WEIGHTED WEAK TYPE $(1,1)$ ESTIMATES FOR OSCILLATORY SINGULAR INTEGRALS WITH DINI KERNELS

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Abstract. We consider A_1-weights and prove weighted weak type $(1,1)$ estimates for oscillatory singular integrals with kernels satisfying a Dini condition.

1. Introduction

We consider an oscillatory singular integral operator of the form:

$$T(f)(x) = \text{p.v.} \int_{\mathbf{R}^n} e^{iP(x,y)} K(x-y) f(y)\, dy = \lim_{\epsilon \to 0} \int_{|x-y|>\epsilon} e^{iP(x,y)} K(x-y) f(y)\, dy,$$

where P is a real-valued polynomial:

$$P(x,y) = \sum_{|\alpha| \le M, |\beta| \le N} a_{\alpha\beta} x^\alpha y^\beta, \tag{1.1}$$

and $f \in \mathfrak{S}(\mathbf{R}^n)$ (the Schwartz space).

Let $K \in C^1(\mathbf{R}^n \setminus \{0\})$ satisfy

$$|K(x)| \le c|x|^{-n}, \qquad |\nabla K(x)| \le c|x|^{-n-1}; \tag{1.2}$$

$$\int_{a<|x|<b} K(x)\, dx = 0 \quad \text{for all } a, b \text{ with } 0 < a < b. \tag{1.3}$$

The smallest constant for which (1.2) holds will be denoted by $C(K)$. The following results are known.

Theorem A. (Ricci-Stein [5]) *Let $1 < p < \infty$. Then, T is bounded on $L^p(\mathbf{R}^n)$ with the operator norm bounded by a constant depending only on the total degree of P, $C(K)$, p and the dimension n.*

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Theorem B. (Chanillo-Christ [1]) *The operator T is bounded from $L^1(\mathbf{R}^n)$ to the weak $L^1(\mathbf{R}^n)$ space with the operator norm bounded by a constant depending only on the total degree of P, $C(K)$ and the dimension n.*

Let w be a locally integrable positive function on $\mathbf{R}^n$. We say that $w \in A_1$ if there is a constant c such that

$$M(w)(x) \le cw(x) \quad \text{a.e.} \tag{1.4}$$

where M denotes the Hardy-Littlewood maximal operator. The smallest constant for which (1.4) holds will be denoted by $C_1(w)$.

It is known that T is bounded from L^1_w to $L^{1,\infty}_w$ (the weak L^1_w space).

Theorem C. ([8]) *There exists a constant c depending only on the total degree of P, $C(K)$, $C_1(w)$ and the dimension n such that*

$$\sup_{\lambda>0} \lambda w\left(\{x \in \mathbf{R}^n : |T(f)(x)| > \lambda\}\right) \le c\|f\|_{L^1_w},$$

where $w(E) = \int_E w(x)\,dx$ and $\|f\|_{L^1_w} = \int |f(x)|w(x)\,dx$.

Let K be locally integrable away from the origin. Put, for $r \ge 1$, $0 < t \le 1$ and $R > 0$,

$$\omega_{r,R}(t) = \sup_{|y|\le Rt/2} \left(R^{-n} \int_{R\le|x|\le 2R} \left|R^n\left(K(x-y) - K(x)\right)\right|^r dx \right)^{1/r}.$$

We say that the kernel K satisfies the D_r-condition if

$$B_r = \int_0^1 \omega_r(t)\,\frac{dt}{t} < \infty \qquad \text{where} \quad \omega_r(t) = \sup_{R>0} \omega_{r,R}(t);$$

$$C_r = \sup_{R>0} \left(R^{-n} \int_{R\le|x|\le 2R} \left|R^n K(x)\right|^r dx \right)^{1/r} < \infty.$$

By the usual modifications we can also define the D_∞-condition. In this note we shall prove the following results, which will improve Theorems B and C.

Theorem 1. *Let $r > 1$ and $1/r + 1/u = 1$. Suppose the kernel K satisfy the D_r-condition and (1.3), and suppose $w^u \in A_1$. Then, there exists a constant c depending only on the total degree of P, B_r, C_r, $C_1(w^u)$, r and the dimension n such that*

$$\sup_{\lambda>0} \lambda w\left(\{x \in \mathbf{R}^n : |T(f)(x)| > \lambda\}\right) \le c\|f\|_{L^1_w}.$$

Theorem 2. *Suppose that K satisfies the D_1-condition and* (1.3). *Then, there exists a constant c depending only on the total degree of P, B_1, C_1 and the dimension n such that*

$$\sup_{\lambda>0} \lambda \,|\{x \in \mathbf{R}^n : |T(f)(x)| > \lambda\}| \le c\|f\|_{L^1}.$$

Every kernel satisfying (1.2) satisfies the D_∞-condition. If $K(x) = |x|^{-n}\Omega(x')$, $x' = x/|x|$, and if Ω satisfies the L^r-Dini condition on S^{n-1}, then K satisfies the D_r-condition.

These theorems will be proved by a double induction as in [5], [1] and [8]. In this note we shall prove only Theorem 1. Theorem 2 can be proved similarly. Let P be a polynomial of the form in (1.1). We assume that there exists α such that $|\alpha| = M$ and $a_{\alpha\beta} \neq 0$ for some β. We write

$$P(x,y) = \sum_{|\alpha|\le M} x^\alpha Q_\alpha(y) \tag{1.5}$$

and define $L = \max\{\deg(Q_\alpha) : Q_\alpha \neq 0, |\alpha| = M\}$. Then $0 \le L \le N$. We assume that $L \ge 1$ and $\max_{|\alpha|=M,|\beta|=L} |a_{\alpha\beta}| = 1$. Under this assumption on a polynomial P, we define

$$T_\infty(f)(x) = \int_{|x-y|>1} e^{iP(x,y)} K(x-y) f(y)\, dy.$$

To prove Theorem 1, we shall use the following result in the induction.

Proposition 1. *Let η, $\rho > 0$ and let the kernel K, the weight w and the exponents r, u be as in Theorem* 1. *Then, there exists a constant c depending only on η, ρ, the total degree of P, r and the dimension n such that if $C_1(w^u) \le \eta$, B_r, $C_r \le \rho$,*

$$\sup_{\lambda>0} \lambda w\left(\{x \in \mathbf{R}^n : |T_\infty(f)(x)| > \lambda\}\right) \le c\|f\|_{L^1_w}.$$

Let $A(f)(x) = \text{p.\,v.}\, K * f(x)$. We need the following result for the first step of induction for the proof of Theorem 1.

Proposition 2. *Let the kernel K, the weight w and the exponents r, u be as in Theorem* 1. *Let η, $\rho > 0$. There exists a constant c depending only on η, ρ, r and the dimension n such that if $C_1(w^u) \le \eta$, B_r, $C_r \le \rho$, then*

$$\sup_{\lambda>0} \lambda w\left(\{x \in \mathbf{R}^n : |A(f)(x)| > \lambda\}\right) \le c\|f\|_{L^1_w}.$$

Since A is bounded on L^2 (see [6, pp. 25–26]), if A is as in Proposition 2, we see that A is a singular integral operator considered in [6, p. 13]. Hence the conclusion of Proposition 2 will follow from [6, p. 15, Theorem 1.6].

We shall give the outlines of the proofs of Theorem 1 and Proposition 1 in Sections 2 and 4, respectively. Our proof of Proposition 1 is based on the techniques in Christ [3] for the proofs of the weak (1, 1) estimates for rough operators (see also Christ-Rubio [4] and Sato [7]). We also use the geometrical argument of Chanillo-Christ [1]. We have to prove a key estimate (Lemma 8 in §5) in the unweighted case in order to apply the method of Vargas [9] involving an interpolation with change of measure. To prove Lemma 8, we need a geometrical result for polynomials (Lemma 6 in §5). We shall prove Lemma 6 in §6 by using the results appearing in the proof of Chanillo-Christ [1, Lemma 4.1]. Lemmas 6 and 8 have been proved in [8]. We include the proofs and some other parts of [8] almost verbatim for the sake of completeness.

2. Outline of proof of Theorem 1

To apply the induction argument of [5] we need some preparation. We may assume that $M \geq 1$ and $N \geq 1$; otherwise Theorem 1 reduces to Proposition 2.

