.4) *with* α *and* Φ *as in Theorem* 1.1. *Then*

$$\|S_{\psi^{(\alpha)}}(f)\|_1 \simeq \|f\|_{H^1}, \quad f \in \mathcal{S}_0(\mathbb{R}^n).$$

Proof. The inequality $\|f\|_{H^1} \leq C\|S_{\psi^{(\alpha)}}(f)\|_1$ follows from the estimate of Lemma 2.2 and duality of H^1 and BMO by taking supremum over $g \in \mathrm{BMO}$ with $\|g\|_{\mathrm{BMO}} \leq 1$.

The reverse inequality will be proved by applying a result of [7]. We first note that $S_{\psi^{(\alpha)}}$ is bounded on $L^2(\mathbb{R}^n)$. To see this, by the Plancherel theorem we observe that

$$\int_{\mathbb{R}^n} S_{\psi^{(\alpha)}}(f)^2(x)\,dx = |B_0| \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 \int_0^\infty |\widehat{\psi^{(\alpha)}}(t\xi)|^2 \,\frac{dt}{t}\,d\xi$$

From this we can deduce the L^2 boundedness, since $|\widehat{\psi^{(\alpha)}}(\xi)| \leq C|\xi|^{-\alpha}$ and $|\widehat{\psi^{(\alpha)}}(\xi)| \leq C|\xi|^{[\alpha]+1-\alpha}$, which is a consequence of the hypothesis $\Phi \in \mathcal{M}^\alpha$ and (2.9). In Lemma 2.1 we have checked the Hörmander condition which is required for [7, Chapter V, Corollary 3.10] to apply. Thus the reverse inequality follows from Corollary 3.10 of [7, Chapter V] for $\mathcal{H}$-valued singular integrals, where $\mathcal{H}$ is the Hilbert space with the norm $\|g\|_{\mathcal{H}} = \left(\int_0^\infty \int_{B_0} |g(z,t)|^2\,dz\,dt/t\right)^{1/2}$. $\square$

We also need the next result for the proof of Theorem 1.1.

Lemma 2.4. *Let* $\alpha > 0$ *and* $w \in A_1$. *There exist Fourier multipliers* ℓ *and* m *for* $H^1_w(\mathbb{R}^n)$ *such that*

$$(2\pi|\xi|)^\alpha = \ell(\xi)(1+4\pi^2|\xi|^2)^{\alpha/2}, \tag{2.15}$$

$$(1+4\pi^2|\xi|^2)^{\alpha/2} = m(\xi)(1+(2\pi|\xi|)^\alpha). \tag{2.16}$$

This follows from [26, Chapter XI, Theorem 14].

Proof of Theorem 1.1. Let $n/2 < \alpha < n$ and let U_α be as in Theorem 1.1. We show that

$$\|U_\alpha(J_\alpha(g))\|_1 + \|J_\alpha(g)\|_{H^1} \simeq \|g\|_{H^1} \tag{2.17}$$

for $g \in H^1$. First we assume that $g \in \mathcal{S}_0(\mathbb{R}^n)$. Then $U_\alpha(J_\alpha(g)) = S_{\psi^{(\alpha)}}(I_{-\alpha}J_\alpha(g))$ and by Theorem 2.3 we have

$$\|U_\alpha(J_\alpha(g))\|_1 \simeq \|I_{-\alpha}J_\alpha(g))\|_{H^1}. \tag{2.18}$$

Thus by (2.15) with $w = 1$

$$\|U_\alpha(J_\alpha(g))\|_1 \leq C\|g\|_{H^1}. \tag{2.19}$$

Also, we easily see that

$$\|J_\alpha(g)\|_{H^1} \leq C\|K_\alpha\|_1\|g^*\|_1 \leq C\|g\|_{H^1}. \tag{2.20}$$

Next, by (2.16) with $w = 1$ and (2.18) we see that

$$\begin{aligned} \|g\|_{H^1} = \|J_{-\alpha}J_\alpha(g)\|_{H^1} &\leq C\|J_\alpha(g)\|_{H^1} + C\|I_{-\alpha}J_\alpha(g)\|_{H^1} \\ &\leq C\|J_\alpha(g)\|_{H^1} + C\|U_\alpha(J_\alpha(g))\|_1, \end{aligned} \tag{2.21}$$

By (2.19), (2.20) and (2.21), we have (2.17) for $g \in \mathcal{S}_0(\mathbb{R}^n)$.

For $g \in H^1$, take a sequence $\{g_k\}$ in $\mathcal{S}_0(\mathbb{R}^n)$ satisfying $g_k \to g$, $J_\alpha(g_k) \to J_\alpha(g)$ in H^1 and almost everywhere as $k \to \infty$. Fix $x \in \mathbb{R}^n$ and $t > 0$. Then

$$|J_\alpha(g_k)(z) - \Phi_t * J_\alpha(g_k)(z)| \to |J_\alpha(g)(z) - \Phi_t * J_\alpha(g)(z)| \quad \text{for almost every } z \in B(x,t).$$

Thus Fatou's lemma implies that

$$\int_{B(x,t)} |J_\alpha(g)(z)-\Phi_t * J_\alpha(g)(z)|^2\,dz \le \liminf_{k\to\infty} \int_{B(x,t)} |J_\alpha(g_k)(z)-\Phi_t * J_\alpha(g_k)(z)|^2\,dz,$$

and hence by Fatou's lemma it follows that

$$U_\alpha(J_\alpha(g))(x) \le \liminf_{k\to\infty} U_\alpha(J_\alpha(g_k))(x)$$

for all x. By (2.17) for $g \in \mathcal{S}_0(\mathbb{R}^n)$ we have

$$\|U_\alpha(J_\alpha(g_k))\|_1 \le C\|g_k\|_{H^1}.$$

Using this and Fatou's lemma, we have

$$\begin{aligned}\|U_\alpha(J_\alpha(g))\|_1 &\le \liminf_{k\to\infty} \|U_\alpha(J_\alpha(g_k))\|_1\\ &\le C \liminf_{k\to\infty} \|g_k\|_{H^1}\\ &\le C\|g\|_{H^1}.\end{aligned}$$

Applying this we can deduce that

$$\begin{aligned}&\lim_{k\to\infty} \|U_\alpha(J_\alpha(g)) - U_\alpha(J_\alpha(g_k))\|_1 \le \lim_{k\to\infty} \|U_\alpha(J_\alpha(g-g_k))\|_1\\ &\le C \lim_{k\to\infty} \|g-g_k\|_{H^1} = 0.\end{aligned}$$

Therefore, letting $k \to \infty$ in

$$\|U_\alpha(J_\alpha(g_k))\|_1 + \|J_\alpha(g_k)\|_{H^1} \simeq \|g_k\|_{H^1},$$

which follows from (2.17) on $\mathcal{S}_0(\mathbb{R}^n)$, we have (2.17) on the whole H^1.

Since we have shown (2.17), to complete the proof of Theorem 1.1, it suffices to prove that $f \in W^\alpha_{H^1}$ if $f \in H^1$ and $U_\alpha(f) \in L^1$. For this we need the next two lemmas.

Lemma 2.5. *Let $f \in L^1_w, w \in A_1, g \in \mathcal{S}(\mathbb{R}^n)$ and $\alpha > 0$. Then*

(1) *we have*

$$K_\alpha * (f*g)(x) = (K_\alpha * f)*g(x) = (K_\alpha * g)*f(x) \quad \text{for every } x \in \mathbb{R}^n;$$

(2) *also,*

$$\int_{\mathbb{R}^n} (K_\alpha * f)(y)g(y)\,dy = \int_{\mathbb{R}^n} (K_\alpha * g)(y)f(y)\,dy.$$

Proof. By Fubini's theorem, the result is a consequence of the estimate

$$\iint K_\alpha(x-z-y)|f(y)||g(z)|\,dy\,dz < \infty, \quad x \in \mathbb{R}^n.$$

This follows from the inequality

$$\int K_\alpha(x-z-y)|g(z)|\,dz \le C_{x,M}(1+|y|)^{-M}$$

for any $M > 0$, which is a consequence of the fact that $g \in \mathcal{S}(\mathbb{R}^n)$ and K_α is rapidly decreasing as $|x| \to \infty$ (see [23, p.132]), since the assumption $f \in L^1_w$ implies $\int |f(y)|(1+|y|)^{-n}\,dy < \infty$ (see [13, Section 4]).

