A Note on a Homeomorphism Type of an Aspherical Homogeneous Space G/H

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Introduction

Let G be a connected simply connected Lie group and H a closed subgroup of G. In [3] V.V.Gorbacevič studied the structures of H of the compact aspherical homogeneous space G/H on which G acts irreducibly , i.e. no proper subgroup of G acts transitively on G/H.When G is a solvable group,the homeomorphism type of G/H (solvmanifold) is uniquely determined by its fundamental group([4] [6]).In [8] we studied the homeomorphism type of 4 dimensional compact aspherical homogeneous space.

In this note we shall consider the homeomorphism type of G/H where G is a semisimple Lie group and prove the following

Theorem. *let G be a connected simply connected semisimple Lie group without compact factors and H a closed subgroup of G such that G/H is compact and aspherical on which G acts irreducibly. Let H be satisfying $z(G) \subset H \subset z(G)T$, where $z(G)$ and T denote the center of G and the maximal triangular subgroup of G respectively. Then the fundamental group of G/H determines G/H uniquely up to a homeomorphism.*

In section 1,we shall list some basic facts about the structure of G and H ,and in section 2, we shall prove that G/H is a principal T^k bundle over $K\backslash S/\Gamma$, where S is a product $A_1 \times A_1 \times \cdots \times A_1$ (A_1 is the universal covering of $SL(2, R)$), K is a subgroup of S whose Lie algebra is maximal compact , and Γ is a torsion free cocompact discrete subgroup of S. In section 3, we shall prove that $K\backslash S/\Gamma$ is uniquely determined up to a homeomorphism by its fundamental group. We shall prove the Theorem in a final section 4.

In this note we shall use the following notations.

1. For a Lie group G, G° denotes the identity component of G.
2. The short exact sequence of groups

$$1 \longrightarrow H \longrightarrow G \longrightarrow K \longrightarrow 1$$

means an exact sequence , unless the contrary stated explicitely.

3. Let G be a group and H its subgroup. Then $N_G(H)$ denotes the normalizer of H.

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4. T^k denotes a k-dimensional toral group.
5. $[G, G]$ denores the derived subgroup of G.
6. For a finitely generated abelian group A, rkA denotes the rank of A.

1. Preliminaries

Let G and H be as in Theorem. We have the follwing several facts.

Fact 1 ([3]). G is a product of $A's$, where A is the universal covering of $SL(2, R)$, i.e. $G = A_1 \times \cdots \times A_r$ $(A_i = A)$.

Facts 2 ([3]). H° is solvable and contained in $[T, T]$, where T is a connected maximal triangular subgroup of G. Note $T = T_1 \times \cdots \times T_r (T_i$ is a maximal connected triangular subgroup of $A_i)$.

Fact 3 ([3]). Let $\lambda_i : G \to A_i$ be the projection. Then $\dim\lambda_i(H) \leq 1$.

Fact 4 ([3]). $N_G(H^\circ)$ contains some maximal connected triangular subgroup $T = T_1 \times \cdots \times T_r$

The normal subgroup of T is of the form

$$(\times_{i \in I_1} T_i) \times (\times_{j \in I_2} [T_j, T_j]),$$

where $I_1 \cup I_2 \subset \{1, 2, \cdots, r\}$ and $I_1 \cap I_2 = \phi$. As H° being a normal subgroup of T, we have

$$H^\circ = \times_{j \in I_2} [T_j, T_j]$$

By renumbering the indics, if necessary, we may assume

$$H^\circ = \times_{j \in \{s+1, \cdots, r\}} [T_j, T_j].$$

Then we have

$$N_G(H^\circ)^\circ = A_1 \times \cdots \times A_s \times T_{s+1} \times \cdots \times T_r.$$

Fact 5. Let T be a connected maximal triangular subgroup of A. Then $N_A([T, T]) = ZT$, where Z is the center of A. This fact follows from the structure of A.

Fact 6. $N_G(H^\circ) = A_1 \times \cdots \times A_s \times Z_{s+1}T_{s+1} \times \cdots \times Z_r T_r$, where Z_i is the center of A_i. This follows from facts 4 and 5.

Fact 7 ([10]). Let G be a connected semisimple Lie group without compact factors and Γ a subgroup of G which is discrete and G/Γ is compact. Then $N_G(\Gamma)$ is also discrete , $G/N_G(\Gamma)$ is compact and $N_G(\Gamma)/\Gamma$ is finite.