We write a polynomial in (1.1) as follows:

$$P(x,y) = \sum_{j=0}^{M} \sum_{|\alpha|=j} x^\alpha Q_\alpha(y) =: \sum_{j=0}^{M} P_j(x,y).$$

We further decompose P_j as follows:

$$P_j(x,y) = \sum_{t=0}^{N} \sum_{\substack{|\alpha|=j \\ |\beta|=t}} a_{\alpha\beta} x^\alpha y^\beta =: \sum_{t=0}^{N} P_{jt}(x,y).$$

For $j = 1, 2, \ldots, M$ and $k = 0, 1, \ldots, N$, define

$$R_{jk}(x,y) = \sum_{s=0}^{j-1} P_s(x,y) + \sum_{t=0}^{k} P_{jt}(x,y). \tag{2.1}$$

Note that $R_{jN} = \sum_{s=0}^{j} P_s \quad (j = 1, 2, \ldots, M)$.

For $j = 1, 2, \ldots, M$ and $k = 0, 1, \ldots, N$, we consider the following propositions.

Proposition $A(j,k)$**.** *Let η, $\rho > 0$. There exists a constant c depending only on η, ρ, j, N, r and the dimension n such that if $C_1(w^u) \leq \eta$, B_r, $C_r \leq \rho$ and if R_{jk} is a polynomial of the form in* (2.1)*, then*

$$\sup_{\lambda>0} \lambda w\left(\{x \in \mathbf{R}^n : |T_{jk}(f)(x)| > \lambda\}\right) \leq c\|f\|_{L^1_w},$$

where

$$T_{jk}(f)(x) = \text{p.v.} \int_{\mathbf{R}^n} e^{iR_{jk}(x,y)} K(x-y) f(y)\, dy.$$

Then, Theorem 1 follows from Proposition $A(M,N)$. We shall prove it by double induction. We first note that $A(1,0)$ follows from the boundedness of the operator A.

Next, we observe that if $M \geq 2$ and if $A(j,N)$ $(1 \leq j \leq M-1)$ is true, so is $A(j+1,0)$ since

$$R_{j+1,0}(x,y) = R_{jN}(x,y) + \sum_{|\alpha|=j+1} a_{\alpha 0} x^\alpha$$

and hence $|T_{j+1,0}(f)(x)| = |T_{jN}(f)(x)|$. Thus, to complete the induction starting from $A(1,0)$ and arriving at $A(M,N)$, it is sufficient to prove $A(j,k+1)$ assuming $A(j,k)$ $(0 \leq k < N,\ 1 \leq j \leq M)$. To achieve this, put $R = R_{j,k+1}$, $R_0 = R_{jk}$, $T_{j,k+1} = S$. We note that

$$R(x,y) = R_0(x,y) + \sum_{\substack{|\alpha|=j \\ |\beta|=k+1}} a_{\alpha\beta} x^\alpha y^\beta.$$

We have only to deal with the case $C_{jk} = \max_{|\alpha|=j,|\beta|=k+1} |a_{\alpha\beta}| \neq 0$. Then, by a suitable dilation we may assume $C_{jk} = 1$. This can be seen as follows. We first note that, for $a > 0$,

$$S(f)(ax) = \mathrm{p.\,v.}\int e^{iR(ax,ay)} K_a(x-y) f(ay)\,dy,$$

where $K_a(x) = a^n K(ax)$. Assume the boundedness of S for the case $C_{jk} = 1$. Then, choosing a to satisfy $a^{j+k+1}C_{jk} = 1$, and using the dilation invariance of both the class A_1 and the class of the kernels considered in Theorem 1, we get

$$\begin{aligned} w\left(\{x \in \mathbf{R}^n : |S(f)(x)| > \lambda\}\right) &= w_a\left(\{x \in \mathbf{R}^n : |S(f)(ax)| > \lambda\}\right) \\ &\leq c\lambda^{-1}\int |f(ax)| a^n w(ax)\,dx \\ &= c\lambda^{-1}\|f\|_{L^1_w}. \end{aligned}$$

We split the kernel K as $K = K_0 + K_\infty$, where $K_0(x) = K(x)$ if $|x| \leq 1$ and $K_\infty(x) = K(x)$ if $|x| > 1$. Assuming $C_{jk} = 1$, we consider the corresponding splitting $S = S_0 + S_\infty$:

$$S_0(f)(x) = \mathrm{p.\,v.}\int e^{iR(x,y)} K_0(x-y) f(y)\,dy,$$

$$S_\infty(f)(x) = \int e^{iR(x,y)} K_\infty(x-y) f(y)\,dy.$$

In the next section, we shall prove

$$\sup_{\lambda>0} \lambda w\left(\{x \in \mathbf{R}^n : |S_0(f)(x)| > \lambda\}\right) \leq c\|f\|_{L^1_w}, \tag{2.2}$$

while by Proposition 1 we have

$$\sup_{\lambda>0} \lambda w\left(\{x \in \mathbf{R}^n : |S_\infty(f)(x)| > \lambda\}\right) \leq c\|f\|_{L^1_w}. \tag{2.3}$$

Combining (2.2) and (2.3), we shall complete the proof of $A(j, k+1)$, which will finish the proof of Theorem 1.

3. Estimate for S_0

In this section, we shall prove, under the assumption made in §2, that if $C_1(w) \leq \eta$, B_r, $C_r \leq \rho$ (η, $\rho > 0$), then S_0 is bounded from L^1_w to $L^{1,\infty}_w$ with the operator norm bounded by a constant depending only on j, N, η, ρ, r and n ((2.2)).

First, we shall prove

$$w\left(\{x \in B(0,1) : |S_0(f)(x)| > \lambda\}\right) \leq c\lambda^{-1}\int_{|y|<2} |f(y)| w(y)\,dy, \tag{3.1}$$

where $B(x, r)$ denotes the closed ball with center x and radius $r > 0$.

Lemma 1. *Let $w, w^u (1 \le u < \infty) \in A_1$. Let T be an operator of the form:*

$$T(f)(x) = \mathrm{p.\,v.} \int_{\mathbf{R}^n} K(x,y) f(y)\, dy = \lim_{\epsilon \to 0} \int_{|x-y|>\epsilon} K(x,y) f(y)\, dy$$

for $f \in \mathfrak{S}(\mathbf{R}^n)$. Let $1/r + 1/u = 1$ and consider a non-negative function L on $\mathbf{R}^n \setminus \{0\}$ satisfying $J_r < \infty$, where

$$J_r = \sup_{R>0} \left(R^{-n} \int_{R \le |x| \le 2R} \left(R^n L(x)\right)^r \, dx \right)^{1/r}$$

for $r < \infty$ and J_∞ can be defined by the usual modification. Suppose the kernel K satisfies $|K(x,y)| \le L(x-y)$. For $\epsilon > 0$, put

$$T_\epsilon(f)(x) = \mathrm{p.\,v.} \int_{|x-y|<\epsilon} K(x,y) f(y)\, dy.$$

Suppose

$$\sup_{\lambda>0} \lambda w\left(\{x \in \mathbf{R}^n : |T(f)(x)| > \lambda\}\right) \le c_w \|f\|_{L^1_w}.$$

Then

$$\sup_{\lambda>0} \lambda w\left(\{x \in \mathbf{R}^n : |T_\epsilon(f)(x)| > \lambda\}\right) \le c(c_w + J_r C_1(w^u)^{1/u}) \|f\|_{L^1_w}.$$

Proof. The proof is similar to that of Lemma in [5, p. 187]. We shall prove

$$(3.2) \quad w\left(\{x \in B(h,\epsilon/4) : |T_\epsilon(f)(x)| > \lambda\}\right) \le c(c_w + J_r C_1(w^u)^{1/u}) \lambda^{-1} \int_{|y-h|<6\epsilon/4} |f(y)| w(y)\, dy$$

uniformly in $h \in \mathbf{R}^n$. Integrating both sides of the inequality in (3.2) with respect to h, we get the conclusion of Lemma 1.

Split f into 3 pieces: $f = f_1 + f_2 + f_3$, where $f_i \in \mathfrak{S}(\mathbf{R}^n)$, $|f_i| \le c|f|$ $(i = 1,2,3)$; $\mathrm{supp}(f_1) \subset B(h, \epsilon/2)$, $\mathrm{supp}(f_2) \subset B(h, 11\epsilon/8) \setminus B(h, 3\epsilon/8)$, $\mathrm{supp}(f_3) \subset \{x : |x-h| \ge 5\epsilon/4\}$. Note that if $|x-h| \le \epsilon/4$, then $T_\epsilon(f_1)(x) = T(f_1)(x)$; since $|y-h| \le \epsilon/2$ and $|x-h| \le \epsilon/4$ imply $|x-y| < \epsilon$. So by the assumption on T, we have

$$w\left(\{x \in B(h,\epsilon/4) : |T_\epsilon(f_1)(x)| > \lambda\}\right) \le c_w \lambda^{-1} \int_{|y-h|<6\epsilon/4} |f(y)| w(y)\, dy.$$

Next, by Chebyshev's inequality, Hölder's inequality and the fact $w^u \in A_1$ we easily see that

$$w\left(\{x \in B(h,\epsilon/4) : |T_\epsilon(f_2)(x)| > \lambda\}\right) \le c J_r C_1(w^u)^{1/u} \lambda^{-1} \int_{|y-h|<6\epsilon/4} |f(y)| w(y)\, dy.$$

Finally, if $|x-h| \le \epsilon/4$ and $|y-h| \ge 5\epsilon/4$, then $|x-y| \ge \epsilon$, and so $T_\epsilon(f_3)(x) = 0$. Combining these results, we get (3.2). This completes the proof of Lemma 1.