$\square$

Lemma 2.6. *Let $w \in A_1$ and let $\{g_m\}_{m=1}^{\infty}$ be a sequence of functions in H_w^1 satisfying $\sup_{m\geq 1} \|g_m\|_{H_w^1} < \infty$. Then there exist a subsequence $\{g_{m_k}\}_{k=1}^{\infty}$ and $g \in H_w^1$ such that*

$$\int_{\mathbb{R}^n} g_{m_k}(x)v(x)\,dx \to \int_{\mathbb{R}^n} g(x)v(x)\,dx \quad \text{as } k \to \infty \text{ for } v \in C_c(\mathbb{R}^n),$$

where $C_c(\mathbb{R}^n)$ is the space of all continuous functions with compact support on $\mathbb{R}^n$; also this is valid for all $v \in \mathcal{S}(\mathbb{R}^n)$.

Proof. Let $O_l = B(0,l)$, $l = 1,2,3,\dots$. We note that $w(x) \geq C_w(1+|x|)^{-n}$ for $w \in A_1$, which follows from the property $M(w)(x) \leq Cw(x)$ valid for A_1 weights (see [13, Section 4]). Thus we have $\sup_{m\geq 1} \int_{O_l} |g_m|\,dx < \infty$ for every l.

It is well-known that there exist a subsequence $\{g_{m_k}\}_{k=1}^{\infty}$ and a regular Borel measure μ_l such that

$$\int_{\mathbb{R}^n} \chi_{O_{l+1}}(y)g_{m_k}(y)v(y)\,dy \to \int_{\mathbb{R}^n} v(y)\,d\mu_l(y) \quad \text{as } k \to \infty \text{ for } v \in C_c(\mathbb{R}^n) \tag{2.22}$$

(see [4, 1.9, Theorem 2]). We note that the subsequence can be chosen independent of l by the diagonal process. Taking $\varphi_\epsilon(x-y)$ in (2.22) in place of $v(y)$ for $x \in O_l, \epsilon \in (0,1)$, we easily see that

$$\left|\int_{\mathbb{R}^n} \varphi_\epsilon(x-y)\,d\mu_l(y)\right| \leq \liminf_{k\to\infty} \left|\int_{\mathbb{R}^n} \varphi_\epsilon(x-y)g_{m_k}(y)\,dy\right| \leq \liminf_{k\to\infty} g_{m_k}^*(x),$$

since $y \in O_{l+1}$ if $\varphi_\epsilon(x-y) \neq 0$, and hence

$$\int_{O_l} \mu_l^\dagger(x)\,dx \leq \liminf_{k\to\infty} \|g_{m_k}^*\|_1 \leq C,$$

where

$$\mu_l^\dagger(x) = \sup_{\epsilon\in(0,1)} \left|\int_{\mathbb{R}^n} \varphi_\epsilon(x-y)\,d\mu_l(y)\right|, \quad g_{m_k}^*(x) = \sup_{\epsilon>0} \left|\int_{\mathbb{R}^n} \varphi_\epsilon(x-y)g_{m_k}(y)\,dy\right|.$$

Also, for $v \in C_c(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \varphi_\epsilon(x-y)\,d\mu_l(y)\right) v(x)\,dx \to \int_{\mathbb{R}^n} v(y)\,d\mu_l(y) \quad \text{as } \epsilon \to 0.$$

Thus

$$\left|\int_{\mathbb{R}^n} v(y)\,d\mu_l(y)\right| \leq \int_{O_l} |v(x)|\mu_l^\dagger(x)\,dx$$

for all $v \in C_c(\mathbb{R}^n)$ with support in O_l, which implies

$$|\mu_l|(O) \leq \int_O \mu_l^\dagger(x)\,dx \leq \int_{O_l} \mu_l^\dagger(x)\,dx$$

for any open set O in O_l, where $|\mu_l|$ is the total variation of μ_l. Thus μ_l is absolutely continuous when restricted to O_l and there exists $g_l \in L^1(\mathbb{R}^n)$ such that

$$\int v(x)\,d\mu_l(x) = \int v(x)g_l(x)\,dx \quad \text{for } v \in C_c(\mathbb{R}^n) \text{ with support in } O_l. \tag{2.23}$$

By (2.22) and (2.23), we can see that there is a locally integrable function g on $\mathbb{R}^n$ such that $g = g_l$ on O_l and

$$\int_{\mathbb{R}^n} g_{m_k}(y)v(y)\,dy \to \int_{\mathbb{R}^n} g(y)v(y)\,dy \quad \text{as } k \to \infty \text{ for } v \in C_c(\mathbb{R}^n).$$

Applying this with $v(y) = \varphi_\epsilon(x-y)$, as above, we have $g^*(x) \le \liminf_{k\to\infty} g^*_{m_k}(x)$, which combined with the assumption that $\{g^*_k\}$ is L^1_w bounded implies $g^* \in L^1_w$ and hence $g \in H^1_w$. The result for $v \in \mathcal{S}(\mathbb{R}^n)$ follows from that for $v \in C_c(\mathbb{R}^n)$.

□

Now we can finish the proof of Theorem 1.1. Suppose that $f \in H^1$ and $U_\alpha(f) \in L^1$. Let $f^{(\epsilon)}(x) = \varphi_\epsilon * f(x)$ and $g^{(\epsilon)}(x) = J_{-\alpha}(\varphi_\epsilon) * f(x)$. Then $g^{(\epsilon)} \in H^1$ and by part (1) of Lemma 2.5 with $w = 1$, $f^{(\epsilon)} = J_\alpha(g^{(\epsilon)})$ and (2.17) implies

$$\|U_\alpha(f^{(\epsilon)})\|_1 + \|f^{(\epsilon)}\|_{H^1} \simeq \|g^{(\epsilon)}\|_{H^1}. \tag{2.24}$$

We easily see that

$$\sup_{\epsilon>0} \|f^{(\epsilon)}\|_{H^1} \le C\|f\|_{H^1}. \tag{2.25}$$

Further, using Minkowski's inequality, we have

$$\begin{aligned} U_\alpha(f^{(\epsilon)})(x) &= \left(\int_0^\infty \int_{B(x,t)} |\varphi_\epsilon * f(z) - \Phi_t * \varphi_\epsilon * f(z)|^2 \, dz \, t^{-2\alpha-n} \frac{dt}{t}\right)^{1/2} \\ &\le \int_{\mathbb{R}^n} \varphi_\epsilon(y) \left(\int_0^\infty \int_{B(x,t)} |f(z-y) - \Phi_t * f(z-y)|^2 \, dz \, t^{-2\alpha-n} \frac{dt}{t}\right)^{1/2} dy \\ &= \int_{\mathbb{R}^n} \varphi_\epsilon(y) U_\alpha(f)(x-y)\, dy, \end{aligned}$$

which implies

$$\sup_{\epsilon>0} \|U_\alpha(f^{(\epsilon)})\|_1 \le \|U_\alpha(f)\|_1. \tag{2.26}$$

Combining (2.24), (2.25) and (2.26), we have $\sup_{\epsilon>0} \|g^{(\epsilon)}\|_{H^1} < \infty$.

