Branches and multiplicities

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In this note we discuss certain aspects of intersection theory in an old-fashioned style and give some generalizations of related thoughts.

As is well known on the theory of plane algebraic curves, the analytic branches centered at a common point of two plane curves, that is, the parametric or branch representations play a central roll for counting the intersection multiplicity and proving the classical theorem of Bézout. Here we want to take a short glance at the above situation in a historical perspective. As mensioned in Chevalley's considerations[1], everything may start from considering a local ring of the form $B=C[[X,Y]]/(f,g)$, where $f(X,Y)=0$ and $g(X,Y)=0$ define two algebraic curves intersecting at the point $P=(0,0)$ without common branches. We denote by "C" the field of complex numbers and for simplicity we throughout assume that the base field is an algebraically closed field or C in the case of some classical arguments.

Firstly choose either of the two curves, say $f=0$. Then consider the 1-dimensional local ring $R=C[[X,Y]]/(f)=C[[x,y]]$ which is Macaulay complete.

Each irreducible factor p of f with a certain multiplicity l_p in $C[[X,Y]]$ corresponds to the derived normal ring $V(p)$ of $A(p)=C[[X,Y]]/(p)$. The ring $A(p)$ or $Spec\,A(p)$ is called an analytic branch centered at P. Let v_p be the associated discrete valuation defined by $V(p)$.

To motivate the discussion, we take up the concept of class of an algebraic curve and prove a generalization of Plücker formula which may explain a nice interaction between branches and multiplicities.

(a) With the notation as above, suppose that R is a domain, i.e., f is irreducible in $C[[X,Y]]$, which is equivalent to say that $P=(0,0)$ is a unibranched point of the curve $f=0$. Assume that $R\neq V$, hence $xy\neq 0$ where V is the derived normal ring of R. Then $R=C[[X,Y]]/(f)=C[[x,y]]$ is a subring of V which is isomorphic to a power series ring $C[[t]]$, and v is the order function of t. Letting $A_y=C[[y]]$, $R=A_y[x]$ is finite over A_y, and if J is the conductor of V in R, $(f_X(x,y))=J\,D_{V/A_y}$ in V, where D_{V/A_y} is the different of V over A_y. We note that if $\mathrm{ord}_X f(0,X)=n$, there is a unit q in $C[[X,Y]]$ such that qf is a monic polynomial in X of degree n with the coefficients of $qf-X^n$ in $YC[[Y]]$(cf.[2],45).

Clearly, $(f_X(x,y))$ is the different of R over A_y. Similarly, $(f_Y(x,y))=JD_{V/A_X}$ in V, where $A_X=C[[x]]$. Put $I=(f_X, f_Y)$. Calculating the different, we have $IM(V)=M(R)J$ in case of characteristic 0.

For an example, let $f=Y^m-X^{m+1}+Y^{m+2}, m\geq 2$, which is treated in this series No.49(there on page 24, line 7 b : insert "not" after "should", and line 5 b : read "$=(m-1)m$" instead of "$=(m-1)(m+1)$").Then the curve $f=0$ has only one singular point(a unibranched point)$P=(0,0)$ and has no singularity at infinity. The equality $IM(V)=M(R)\,J$ implies $v(I)=v(J)+v(M(R))-1$, and hence, in general $v(I)\neq v(J)$.

Let X be an irreducible curve of degree n in the projective plane. For a point P on X, there correspond the analytic branches $A(p_i)$, $i=1,2,...,r$ centered at P with derived normal rings $V(p_i)$. Let $J(p)$ be the conductor of $V(p)$ in $A(p)$ and let $c_P=\Sigma\ v_{pi}(J(p_i))$. Let $t(i,j)=length\ C[[X,Y]]/(p_i,p_j)$, $t_P=\Sigma\ t(i,j)\ (i<j)$, e_P the multiplicity of X

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at P and let $s_P=r$ the number of analytic branches $A(p)$ at P. Put $t_P=0$ if $s_P=1$.Then we have the following extension of Plücker formula :

(1) *The class of X is equal to* $n(n-1)-\Sigma(c_P+e_P+2\,t_P-s_P)$.*In paticular, if the singularity at P is ordinary,* $c_P+e_P+2\,t_P-s_P=e_P(e_P-1)$, *and if that is cuspidal, resolved by just one blowingup, then this value is equal to* $(e_P+1)(e_P-1)$.