Fact 8 ([11]). Let S be a connected semisimple Lie group without compact factors. If Γ is a discrete and cocompact subgroup of $S \times R^n$, then the image of Γ by the projection $S \times R^n \to S$ is also discrete and cocompact.

2. A principal torus fibration

Let G and H be as in Theorem. Then we may assume

$$G = A_1 \times \cdots \times A_r, \quad H^\circ = [T_{s+1}, T_{s+1}] \times \cdots \times [T_r, T_r].$$

Let$S_1 = A_1 \times \cdots \times A_s$, $S_2 = A_{s+1} \times \cdots \times A_r$ and $p_1 : G \to S_1$ the projection. Then we have the following

Proposition 1. $p_1(H)$ *is a discrete and cocompact subgroup* S_1.

PROOF. Define $H_1 = HN_G(H^\circ)^\circ$ and $\Gamma = H \cap H_1^\circ/(H \cap H_1^\circ)^\circ$. Then Γ is a discrete and cocompact subgroup of $G_1 = H_1^\circ/(H \cap H_1^\circ)^\circ$. Clearly G_1 is $S_1 \times R^{r-s}$. It follows from fact 8 that $q(\Gamma)$ is a discrete and cocompact subgroup of S_1, where q is the projection $G_1 \to S_1$. Note that $q(\Gamma) = p_1(H \cap H_1^\circ)$. It is easy to see that $p_1(H) \subset N_{S_1}(p_1(H \cap H_1^\circ))$. Fact 7 implies that $p_1(H)$ is discrete and cocompact in S_1. This completes the proof of Proposition 1.

Q.E.D.

We have the following

Proposition 2. *We have homeomorphisms*

$$\begin{aligned} G/H &\cong (G/H^\circ)/(H/H^\circ) \\ &\cong ((S_2/H^\circ) \times S_1)/(H/H^\circ) \\ &\cong ((S_2/H \cap S_2) \times S_1)/p_1(H). \end{aligned}$$

PROOF. Define a map $h_1 : G/H \to (G/H^\circ)/(H/H^\circ)$ by $h_1(gH) = [gH^\circ]$, where $[gH^\circ]$ is the orbit of gH° under the action of H/H° on G/H° defined by $(hH^\circ)(gH^\circ) = gh^{-1}H^\circ$.This is well defined. In fact, we have

$$\begin{aligned} gH = g'H &\Rightarrow g^{-1}g' \in H \\ &\Rightarrow g' = gh^{-1} \quad (h \in H) \\ &\Rightarrow g'H^\circ = gh^{-1}H^\circ = (hH^\circ)(gH^\circ). \end{aligned}$$

Clearly h_1 is a homeomorphism.

Define an action of H/H° on $S_2/H^\circ \times S_1$ by

$$(hH^\circ)(g_2H^\circ, g_1) = (g_2h_2^{-1}H^\circ, g_1h_1^{-1}),$$

where $h = (h_1, h_2) \in H \subset G = S_1 \times S_2$.

Define a map $h'_2 : G/H^\circ \to S_2/H^\circ \times S_1$ by $h'_2(gH^\circ) = (g_2H^\circ, g_1)$, where $g = (g_1, g_2)$. Clearly h'_2 is a homeomorphism and H/H°-equivariant.

Let $h'_3 : S_2/H^\circ \times S_1 \to S_2/H \cap S_2 \times S_1$ be the map projection $\times$ identity. Note that $p_1(H) = H/S_2 \cap H$. Define an action of $p_1(H)$ on $S_2/H^\circ \times S_1$ by

$$h(S_2 \cap H)(g_2(H \cap S_2), g_1) = (g_2h_2^{-1}(H \cap S_2), g_1h_1^{-1}),$$

where $h = (h_1, h_2) \in H$.

The map h'_3 is an equivariant map between H/H°-manifold $S_2/H^\circ \times S_1$ and $p_1(H)$-manifold $S_2/(H \cap S_2) \times S_1$. In fact, we have

$$\begin{aligned} h'_3((hH^\circ)(g_2H^\circ, g_1)) &= h'_3(g_2h_2^{-1}(H \cap S_2), g_1h_1^{-1}) \\ &= h(S_2 \cap H)(h'_3(g_2H^\circ, g_1)), \end{aligned}$$

where $h = (h_1, h_2) \in H$.