Thus by Lemma 2.6 for $w = 1$ we have a sequence $\{g^{(\epsilon_k)}\}$ with $\epsilon_k \to 0$ and $g \in H^1$ such that

$$\int_{\mathbb{R}^n} g^{(\epsilon_k)}(x)v(x)\, dx \to \int_{\mathbb{R}^n} g(x)v(x)\, dx \quad \text{as } k\to\infty \text{ for } v \in \mathcal{S}(\mathbb{R}^n).$$

Also, $\{f^{(\epsilon_k)}\}$ converges to f in L^1. Thus, we can see that $f = J_\alpha(g)$. We show this in more detail as follows. Let $v \in \mathcal{S}(\mathbb{R}^n)$. Then, using part (2) of Lemma 2.5 with $w = 1$ and the fact that $J_\alpha(v) \in \mathcal{S}(\mathbb{R}^n)$, we see that

$$\begin{aligned} \int f(x)v(x)\, dx &= \lim_{k\to\infty} \int f^{(\epsilon_k)}(x)v(x)\, dx = \lim_{k\to\infty} \int J_\alpha(g^{(\epsilon_k)})(x)v(x)\, dx \\ &= \lim_{k\to\infty} \int g^{(\epsilon_k)}(x) J_\alpha(v)(x)\, dx = \int g(x) J_\alpha(v)(x)\, dx \\ &= \int J_\alpha(g)(x)v(x)\, dx. \end{aligned} \tag{2.27}$$

It follows that $f = J_\alpha(g)$. Thus $f \in W^\alpha_{H^1}(\mathbb{R}^n)$.

□

3. Estimates for Littlewood-Paley functions on the weighted Hardy spaces

To prove Theorem 1.2, we need Theorem 3.2 below with $p = 1$ and $n = 1$ (see [19] and [20] for the unweighted case).

Definition 3.1. Let $\psi \in L^1(\mathbb{R}^n)$. We say $\psi \in \mathcal{B}$ if

(1) $\hat{\psi} \in C^\infty(\mathbb{R}^n \setminus \{0\})$;
(2) $\sup_{t>0} |\hat{\psi}(t\xi)| > 0$ for all $\xi \neq 0$;
(3) $\psi \in C^1(\mathbb{R}^n)$, $\partial_k \psi \in L^1(\mathbb{R}^n)$, $1 \le k \le n$;
(4) $|\hat{\psi}(\xi)| \le C|\xi|^\epsilon$ for some $\epsilon > 0$;
(5) $|\partial^\gamma \hat{\psi}(\xi)| \le C_{\gamma,\tau}|\xi|^{-\tau}$ outside a neighborhood of the origin for all multi-indices γ and $\tau > 0$.

Theorem 3.2. *Let $0 < p \le 1$, $w \in A_1$ and $\psi \in \mathcal{B}$. Then we have*

$$\|f\|_{H^p_w} \le C_p \|g_\psi(f)\|_{p,w}$$

for $f \in \mathcal{S}_0(\mathbb{R}^n)$ with a positive constant C_p independent of f, where we recall that $\|f\|_{H^p_w} = \|f^*\|_{p,w}$.

Let $\mathcal{H}$ be the Hilbert space of functions $u(t)$ on $(0,\infty)$ such that $\|u\|_{\mathcal{H}} = \left(\int_0^\infty |u(t)|^2\, dt/t\right)^{1/2} < \infty$. Let $w \in A_1$. We consider the weighted Lebesgue space $L^q_{\mathcal{H},w}(\mathbb{R}^n)$ of functions $h(y,t)$ with the norm

$$\|h\|_{q,\mathcal{H},w} = \left(\int_{\mathbb{R}^n} \|h^y\|^q_{\mathcal{H}} w(y)\, dy\right)^{1/q},$$

where $h^y(t) = h(y,t)$. When $w = 1$, we write simply $L^q_{\mathcal{H},w}(\mathbb{R}^n) = L^q_{\mathcal{H}}(\mathbb{R}^n)$.

Let $0 < p \le 1$. We consider the weighted Hardy space of functions on $\mathbb{R}^n$ with values in $\mathcal{H}$, which is denoted by $H^p_{\mathcal{H},w}(\mathbb{R}^n)$. We say that $h \in H^p_{\mathcal{H},w}(\mathbb{R}^n)$ if $h \in L^2_{\mathcal{H}}(\mathbb{R}^n)$ and $\|h\|_{H^p_{\mathcal{H},w}} = \|h^*\|_{p,w} < \infty$ with

$$h^*(x) = \sup_{s>0} \left(\int_0^\infty |\varphi_s * h^t(x)|^2 \frac{dt}{t}\right)^{1/2},$$

where we write $h^t(x) = h(x,t)$ and we recall that φ is the function in $\mathcal{S}(\mathbb{R}^n)$ fixed in Section 1.

In proving Theorem 3.2, we need the next result.

Lemma 3.3. *Let $\hat{\psi} \in \mathcal{S}(\mathbb{R}^n)$ be a radial function such that $\operatorname{supp}(\hat{\psi}) \subset \{1 \le |\xi| \le 2\}$ and*

$$\int_0^\infty |\hat{\psi}(t\xi)|^2 \frac{dt}{t} = 1 \quad \textit{for all } \xi \neq 0.$$

Let $F(y,t) = f * \psi_t(y)$ with $f \in \mathcal{S}_0(\mathbb{R}^n)$. Let $w \in A_1$. Then $F \in H^p_{\mathcal{H},w}(\mathbb{R}^n)$, $0 < p \le 1$, and

$$\|f\|_{H^p_w} \le C\|F\|_{H^p_{\mathcal{H},w}}.$$

Let ψ be as in Lemma 3.3 and

$$E^\epsilon_\psi(h)(x) = \int_0^\infty \int_{\mathbb{R}^n} \psi_t(x-y) h_{(\epsilon)}(y,t)\, dy\, \frac{dt}{t},$$

where $h \in L^2_{\mathcal{H}}$ and $h_{(\epsilon)}(y,t) = h(y,t)\chi_{(\epsilon,\epsilon^{-1})}(t)$, $0 < \epsilon < 1$.

To prove Lemma 3.3 we apply the following.

Lemma 3.4. *Suppose that $w \in A_1$. Then*

$$\sup_{\epsilon\in(0,1)} \|E_\psi^\epsilon(h)\|_{H_w^p} \le C\|h\|_{H^p_{\mathcal{H},w}}, \quad 0 < p \le 1.$$

To prove Lemma 3.4 we use the atomic decomposition.

Definition 3.5. We say that a is a (p,∞) atom in $H^p_{\mathcal{H},w}(\mathbb{R}^n)$, $w \in A_1$, $0 < p \le 1$, if

(i) $\left(\int_0^\infty |a(x,t)|^2\, dt/t\right)^{1/2} \le w(B)^{-1/p}$, where B is a ball in $\mathbb{R}^n$ and $w(B) = \int_B w(x)\, dx$;
(ii) $\operatorname{supp}(a(\cdot,t)) \subset B$ for all $t > 0$, where B is the same as in (i);
(iii) $\int_{\mathbb{R}^n} a(x,t)x^\gamma\, dx = 0$ for all $t > 0$ and multi-indices γ such that $|\gamma| \le [n(1/p-1)]$.

Lemma 3.6. *Let $w \in A_1$, $0 < p \le 1$. Suppose that $h \in H^p_{\mathcal{H},w}(\mathbb{R}^n)$. Then there exist a sequence $\{a_k\}$ of (p,∞) atoms in $H^p_{\mathcal{H},w}(\mathbb{R}^n)$ and a sequence $\{\lambda_k\}$ of positive numbers such that $\sum_{k=1}^\infty \lambda_k^p \le C\|h\|^p_{H^p_{\mathcal{H},w}}$, where C is a constant independent of h, and $h = \sum_{k=1}^\infty \lambda_k a_k$ in $H^p_{\mathcal{H},w}(\mathbb{R}^n)$ and in $L^2_{\mathcal{H}}(\mathbb{R}^n)$.*

A proof of the atomic decomposition for $H^p_w(\mathbb{R}^n)$ can be found in [26, Chapter VIII] and the proof for the vector valued case is similar; we can apply the same arguments as in the case of the scalar valued case by replacing absolute values with $\mathcal{H}$-norms in appropriate places.

Also, we need the following result in proving Lemma 3.4.