Now we return to the proof of (3.1). If $|x| \le 1$ and $|y| \le 2$, then

$$\left|\exp\left(iR(x,y)\right) - \exp\left(i\left(R_0(x,y) + \sum_{\substack{|\alpha|=j\\|\beta|=k+1}} a_{\alpha\beta}y^{\alpha+\beta}\right)\right)\right| \le c|x-y|,$$

where c depends only on k, j and n.

Hence, if $|x| \le 1$,

$$|S_0(f)(x)| \le \left|U\left(\exp\left(i\sum_{\substack{|\alpha|=j\\|\beta|=k+1}} a_{\alpha\beta}y^{\alpha+\beta}\right)f(y)\right)(x)\right| + cI(f)(x),$$

where

$$U(f)(x) = \text{p. v.}\int e^{iR_0(x,y)}K_0(x-y)f(y)\,dy,\ I(f)(x) = \int_{|x-y|<1}|x-y|L(x-y)|f(y)|\,dy.$$

Note that $U(f)(x) = U(f\chi_{B(0,2)})(x)$, $I(f)(x) = I(f\chi_{B(0,2)})(x)$ if $|x| < 1$. By the induction hypothesis $A(j,k)$ and Lemma 1, we see that U is bounded from L^1_w to $L^{1,\infty}_w$. On the other hand, it is easy to see that

$$\int_{|x-y|<1}|x-y|L(x-y)w(x)\,dx \le \sum_{j\le 0}2^j\int_{2^{j-1}\le|x-y|\le 2^j}L(x-y)w(x)\,dx \le cJ_rM_u(w)(y),$$

where $M_u(w) = M(w^u)^{1/u}$. Thus, by Chebyshev's inequality and the fact $w^u \in A_1$ we have

$$w(\{x \in B(0,1) : I(f)(x) > \lambda\}) \le cJ_rC_1(w^u)^{1/u}\lambda^{-1}\int_{|y|<2}|f(y)|w(y)\,dy.$$

Combining these results, we get (3.1).

Similarly we can prove

$$w\left(\{x \in B(h,1) : |S_0(f)(x)| > \lambda\}\right) \le c\lambda^{-1}\int_{|y-h|<2}|f(y)|w(y)\,dy, \tag{3.3}$$

where c is independent of $h \in \mathbf{R}^n$. To see this, we first note that

$$S_0(f)(x+h) = \text{p. v.}\int e^{iR(x+h,y+h)}K_0(x-y)f(y+h)\,dy$$

and

$$R(x+h,y+h) = R_1(x,y,h) + \sum_{\substack{|\alpha|=j\\|\beta|=k+1}} a_{\alpha\beta}x^\alpha y^\beta.$$

We can apply the induction hypothesis $A(j,k)$ to the operator

$$\text{p. v.} \int e^{iR_1(x,y,h)} K(x-y) f(y)\, dy$$

to get its boundedness from L^1_w to $L^{1,\infty}_w$. Thus, by the same argument that leads to (3.1) we get

$$\begin{aligned} w\left(\{x \in B(h,1) : |S_0(f)(x)| > \lambda\}\right) &= \tau_h w\left(\{x \in B(0,1) : |S_0(f)(x+h)| > \lambda\}\right) \\ &\le c\lambda^{-1} \int_{|y|<2} |f(y+h)| w(y+h)\, dy \\ &\le c\lambda^{-1} \int_{|y-h|<2} |f(y)| w(y)\, dy, \end{aligned}$$

where $\tau_h w(x) = w(x+h)$, and we have used the translation invariance of the class A_1. Integrating both sides of the inequality (3.3) with respect to h, we get (2.2).

4. Outline of proof of Proposition 1

Let $f \in \mathfrak{S}(\mathbb{R}^n)$. By Calderón-Zygmund decomposition at height $\lambda > 0$ we have a collection $\{Q\}$ of non-overlapping closed dyadic cubes and functions g, b such that

$$f = g + b; \tag{4.1}$$

$$\lambda \le |Q|^{-1} \int_Q |f| \le c\lambda; \tag{4.2}$$

$$v(\cup Q) \le c_v \|f\|_{L^1_v}/\lambda \qquad \text{for all } v \in A_1; \tag{4.3}$$

$$\|g\|_\infty \le c\lambda, \qquad \|g\|_{L^1_v} \le c_v \|f\|_{L^1_v} \qquad \text{for all } v \in A_1; \tag{4.4}$$

$$b = \sum_Q b_Q, \quad \operatorname{supp}(b_Q) \subset Q, \quad \|b_Q\|_{L^1} \le c\lambda |Q|. \tag{4.5}$$

Let a polynomial P be as in Proposition 1. We assume as we may that $M \ge 1$ as in the outline of the proof of Theorem 1 in §2. We write P as in (1.5). Then, let $q(y) = \sum_{|\beta| \le L} c_\beta y^\beta$ be the coefficient of x_1^M. By a rotation of coordinates and a normalization, to prove Proposition 1 we may assume $\max_{|\beta|=L} |c_\beta| = 1$ (see [1, p. 151] and Sublemma 2 in §6).

We take a non-negative $\varphi \in C_0^\infty(\mathbf{R}^n)$ such that

$$\operatorname{supp}(\varphi) \subset \{1/2 \le |x| \le 2\}, \qquad \sum_{j=0}^{\infty} \varphi(2^{-j}x) = 1 \quad \text{if} \quad |x| \ge 1.$$

Put $K_j(x,y) = \varphi(2^{-j}(x-y)) K_\infty(x,y)$, where $K_\infty(x,y) = e^{iP(x,y)} K_\infty(x-y)$ ($K_\infty(x)$ is as in §2) and decompose $K_\infty(x,y)$ as $K_\infty(x,y) = \sum_{j=0}^\infty K_j(x,y)$.

Define

$$V_j(f)(x) = \int K_j(x,y) f(y)\, dy \qquad \text{for} \qquad j \ge 0$$

and put

$$V(f)(x) = \sum_{j=1}^{\infty} V_j(f)(x).$$

Then $T_\infty = V_0 + V$. We have only to deal with V since we easily see that V_0 is bounded on L^1_w ($w^u \in A_1$).

We set (see [3, 4])

$$B_i = \sum_{|Q|=2^{in}} b_Q \quad (i \geq 1), \qquad B_0 = \sum_{|Q|\leq 1} b_Q.$$

Put $\mathcal{U} = \cup \tilde{Q}$, where $\tilde{Q}$ denotes the cube with the same center as Q and with sidelength 100 times that of Q. (Throughout this note we consider the cubes with sides parallel to the coordinate axes.)

When $x \in \mathbf{R}^n \setminus \mathcal{U}$, we observe that

$$\begin{aligned}(4.6)\quad V(b)(x) &= V\left(\sum_{i\geq 0} B_i\right)(x)\\ &= \sum_{i\geq 0}\sum_{j\geq 1}\int K_j(x,y)B_i(y)\,dy = \sum_{i\geq 0}\sum_{j\geq i+1}\int K_j(x,y)B_i(y)\,dy\\ &= \sum_{s\geq 1}\sum_{j\geq s}\int K_j(x,y)B_{j-s}(y)\,dy = \sum_{s\geq 1}\sum_{j\geq s} V_j(B_{j-s})(x).\end{aligned}$$

To prove Proposition 1 we need the following results (Lemmas 2, 3 and 4).

Lemma 2. *Suppose $w \in A_1$. Let $\{L_j\}_{j\geq 1}$ be a family of kernels satisfying*

$$\operatorname{supp}(L_j) \subset \{2^{j-6} \leq |x| \leq 2^{j+6}\}, \quad |L_j(x)| \leq c_1|x|^{-n}, \quad |\nabla L_j(x)| \leq c_2|x|^{-n-1}.$$

Let

$$G_j(f)(x) = \int_{\mathbf{R}^n} e^{iP(x,y)} L_j(x-y) f(y)\,dy.$$

Put

$$E_\lambda^s = \left\{ x \in \mathbf{R}^n : \left|\sum_{j\geq s} G_j(B_{j-s})(x)\right| > \lambda \right\}.$$

Then there exists $\epsilon, \eta > 0$ such that, for any positive integer s,

$$w\left(E^s_{\lambda c_\eta 2^{-\eta s}}\right) \leq c 2^{-\epsilon s}\lambda^{-1}\|f\|_{L^1_w},$$

where c_η is a positive constant satisfying $\sum_{s=1}^\infty c_\eta 2^{-\eta s/2} = 1$.