Hence h'_3 induces a homeomorphism

$$h_3 : (S_2/H^\circ \times S_1)/(H/H^\circ) \cong (S_2/(H \cap S_2) \times S_1)/p_1(H).$$

This completes the proof of Proposition 2.

Q.E.D.

In the following sections we shall assume $z(G) \subset H \subset z(G)T$ and prove the following

Theorem 1. *G/H admits a free $T^{s+2(r-s)}$-action.*

PROOF. We have gotten the homeomorphism

$$G/H \cong (S_2/S_2 \cap H \times S_1)/p_1(H).$$

Put $H_2 = H \cap S_2$.

Let K_i be the subgroup of S_i whose Lie algebra is maximal compact for $i = 1, 2$. We show that G/H admits a free action of $K_1/z(p_1(H))$. In fact , define an action of K_1 on G/H by

$$k[g_2H_2, g_1] = [g_2H_2, kg_1],$$

where $[,]$ denotes an equivalence class of $(S_2/S_2 \cap H \times S_1)$ under the action of $p_1(H)$. We first prove this action induces an effective action of $K_1/z(p_1(H))$. Let $[g_2H_2, g_1] = k[g_2H_2, g_1]$ for any $g_1 \in S_1$ and any $g_2 \in S_2$. Then there exists $h_1 \in p_1(H), h = (h_1, h_2) \in H$ satisfying $(g_2h_2^{-1}H_2, g_1h_1^{-1}) = (g_2H_2, kg_1)$. So we obtain $h_2 \in H_2$ and $g_1h_1^{-1}g_1^{-1} = k$. This induces $h_1 \in z(S_1)$ and $k \in z(p_1(H))$. This proves an effectiveness of the action by $K_1/z(p_1(H))$.

This action induces a free action of $K_1/z(p_1(H))$. In fact,

$$\begin{aligned} [g_2H_2, kg_1] &= [g_2H_2, g_1] \\ \Rightarrow (g_2H_2, kg_1) &= (hH_2)(g_2H_2, g_1) \\ &= (g_2h_2^{-1}H_2, g_1h_1^{-1}) \end{aligned}$$

$$\Rightarrow h_2 \in H_2,\ k = g_1h_1^{-1}g_1^{-1} \in K_1 \Rightarrow h_1 \in p_1(H) \cap K_1 = z(S_1)$$
$$\Rightarrow k \in z(p_1(H)),$$

where $h = (h_1, h_2)$.

Next define an action of K_2 on S_2/H_2 by

$$k(g_2H_2) = (kg_2)H_2.$$

The restriction of this action to $H \cap K_2$ is trivial, because $H \cap K_2 \subset z(S_2)$. Moreover we can define an action of T_2 (a maximal connected triangular subgroup of S_2 such that $K_2T_2 = S_2$) on S_2/H_2 by

$$n(g_2H_2) = (g_2n^{-1})H_2.$$

This is well defined. In fact,

$$g_2^{-1}g_2' \in H \cap S_2 \Rightarrow n(g_2^{-1}g_2')n^{-1} \in H \cap S_2$$

because $H \cap S_2 \subset z(S_2)T_2$. Naturally this action induces an action of $T_2/T_2 \cap H$.

Now we define the action of $T^{2(r-s)}$ on $S_2/H_2 \times S_1$ where $T^{2(r-s)} = K_2/H \cap K_2 \times T_2/T_2 \cap H$, by

$$(\overline{n_1}, \overline{n_2})(g_2H_2, g_1) = (n_1g_2n_2^{-1}H_2, g_1),$$

where $\overline{n_1} \in K_2/H \cap K_2$ and $\overline{n_2} \in T_2/T_2 \cap H$. This is compatible with the action of $p_1(H)$. In fact, note that $H^\circ \subset [T,T]$, $H \subset z(G)T = z(S_1)T_1 \times z(S_2)T_2$. Any element $h = (h_1, h_2) \in H$ is written as a product $h_2 = z_2t_2$, where $z_2 \in z(S_2)$ and $t_2 \in T_2$.