Lemma 3.7. *Let $w \in A_2$ and $h \in L^2_{\mathcal{H},w}(\mathbb{R}^n) \cap L^2_{\mathcal{H}}(\mathbb{R}^n)$. Let ψ be as in Lemma 3.3. Then*

$$\sup_{\epsilon\in(0,1)} \|E_\psi^\epsilon(h)\|_{2,w} \le C\|h\|_{2,\mathcal{H},w}.$$

Proof. Let $\Psi \in C_0^\infty(\mathbb{R}^n)$ satisfy $\operatorname{supp}(\Psi) \subset \{1/2 \le |\xi| \le 2\}$ and $\sum_{m\in\mathbb{Z}} \Psi(2^{-m}\xi) = 1$ for $\xi \ne 0$. Define

$$\Delta_m(h)(x,t) = \int \Psi(2^{-m}\xi)\hat{h}(\xi,t)e^{2\pi i\langle \xi,x\rangle}\, d\xi,$$

where the Fourier transform $\hat{h}$ is with respect to x variable, and

$$A_m(h_{(\epsilon)})(x) = \int_0^\infty \int_{\mathbb{R}^n} \psi_t(x-y)\Delta_m(h_{(\epsilon)})(y,t)\, dy\, \frac{dt}{t}.$$

Then by taking the Fourier transform we see that

$$A_m(h_{(\epsilon)})(x) = \int_{2^{-m-1}}^{2^{-m+2}} \int_{\mathbb{R}^n} \psi_t(x-y)\Delta_m(h_{(\epsilon)})(y,t)\, dy\, \frac{dt}{t}.$$

We note that

$$\left|\int_{\mathbb{R}^n} \psi_t(x-y)\Delta_m(h_{(\epsilon)})(y,t)\, dy\right| \le CM(\Delta_m(h_{(\epsilon)})(\cdot,t))(x),$$

since the least decreasing radial majorant of ψ is integrable (see [23, Chapter III, Theorem 2]). Similarly,

$$\left|\Delta_m(h_{(\epsilon)})(x,t)\right| \le CM(h_{(\epsilon)}(\cdot,t))(x).$$

It follows that
$$|A_m(h_{(\epsilon)})(x)|^2 \le C\int_{2^{-m-1}}^{2^{-m+2}} M^2(h_{(\epsilon)}(\cdot,t))(x)^2\,\frac{dt}{t}.$$
Thus, since the maximal operator M is bounded on L^2_w for $w \in A_2$,
$$\int |A_m(h_{(\epsilon)})(x)|^2 w(x)\,dx \le C\int_{2^{-m-1}}^{2^{-m+2}} \int |h(x,t)|^2 w(x)\,dx\,\frac{dt}{t}.$$
Using this and applying Littlewood-Paley inequality with $w \in A_2$, we have
$$\begin{aligned}\|\sum_{m\in\mathbb{Z}} A_m(h_{(\epsilon)})\|_{2,w}^2 &\le C\sum_{m\in\mathbb{Z}} \|A_m(h_{(\epsilon)})\|_{2,w}^2\\ &\le C\|h\|_{2,\mathcal{H},w}^2.\end{aligned}$$
This completes the proof, since $E^\epsilon_\psi(h) = \sum_{m\in\mathbb{Z}} A_m(h_{(\epsilon)})$. □

Proof of Lemma 3.4. The proof is analogous to the one for Lemma 3.5 of [19]. So we put it briefly. Let a be a (p,∞) atom in $H^p_{\mathcal{H},w}(\mathbb{R}^n)$ supported on the ball B of Definition 3.5. If $\widetilde{B}$ is a concentric enlargement of B such that $2|y-y'| < |x-y'|$ for $y, y' \in B$ and $x \in \mathbb{R}^n \setminus \widetilde{B}$. Then, as in the proof of Lemma 3.5 of [19], using properties of an atom, for $x \in \mathbb{R}^n \setminus \widetilde{B}$ and $y' \in B$ we have
$$\left|\varphi_s * E^\epsilon_\psi(a)(x)\right| \le Cw(B)^{-1/p}|x-y'|^{-n-M-1}\int_B |y-y'|^{M+1}\,dy \tag{3.1}$$
where $M = [n(1/p-1)]$.

To prove (3.1), let $\Psi_{s,t} = \varphi_s * \psi_t$, $s,t>0$. Let $P_x(y,y')$ be the Taylor polynomial in y of order $M = [n(1/p-1)]$ at y' for $\varphi_{s/t} * \psi(x-y)$. Then, if $|x-y| > 2|y'-y|$, we see that
$$|\Psi_{s,t}(x-y) - t^{-n}P_{x/t}(y/t, y'/t)| \le Ct^{-n-M-1}|y-y'|^{M+1}(1+|x-y'|/t)^{-L},$$
where $L > n+M+1$ and the constant C is independent of s,t,x,y',y. Thus, using the properties of an atom and the Schwarz inequality, for $x \in \mathbb{R}^n \setminus \widetilde{B}$ we have
$$\begin{aligned}&\left|\varphi_s * E^\epsilon_\psi(a)(x)\right| = \left|\iint_{\mathbb{R}^n\times(0,\infty)} \left(\Psi_{s,t}(x-y) - t^{-n}P_{x/t}(y/t,y'/t)\right) a_{(\epsilon)}(y,t)\,dy\,\frac{dt}{t}\right|\\ &\le \int_B \left(\int_0^\infty \left|\Psi_{s,t}(x-y) - t^{-n}P_{x/t}(y/t,y'/t)\right|^2\frac{dt}{t}\right)^{1/2}\left(\int_0^\infty |a(y,t)|^2\,\frac{dt}{t}\right)^{1/2} dy\\ &\le Cw(B)^{-1/p}\int_B \left(\int_0^\infty \left|\Psi_{s,t}(x-y) - t^{-n}P_{x/t}(y/t,y'/t)\right|^2\frac{dt}{t}\right)^{1/2} dy\\ &\le Cw(B)^{-1/p}\int_B |y-y'|^{M+1}|x-y'|^{-n-M-1}\,dy,\end{aligned}$$
which proves (3.1).

Since $p > n/(n+M+1)$, by a straightforward computation, using (3.1), we see that
$$\begin{aligned}\int_{\mathbb{R}^n\setminus\widetilde{B}} \sup_{s>0}\left|\varphi_s * E^\epsilon_\psi(a)(x)\right|^p w(x)\,dx &\le Cw(B)^{-1}|B|\inf_{y'\in B} M(w)(y')\\ &\le Cw(B)^{-1}\int_B w(y)\,dy \le C.\end{aligned} \tag{3.2}$$

By Hölder's inequality, the L^2_w-boundedness of M, Lemma 3.7 and the properties (i), (ii) of Definition 3.5, we get

(3.3)
$$\begin{aligned}
\int_{\widetilde{B}} \sup_{s>0} \left|\varphi_s * E^\epsilon_\psi(a)(x)\right|^p w(x)\,dx &\le Cw(B)^{1-p/2}\left(\int_{\widetilde{B}} |M(E^\epsilon_\psi(a))(x)|^2 w(x)\,dx\right)^{p/2}\\
&\le Cw(B)^{1-p/2}\left(\int_{\widetilde{B}} |E^\epsilon_\psi(a)(x)|^2 w(x)\,dx\right)^{p/2}\\
&\le Cw(B)^{1-p/2}\left(\int_B\int_0^\infty |a(y,t)|^2 w(y)\,\frac{dt}{t}\,dy\right)^{p/2}\\
&\le C,
\end{aligned}$$

where we have used the estimate $\left|\varphi_s * E^\epsilon_\psi(a)\right| \le CM(E^\epsilon_\psi(a))$. Combining (3.2) and (3.3), we have

$$\int_{\mathbb{R}^n} \sup_{s>0} \left|\varphi_s * E^\epsilon_\psi(a)(x)\right|^p w(x)\,dx \le C. \tag{3.4}$$

By Lemma 3.6 and (3.4) we can prove

$$\int_{\mathbb{R}^n} \sup_{s>0} \left|\varphi_s * E^\epsilon_\psi(h)(x)\right|^p w(x)\,dx \le C\|h\|^p_{H^p_{\mathcal{H},w}}.$$