In fact, by [8], (6.2), everything depends on the calculation of the number $\Sigma\ \delta(P)$. The above consideration is also valid if R is replaced by an analytic branch $A(p)=C[[X,Y]]/(p)=C[[x,y]]$. Then we have

$$p_X(x,y)V(p)=J(p)D_{V(p)/Ay}=J(p)(y)V(p)M(V(p))^{-1}$$

(cf.[5],p.56). Adding this to the similar equation for $p_Y(x,y)V(p)$, we have

$$(p_X,p_Y)M(V(p))=J(p)(x,y)V(p)=J(p)M(A(p))V(p).$$

Letting $f=p(X,Y)h(X,Y)$, we have $f_X=p_Xh+ph_X$, hence $f_XV(p)=p_XhV(p)$, and similarly $f_YV(p)=p_YhV(p)$.Then multiplying the above equation by h, we have $(f_X,f_Y)M(V(p))=J(p)M(A(p))hV(p)$. If $p=p_1$, then $h=p_2...p_r$, and $v_p(p_i)$ $=length\ C[[X,Y]]/(p_1,p_i)=t(1,i)$, $i=2,...,r$. Thus taking the value v_p on the above equation, we have

$$v_p((f_X,f_Y))+1=v_p(J(p))+v_p(M(A(p)))+t(1,2)+...+t(1,r).$$

Adding these equations for $p=p_1,...,p_r$, we have $\delta(P)=\Sigma\ v_p((f_X,f_Y))=c_P+e_P+2\,t_P-s_P$. Since the class of the curve X is $n(n-1)-\Sigma\ \delta(P)$, our assertion is completely proved.

(b) Now we return to the discussion on the concept of intersection multiplicity. As is well known, the intersection multiplicity at the origin P, $i(P\ ;f=0,g=0)$ is defined to be the length of $B=C[[X,Y]]/(f,g)$ as a $B-$ module, which is equal to the dimension of B as a vector space over C in our case.

The next part is concerned with the use of branches, that is, the local prametrizations of an algebraic curve. With the notation as stated above, this classical case is described concisely as follows :

(2) *With the same notation as above,* $i(P\ ;f=0,g=0)=\Sigma\ l_pv(gV)$, *where the sum is taken over all branches V centered at* P *on the curve* $f=0$. *The similar formula holds by reversing the roles of* $f=0$ *and* $g=0$. *Further, suppose that* $f(0,Y)\neq 0$ *and* $g(0,Y)\neq 0$. *Let* f^* *be the monic polynomial in Y over* $C[[X]]$ *which is obtained as in* [2],45, *exer.* 1. *Then as a local version of the Zeuthen's rule (cf.* [6],*p.74 or* [7]), *this value is caluculated as* $ord_Xr^*(X)$, *where* $r^*(X)\in C[[X]]$ *is the resultant of* f^* *and g with respect Y(the modification for g is superfluous). More simply, if* $r(X)=Res_Y(f(X+uY,Y),g(X+uY,Y))$ *is the resultant with respect to* Y, *this value is the multiplicity of zero as a root of* $r(X)$, *where u is an indeterminate*.

In fact, $I(P\ ;f=0,g=0)$is the multiplicity of the ideal (f,g) in $C[[x,y]]$ which is equal the multiplicity of the ideal (g) in $C[[x,y]]/(f)$. The additive formula(cf.[2],23)implies that this value is equal to $\Sigma\ l_pm_p(g)$, where $m_p(g)$ is the multiplicity of (g) in $A=A(p)=C[[x,y]]/(p)$. We know that the branch V is a finite A-module and

$$m_p(g)=length_A(A/gA)=length_A(V/gV)=v(gV)$$

(the residue fields of A and V are equal to $\boldsymbol{C}$).

The second assertion is easy. The following is a slight variation of the proof given in [6],p.74.
By solving $f^*(Y)=0$ in the field $C(X)^*$ of fractional power series, we obtain the roots $\overline{Y}$, and each root corresponds to an irreducible parametrization of the curve $f=0$ at the origin and conversely. Each equivalent(or conjugate) class of the roots corresponds to a unique analytic branch p_i of the curve at the origin, which is represented by an equivalent parametrization :

$$(t^{m_i},\ a_{1i}t^{n_{1i}}+a_{2i}t^{n_{2i}}+....),\ a_{1i}\neq 0,\ 1\leq i\leq r.$$

Clearly $v_{pi}(g)=m_i\ ord_X\ g(X,\overline{Y}_1)$, where$(\overline{Y}_1,...,\overline{Y}_{m_i})$ is the conjugate class of roots corresponding to the branches

p_i and v_{p_i} is the associated valuation. The last assertion is clear from a usual representation of the resultant. We just used a linear change of variables to satisfy a certain condition which enables $r(X)$ to calculate the intersection multiplicity only at the origin.