Then we have

$$\begin{aligned} &(\overline{n_1}, \overline{n_2})\{hH_2(g_2H_2, g_1)\} \\ &= (\overline{n_1}, \overline{n_2})\{(g_2h_2^{-1}H_2, g_1h_1^{-1})\} \\ &= (n_1g_2h_2^{-1}n_2^{-1}H_2, g_1h_1^{-1}) \\ &= (n_1g_2z_2^{-1}t_2^{-1}n_2^{-1}H_2, g_1h_1^{-1}) \\ &= (n_1g_2t_2^{-1}n_2^{-1}z_2^{-1}H_2, g_1h_1^{-1}) \end{aligned}$$

Since $[T_2, T_2] \subset H_2$, we have

$$ht_2n_2 = n_2t_2,\ h \in H_2 \quad \text{for every } t_2, n_2 \in T_2.$$

We have then

$$\begin{aligned}&(n_1g_2t_2^{-1}n_2^{-1}z_2^{-1}H_2, g_1h_1^{-1})\\&= (n_1g_2n_2^{-1}t_2^{-1}z_2^{-1}H_2, g_1h_1^{-1})\\&= hH_2\{(\overline{n_1}, \overline{n_2})(g_2H_2, g_1)\}.\end{aligned}$$

This implies the compatibility.

Next we shall prove the freeness of the action. Assume $(\overline{n_1}, \overline{n_2})(g_2H_2, g_1) = (g_2H_2, g_1)$. Then we have

$$n_1g_2n_2^{-1} = g_2h,\ (h \in H_2)$$

Since $g_2 = k_2t_2, h = z_2s_2\,(k_2 \in K_2,\ t_2 \in T_2,\ s_2 \in T_2,\ z_2 \in z(S_2))$, we have

$$\begin{aligned}n_1k_2t_2 &= k_2t_2z_2s_2n_2\\ \Rightarrow z_2^{-1}n_1k_2 &= k_2t_2s_2n_2t_2^{-1}\\ \Rightarrow k_2^{-1}z_2^{-1}n_1k_2 &= t_2s_2n_2t_2^{-1}.\end{aligned}$$

All elements of left hand side being in K_2, we have

$$\begin{aligned}z_2^{-1}n_1 &= t_2s_2t_2^{-1}n_2h', \quad (h' \in H_2^\circ)\\ &= s_2n_2h'', \quad (h'' \in H_2^\circ)\end{aligned}$$

Hence we have $z_2^{-1}n_1 = s_2n_2h''$ in $K_2 \cap T_2$, which implies $z_2^{-1}n_1 \in z(S_2)$. Thus we have shown that $\overline{n_1} = 1$, which induces $\overline{n_2} = 1$. This completes the proof of Theorem 1.

Q.E.D.

It follows from the Theorem 1 that we have a principal bundle;

$$T^{s+2(r-s)} \to G/H \to M.$$

We have the following

Theorem 2. $M \cong K_1 \backslash S_1 / p_1(H)$.

PROOF. We have proved that

$$G/H \cong (S_2/H_2 \times S_1)/p_1(H),$$

and the action of $T^{2(r-s)}$ on G/H is induced by the action of $T^{2(r-s)}$ on S_2/H_2. Since $\dim S_2/H_2 = 2(r-s)$, this action is transitive. Thus we have

$$\begin{aligned}(G/H)/(T^s \times T^{2(r-s)}) &\cong ((G/H)/T^{2(r-s)})/T^s\\ &\cong (S_1/p_1(H))/T^s.\end{aligned}$$

Moreover from the definition of the action of T^s on $S_1/p_1(H)$, it follows that

$$(S_1/p_1(H))/T^s \cong K_1\backslash S_1/p_1(H).$$

This proves the Theorem 2.

Q.E.D.

Thus we have the fiber bundle;

$$T^{s+2(r-s)} \to G/H \to K_1\backslash S_1/p_1(H).$$

Hence we have the following exact sequence of the fundamental groups;

$$1 \to Z^{s+2(r-s)} \xrightarrow{i_*} \pi_1(G/H) \xrightarrow{p_*} \pi_1(M) \to 1.$$

It follows from a result in[1] that the image of i_* is contained in the center $z(\pi_1(G/H))$. We have the following

Proposition 3. rank $z(\pi_1(G/H)) = s + 2(r-s)$.