This completes the proof. □

Proof of Lemma 3.3. It can be shown that $F \in H^p_{\mathcal{H},w}(\mathbb{R}^n)$ similarly to the proof of Lemma 3.4 by using the atomic decomposition for $f \in H^p_w(\mathbb{R}^n)$; recall that $f \in \mathcal{S}_0$ and that $\mathcal{S}_0$ is a subspace of $H^p_w(\mathbb{R}^n)$. We give a sketch of the proof. First we can prove an estimate analogous to (3.1):

$$\begin{aligned}
\sup_{s>0}\left(\int_0^\infty |\varphi_s * \psi_t * a(x)|^2\,\frac{dt}{t}\right)^{1/2}&\\
\le Cw(B)^{-1/p}|x-y'|^{-n-M-1}&\int_B |y-y'|^{M+1}\,dy,
\end{aligned} \tag{3.5}$$

where a is a (p,∞) atom for $H^p_w(\mathbb{R}^n)$ supported on the ball B with properties analogous to those for the atom in (3.1) and $y' \in B$ and $x \in \mathbb{R}^n \setminus \widetilde{B}$ with $\widetilde{B}$ denoting a concentric enlargement of B as in the case of (3.1); further $M = [n(1/p-1)]$. Also, we have the following L^2_w-boundedness:

$$\int_{\mathbb{R}^n} \sup_{s>0} \int_0^\infty |\varphi_s * \psi_t * f(x)|^2\,\frac{dt}{t}\,w(x)\,dx \le C\|f\|^2_{L^2_w}. \tag{3.6}$$

This can be shown by using the L^2_w-boundedness of M and g_ψ as follows.

$$\begin{aligned}
\int_{\mathbb{R}^n} \sup_{s>0} \int_0^\infty |\varphi_s * \psi_t * f(x)|^2\,\frac{dt}{t}\,w(x)\,dx &\le C\int_0^\infty \int_{\mathbb{R}^n} |M(\psi_t * f)(x)|^2\,w(x)\,dx\,\frac{dt}{t}\\
&\le C\int_0^\infty \int_{\mathbb{R}^n} |\psi_t * f(x)|^2\,w(x)\,dx\,\frac{dt}{t}\\
&= C\int_{\mathbb{R}^n} g_\psi(f)(x)^2\,w(x)\,dx\\
&\le C\|f\|^2_{L^2_w}.
\end{aligned}$$

Using (3.5) and (3.6), we can show $F \in H^p_{\mathcal{H},w}(\mathbb{R}^n)$ as in the proof of Lemma 3.4.

Let ψ, F be as in Lemma 3.3 and let $\bar{\psi}$ denote the complex conjugate. Then

$$E^{\epsilon}_{\bar{\psi}}(F)(x) = \int_{\epsilon}^{\epsilon^{-1}} \int_{\mathbb{R}^n} \psi_t * f(y)\bar{\psi}_t(y-x)\,dy\,\frac{dt}{t} = \int_{\mathbb{R}^n} \Psi^{(\epsilon)}(x-z)f(z)\,dz,$$

where

$$\Psi^{(\epsilon)}(x) = \int_{\epsilon}^{\epsilon^{-1}} \int_{\mathbb{R}^n} \psi_t(x+y)\bar{\psi}_t(y)\,dy\,\frac{dt}{t}.$$

There exists $\epsilon_0 \in (0,1)$ such that

$$\widehat{\Psi^{(\epsilon_0)}}(\xi) = \int_{\epsilon_0}^{\epsilon_0^{-1}} \hat{\psi}(t\xi)\hat{\bar{\psi}}(-t\xi)\,\frac{dt}{t} = \int_{\epsilon_0}^{\epsilon_0^{-1}} |\hat{\psi}(t\xi)|^2\,\frac{dt}{t} = 1 \quad \text{for } \xi \in \operatorname{supp}(\hat{f}).$$

Thus

$$E^{\epsilon_0}_{\bar{\psi}}(F)(x) = \int \widehat{\Psi^{(\epsilon_0)}}(\xi)\hat{f}(\xi)e^{2\pi i\langle x,\xi\rangle}\,d\xi = \int \hat{f}(\xi)e^{2\pi i\langle x,\xi\rangle}\,d\xi = f(x)$$

and hence by Lemma 3.4

$$\|f\|_{H^p_w} = \|E^{\epsilon_0}_{\bar{\psi}}(F)\|_{H^p_w} \le C\|F\|_{H^p_{\mathcal{H},w}}.$$

$\square$

Along with Lemma 3.3, the next two results (Lemmas 3.8, 3.9) are used in proving Theorem 3.2.

Lemma 3.8. *Let $w \in A_\infty = \cup_{p>1}A_p$. Suppose that $\eta \in \mathcal{S}(\mathbb{R}^n)$, $\operatorname{supp}(\hat{\eta}) \subset \{1/2 \le |\xi| \le 4\}$ and $\hat{\eta}(\xi) = 1$ on $\{1 \le |\xi| \le 2\}$. Let ψ be as in Lemma 3.3. Then for $p, q > 0$ and $f \in \mathcal{S}_0(\mathbb{R}^n)$ we have*

$$\left\| \left(\int_0^\infty \sup_{s>0} |\varphi_s * \psi_t * f|^q \,\frac{dt}{t}\right)^{1/q} \right\|_{p,w} \le C \left\| \left(\int_0^\infty |\eta_t * f|^q \,\frac{dt}{t}\right)^{1/q} \right\|_{p,w}.$$

This can be established by the proof of Lemma 3.3 of [19], where only the unweighted version of Lemma 3.8 is explicitly treated but the proof is exactly the same in the weighted cases.

Lemma 3.9. *Let $\Psi \in \mathcal{B}$ and $w \in A_\infty$. Suppose that $0 < p, q < \infty$. Let $\eta \in \mathcal{S}(\mathbb{R}^n)$ satisfy $\hat{\eta} = 0$ in a neighborhood of the origin. Then*

$$\left\| \left(\int_0^\infty |f * \eta_t|^q \,\frac{dt}{t}\right)^{1/q} \right\|_{p,w} \le C \left\| \left(\int_0^\infty |f * \Psi_t|^q \,\frac{dt}{t}\right)^{1/q} \right\|_{p,w}$$

for $f \in \mathcal{S}_0(\mathbb{R}^n)$ with a positive constant C independent of f.

This is a particular case of Theorem 2.4 of [20] (also, results of [19] imply Lemma 3.9).

Now we can complete the proof of Theorem 3.2.

Proof of Theorem 3.2. Let η and ψ be as in Lemma 3.8 and $w \in A_1$. Applying successively Lemma 3.3 and Lemma 3.8 with $q = 2$, we have

$$\begin{aligned}\|f\|_{H^p_w} &\le C \left\| \sup_{s>0} \left(\int_0^\infty |\varphi_s * \psi_t * f|^2 \, \frac{dt}{t} \right)^{1/2} \right\|_{p,w} \\ &\le C \, \|g_\eta(f)\|_{p,w}\end{aligned}$$

for $f \in \mathcal{S}_0(\mathbb{R}^n)$. By this and Lemma 3.9 with $q = 2$ we can finish the proof of Theorem 3.2.

□

4. Proof of Theorem 1.2

We first note the following.

Lemma 4.1. *Let $W^1_{H^1_w}(\mathbb{R})$, $w \in A_1$, be as in Section* 1. *Then*

$$\|f\|_{W^1_{H^1_w}} \simeq \|f\|_{H^1_w} + \|f'\|_{H^1_w}, \quad f \in \mathcal{S}_0(\mathbb{R}).$$

Proof. Let $f \in \mathcal{S}_0(\mathbb{R})$. Then $f \in H^1_w$ and $f = J_1(g)$ for $g \in \mathcal{S}_0(\mathbb{R})$. Applying integration by parts, we have

$$\int f'(x)\hat{\eta}(x)\, dx = -\int f(x)(\hat{\eta})'(x)\, dx = \int \widehat{K}_1(\xi)\hat{g}(\xi) 2\pi i \xi \eta(\xi)\, d\xi$$

for $\eta \in \mathcal{S}(\mathbb{R})$. Since $\xi = (\operatorname{sgn} \xi)|\xi|$, by Lemma 2.4 (2.15) and the fact that the Hilbert transform is bounded on H^1_w, we see that $2\pi i \xi \widehat{K}_1(\xi)\hat{g}(\xi) = \hat{h}(\xi)$ for $h \in \mathcal{S}_0(\mathbb{R})$ with $\|h\|_{H^1_w} \le C\|g\|_{H^1_w}$ (see [26, Chapter XI, Theorem 14]). Thus we have

$$\int f'(x)\hat{\eta}(x)\, dx = \int \hat{h}(\xi)\eta(\xi)\, d\xi = \int h(x)\hat{\eta}(x)\, dx,$$

which implies that $f' = h$ and hence $\|f'\|_{H^1_w} \le C\|f\|_{W^1_{H^1_w}}$. Also, a straightforward computation implies that

$$\|f\|_{H^1_w} = \|(J_1(g))^*\|_{1,w} \le C\|g^*\|_{1,M(w)} \le C\|g^*\|_{1,w} = C\|f\|_{W^1_{H^1_w}}.$$

Here we give a proof of the first inequality for completeness. As in Section 1, we have $(J_1(g))^* \le K_1 * g^*$ and $K_1 * w \le CM(w)$. Thus

$$\|(J_1(g))^*\|_{1,w} \le \int K_1 * g^*(x) w(x)\, dx = \int g^*(y) K_1 * w(y)\, dy \le C \int g^*(y) M(w)(y)\, dy,$$

where we have used the fact that K_1 is even.