We shall prove this in §5.

Lemma 3. *Let L_j and G_j be as in Lemma 2. Then, for $j \geq 1$,*

$$\|G_j\|_2 \leq c2^{-j\epsilon} \quad \textit{for some } \epsilon > 0,$$

where $\|G_j\|_2$ denotes the operator norm on L^2.

This follows from Ricci-Stein [5]. See also [8] for an alternative proof.

Lemma 4. *If $w^u \in A_1$, then the operator V is bounded on L^2_w.*

Proof. Let

$$N_j(x) = \varphi(2^{-j}x)K(x), \quad L_j(x) = N_j * \psi_{2^{-j+\delta j}}(x) \quad (\delta > 0),$$

where $\psi \in C^\infty(\mathbf{R}^n)$ which is supported in $\{|x| < 2^{-10}\}$ and satisfying $\int \psi = 1$. Then L_j satisfies all the conditions of Lemma 2 with $c_1 = c2^{n\delta j}$, $c_2 = c2^{(n+1)\delta j}$, and we find

$$\|L_j\|_{L^1} \leq c\,C_1, \tag{4.7}$$

$$\|L_j\|_{L^r} \leq c\,C_r 2^{-jn/u}. \tag{4.8}$$

Put

$$R_j(x) = N_j(x) - L_j(x) = \int (N_j(x) - N_j(x-y))\,\psi_{2^{-j+\delta j}}(y)\,dy.$$

Then, it is easy to see that

$$\|R_j\|_{L^1} \leq c\,\omega_1(2^{-\delta j}) + c2^{-\delta j} \leq c\omega_r(2^{-\delta j}) + c2^{-\delta j}, \tag{4.9}$$

$$\|R_j\|_{L^r} \leq c(\omega_r(2^{-\delta j}) + c2^{-\delta j})2^{-jn/u}. \tag{4.10}$$

Put

$$U_j(f)(x) = \int_{\mathbf{R}^n} e^{iP(x,y)} L_j(x-y)f(y)\,dy, \qquad W_j(f)(x) = \int_{\mathbf{R}^n} e^{iP(x,y)} R_j(x-y)f(y)\,dy.$$

First we estimate U_j. By Hölder's inequality and (4.7), (4.8) we have

$$\|U_j(f)\|^2_{L^2_w} \leq c\int \left(\int |L_j(x-y)|w(x)\,dx\right)|f(y)|^2\,dy \leq c\int |f(y)|^2 M_u(w)(y)\,dy. \tag{4.11}$$

On the other hand, if δ is small enough, by Lemma 3

$$\|U_j(f)\|^2_{L^2} \leq c2^{-\epsilon j}\|f\|^2_2 \quad \text{for some } \epsilon > 0. \tag{4.12}$$

Interpolating between the estimates (4.11) and (4.12), we get

$$\|U_j(f)\|^2_{L^2_{w^\theta}} \leq c2^{-\epsilon(1-\theta)j}\int |f(y)|^2 M_u(w)(y)^\theta\,dy,$$

for $\theta \in (0,1)$. Substituting $w^{1/\theta}$ for w, we have

$$\|U_j(f)\|^2_{L^2_w} \leq c2^{-\epsilon(s-u)j/s}\int |f(y)|^2 M_s(w)(y)\,dy \quad \text{for all } s > u. \tag{4.13}$$

Next we estimate W_j. By Hölder's inequality and (4.9), (4.10)

(4.14)

$$\begin{aligned}\|W_j(f)\|_{L^2_w}^2 &\le c(\omega_r(2^{-\delta j}) + c2^{-\delta j}) \int \left(\int |R_j(x-y)| w(x)\, dx \right) |f(y)|^2\, dy \\ &\le c(\omega_r(2^{-\delta j}) + c2^{-\delta j})^2 \int |f(y)|^2 M_u(w)(y)\, dy.\end{aligned}$$

By (4.13) and (4.14), for all $s > u$,

$$\|V(f)\|_{L^2_w} \le c \sum_{j \ge 1} (\omega_r(2^{-\delta j}) + 2^{-\delta j} + 2^{-\epsilon(s-u)j/(2s)}) \|f\|_{L^2_{M_s(w)}} \le c_s \|f\|_{L^2_{M_s(w)}}.$$

From this we get the conclusion of Lemma 4, since $w^s \in A_1$ for some $s > u$.

Using these results, we can prove Proposition 1. Let N_j and ψ be as in the proof of Lemma 4. For a positive integer s let

$$L_j^{(s)}(x) = N_j * \psi_{2^{-j+\delta s}}(x) \quad (\delta > 0).$$

Put

$$R_j^{(s)}(x) = N_j(x) - L_j^{(s)}(x) = \int (N_j(x) - N_j(x-y))\, \psi_{2^{-j+\delta s}}(y)\, dy.$$

Then $L_j^{(s)}$ is supported in $\{2^{j-6} \le |x| \le 2^{j+6}\}$ and satisfies

$$|L_j^{(s)}(x)| \le c2^{n\delta s}|x|^{-n}, \qquad |\nabla L_j^{(s)}(x)| \le c2^{(n+1)\delta s}|x|^{-n-1}.$$

Set

$$U_j^{(s)}(f)(x) = \int_{\mathbf{R}^n} e^{iP(x,y)} L_j^{(s)}(x-y) f(y)\, dy, \;\; W_j^{(s)}(f)(x) = \int_{\mathbf{R}^n} e^{iP(x,y)} R_j^{(s)}(x-y) f(y)\, dy.$$

Put

$$F_\lambda^s = \left\{ x \in \mathbf{R}^n : \left| \sum_{j \ge s} U_j^{(s)}(B_{j-s})(x) \right| > \lambda \right\}.$$

Then, if $(n+1)\delta < \eta/2$ by Lemma 2

$$w\left(F^s_{c_\eta 2^{-\eta s/2}\lambda}\right) \le c2^{-\epsilon s}\lambda^{-1}\|f\|_{L^1_w}, \tag{4.15}$$

where ϵ, η and c_η are as in Lemma 2. Since $\sum_{s=1}^\infty c_\eta 2^{-\eta s/2} = 1$, we have

$$\left\{ x \in \mathbf{R}^n : \left| \sum_{s \ge 1} \sum_{j \ge s} U_j^{(s)}(B_{j-s})(x) \right| > \lambda \right\} \subset \bigcup_{s \ge 1} F^s_{c_\eta 2^{-\eta s/2}\lambda}.$$

Thus by (4.15)

(4.16)

$$w\left(\left\{x \in \mathbf{R}^n : \left|\sum_{s\geq 1}\sum_{j\geq s} U_j^{(s)}(B_{j-s})(x)\right| > \lambda\right\}\right) \leq \sum_{s\geq 1} w\left(F^s_{c_\eta 2^{-\eta s/2}\lambda}\right) \leq c\lambda^{-1}\|f\|_{L^1_w}.$$

Since

$$\|R_j^{(s)}\|_{L^r} \leq c(\omega_r(2^{-\delta s}) + 2^{-\delta s})2^{-jn/u},$$

by Hölder's inequality and the condition that $w^u \in A_1$ we find

$$\left\|\sum_{j\geq s} W_j^{(s)}(B_{j-s})\right\|_{L^1_w} \leq c\left(\omega_r(2^{-\delta s}) + 2^{-\delta s}\right)\|f\|_{L^1_w}.$$

Thus, by Chebyshev's inequality we have

$$w\left(\left\{x \in \mathbf{R}^n : \left|\sum_{s\geq 1}\sum_{j\geq s} W_j^{(s)}(B_{j-s})(x)\right| > \lambda\right\}\right) \leq c\left(\sum_{s\geq 1}\left(\omega_r(2^{-\delta s}) + 2^{-\delta s}\right)\right)\lambda^{-1}\|f\|_{L^1_w}. \tag{4.17}$$

By (4.6), (4.16) and (4.17) we have

$$w\left(\{x \in \mathbf{R}^n \setminus \mathcal{U} : |V(b)(x)| > 2\lambda\}\right) \leq c\lambda^{-1}\|f\|_{L^1_w}. \tag{4.18}$$

By (4.3) we see that

$$w(\mathcal{U}) \leq c_w\lambda^{-1}\|f\|_{L^1_w}. \tag{4.19}$$

By Lemma 4 and (4.4)

$$w\left(\{x \in \mathbf{R}^n : |V(g)(x)| > \lambda\}\right) \leq c\lambda^{-1}\|f\|_{L^1_w}. \tag{4.20}$$

Combining (4.18), (4.19) and (4.20), we conclude the proof of Proposition 1.