PROOF. We know the following facts (see [3]);
(1) $H^\circ = 1 \times [T_2, T_2]$
(2) $N_G(H^\circ)^\circ = S_1 \times T_2$
(3) $N_G(H^\circ) = S_1 \times Z_2T_2$
(4) $H \subset N_G(H^\circ)$.

Consider the following exact sequence;

$$\begin{array}{ccccccccc} 1 & \longrightarrow & H_2/H_2 \cap H^\circ & \longrightarrow & H/H^\circ & \xrightarrow{p_1} & p_1(H/H^\circ) & \longrightarrow & 1 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & (H_2/H_2 \cap H^\circ) \cap z(H/H^\circ) & \longrightarrow & z(H/H^\circ) & \longrightarrow & p_1(z(H/H^\circ)) & \longrightarrow & 1 \end{array}$$

Snce $(H_2/H_2 \cap H^\circ) \cap z(H/H^\circ) \subset z(H_2/H_2 \cap H^\circ)$ and $p_1(z(H/H^\circ)) \subset z(p_1(H/H^\circ))$, we have

$$\begin{aligned} \operatorname{rank} z(H/H^\circ) &= \operatorname{rank} p_1(z(H/H^\circ)) + \operatorname{rank}((H_2/H_2 \cap H^\circ) \cap z(H/H^\circ)) \\ &\leq s + 2(r-s). \end{aligned}$$

This completes the proof.

Q.E.D.

Since $K_1\backslash S_1/p_1(H)$ is a closed aspherical manifold, its fundamental group is torsion free. Hence we have $i_*(Z^{s+2(r-s)}) = z(\pi_1(G/H))$.

Let H_1, H_2 be the subgroups of G as H in Theorem. Assume $\pi_1(G/H_1) \cong \pi_1(G/H_2)$. We have the following

Proposition 4. dim $H_1 =$ dim H_2.

PROOF. By the assumption, we have $\mathrm{cd}(H_1/H_1^\circ) = \mathrm{cd}(H_2/H_2^\circ)$, where cd( ) is the cohomological dimension of a group. This means that $\dim G/H_1^\circ = \dim G/H_2^\circ$. Then $\dim G/H_1 = \dim G/H_2$. This completes the proof.

Q.E.D.

It follows from the arguments in Fact 4 that

$$H_i^\circ = \times_{j \in I_i}[T_j, T_j] \quad (i = 1, 2) \text{ and } \sharp I_1 = \sharp I_2.$$

Then we may assume that $H_1^\circ = H_2^\circ$, and that the decomposition of G into the product $S_1 \times S_2$ in the above argument for H_2 is the same one for H_1. Thus we have the following commutative diagram;

$$(*) \quad \begin{array}{ccccccccc} 1 & \rightarrow & Z^{s+2(r-s)} & \rightarrow & \pi_1(G/H_1) & \rightarrow & \pi_1(K_1\backslash S_1/p_1(H_1)) & \rightarrow 1 \\ & & \gamma\downarrow & & \alpha\downarrow & & \beta\downarrow & \\ 1 & \rightarrow & Z^{s+2(r-s)} & \rightarrow & \pi_1(G/H_2) & \rightarrow & \pi_1(K_1\backslash S_1/p_1(H_2)) & \rightarrow 1, \end{array}$$

where α is the given isomorphism, γ is the restriction (note that $Z^{s+2(r-s)} = z(\pi_1(G/H_i))$ and β is the isomorphism induced by α. We have the following

Proposition 5. $\pi_1(K_1\backslash S_1/p_1(H_1)) \cong \pi_1(K_1\backslash S_1/p_1(H_2))$

3. The structure of $K\backslash S/\Gamma$

In this section, we shall prove the following

Theorem 3. *Let G be a product $A_1 \times \cdots \times A_1$, K a subgroup of G whose Lie algebra is maximal compact and Γ a torsion free cocompact discrete subgroup of G. Then $K\backslash S/\Gamma$ is uniquely determined, up to a homeomorphism, by its fundamental group.*

This Theorem is true when dim $K\backslash S/\Gamma \neq 4$. In fact, if dim $K\backslash S/\Gamma = 2$, this is a classical result. Assume dim $K\backslash S/\Gamma \geq 5$. Then $K\backslash S/\Gamma$ supports the structure of nonpositively curved Riemannian manifold. By a result in [2] the homeomorphism type of such a manifold is uniquely determined by its fundamental group.