(c) The second part is to describe a general relationship between branches and intersection multiplicities. Let U and W be subvarieties of an algebraic variety V. Let X be a proper component of the intersection of U and W with respect to V, and R the local ring of V along X with $dim\, R=r$.

In [1], an intersection theory of algebraic varietis was established , and there the term " the sheets of U at a point $P \in X$", was introduced for the study of connections between algebraic varieties and algebroid varieties. Instead of the concept of sheets at points, we introduce a new term "the analytic branches of a variety U along a subvariety X". Namely, they are $U_X(p)=Spec\, A^*/p$, where A^* is the completion of the local ring A of U along X and p runs over all prime divisors of the zero ideal (0) of A^*, which is semiprime in our case. Let U_i and W_k be analytic branches of U and W along X respectively. They correspond to the prime divisors $p_i(U)$ of $p(U)R^*$and $p_k(W)$ of $p(W)R^*$ respectively, where $p(U)$ and $p(W)$ are the ideals of R which correspond to U and W respectively. Put $D=R \otimes_k R'$, where R' is the copy of R. Let δ be the ideal of D which corresponds to the diagonal Δ of $V \times V$. Then $N=M(R)+\delta$ is a maximal ideal of D. Here we note that there is a simple point Q on X which is simple on V. Let $x_1,...,x_n$ be a regular system of parameters of the local ring S of V at Q, and suppose that $(x_1,...,x_r)$ is the ideal of X in S.The completion of D_N is isomorphic to a power series ring $\overline{D}= K[[x_1,...,x_r,y_1,...,y_r,z_{r+1},...,z_{r+d}]]$, where K is a coefficient field of R^* with $k \subseteq K$, $d=dim\, X$ and $z_j=x_j-y_j$, $j=1,...,n=r+d$. Let q_{ik} be the ideal generated by $p_i(U)$ and the copy of $p_k(W)$ in $B=K[[x_1,...,x_r,y_1,...,y_r]]$, and let $B_{ik}=B/q_{ik}$. As in the case of algebraic curve, we define the intersection multiplicity e_{ik} of the analytic branches U_i and W_k by the multiplicity of $(z_1,...,z_r)B_{ik}$ in B_{ik}.

(3) *The intersection multiplicity of U and W along X in V is equal to* $\Sigma\, e_{ik}$, *where* U_i *and* W_k *run over all analytic branches of U and W along X respectively. Further, the value* e_{ik} *is equal to* $\Sigma(-1)^j$ *length* $Tor_j^{R^*}(R^*/p_i(U),R^*/p_k(W))$.

In the above statement, the number of branches may be different from that of sheets defined in[1], but multiplicities may be calculated somewhat directly in certain cases.

The proof of our assertion is easy. Let $H = D_N/(p(U), p(W)')\, D_N$, which is a domain in our case. From the definition of intersection multiplicity, $i(X\, ;\, U \cdot W)$ is equal to the multiplicity of δH. The completion $H^*$of H is isomorphic to $\overline{D}/(p(U), p(W)')\,\overline{D}$ which is reduced, and $i(X\, ;\, U \cdot W)$ is equal to the multiplicity of $(z_1,...,z_n)H^*$. The completion of $R \rightarrow D_N$ defines a inclusion $R_K=K[[x_1,...,x_r]] \subseteq \overline{D}$, and the induced inclusion $B \rightarrow \overline{D}$ defines the faithfully flat extension $\overline{B} \rightarrow H^*$ with $\overline{B}=B/(p(U),p(W)')B$. Clearly, $H^*/(z_{r+1},...,z_n)$ is isomorphic to $\overline{B}$ which is reduced and unmixed (or the radical of $(z_1,...,z_r)\, H^*$is $q=M(\overline{B})H^*$, a prime ideal, and H^*/q is isomorphic to $K[[z_{r+1},...,z_n]]$, regular local).