5. Proof of Lemma 2

In this section we shall prove Lemma 2 in §4. For $k, m \geq 1$, put

$$H_{km}(x,y) = \int e^{-iP(z,x)+iP(z,y)}\overline{L}_k(z-x)L_m(z-y)\,dz. \tag{5.1}$$

Then $G_k^*G_m(f)(x) = \int H_{km}(x,y)f(y)\,dy$, where G_k^* denotes the adjoint of G_k.

Lemma 5. *Let $k \geq m \geq 1$. Then, $H_{km}(x,y) = 0$ unless $|x-y| \leq 2^{k+7}$; and*

(1) $$|H_{km}(x,y)| \leq c2^{-kn},$$

(2) $$|H_{km}(x,y)| \leq c2^{-kn}2^{-m}|q(x)-q(y)|^{-1/M}.$$

Proof. We prove only the estimate of (2) since the other assertions immediately follow from the definition of H_{km} in (5.1). We first note that

$$(\partial/\partial z_1)^M (P(z,x) - P(z,y)) = M!(q(x)-q(y)).$$

Hence, from van der Corput's lemma it follows that

$$\left| \int_a^b e^{i(P(z,x)-P(z,y))}\, dz_1 \right| \leq c|q(x)-q(y)|^{-1/M},$$

for any a and b (see [1, p.152]).

Therefore by integration by parts in variable z_1 in the formula of (5.1) we get the conclusion.

For the rest of this note, we denote by $P(x)$ a real-valued polynomial on $\mathbf{R}^n$.

Definition 1. For a polynomial $P(x) = \sum_{|\alpha| \leq N} a_\alpha x^\alpha$ of degree N, define

$$\|P\| = \max_{|\alpha|=N} |a_\alpha|.$$

Definition 2. For a polynomial P and $\beta > 0$, let

$$\mathcal{R}(P,\beta) = \{x \in \mathbf{R}^n : |P(x)| \leq \beta\}.$$

Let $d(E,F)$ denote the distance between sets E and F. We now state a geometrical lemma for polynomials, which will be proved in §6.

Lemma 6. *Let k, m be integers such that $k \geq m$. Suppose $N \geq 1$. Then, for any polynomial P of degree N satisfying $\|P\| = 1$ and for any $\gamma > 0$, there exists a positive constant $C_{n,N,\gamma}$ depending only on n, N and γ such that*

$$\left|\left\{x \in B(a,2^k) : d\left(x, \mathcal{R}(P, 2^{Nm})\right) \leq \gamma 2^m\right\}\right| \leq C_{n,N,\gamma} 2^{(n-1)k} 2^m$$

uniformly in $a \in \mathbf{R}^n$.

Let $\lambda > 0$ and let $\{\mathcal{B}_j\}_{j \geq 0}$ be a family of measurable functions such that

(5.2) $$\int_Q |\mathcal{B}_j| \leq \lambda |Q|$$

for all cubes Q in $\mathbf{R}^n$ with sidelength $\ell(Q) = 2^j$.

Then we have the following.

Lemma 7. *Let the kernels H_{ji} be as in Lemma 5. Then, we can find a constant c such that*

$$\sum_{i=s}^{j} \sup_{x\in\mathbf{R}^n} \left| \int \mathcal{B}_{i-s}(y) H_{ji}(x,y)\, dy \right| \le c\lambda 2^{-s}$$

for all integers j and s such that $0 < s \le j$.

Definition 3. For $m \in \mathbf{Z}$ (the set of all integers), let $\mathcal{D}_m$ be the family of all closed dyadic cubes Q with sidelength $\ell(Q) = 2^m$.

Proof of Lemma 7. Fix $x \in \mathbf{R}^n$. Let

$$\mathcal{F} = \left\{ Q \in \mathcal{D}_{i-s} : Q \cap B(x, 2^{j+2}) \ne \emptyset \right\} \quad (0 < s \le i \le j).$$

Then clearly $\sum_{Q\in\mathcal{F}} |Q| \le c2^{jn}$.

Decompose $\mathcal{F} = \mathcal{F}_0 \cup \mathcal{F}_1$, where

$$\mathcal{F}_0 = \left\{ Q \in \mathcal{F} : Q \cap \mathcal{R}(q(\cdot) - q(x), 2^{L(i-s)}) \ne \emptyset \right\}$$

and $\mathcal{F}_1 = \mathcal{F} \setminus \mathcal{F}_0$. Then by Lemma 6 we have

$$\sum_{Q\in\mathcal{F}_0} |Q| \le c2^{(n-1)j} 2^{i-s}. \tag{5.3}$$

By Lemma 5 (1), (5.2) and (5.3), we see that

$$\begin{aligned} \sum_{Q\in\mathcal{F}_0} \int_Q |\mathcal{B}_{i-s}(y) H_{ji}(x,y)|\, dy &\le c2^{-jn} \sum_{Q\in\mathcal{F}_0} \int_Q |\mathcal{B}_{i-s}(y)|\, dy \\ &\le c2^{-jn}\lambda \sum_{Q\in\mathcal{F}_0} |Q| \le c2^{-jn}\lambda 2^{(n-1)j}2^{i-s} = c\lambda 2^{i-j-s}. \end{aligned} \tag{5.4}$$

Next, by Lemma 5 (2), (5.2) and the estimate $\sum_{Q\in\mathcal{F}_1} |Q| \le c2^{jn}$, we have

$$\begin{aligned} \sum_{Q\in\mathcal{F}_1} \int_Q |\mathcal{B}_{i-s}(y) H_{ji}(x,y)|\, dy &\le c2^{-jn}2^{-i}2^{-L(i-s)/M} \sum_{Q\in\mathcal{F}_1} \int_Q |\mathcal{B}_{i-s}(y)|\, dy \\ &\le c2^{-jn}2^{-i}2^{-L(i-s)/M}\lambda \sum_{Q\in\mathcal{F}_1} |Q| \le c\lambda 2^{-i}2^{-L(i-s)/M}. \end{aligned} \tag{5.5}$$

From (5.4) and (5.5) it follows that

$$\begin{aligned} \int |\mathcal{B}_{i-s}(y) H_{ji}(x,y)|\, dy &= \sum_{Q\in\mathcal{F}} \int_Q |\mathcal{B}_{i-s}(y) H_{ji}(x,y)|\, dy \\ &= \sum_{\nu=0}^{1} \sum_{Q\in\mathcal{F}_\nu} \int_Q |\mathcal{B}_{i-s}(y) H_{ji}(x,y)|\, dy \le c\lambda \left(2^{i-j-s} + 2^{-i}2^{-L(i-s)/M} \right). \end{aligned}$$

Thus we see that

$$\sum_{i=s}^{j} \sup_{x\in\mathbf{R}^n} \int |\mathcal{B}_{i-s}(y) H_{ji}(x,y)|\, dy \le c\lambda \sum_{i=s}^{j} \left(2^{i-j-s} + 2^{-i}2^{-L(i-s)/M} \right) \le c\lambda 2^{-s}.$$

This completes the proof of Lemma 7.

By Lemma 7 we readily get the following.

Lemma 8. *Let $\{\mathcal{B}_j\}_{j\geq 0}$ be as in Lemma 7. Suppose $\sum_{j\geq 0}\|\mathcal{B}_j\|_{L^1}<\infty$. Let G_j be as in Lemma 2. Then, for any positive integer s, we have*

$$\left\|\sum_{j\geq s} G_j(\mathcal{B}_{j-s})\right\|_{L^2}^2 \leq c\lambda 2^{-s}\sum_{j\geq 0}\|\mathcal{B}_j\|_{L^1}.$$

Proof. Let $\langle\cdot,\cdot\rangle$ denote the inner product in L^2. Using Lemma 7, we see that

$$\begin{aligned}\left\|\sum_{j\geq s} G_j(\mathcal{B}_{j-s})\right\|_{L^2}^2 &\leq 2\sum_{j\geq s}\sum_{i=s}^{j}|\langle G_j(\mathcal{B}_{j-s}), G_i(\mathcal{B}_{i-s})\rangle| \leq 2\sum_{j\geq s}\sum_{i=s}^{j}\left|\langle \mathcal{B}_{j-s}, G_j^*G_i(\mathcal{B}_{i-s})\rangle\right| \\ &\leq 2\sum_{j\geq s}\sum_{i=s}^{j}\|\mathcal{B}_{j-s}\|_{L^1}\|G_j^*G_i(\mathcal{B}_{i-s})\|_{L^\infty} \leq c\lambda 2^{-s}\sum_{j\geq s}\|\mathcal{B}_{j-s}\|_{L^1}.\end{aligned}$$

This completes the proof of Lemma 8.