Now we suppose that $G = A_1 \times A_1$. If Γ is irreducible, it follows from results in [7] that the homeomorphism type of $K\backslash S/\Gamma$ is uniquely determined by its fundamental group. Then we may assume that Γ is not irreducible, i.e. the image of Γ by any projection $G \to A_1$ is cocompact discrete, and hence the intersection $\Gamma \cap A_1$is also cocompact discrete.

We denote $G = A_1^{(1)} \times A_1^{(2)}$ where $A_1^{(i)} = A_1$, $\Gamma_i = \Gamma \cap A_1^{(i)}$ and $\Gamma^i = p_i(\Gamma)$ where $p_i : G \to A_1^{(i)}$ is the projection. We put $\overline{U} = U/z(U)$ for any group U. Then we have

Proposition 6. 1. $\overline{\Gamma_i} = \overline{\Gamma} \cap A_1^{(i)}$.
2. $\overline{\Gamma^i} = \overline{p_i}(\overline{\Gamma})$, *where* $\overline{p_i}$ *is the natural map induced by* p_i.
3. $\overline{\Gamma_1} \times \overline{\Gamma_2} \subset \overline{\Gamma^1} \times \overline{\Gamma^2}$
4. *We have the fiber bundles*;

$$K_1\backslash A_1^{(i)}/\Gamma_i \to K\backslash G/\Gamma \to K_1\backslash A_1^{(3-i)}/\Gamma^{3-i} \qquad i = 1, 2,$$

where K_1 *is a subgroup of* A_1 *whose Lie algebra is maximal compact.*
We note that $K_1\backslash A_1^{(i)}/\Gamma_i \cong \overline{K_1}\backslash\overline{A_1^{(i)}}/\overline{\Gamma_i}$ *and* $K_1\backslash A_1^{(i)}/\Gamma^i \cong \overline{K_1}\backslash\overline{A_1^{(i)}}/\overline{\Gamma^i}$.

PROOF. We omit the proof since it is not difficult to prove this Proposition.

Q.E.D.

Since $K\backslash G/\Gamma$ is aspherical, the fiber and the base are also aspherical and hence $\overline{\Gamma_i}$, $\overline{\Gamma^i}$ are torsion free. Let Γ' be an another cocompact discrete subgroup of G which is isomorphic to Γ and θ the isomorphism. In the following the natural isomorphism $\overline{\Gamma'} \to \overline{\Gamma}$ induced by θ is also denoted by θ. It follows from a result in[7] (Theorem 4.1 in[7]) that Γ' is also not irreducible. $\overline{\Gamma'}$ and $\overline{\Gamma'^i}$ are also torsion free.

We have the following exact sequence;

$$1 \longrightarrow \overline{\Gamma_i} \longrightarrow \overline{\Gamma} \longrightarrow \overline{\Gamma^{(3-i)}} \longrightarrow 1$$

and

$$1 \longrightarrow \overline{\Gamma'_i} \longrightarrow \overline{\Gamma'} \longrightarrow \overline{\Gamma'^{(3-i)}} \longrightarrow 1.$$

Put $\overline{\Gamma''_i} = \theta(\overline{\Gamma'_i})$. We have the following commutative diagram;

$$
\begin{array}{cccccccccc}
 & & & 1 & & 1 & & 1 & & \\
 & & & \downarrow & & \downarrow & & \downarrow & & \\
 & 1 & \rightarrow & \overline{\Gamma_i} \cap \overline{\Gamma''_j} & \rightarrow & \overline{\Gamma''_j} & \rightarrow & \overline{p}_{3-i}(\overline{\Gamma''_j}) & \rightarrow & 1 \\
 & & & \downarrow & & \downarrow & & \downarrow & & \\
(*) & 1 & \rightarrow & \overline{\Gamma_i} & \rightarrow & \overline{\Gamma} & \rightarrow & \overline{\Gamma^{(3-i)}} & \rightarrow & 1 \\
 & & & \downarrow & & \downarrow & & \downarrow & & \\
 & 1 & \rightarrow & \overline{\Gamma_i}/\overline{\Gamma_i} \cap \overline{\Gamma''_j} & \rightarrow & \overline{\Gamma}/\overline{\Gamma''_j} & \rightarrow & \overline{\Gamma^{(3-i)}}/\overline{p}_{3-i}(\overline{\Gamma''_j}) & \rightarrow & 1 \\
 & & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & 1 & & 1 & & 1 & & .
\end{array}
$$