By the associative law, we conclude that the multiplicity of $(z_1,...,z_n)H^*$ is(equal to that of $(z_1,...,z_r)H_q^*$, hence by the theorem of transition,) equal to the multiplicity of $(z_1,...,z_r)\overline{B}$. We identify $\overline{B}$ with the analytic tensor product of $R_K/p(U)R_K$ and $R_K'/p(W)'R_K'$ over K. Then it is faithfully flat over each factor and $(p(U),p(W)')B=\cap q_{ij}$. In fact, this intersection is irredundant. Suppose that q_{ij} contains the intersection of all $q_{kl} \neq q_{ij}$. Then a prime divisor Q of q_{ij} contain some q_{kl}, and their restrictions to R_K(resp. R'_K)are $p_i(U)$ and $p_k(U)$ (resp. $q_j(W)$ and $q_l(W)$) respectively. It is imposible since the restrictions must be same on either side.

The additive law implies that the multiplicity of$(z_1,\dots,z_r)\overline{B}$ is the sum of the multiplicities e_{ik} of$(z_1,\dots,z_r)B_{ik}$ in B_{ik}.

Here we recall Serre's definition of intersection multiplicity, which is described by using the Tor functors(cf. [4]).With the same notation as above, it takes the following expression :

$$i(X\,;\,U\cdot W)=\Sigma(-1)^j\,length\,Tor_j^R(R/p(U),\,R/p(W)).$$

Clearly,

$$Tor_j^R(R/p(U),R/p(W))=Tor_j^R(R^*/p(U)R^*,R^*/p(W)R^*),$$

and R^* is identified with $K[[x_1,\dots,x_r]]$ as is stated above. These are identified with the homology modules of Kosuzul complex

$$H_j(K^B((z_1,\dots,z_r),\,R^*/p(U)R^*\overline{\otimes}_K R^*/p(W)R^*))$$

with $B=K[[x_1,\dots,x_r,y_1,\dots,y_r]]$, where $\overline{\otimes}$ denotes the complete tensor product and it is identified with the analytic tensor product(cf.[2],47).Thus we have

$$length\ Tor_j^R(R/p(U),\,R/p(W))=length\,H_j(K^B(((z_1,\dots,z_r),\,B/(p(U),p(W))B)$$

and the Euler-Poincare characteristic of the complex is equal to the multiplicity of the ideal$(z_1,\dots,z_r)$of B , hence equal to $i(X\,;\,U\cdot W)$. Let $q_i(U)$ be the intersection of all $p_j(U)$, $j\neq i$.Then $p_i(U)\cap q_i(U)=p(U)R^*$, $\dim(p_i(U),\,q_i(U))<dim\,p(U)R^*$. By considering certain exact sequences associated with

$$R^*/p_i(U)\cap q_i(U),\ R^*/p_i(U),\ R^*/q_i(U)\ \text{and}\ R^*/(p_i(U),q_i(U)),\text{etc.},$$

we have, letting $E(I,J)=\Sigma(-1)^j\,length\,Tor_j^{R^*}(R^*/I,R^*/J)$,

$$i(X\,;\,U\cdot W)=E(p(U)R^*,p(W)R^*)=E(p_i(U),p(W)R^*)+E(q_i(U),p(W)R^*)$$
$$=\Sigma\ E(p_i(U),p(W))R^*)=\Sigma\ E(p_i(U),p_k(W))$$

from long exact sequences of Tor functors. Here we remark that $E((p_i(U),\,q_i(U)),\,p(W)R^*)=0$, etc. Since $E(p_i(U),p_k(W))$ is equal to $e_{i,k}$, the intersecion multiplicty of analytic branches U_i and W_k, the assertion is proved again.

(d) Finally we discuss some expressions of multiplicities of local rings.

For simplicity, let(R,M)be a local ring of dimensin $d>0$ with infinite residue field $k=R/M$ and let $J=(a_1,\dots,a_d)$ be a reduction of an M-primary ideal I. We give here some elementary remarks on reductions of ideals and superficial elements.