Definition 4. For each $j\geq 0$, let $\mathcal{G}_j$ be a family of non-overlapping closed dyadic cubes Q such that $\ell(Q)\leq 2^j$. We suppose that if $Q\in\mathcal{G}_j$, $R\in\mathcal{G}_k$ and $j\neq k$, then Q and R are non-overlapping and that $\sum_{j\geq 0}\sum_{Q\in\mathcal{G}_j}|Q|<\infty$. Put $\mathcal{G}=\cup_{j\geq 0}\mathcal{G}_j$.

Let $\lambda>0$. To each $Q\in\mathcal{G}$ we associate $f_Q\in L^1$ such that

$$\int |f_Q|\leq \lambda|Q|, \qquad \mathrm{supp}(f_Q)\subset Q.$$

We define $\mathcal{A}_i=\sum_{Q\in\mathcal{G}_i} f_Q$.

Lemma 9. *Let G_j be as in Lemma 2 and let v be a locally integrable positive function. Then for a positive integer s we have*

$$\left\|\sum_{j\geq s} G_j(\mathcal{A}_{j-s})\right\|_{L^1_v} \leq c\lambda\sum_{Q\in\mathcal{G}}|Q|\inf_Q M(v),$$

where $\inf_Q M(v)=\inf_{x\in Q} M(v)(x)$.

Proof. We easily see that

$$\begin{aligned}\left\|\sum_{j\geq s} G_j\left(\mathcal{A}_{j-s}\right)\right\|_{L^1_v} &\leq \sum_j\int|\mathcal{A}_{j-s}(y)|\left(\int|L_j(x-y)|v(x)\,dx\right)\,dy \\ &\leq \sum_j\sum_{Q\in\mathcal{G}_{j-s}}\int|f_Q(y)|\inf_{z\in Q}M(v)(z)\,dy \leq c\sum_{Q\in\mathcal{G}}\lambda|Q|\inf_Q M(v).\end{aligned}$$

We prove Lemma 2 by the estimates of Lemma 8 and Lemma 9. We slightly modify the interpolation argument of [9].

Lemma 10. *Let $\mathcal{F}$ denote the family of dyadic cubes arising from the Calderón-Zygmund decomposition in §4. Define a set E_λ^s as in Lemma 2. Then, for all $t > 0$, we have*

$$\int_{E_\lambda^s} \min(v(x), t)\,dx \le c \sum_{Q\in\mathcal{F}} |Q| \min\left(t2^{-s}, \inf_Q M(v)\right), \tag{5.6}$$

where s is a positive integer and v is a locally integrable positive function.

Proof. For $t > 0$, set $\mathcal{F}_t = \{Q \in \mathcal{F} : \inf_Q M(v) < t2^{-s}\}$ and $\mathcal{F}_t^* = \mathcal{F} \setminus \mathcal{F}_t$. Put

$$B_j' = \sum_{\substack{\ell(Q)=2^j\\ Q\in\mathcal{F}_t}} b_Q, \quad B_j'' = \sum_{\substack{\ell(Q)=2^j\\ Q\in\mathcal{F}_t^*}} b_Q \quad (j \ge 1); \qquad B_0' = \sum_{\substack{|Q|\le 1\\ Q\in\mathcal{F}_t}} b_Q, \quad B_0'' = \sum_{\substack{|Q|\le 1\\ Q\in\mathcal{F}_t^*}} b_Q.$$

Define

$$E_\lambda' = \left\{ \left| \sum_{j\ge s} G_j\left(B_{j-s}'\right) \right| > \lambda \right\}, \quad E_\lambda'' = \left\{ \left| \sum_{j\ge s} G_j\left(B_{j-s}''\right) \right| > \lambda \right\}.$$

Then we find $E_\lambda^s \subset E_{\lambda/2}' \cup E_{\lambda/2}''$, since $B_j = B_j' + B_j''$, and so

$$\begin{aligned}\int_{E_\lambda^s} \min(v(x), t)\,dx &\le \int_{E_{\lambda/2}'} \min(v(x), t)\,dx + \int_{E_{\lambda/2}''} \min(v(x), t)\,dx\\ &\le \int_{E_{\lambda/2}'} v(x)\,dx + \int_{E_{\lambda/2}''} t\,dx =: I + II.\end{aligned}$$

By Lemma 9 with $\mathcal{A}_j = cB_j'$, we get

$$I \le c \sum_{Q\in\mathcal{F}_t} |Q| \inf_Q M(v) = c \sum_{Q\in\mathcal{F}_t} |Q| \min\left(t2^{-s}, \inf_Q M(v)\right).$$

By Lemma 8 with $\mathcal{B}_j = cB_j''$, we have

$$II \le ct2^{-s} \sum_{Q\in\mathcal{F}_t^*} |Q| = c \sum_{Q\in\mathcal{F}_t^*} |Q| \min\left(t2^{-s}, \inf_Q M(v)\right).$$

Combining the estimates for I and II, we conclude the proof of Lemma 10.

Now we finish the proof of Lemma 2. Multiplying both sides of the inequality (5.6) by $t^{-\theta}$ ($\theta \in (0,1)$), then integrating them on $(0,\infty)$ with respect to the measure dt/t, and using

$$\int_0^\infty \min(A, t)t^{-\theta}\,\frac{dt}{t} = c_\theta A^{1-\theta} \quad (A > 0),$$

we get

$$\begin{aligned}\int_{E_\lambda^s} v(x)^{1-\theta}\,dx &\le c \sum_{Q\in\mathcal{F}} |Q| 2^{-\theta s} \inf_Q M(v)^{1-\theta}\\ &\le c\lambda^{-1}2^{-\theta s} \sum_{Q\in\mathcal{F}} \inf_Q M(v)^{1-\theta} \int_Q |f(x)|\,dx \le c\lambda^{-1}2^{-\theta s} \int |f(x)| M(v)(x)^{1-\theta}\,dx,\end{aligned} \tag{5.7}$$

where the second inequality follows from (4.2).

If $w \in A_1$, then $w^{1+\delta} \in A_1$ for some $\delta > 0$; so substituting $w^{1+\delta}$ for v and taking θ such that $1-\theta = (1+\delta)^{-1}$ in (5.7), we get

$$w\left(E_\lambda^s\right) \le c\lambda^{-1} 2^{-s\delta/(1+\delta)} \|f\|_{L^1_w}. \tag{5.8}$$

Checking the constants appearing in the proof of (5.8) and replacing L_j by $c2^{\eta s} L_j$, we get the desired estimate of Lemma 2.

6. Proof of Lemma 6

Our proof is an application of the method for the proof of [1, Lemma 4.1]. We use some tools and results given in [1].

Definition 5. Suppose $n \ge 2$. Let

$$S_m = \{Q_m + (0,0,\dots,0,j) : j \in \mathbf{Z}\},$$

where $m = (m_1, m_2, \dots, m_{n-1}) \in \mathbf{Z}^{n-1}$ and $Q_m = [0,1]^n + (m_1, m_2, \dots, m_{n-1}, 0)$. We call S_m a strip.

Definition 6. Suppose $n \ge 2$. For $m \in \mathbf{Z}^{n-1}$, we define

$$I_m = \{Q_m + (0,0,\dots,0,j) : j_1 < j < j_2\},$$

where $j_1, j_2 \in \mathbf{Z} \cup \{-\infty, \infty\}$ and Q_m is as in Definition 5. We call I_m an interval.

Definition 7. For a set $E \subset \mathbf{R}^n$, we put

$$\mathcal{N}(E) = \{x \in \mathbf{R}^n : d(x,E) \le 1\}.$$

Let P be a polynomial of degree N as in Lemma 6. We consider $\mathcal{R}(P,\beta)$ for $\beta > 0$ (see Definition 2).