On the other hand, let $g = J_{-1}(f) \in \mathcal{S}_0(\mathbb{R})$. Then, by (2.16)

$$\hat{g}(\xi) = \hat{f}(\xi)\widehat{K}_{-1}(\xi) = m(\xi)\hat{f}(\xi) + m(\xi)(-i \operatorname{sgn}(\xi))\widehat{f'}(\xi).$$

Using again the boundedness of the Hilbert transform together with Lemma 2.4, we get that $\|f\|_{W^1_{H^1_w}} = \|g\|_{H^1_w} \le C\|f\|_{H^1_w} + C\|f'\|_{H^1_w}$.

□

Also, we require the next result to prove Theorem 1.2.

Lemma 4.2. *Let ν be as in* (1.2) *and $w \in A_1$. Then*

$$\|\nu(f)\|_{1,w} \simeq \|f'\|_{H^1_w}, \quad f \in \mathcal{S}_0(\mathbb{R}).$$

Proof. It suffices to show that $\|\mu(f)\|_{1,w} \simeq \|f\|_{H^1_w}$ for $f \in \mathcal{S}_0(\mathbb{R})$. We recall a Littlewood-Paley function g_0 defined as

$$g_0(f)(x) = \left(\int_0^\infty |(\partial/\partial x)u(x,t)|^2 t\, dt\right)^{1/2},$$

where $u(x,t)$ is the Poisson integral of f: $u(x,t) = P_t * f(x)$, $\hat{P}(\xi) = e^{-2\pi|\xi|}$.

Let $\hat{R}(\xi) = 2\pi i\xi e^{-2\pi|\xi|}$. We note that $R = P'$ and that $R \in \mathcal{B}$; the condition (3) of Definition 3.1 is obvious from the explicit forms:

$$P'(x) = \frac{-2x}{\pi(1+x^2)^2}, \quad P''(x) = \frac{2(3x^2-1)}{\pi(1+x^2)^3}.$$

We see that $g_0(f) = g_R(f)$ and, since $R \in \mathcal{B}$, by Theorem 3.2 with $p = 1$ we have $\|f\|_{H^1_w} \le C\|g_R(f)\|_{1,w}$ for $f \in \mathcal{S}_0(\mathbb{R})$ and $w \in A_1$ (see [6, 27] for the unweighted case). Also we have the pointwise relation $g_0(f) \le C\mu(f)$ for $f \in \mathcal{S}_0(\mathbb{R})$ (see [14, Theorem 5]). Combining results, we see that $\|f\|_{H^1_w} \le C\|\mu(f)\|_{1,w}$.

In proving the reverse inequality, we apply the pointwise equivalence between g_3^* and μ to get

$$\|\mu(f)\|_{1,w} \le C\|g_3^*(f_1)\|_{1,w} + C\|g_3^*(f_2)\|_{1,w},$$

where

$$g_\lambda^*(f)(x) = \left(\iint_{\mathbb{R}\times(0,\infty)} \left(\frac{t}{t+|x-y|}\right)^\lambda |\nabla u(y,t)|^2\, dy\, dt\right)^{1/2}$$

is another Littlewood-Paley function and $\hat{f}_1 = \hat{f}\chi_{[0,\infty)}$, $\hat{f}_2 = \hat{f}\chi_{(-\infty,0]}$ (see [14, Theorems 1, 2 and Remark 1] for the pointwise equivalence). It follows that

$$\|\mu(f)\|_{1,w} \le C\|g_3^*(f)\|_{1,w} + C\|g_3^*(Hf)\|_{1,w},$$

where H denotes the Hilbert transform; this can be seen by noting

$$\chi_{[0,\infty)}(\xi) = \frac{1}{2}\left(\chi_{(-\infty,\infty)}(\xi) + \operatorname{sgn}(\xi)\right), \quad \chi_{(-\infty,0]}(\xi) = \frac{1}{2}\left(\chi_{(-\infty,\infty)}(\xi) - \operatorname{sgn}(\xi)\right)$$

for $\xi \ne 0$. Then, we apply the $H^1_w - L^1_w$ boundedness with $w \in A_1$ of g_3^* due to [12] and the boundedness of H on H^1_w. Here we would like to recall the following. In [12] H^1_w norm is defined as $\|N(u)\|_{1,w}$ for $u(x,t) = f * P_t(x)$, where $N(u)$ denotes the non-tangential maximal function and we have

$$\|N(u)\|_{1,w} \le C\|f^*\|_{1,w},$$

which can be shown, for example, by applying the atomic decomposition for $f \in H^1_w$. □

To prove $\|\mu(f)\|_{1,w} \le C\|f\|_{H^1_w}$, alternatively, we can apply an argument similar to the one in the proof of Lemma 3.4, using an atomic decomposition for H^1_w and an estimate from (4.7) of [17]:

$$\left(\int_0^\infty |\psi_t(x-y) - \psi_t(x)|^2 \frac{dt}{t}\right)^{1/2} \le C\frac{|y|^{1/2}}{|x|^{3/2}} \quad \text{for } 2|y| < |x|,$$

where $\psi(x) = \chi_{[0,1]}(x) - \chi_{[-1,0]}(x)$ (see the proof of Theorem 4.5 of [17]).

Proof of Theorem 1.2. The proof is analogous to that of Theorem 1.1. Lemmas 4.1 and 4.2 imply that

$$\|J_1(g)\|_{H^1_w} + \|\nu(J_1(g))\|_{1,w} \simeq \|g\|_{H^1_w} \tag{4.1}$$

for $g \in \mathcal{S}_0(\mathbb{R})$. We prove (4.1) for all $g \in H^1_w$. Let $g \in H^1_w$ and take a sequence $\{g_k\}$ in $\mathcal{S}_0(\mathbb{R})$ such that $g_k \to g$, $J_1(g_k) \to J_1(g)$ in H^1_w and almost everywhere as $k \to \infty$. We note that, if x is fixed,

$$|J_1(g_k)(x+t) + J_1(g_k)(x-t) - 2J_1(g_k)(x)| \to |J_1(g)(x+t) + J_1(g)(x-t) - 2J_1(g)(x)|$$

for a.e. $t \in (0,\infty)$, which implies that

$$\nu(J_1(g))(x) \le \liminf_{k\to\infty} \nu(J_1(g_k))(x)$$

for every x by Fatou's lemma. By (4.1) established with $g \in \mathcal{S}_0(\mathbb{R}^n)$, for $g_k \in \mathcal{S}_0(\mathbb{R}^n)$ we have

$$\|\nu(J_1(g_k))\|_{1,w} \le C\|g_k\|_{H^1_w},$$

from which and Fatou's lemma, it follows that

$$\begin{aligned}\|\nu(J_1(g))\|_{1,w} &\le \liminf_{k\to\infty} \|\nu(J_1(g_k))\|_{1,w} \\ &\le C \liminf_{k\to\infty} \|g_k\|_{H^1_w} \\ &\le C\|g\|_{H^1_w}.\end{aligned}$$

Therefore

$$\begin{aligned}\lim_{k\to\infty} \|\nu(J_1(g)) - \nu(J_1(g_k))\|_{1,w} &\le \lim_{k\to\infty} \|\nu(J_1(g-g_k))\|_{1,w} \\ &\le C \lim_{k\to\infty} \|g-g_k\|_{H^1_w} = 0.\end{aligned}$$

Thus, letting $k \to \infty$ in the relation

$$\|J_1(g_k)\|_{H^1_w} + \|\nu(J_1(g_k))\|_{1,w} \simeq \|g_k\|_{H^1_w}, \quad g_k \in \mathcal{S}_0(\mathbb{R}),$$

which is already shown, we have (4.1) for $g \in H^1_w$.