(4) *Let* $R(J)=\oplus J^n$. *Then* $R(J)/MR(J)\cong k[X_1,\dots,X_d]$, *and hence,* $MR(J)$*is a unique minimal prime divisor of* $JR(J)$. *In particular, if* R *is Cohen-Macaulay,* $JR(J)$ *is a* $MR(J)$*-primary ideal and* $a_1,\dots,a_d$ *are superficial elements of* J*(cf.* [2],22*).*

Clearly, $R(I)$ is integral over $R(J)$ .Further, there is an element $a\in J$ which is a superficial element of I and is a menber of a minimal basis for J. In this case, letting $T=R[Ja^{-1}]$, T/MT is isomorphic to a polynomial ring $k[Z_1,\dots,Z_{d-1}]$.

Letting $X_i=a_i\ mod\ MJ$, the first asertion is clear since $a_1,\dots,a_d$ are analytically independent. If R is Macaulay, $a_1,\dots,a_d$ is a regular sequence and hence, $R(J)/JR(J)\cong R/J[Y_1,\dots,Y_d]$ ($Y_i=a_i\ mod\ J^2$).Thus $R(J)/MR(J)\cong(R/J[Y_1,\dots,Y_d])/(M/J[Y_1,\dots,Y_d])$, and hence $JR(J)$ is $MR(J)$-primary and $a_1,\dots,a_d$ are superficial elements of J. Let P be a prime divisor of $IR(I)$ with $It\not\subseteq P$. Then $Jt\not\subseteq P\cap R(J)$ since$(Jt)(It)^n=(It)^{n+1}$ for all large n. Let $Q_1,\dots,Q_t$ be the set of all prime divisors of $JR(J)$ and restrictions of prime divisors P of $IR(I)$ as stated above. Then each $Q_i\cap Jt$ is properly contained in Jt and hence, $Q_i\cap Jt+MJt/MJt$ are proper subspaces of Jt/MJt. Since k is infinite, after a usual argument, we find an $at\in Jt$ with $at\not\in Q_i\cap Jt+MJt$. Then $a\in J$ is a required element(cf.[2],22). Now let $a=f(a_1,...,a_d)$ with $f(X_1,...,X_d)$ a linear form over R. Clearly, T/MT is the degree zero part of $R[Jt][1/at]/MR[Jt][1/at]$ and is canonically isomorphic to $k[(X_1,...,X_d)(\bar{f})^{-1}] \cong k[Z_1,...,Z_{d-1}]$.

As stated above, let I be an M-primary ideal and J be an reduction of I. Let L(resp. L_s)be the set of superficial elements(resp. superficial elements of degree s) of I. By R^I we denote the first neighbourhood ring of R with respect to I(cf.[3]), that is, the set of elements a/b with $a \in I^s$ and $b \in L_s$ in the localization $L^{-1}R$.

(5) *For simplicity, suppose that (R,M) is a local ring of an algebraic variety along some subvariety. Let* $P_1,...,P_t$ *be the prime divisors P of* IR^I *such that* $trans.deg_k\, R^I/P=d-1$. *Let* $R^{<I>}$ *be the localization* $S^{-1}R^I$ *with* $S=\bigcap(R^I-P_i)$ *and let* $R_o=(R^J)_{MR^J}$.

Then the residue field $k_o=k(R_0)$*is a purely transcendental extensin of degree* d-1 *over k, and*

$$e_R(I)=length_{R_0}\, R^{<I>}/IR^{<I>}=\Sigma\, e_{A_i}(IA_i)|k(A_i):k_o)|,\ where\ A_i=(R^{<I>})_{P_i},\ i=1,...,t.$$

We note that R/M is infinite in this case, and we can use an I-superficial element of degree 1 for the construction of R^J,R_o, R^I, etc. Clearly, $trans.deg_k\, R^I/P \leq d-1$, where P runs over all prime divisors of IR^I, and the equality holds just for $P_1,...,P_t$. From (4) and our discussions above, we conclude that $R^{<I>}$ is integral over its subring R_o which is a 1-dimensional local ring. Since $R(J)/MR(J) \cong k[X_1,...,X_d]$, we see that the multiplicity of J is equal to that of the principal ideal JR_o of R_o by a usual dicussion of degree zero localizations. Then the first assertion is clear. Since J is a reduction of I, their multiplicities are equal and the extension law of multiplicities(cf.[2],23) implies the second assertion.

In particular, if $e_R(I)=e_{R^{<I>}}(IR^{<I>})$, then each quotient field of R^I/P_i is purely transcendental of degree $d-1$ over k. For a general local ring, the similar assertions in (5) may be also proved by a certain technical device.

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