We have the following

Lemma 1. $\overline{\Gamma}/\overline{\Gamma''_j}$ *is torsion free.*

PROOF. This follows immediately from the following commutative diagram;

$$
\begin{array}{ccccccccc}
1 & \rightarrow & \overline{\Gamma'_j} & \rightarrow & \overline{\Gamma'} & \rightarrow & \overline{\Gamma'^{(3-i)}} & \rightarrow & 1 \\
 & & \tilde{\theta} \downarrow & & \theta \downarrow & & \bar{\theta} \downarrow & & \\
1 & \rightarrow & \overline{\Gamma''_j} & \rightarrow & \overline{\Gamma} & \rightarrow & \overline{\Gamma}/\overline{\Gamma''_j} & \rightarrow & 1
\end{array}
$$

where $\tilde{\theta}$ is the restriction of θ.

Q.E.D.

Since the surface group $\overline{\Gamma^{(3-i)}}$ has a trivial center, it contains no non-trivial normal finitely generated subgroup of infinite index,and hence $\overline{p}_{3-i}(\overline{\Gamma''_j})$ is trivial or a finite index in $\overline{\Gamma^{(3-i)}}$.

Suppose the $\overline{p}_2(\overline{\Gamma''_1})$ is trivial. Then we have $\overline{\Gamma_1} \cap \overline{\Gamma''_1} = \overline{\Gamma''_1}$. The group $\overline{\Gamma_1}/\overline{\Gamma_1} \cap \overline{\Gamma''_1}$ is also finite. It follows from Lemma 1 that this group is trivial. Thus we have the following commutative diagram;

$$
\begin{array}{cccccccccc}
 & 1 & \rightarrow & \overline{\Gamma'_1} & \rightarrow & \overline{\Gamma'} & \rightarrow & \overline{\Gamma'^2} & \rightarrow & 1 \\
(\sharp) & & & \tilde{\theta} \downarrow & & \theta \downarrow & & \bar{\theta} \downarrow & & \\
 & 1 & \rightarrow & \overline{\Gamma_1} & \rightarrow & \overline{\Gamma} & \rightarrow & \overline{\Gamma}/\overline{\Gamma_1} & \rightarrow & 1,
\end{array}
$$

where $\tilde{\theta}$ is the restriction of θ and the vertical maps are isomorphisms.

Note that manifolds $K\backslash G/\Gamma$ and $K\backslash G/\Gamma'$ are considered as 4 dimensional codimension 2 foliated manifolds with all leaves compact. The diagram ($\sharp$) shows that the isomorphism $\theta : \pi_1(K\backslash G/\Gamma') \to \pi_1(K\backslash G/\Gamma)$ restricts to an isomorphism $\tilde{\theta} : \pi_1(F') \to \pi_1(F)$, where F, F' are typical fibers. Then the argument of the proof of Theorem(5.3) and Theorem (4.1) in [9] shows that $K\backslash G/\Gamma$ and $K\backslash G/\Gamma'$ are homeomorphic.

Next we suppose that $\bar{p}_2(\overline{\Gamma''_1}) \neq 1$. Since $\overline{\Gamma^2}$ is a surface group of genus > 1, $\bar{p}_2(\overline{\Gamma''_1})$ is a subgroup of $\overline{\Gamma^2}$ of finite index and $\overline{\Gamma_1} \cap \overline{\Gamma''_1}$ is trivial. Then the diagram ($*$) is now as follows;

$$\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 1 & \to & \overline{\Gamma''_2} & \overset{\bar{p}_2}{\to} & \bar{p}_2(\overline{\Gamma''_2}) & \to & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & \overline{\Gamma_1} & \overset{i}{\to} & \overline{\Gamma} & \overset{\bar{p}_2}{\to} & \overline{\Gamma^2} & \to & 1 \\
 & & \tilde{q}\downarrow & & q\downarrow & & \bar{q}\downarrow & & \\
1 & \to & \overline{\Gamma_1} & \overset{\bar{i}}{\to} & \overline{\Gamma}/\overline{\Gamma''_2} & \overset{\tilde{p}_2}{\to} & \overline{\Gamma^2}/\bar{p}_2(\overline{\Gamma''_2}) & \to & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array}$$