Thus, to complete the proof of Theorem 1.2, it suffices to prove that if $f \in H^1_w$ and $\nu(f) \in L^1_w$, then $f \in W^1_{H^1_w}$. Suppose that $f \in H^1_w$ and $\nu(f) \in L^1_w$. Let $f^{(\epsilon)}(x) = \varphi_\epsilon * f(x)$ and $g^{(\epsilon)}(x) = J_{-1}(\varphi_\epsilon) * f(x)$. Then $g^{(\epsilon)} \in H^1_w$ and $f^{(\epsilon)} = J_1(g^{(\epsilon)})$ (see Lemma 2.5). The relation (4.1) implies

$$\|f^{(\epsilon)}\|_{H^1_w} + \|\nu(f^{(\epsilon)})\|_{1,w} \simeq \|g^{(\epsilon)}\|_{H^1_w}. \tag{4.2}$$

We see that

$$\sup_{\epsilon>0} \|f^{(\epsilon)}\|_{H^1_w} \le C\|f^*\|_{1,M(w)} \le C\|f^*\|_{1,w} = C\|f\|_{H^1_w}, \tag{4.3}$$

since $w \in A_1$. By Minkowski's inequality,

$$\begin{aligned}\nu(f^{(\epsilon)})(x) &= \left(\int_0^\infty |\varphi_\epsilon * f(x+t) + \varphi_\epsilon * f(x-t) - 2\varphi_\epsilon * f(x)|^2 \, \frac{dt}{t^3}\right)^{1/2} \\ &\le \int_{\mathbb{R}} \varphi_\epsilon(y) \left(\int_0^\infty |f(x+t-y) + f(x-t-y) - 2f(x-y)|^2 \, \frac{dt}{t^3}\right)^{1/2} dy \\ &= \int_{\mathbb{R}} \varphi_\epsilon(y)\nu(f)(x-y)\, dy.\end{aligned}$$

Thus

$$\sup_{\epsilon>0} \|\nu(f^{(\epsilon)})\|_{1,w} \le C\|\nu(f)\|_{1,M(w)} \le C\|\nu(f)\|_{1,w}. \tag{4.4}$$

Consequently, it follows that $\sup_{\epsilon>0} \|g^{(\epsilon)}\|_{H^1_w} < \infty$ from (4.2), (4.3) and (4.4).

Applying Lemma 2.6, we choose a sequence $\{g^{(\epsilon_k)}\}$ in H^1_w with $\epsilon_k \to 0$ and $g \in H^1_w$ such that

$$\int_{\mathbb{R}} g^{(\epsilon_k)}(x)v(x)\,dx \to \int_{\mathbb{R}} g(x)v(x)\,dx \quad \text{as } k \to \infty \text{ for } v \in \mathcal{S}(\mathbb{R}).$$

Further, $\{f^{(\epsilon_k)}\}$ converges to f in L^1_w. Thus, (2.27) applies and $f = J_1(g)$, so $f \in W^1_{H^1_w}(\mathbb{R})$. □

Remark 4.3. The function of Marcinkiewicz $\mu(f)$ is generalized. Let $\mu_\beta(f) = g_{\eta^{(\beta)}}(f)$, $\beta > 0$, where

$$\eta^{(\beta)}(x) = \beta|1-|x||^{\beta-1}\operatorname{sgn}(x)\chi_{(-1,1)}(x).$$

Then μ_β generalizes μ in the sense that $\mu_1 = \mu$. See [14] for properties of μ_β.

5. Proofs of results in Remarks 1.4 and 1.5

Here we give proofs of Remarks 1.4 and 1.5 for completeness.

Proof of Remark 1.4. We prove that if $0 < \alpha < n/2$, $1 \le p \le 2$ and $S_{\psi^{(\alpha)}}$ is bounded on $L^p(\mathbb{R}^n)$, then $p \ge 2n/(n+2\alpha)$. Let $f \in \mathcal{S}_0(\mathbb{R}^n)$, $f \ne 0$. We estimate $S_{\psi^{(\alpha)}}(f)$ as

$$\begin{aligned}
2^{\alpha+(n+1)/2}S_{\psi^{(\alpha)}}(f)(x) &\ge \left(\int_1^2\int_{B(x,1)} |I_\alpha(f)(z) - \Phi_t * I_\alpha(f)(z)|^2\,dz\,dt\right)^{1/2} \\
&\ge \left(\int_1^2\int_{B(x,1)} |I_\alpha(f)(z)|^2\,dz\,dt\right)^{1/2} - \left(\int_1^2\int_{B(x,1)} |\Phi_t * I_\alpha(f)(z)|^2\,dz\,dt\right)^{1/2} \\
&= \left(\int_{B(x,1)} |I_\alpha(f)(z)|^2\,dz\right)^{1/2} - \left(\int_1^2\int_{B(x,1)} |\Phi_t * I_\alpha(f)(z)|^2\,dz\,dt\right)^{1/2}.
\end{aligned}$$

We easily see that

$$\int_1^2\int_{B(x,1)} |\Phi_t * I_\alpha(f)(z)|^2\,dz\,dt \le C\|\Phi\|_\infty^2 \left(\chi_{B(0,C_1)} * |I_\alpha(f)|(x)\right)^2.$$

Thus

$$\left(\int_{B(x,1)} |I_\alpha(f)(z)|^2\,dz\right)^{1/2} \le CS_{\psi^{(\alpha)}}(f)(x) + C\chi_{B(0,C_1)} * |I_\alpha(f)|(x). \tag{5.1}$$

On the other hand we will show that

$$\left(\int_{\mathbb{R}^n} |I_\alpha(f)(x)|^2\,dx\right)^{p/2} \le C\int_{\mathbb{R}^n}\left(\int_{B(x,1)} |I_\alpha(f)(z)|^2\,dz\right)^{p/2} dx. \tag{5.2}$$

To see this, we consider a covering of $\mathbb{R}^n$: $\cup_{j=1}^{\infty} B(c_j, 1) = \mathbb{R}^n$. We assume that there exists $\tau > 0$ such that $\cup_{j=1}^{\infty} B(x_j(y), 1) = \mathbb{R}^n$ for all $y \in B(0, \tau)$ with $x_j(y) = c_j + y$ and $B(c_j, \tau) \cap B(c_k, \tau) = \emptyset$ if $j \neq k$. Then, since $p/2 \leq 1$, we have

$$\left(\int_{\mathbb{R}^n} |I_\alpha(f)(x)|^2\, dx\right)^{p/2} \leq \sum_{j=1}^{\infty} \left(\int_{B(x_j(y),1)} |I_\alpha(f)(x)|^2\, dx\right)^{p/2}$$

for all $y \in B(0, \tau)$. Thus

$$\begin{aligned}
\left(\int_{\mathbb{R}^n} |I_\alpha(f)(x)|^2\, dx\right)^{p/2} &\leq \inf_{y\in B(0,\tau)} \sum_{j=1}^{\infty} \left(\int_{B(x_j(y),1)} |I_\alpha(f)(x)|^2\, dx\right)^{p/2} \\
&\leq C_\tau \int_{B(0,\tau)} \sum_{j=1}^{\infty} \left(\int_{B(x_j(y),1)} |I_\alpha(f)(x)|^2\, dx\right)^{p/2} dy \\
&= C_\tau \sum_{j=1}^{\infty} \int_{B(0,\tau)} \left(\int_{B(x_j(y),1)} |I_\alpha(f)(x)|^2\, dx\right)^{p/2} dy \\
&= C_\tau \sum_{j=1}^{\infty} \int_{B(c_j,\tau)} \left(\int_{B(y,1)} |I_\alpha(f)(x)|^2\, dx\right)^{p/2} dy \\
&\leq C_\tau \int_{\mathbb{R}^n} \left(\int_{B(y,1)} |I_\alpha(f)(x)|^2\, dx\right)^{p/2} dy,
\end{aligned}$$

which proves (5.2). From (5.1) and (5.2), it follows that

$$\|I_\alpha(f)\|_2 \leq C\|S_{\psi^{(\alpha)}}(f)\|_p + C\|\chi_{B(0,C_1)} * |I_\alpha(f)|\|_p.$$