Lemma 11. *Suppose that $n \ge 2$ and $N \ge 1$. There exists a positive integer $C_{n,N}$ depending only on n and N such that for $i = 1, 2, \dots, C_{n,N}$ we can find $U_i \in O(n)$ (the orthogonal group) and families of cubes $J_{m,i} \subset S_m$ ($m \in \mathbf{Z}^{n-1}$) so that*

(1) $\mathcal{N}(\mathcal{R}(P,\beta)) \subset \bigcup_{i=1}^{C_{n,N}} U_i(\mathcal{L}_i)$, *where*

$$\mathcal{L}_i = \cup \left\{ Q : Q \in \bigcup_{m \in \mathbf{Z}^{n-1}} J_{m,i} \right\};$$

(2) $\operatorname{card}(J_{m,i}) \le c$ *for some constant c depending only on n, N and β.*

Remark 1. If Lemma 11 holds, then we have, for any $\gamma > 0$,

$$\{x : d(x, \mathcal{R}(P,\beta)) \le \gamma\} \subset \bigcup_{i=1}^{C_{n,N,\gamma}} U_i(\mathcal{L}_i)$$

for some positive integer $C_{n,N,\gamma}$ depending only on n, N and γ, where U_i and $\mathcal{L}_i$ are as in Lemma 11. This can be proved by considering a finite number of polynomials which are defined by translating P and by applying Lemma 11 to each of them. (See [1, p. 149].)

To prove Lemma 11, we need the following results given in [1].

Sublemma 1. *Suppose $n \geq 2$. For any positive integer N, there exists a positive integer $C_{n,N}$ depending only on n and N such that for any strip S, any polynomial P of degree N and any $\gamma > 0$*

$$\{Q \in S : Q \cap \mathcal{R}(P, \gamma) \neq \emptyset\}$$

is a union of at most $C_{n,N}$ intervals. (See Lemma 4.2 of [1].)

Sublemma 2. *Suppose $n \geq 2$. For any positive integer N, there exist positive constants $A_{n,N}$ and $B_{n,N}$ depending only on n and N such that*

$$A_{n,N}\|P\| \leq \|P \circ \Xi\| \leq B_{n,N}\|P\|$$

for all polynomial P of degree N and all $\Xi \in O(n)$, where $P \circ \Xi(x) = P(\Xi x)$.

Sublemma 3. *Suppose $n \geq 2$. For any positive integer N, there exists a positive constant $C_{n,N}$ depending only on n and N such that for any polynomial P of degree N we can find $\Theta \in O(n)$ so that*

$$\min_{1 \leq j \leq n} \|D_j(P \circ \Theta)\| \geq C_{n,N}\|P \circ \Theta\|,$$

where $D_j = \partial / \partial x_j$.

Now we prove Lemma 11. We use induction on the polynomial degree N. Let $A(N)$ be the assertion of Lemma 11 for polynomials of degree N.

Proof of $A(1)$. Let $P(x) = \sum_{i=1}^{n} a_i x_i + b$. First, we consider the case $|a_n| = 1$. Now we show that if I is an interval such that each cube of I intersects $\mathcal{R}(P, \beta)$, then $\mathrm{card}(I) \leq c$ for some c depending only on n and β. Let $y \in Q \in I$ satisfy $|P(y)| \leq \beta$. We note that

$$P(y + de_n) - P(y) = da_n \quad \text{for} \quad d \in \mathbf{R},$$

where e_j is the element of $\mathbf{R}^n$ whose jth coordinate is 1 and whose other coordinates are all 0. Therefore, if $y + de_n \in Q' \in I$, we see that

$$\inf_{z \in Q'} |P(z)| \geq |P(y + de_n)| - \sum_{i=1}^{n} |a_i| \geq |da_n| - \beta - \sum_{i=1}^{n} |a_i| \geq |d| - \beta - n.$$

This easily implies that $\mathrm{card}(I) \leq c$.

By this and Sublemma 1, there exists a constant c depending only on n and β such that

$$\mathrm{card}\left(\{Q \in S : Q \cap \mathcal{R}(P, \beta) \neq \emptyset\}\right) \leq c$$

for all strips S.

Therefore, if we put

$$J_m = \{Q \in S_m : d(Q, \mathcal{R}(P, \beta)) \leq 1\},$$

then $\mathrm{card}(J_m) \leq c$ for some c depending only on n and β ; and $\mathcal{N}(\mathcal{R}(P, \beta)) \subset \mathcal{L}$, where

$$\mathcal{L} = \cup \left\{Q : Q \in \bigcup_{m \in \mathbf{Z}^{n-1}} J_m\right\}.$$

Next, we consider any polynomial P of degree 1 such that $\|P\| = 1$. Then if $P_1(x) = P(Ux)$ for a suitable $U \in O(n)$, we have $D_n P_1 = 1$. Hence, by what we have already proved we get $\mathcal{N}(\mathcal{R}(P_1, \beta)) \subset \mathcal{L}$. It follows that $\mathcal{N}(\mathcal{R}(P, \beta)) \subset U(\mathcal{L})$ since $\mathcal{N}(\mathcal{R}(P \circ U, \beta)) = U^{-1}\mathcal{N}(\mathcal{R}(P, \beta))$. This completes the proof of $A(1)$.

Now we assume $A(N-1)$ ($N \geq 2$) and prove $A(N)$. For a polynomial P of degree N such that $\|P\| = 1$, we take $\Theta \in O(n)$ as in Sublemma 3. Put

$$E_0 = \mathcal{R}(P \circ \Theta, \beta) \cap \left(\bigcup_{j=1}^{n} \mathcal{R}(D_j(P \circ \Theta), \beta) \right);$$

and for $\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_n) \in \{-1, 1\}^n$ put

$$E_\kappa = \{x \in \mathcal{R}(P \circ \Theta, \beta) : \kappa_j D_j(P \circ \Theta)(x) > \beta \quad \text{for} \quad j = 1, 2, \ldots, n\}.$$

Then

$$\mathcal{R}(P \circ \Theta, \beta) = E_0 \cup \left(\bigcup_{\kappa \in \{-1,1\}^n} E_\kappa \right)$$

and so

$$\mathcal{N}(\mathcal{R}(P \circ \Theta, \beta)) = \mathcal{N}(E_0) \cup \left(\bigcup_{\kappa \in \{-1,1\}^n} \mathcal{N}(E_\kappa) \right). \tag{6.1}$$

We separately treat the $2^n + 1$ sets of the right hand side.

First, clearly

$$\mathcal{N}(E_0) \subset \bigcup_{j=1}^{n} \mathcal{N}(\mathcal{R}(D_j(P \circ \Theta), \beta)). \tag{6.2}$$

Since $C_j = \|D_j(P \circ \Theta)\| \sim 1$ (this means that $c^{-1} \leq \|D_j(P \circ \Theta)\| \leq c$ for some $c > 1$ depending only on n and N) and $\mathcal{R}(D_j(P \circ \Theta), \beta) = \mathcal{R}(C_j^{-1} D_j(P \circ \Theta), C_j^{-1}\beta)$, we can apply the induction hypothesis $A(N-1)$ to the right hand side of (6.2).

Next, we fix κ and consider $\mathcal{N}(E_\kappa)$. Take $O_\kappa \in O(n)$ such that $O_\kappa(e_n) = n^{-1/2}\kappa$. Define

$$\mathcal{D}_0^* = \mathcal{D}_0 \setminus \left\{ Q \in \mathcal{D}_0 : \left(\bigcup_{j=1}^{n} \mathcal{R}((D_j(P \circ \Theta)) \circ O_\kappa, \beta) \right) \cap Q \neq \emptyset \right\}.$$

Since $\|(D_j(P \circ \Theta)) \circ O_\kappa\| \sim 1$ by Sublemmas 2 and 3, we can apply the hypothesis $A(N-1)$ along with Remark 1 to

$$G = \cup \left\{ Q \in \mathcal{D}_0 : \left(\bigcup_{j=1}^{n} \mathcal{R}((D_j(P \circ \Theta)) \circ O_\kappa, \beta) \right) \cap Q \neq \emptyset \right\}$$

to get

$$\mathcal{N}(G) \subset \cup_i U_i'(\mathcal{L}_i') \tag{6.3}$$

for some $U_i' \in O(n)$ and for some $\mathcal{L}_i'$ such that

$$\mathcal{L}_i' = \cup \left\{ Q : Q \in \bigcup_{m \in \mathbf{Z}^{n-1}} J_{m,i}' \right\}$$

for some $J_{m,i}'$ $(\subset S_m)$ satisfying $\mathrm{card}(J_{m,i}') \leq c$.

Therefore we have only to consider $O_\kappa^{-1}(E_\kappa) \cap (\cup \mathcal{D}_0^*)$. First, we note that if $O_\kappa^{-1}(E_\kappa)$ intersects Q, $Q \in \mathcal{D}_0^*$, then

$$\min_{1 \leq j \leq n} \kappa_j D_j(P \circ \Theta)(O_\kappa y) > \beta \quad \text{for all} \quad y \in Q. \tag{6.4}$$

This can be seen as follows. Suppose that there are j_0 and $y_0 \in Q$ such that $\kappa_{j_0} D_{j_0}(P \circ \Theta)(O_\kappa y_0) \leq \beta$. Then, since we have $\kappa_{j_0} D_{j_0}(P \circ \Theta)(O_\kappa x) > \beta$ for some $x \in Q$, by the intermediate value theorem we can find $z \in Q$ such that $|D_{j_0}(P \circ \Theta)(O_\kappa z)| \leq \beta$. This contradicts the fact that $Q \in \mathcal{D}_0^*$.