Assume $\overline{\Gamma_1} \cap \overline{\Gamma''_2} = 1$. Then we have $\bar{i}(\overline{\Gamma_1}) \cap q(\overline{\Gamma''_2}) = 1$. In fact, let $x = \bar{i}(\tilde{q}(x_1)) = q(x_2)$,

i.e. $q(x_2) = q(i(x_1))$, where $x_2 \in \overline{\Gamma''_2}$ and $x_1 \in \overline{\Gamma_1}$. Since $\overline{\Gamma_1} \cap \overline{\Gamma''_2} = 1$ and $\overline{\Gamma''_1} \cap \overline{\Gamma''_2} = 1$, $q\,|\overline{\Gamma''_2}$ and $q\,|i(\overline{\Gamma_1})$ are injective. Hence we have $x_2 = i(x_1)$, which implies $\overline{\Gamma''_2} \cap \overline{\Gamma_1} \neq 1$, which contradicts the assumption. Thus $\tilde{p}_2\,|q(\overline{\Gamma''_2})$ is infinite. But $\overline{\Gamma^2}/\bar{p}_2(\overline{\Gamma'_2})$ is finite. This is a contradiction. Thus we have proved that $\overline{\Gamma''_2} \cap \overline{\Gamma_1} \neq 1$. We have the diagram;

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & \overline{\Gamma_1} \cap \overline{\Gamma''_2} & \to & \overline{\Gamma''_2} & \to & \bar{p}_2(\overline{\Gamma''_2}) & \to & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & \overline{\Gamma_1} & \to & \overline{\Gamma} & \to & \overline{\Gamma^2} & \to & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & \overline{\Gamma_1}/\overline{\Gamma_1} \cap \overline{\Gamma''_2} & \to & \overline{\Gamma}/\overline{\Gamma''_2} & \to & \overline{\Gamma^2}/\bar{p}_2(\overline{\Gamma''_2}) & \to & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array}$$

Since $\overline{\Gamma_1} \cap \overline{\Gamma''_2} \neq 1$, we have $\overline{\Gamma_1}/\overline{\Gamma_1} \cap \overline{\Gamma''_2} = 1$. Thus we have the following diagram;

$$\begin{array}{ccccccccc} 1 & \to & \overline{\Gamma'_1} & \to & \overline{\Gamma'} & \to & \overline{\Gamma'^2} & \to 1 \\ & & \tilde{\theta}\downarrow & & \theta\downarrow & & \bar{\theta}\downarrow & \\ 1 & \to & \overline{\Gamma_1} & \to & \overline{\Gamma} & \to & \overline{\Gamma^2} & \to 1. \end{array}$$

By the same argument as above, we have a homeomorphism $K\backslash G/\Gamma' \cong K\backslash G/\Gamma$. This completes the proof of Theorem 3.

4. The Proof of Theorem

In this section, we shall prove the Theorem in Introduction. We use the same notations in section 2. It follows from Theorem 3 and Proposition 5 that we have the following

Proposition 7.

$$K_1\backslash S_1/p_1(H_1) \cong K_1\backslash S_1/p_1(H_2).$$

Since the principal fibration

$$T^{s+2(r-s)} \to G/H_i \to K_1\backslash S_1/p_1(H_i), i = 1,2$$

is uniquely determined up to an equivalence by its characteristic class

$$c \in H^2(K_1\backslash S_1/p_1(H_i); Z^{s+2(r-s)}).$$

It follows from a result in[5] that this class is naturally identified with the characteristic class corresponding to the exact sequence;

$$1 \to Z^{s+2(r-s)} \to \pi_1(G/H_i) \to \pi_1(K_1\backslash S_1/p_1(H_i)) \to 1$$

The diagram ($\star$) in section 2 shows that two fibrations over $K_1\backslash S_1/p_1(H_i)$

$$T^{s+2(r-s)} \to G/H_i \to K_1\backslash S_1/p_1(H_i), i = 1,2$$

are equivalent, which means G/H_1 and G/H_2 are homeomorphic. This completes the proof of the Theorem in Introduction.

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