Thus if $\|S_{\psi^{(\alpha)}}(f)\|_p \leq C\|f\|_p$, since $\|\chi_{B(0,C_1)} * |I_\alpha(f)|\|_p \leq C\|I_\alpha(f)\|_p$, we have

$$\|I_\alpha(f)\|_2 \leq C\|f\|_p + C\|I_\alpha(f)\|_p.$$

From this with f_ρ in place of f, by homogeneity, it readily follows that

$$\rho^{\alpha-n/2} \leq C\rho^{-n+n/p} + C\rho^{\alpha+n(1/p-1)} \leq C\rho^{-n+n/p}$$

for all $\rho \in (0, 1)$, which implies that $p \geq 2n/(n + 2\alpha)$ as claimed. □

Proof of Remark 1.5. Suppose that $0 < \alpha < n/2$, $1 < p < 2n/(2\alpha + n)$. We see that $S_{\psi^{(\alpha)}}$ is not bounded from H^1 to L^1; otherwise $S_{\psi^{(\alpha)}}$ would be bounded on L^p by interpolation between the $H^1 - L^1$ and L^2 boundedness of $S_{\psi^{(\alpha)}}$ (for the L^2 boundedness see the proof of Theorem 2.3 in Section 2). This contradicts Remark 1.4.

However, if U_α was bounded from $W^\alpha_{H^1}$ to L^1, then $\|U_\alpha(J_\alpha(g))\|_1 \leq C\|g\|_{H^1}$ for $g \in \mathcal{S}_0(\mathbb{R}^n)$. Since $U_\alpha(f) = S_{\psi^{(\alpha)}}(I_{-\alpha}(f))$, it follows that $\|S_{\psi^{(\alpha)}}(g)\|_1 \leq C\|I_\alpha J_{-\alpha}(g)\|_{H^1}$. Thus by Lemma 2.4 with $w = 1$

$$\|S_{\psi^{(\alpha)}}(g)\|_1 \leq C\|I_\alpha(g)\|_{H^1} + C\|g\|_{H^1}, \quad g \in \mathcal{S}_0(\mathbb{R}^n). \tag{5.3}$$

Since $S_{\psi^{(\alpha)}}(g_\rho) = (S_{\psi^{(\alpha)}}(g))_\rho$, $\|I_\alpha(g_\rho)\|_{H^1} = \rho^\alpha\|I_\alpha(g)\|_{H^1}$ and $\|g_\rho\|_{H^1} = \|g\|_{H^1}$, by (5.3) with g_ρ in place of g we have

$$\|S_{\psi^{(\alpha)}}(g)\|_1 \leq C\rho^\alpha\|I_\alpha(g)\|_{H^1} + C\|g\|_{H^1}$$

for all $\rho > 0$. Thus, letting $\rho \to 0$, we see that $\|S_{\psi^{(\alpha)}}(g)\|_1 \le C\|g\|_{H^1}$, from which the $H^1 - L^1$ boundedness of $S_{\psi^{(\alpha)}}$ follows. This contradicts what we have already observed.

□

References

[1] R. Alabern, J. Mateu and J. Verdera, *A new characterization of Sobolev spaces on* $\mathbb{R}^n$, Math. Ann. **354** (2012), 589–626.

[2] A. P. Calderón and A. Torchinsky, *Parabolic maximal functions associated with a distribution*, Advances in Math. **16** (1975), 1-64.

[3] F. Dai, J. Liu, D. Yang and W. Yuan, *Littlewood-Paley characterizations of fractional Sobolev spaces via averages on balls*, arXiv: 1511.07598.

[4] L. C. Evans and R. F. Gariepy, *Measure Theory and Fine Properties of Functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1992.

[5] C. Fefferman, *Inequalities for strongly singular convolution operators*, Acta Math. **124** (1970), 9–36.

[6] C. Fefferman and E. M. Stein, *H^p spaces of several variables*, Acta Math. **129** (1972), 137–193.

[7] J. Garcia-Cuerva and J.L. Rubio de Francia, *Weighted Norm Inequalities and Related Topics*, North-Holland, Amsterdam, New York, Oxford, 1985.

[8] P. Hajłasz and Z. Liu, *A Marcinkiewicz integral type characterization of the Sobolev space*, Publ. Mat. **61** (2017), 83–104.

[9] Z. He, D. Yang and W. Yuan, *Littlewood-Paley characterizations of second order Sobolev spaces via averages on balls*, Canadian Math. Bull. **59** (2016), 104–118.

[10] J.-L. Journé, *Calderón-Zygmund Operators, Pseudo-Differential Operators and the Cauchy Integral of Calderón*, Lecture Notes in Math. vol. 994, Springer-Verlag, 1983.

[11] J. Marcinkiewicz, *Sur quelques intégrales du type de Dini*, Annales de la Société Polonaise **17** (1938), 42–50.

[12] B. Muckenhoupt and R. L. Wheeden, *Norm inequalities for the Littlewood-Paley function* g_λ^*, Trans. Amer. Math. Soc. **191** (1974), 95–111.

[13] B. Muckenhoupt and R. L. Wheeden, *On the dual of weighted H^1 of the half-space*, Studia Math. **63** (1978), 57–79.

[14] S. Sato, *Multiparameter Marcinkiewicz integrals and a resonance theorem*, Bull. Fac. Ed. Kanazawa Univ. Natur. Sci. **48** (1999), 1–21. (http://hdl.handle.net/2297/25017)

[15] S. Sato, *Littlewood-Paley operators and Sobolev spaces*, Illinois J. Math. **58** (2014), 1025-1039.

[16] S. Sato, *Square functions related to integral of Marcinkiewicz and Sobolev spaces*, Linear and Nonlinear Analysis **2** (2016), 237–252 (special issue on ISBFS 2015).

[17] S. Sato, *Littlewood-Paley equivalence and homogeneous Fourier multipliers*, Integr. Equ. Oper. Theory **87** (2017), 15–44.

[18] S. Sato, *Spherical square functions of Marcinkiewicz type with Riesz potentials*, Arch. Math. **108** (2017), 415–426.

[19] S. Sato, *Vector valued inequalities and Littlewood-Paley operators on Hardy spaces*, Hokkaido Math. J. (to appear), arXiv:1608.08059v2 [math.CA].

[20] S. Sato, *Characterization of parabolic Hardy spaces by Littlewood-Paley functions*, arXiv:1607.03645v2 [math.CA].

[21] S. Sato, F. Wang, D. Yang and W. Yuan, *Generalized Littlewood-Paley characterizations of fractional Sobolev spaces*, Commun. Contemp. Math. (to appear), https://doi.org/10.1142/S0219199717500778.

[22] E. M. Stein, *The characterization of functions arising as potentials*, Bull. Amer. Math. Soc. **67** (1961), 102–104.

[23] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press, 1970.

[24] E. M. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton Univ. Press, 1971.

[25] R. S. Strichartz, *H^p Sobolev spaces*, Colloq. Math. **60/61** (1990), 129–139.

[26] J. -O. Strömberg and A. Torchinsky, *Weighted Hardy Spaces*, Lecture Notes in Math. 1381, Springer-Verlag, Berlin Heidelberg New York London Paris Tokyo Hong Kong, 1989.

[27] A. Uchiyama, *Characterization of $H^p(\mathbb{R}^n)$ in terms of generalized Littlewood-Paley g-functions*, Studia Math. **81** (1985), 135–158.

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