By (6.4) we have

$$\begin{aligned} O_\kappa^{-1}(E_\kappa) \cap (\cup \mathcal{D}_0^*) \subset \cup \Big\{ Q \in \mathcal{D}_0 : \min_{1 \leq j \leq n} \kappa_j D_j(P \circ \Theta)(O_\kappa y) > \beta \quad \text{for all} \quad y \in Q \\ \text{and} \quad \mathcal{R}(P \circ \Theta \circ O_\kappa, \beta) \cap Q \neq \emptyset \Big\}. \end{aligned} \tag{6.5}$$

For a strip S, put

$$\begin{aligned} \mathcal{E} = \Big\{ Q \in S : \min_{1 \leq j \leq n} \kappa_j D_j(P \circ \Theta)(O_\kappa y) > \beta \quad \text{for all} \quad y \in Q \\ \text{and} \quad \mathcal{R}(P \circ \Theta \circ O_\kappa, \beta) \cap Q \neq \emptyset \Big\}. \end{aligned}$$

We shall show $\mathrm{card}(\mathcal{E}) \leq C_{n,N}$.

We first see that $\mathcal{E}$ is a union of at most $C_{n,N}$ intervals. Put

$$\begin{aligned} \mathcal{E}' = \Big\{ Q \in S : \min_{1 \leq j \leq n} |D_j(P \circ \Theta)(O_\kappa y)| > \beta \quad \text{for all} \quad y \in Q \\ \text{and} \quad \mathcal{R}(P \circ \Theta \circ O_\kappa, \beta) \cap Q \neq \emptyset \Big\}. \end{aligned}$$

Then

$$\begin{aligned} \mathcal{E}' = \left(\bigcap_{j=1}^{n} (S \setminus \{Q \in S : \mathcal{R}((D_j(P \circ \Theta)) \circ O_\kappa, \beta) \cap Q \neq \emptyset\}) \right) \\ \cap \{Q \in S : \mathcal{R}(P \circ \Theta \circ O_\kappa, \beta) \cap Q \neq \emptyset\}. \end{aligned}$$

We observe that the complement of a finite union of intervals in a strip S is also a finite union of intervals, and the intersection of finite unions of intervals is also a finite union of intervals. Hence, by Sublemma 1 we see that $\mathcal{E}'$ is a union of at most $C_{n,N}$ intervals: $\mathcal{E}' = \cup_i J_i$.

Take any J_i. Then by the intermediate value theorem we have either

$$\min_{1\le j\le n} \kappa_j D_j(P\circ\Theta)(O_\kappa y) > \beta \quad \text{for all} \quad y \in \cup\{Q : Q\in J_i\}$$

or

$$\min_{1\le j\le n} \kappa_j D_j(P\circ\Theta)(O_\kappa y) < -\beta \quad \text{for all} \quad y \in \cup\{Q : Q\in J_i\}\,.$$

Thus $\mathcal{E}$ is a union of a subfamily $\{I_i\}$ of $\{J_i\}$: $\mathcal{E} = \cup_i I_i$.

Let I be any interval in $\{I_i\}$. We need the following (see [1, p. 151]).

Sublemma 4. *There exists a constant c_n depending only on n such that if $x, y \in I$ and $y_n - x_n \ge c_n$, then*

$$y - x = \sum_{i=1}^{n} \lambda_i O_\kappa^{-1} e_i$$

for some $\lambda_i \in \mathbf{R}$ such that $\kappa_i\lambda_i \ge 3$.

Proof. We see that

$$\begin{aligned} O_\kappa(y-x) &= \sum_{i=1}^{n}(y_i-x_i)O_\kappa e_i = \sum_{i=1}^{n-1}(y_i-x_i)O_\kappa e_i + (y_n-x_n)n^{-1/2}\kappa \\ &= \sum_{i=1}^{n}\left(n^{-1/2}(y_n-x_n)\kappa_i + b_i\right)e_i \end{aligned}$$

for some $b_i \in \mathbf{R}$ such that $|b_i| \le c$, which is feasible since $|y_i - x_i| \le 1$ for $i = 1, 2, \ldots, n-1$. This readily implies the conclusion.

Put $Y = P\circ\Theta\circ O_\kappa$. Then $\nabla Y(x) = O_\kappa^{-1}(\nabla(P\circ\Theta)(O_\kappa x))$; so, if $x, y \in I$ and $y_n - x_n \ge c_n$, by Sublemma 4 we have

$$\begin{aligned} Y(y) - Y(x) &= \int_0^1 \langle y-x, (\nabla Y)(x+t(y-x))\rangle\, dt \\ &= \int_0^1 \sum_{i=1}^{n} \lambda_i \left\langle O_\kappa^{-1}e_i, O_\kappa^{-1}\left(\nabla(P\circ\Theta)(O_\kappa(x+t(y-x)))\right)\right\rangle dt \\ &= \int_0^1 \sum_{i=1}^{n} \lambda_i D_i(P\circ\Theta)(O_\kappa(x+t(y-x)))\, dt \ge \sum_{i=1}^{n}\lambda_i\kappa_i\beta \ge 3n\beta > 3\beta, \end{aligned}$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product in $\mathbf{R}^n$. Since $\mathcal{R}(Y,\beta)\cap Q \ne \emptyset$ for all $Q \in I$, we can conclude that $\operatorname{card}(I) \le c_n + 3$.

Combining the above results, we have $\operatorname{card}(\mathcal{E}) \le C_{n,N}$ as claimed. From this and (6.5) we easily see that

$$\mathcal{N}\left(O_\kappa^{-1}(E_\kappa)\cap(\cup\mathcal{D}_0^*)\right) \subset \mathcal{L}, \tag{6.6}$$

where $\mathcal{L} = \cup\left\{Q : Q \in \bigcup_{m\in\mathbf{Z}^{n-1}} J_m\right\}$ for some $J_m \subset S_m$ with $\operatorname{card}(J_m) \le C_{n,N}$.

By (6.3) and (6.6) we have

$$\mathcal{N}\left(O_\kappa^{-1}(E_\kappa)\right) \subset \mathcal{N}(G) \cup \mathcal{N}\left(O_\kappa^{-1}(E_\kappa) \cap (\cup \mathcal{D}_0^*)\right) \subset (\cup_i U_i'(\mathcal{L}_i')) \cup \mathcal{L};$$

and so, observing $\mathcal{N}\left(O_\kappa^{-1}(E_\kappa)\right) = O_\kappa^{-1}\mathcal{N}(E_\kappa)$,

$$\mathcal{N}(E_\kappa) \subset (\cup_i O_\kappa U_i'(\mathcal{L}_i')) \cup O_\kappa(\mathcal{L}). \tag{6.7}$$

Since $\mathcal{N}(\mathcal{R}(P \circ \Theta, \beta)) = \Theta^{-1}\mathcal{N}(\mathcal{R}(P, \beta))$, by (6.1), (6.2) with $A(N-1)$ and (6.7) we get $A(N)$. This completes the proof of Lemma 11.

Proof of Lemma 6. We see that $\mathcal{R}(P, 2^{Nm}) = 2^m \mathcal{R}(\tilde{P}, 1)$, where

$$\tilde{P}(x) = 2^{-Nm} P(2^m x).$$

Note that $\|\tilde{P}\| = 1$. (See [1, p. 151].) This observation enables us to assume $m = 0$ to prove Lemma 6. Clearly, we may also assume $\gamma = 1$.

Thus it is sufficient to show, for $k \geq 0$,

$$\left|\left\{x \in B(a, 2^k) : d(x, \mathcal{R}(P, 1)) \leq 1\right\}\right| \leq C_{n,N} 2^{(n-1)k} \tag{6.8}$$

uniformly in $a \in \mathbf{R}^n$.

If $n = 1$, (6.8) easily follows from Chanillo-Christ [1, Lemma 3.2] (see also [2]). Suppose $n \geq 2$. Then, (6.8) follows from Lemma 11 with $\beta = 1$ and the obvious estimate:

$$\left|B(a, 2^k) \cap U_i(\mathcal{L}_i)\right| \leq c 2^{(n-1)k},$$

where $U_i(\mathcal{L}_i)$ is as in Lemma 11. This completes the proof of Lemma 6.

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