Stability estimates and Lagrange-Galerkin schemes for Navier-Stokes type models of flow in non-homogeneous porous media

Dissertation

Stability estimates and Lagrange-Galerkin schemes for Navier-Stokes type models of flow in non-homogeneous porous media

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Abstract

To approach the phenomena in the geothermal reservoir, we deal with the equations of non-steady flow in the non-homogeneous porous media proposed by C.T. Hsu and P. Cheng in (1990). However, any mathematical and numerical analysis for their model has not been studied yet. This thesis aims to prove the L^2-stability estimate of the model, to propose an appropriate numerical method, and to perform simulations of fluid flow in simple and complex structures of the porosity.

The stability estimate is obtained gratitude to the presence of a non-linear drag force term in the model which corresponds to the Forchheimer friction term. We used this term to control the non-linear convection term with the non-homogeneous porosity. The obtained estimate also gives a consistent decay property of the kinetic energy of the fluid due to the viscosity and microscopic friction.

As a numerical scheme, we proposed a characteristic finite element method (Lagrange–Galerkin scheme with the Adams-Bashforth time discretization). We derive the Lagrange–Galerkin scheme by extending the idea of the method of characteristics by introducing the macroscopic average velocity to overcome the difficulty which comes from the non-homogeneous porosity.

To check the order of convergence of the scheme, we constructed an

exact solution and numerically computed the error. The results suggest that our scheme has second-order accuracy both in space and in time. Several numerical simulations in simple and complex structures of porosity were also given by the Lagrange-Galerkin scheme, and qualitatively satisfactory fluid profiles in those structures were reproduced.

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Chapter 1

Introduction

1.1 Motivations

Fluid flow and heat transfer in porous media have received significant attention in many kinds of applications such as in geophysics, petroleum engineering, and geothermal engineering, cf., e.g., [6; 14; 15]. In geothermal engineering, simulation of fluid flow and heat transfer in porous media is a useful tool not only for the pre-exploration process but also during the exploration process. For the pre-exploration process, simulation can be used to predict how much electricity can be produced and also to determine the lifetime of the reservoir. To that simulation, we use physical parameters such as pressure, temperature, density, porosity, size of the reservoir, and the type of reservoir obtained from seismic data as an input parameter. From this simulation, we can determine the feasibility of a reservoir to be explored. During exploration, simulations are used to predict the pressure and temperature changes in the reservoir because of the injection and extraction processes. The injection process is needed to maintain the balance of mass in a reservoir and to supply the water, which will be heated by the reservoir. In the extraction process, the fluid and steam are produced from the reservoir and used to generate electricity.

The Darcy equations give the most standard mathematical model widely employed for the underground water steady flow. These equations arise from Darcy's law [14]. Since the porosity is non-homogeneous and the flow is non-steady flow due to injection and extraction processes, the Darcy law is not appropriate for the geothermal application. Then we need to find another model to approach that phenomenon.

The analysis of fluid flow in porous media was started from H. Darcy. In 1856 he observed the water flow in packed sand. His experiments were performed with a constant temperature, single fluid, and homogeneous porous media. According to his research, he concluded that the fluid velocity is proportional to the pressure gradient. Then resulting Darcy equation in the one-dimensional case is

$$u = -k_D \frac{\partial p}{\partial x},$$

where u is the so called Darcy velocity, cf. (2.9), k_D is the hydraulic conductivity, p is the pressure, and x is the spatial coordinate. To accommodate the thermal effect in Darcy's equation, A. Hazen [11] introduced the specific permeability K and showed that the hydraulic conductivity is given by $k_D = \frac{K}{\mu}$, where μ is the temperature dependent dynamic viscosity. J. Kozeny and P.C. Carman gave a concrete form of the specific permeability K in terms of the porosity ϕ and the particle diameter d_p will be described later.

Darcy's law is the basic equation for modeling steady flow in porous media. This law assumes that the viscous forces dominate over inertial forces in porous media; hence, the inertial forces can be neglected. In the application where the permeability and porosity of the media are small such as in the groundwater and petroleum flows [14; 15], Darcy's law has an excellent performance to describe that phenomenon. However, in the application where the permeability and porosity of the medium are significantly large such as in the geothermal system, Darcy's law failed to describe it [19; 23; 24; 25].

To improve Darcy's law, in 1947, H.C. Brinkman added a viscosity term which represents the shear stress term, and proposed the Darcy–Brinkman equation [5]:

$$\frac{dp}{dx} = \mu \frac{\partial^2 u}{\partial x^2} - \frac{\mu}{K} u.$$

In the case of small porosity and permeability, if the viscosity effect in the pore throats is small, then the Brinkman equation is reduced to Darcy's law [24]. The Brinkman equation describes the transport processes in the porous media more generally than Darcy's equation. However, it can only be applied in a steady state.

J. Dupuit (1863) and P. Forchheimer (1901) found empirically that as the flow rate increases, the inertial forces become significantly large, and the relationship between the pressure drop and velocity becomes non-linear [24]. With that fact, J. Dupuit and P. Forchheimer added a quadratic term of the velocity to represent the microscopic inertial effect, which results in the Darcy–Brinkman–Forchheimer equation :

$$\frac{dp}{dx} = \mu \frac{\partial^2 u}{\partial x^2} - \frac{\mu}{K} u - \beta \rho u^2,$$

where $\beta = \frac{F\phi}{\sqrt{K}}$ is the non-Darcy coefficient, F is the Forchheimer constant, ϕ is the porosity, and ρ is the density of the fluid. This equation is more general than the Darcy–Brinkman equation, but again, it is only applied in steady state.

S. Whitaker (1967) introduced the volume average technique to relate the volume average of the spatial derivative to the spatial derivative of the volume average, and to make the transformation from microscopic equations to macroscopic equations possible [25]. C.T. Hsu and P. Cheng (1990) applied the volume average in the representative elementary volume (REV) to derive the equation for fluid flow in non-homogeneous porous media. In the process of the derivation, they got the expression of total drag force per unit volume due to the presence of solid particles in the integral boundary form.

To overcome this difficulty, they adopted the Darcy-Brinkmann-Forchaimmer model of the drag force [18; 24]. This model, consists of two-terms. The first term is related to Darcy's term and the second term is connected to the Forchaimer term. The Forchaimmer term plays an essential role in establishing the stability energy estimate of the model proposed by C.T.Hsu and P. Cheng in our study.

In reality, the shape of the geothermal reservoir is irregular and complicated. It is known that the finite element method (FEM) is an appropriate numerical method to approach irregular domain. In this method, we have applied a Lagrange–Galerkin (LG) method. The LG method is a finite element method embracing the method of characteristics. Two main advantages are using in LG method, and there are robustness and symmetry of the resulting matrix. Many authors have studied LG schemes for convection-diffusion problems [2] and the Navier-Stokes equations, Oseen and natural convection problems [1]. We applied a characteristic finite element method (Lagrange–Galerkin scheme with the Adams-Bashforth method) to solve the model proposed by C.T. Hsu and P. Cheng numerically.

1.2 Objective

C.T.Hsu and P. Cheng have proposed the equations of non-steady flow in the porous media by applying the averaging technique to the Navier–Stokes equations. However, any mathematical and numerical analysis for their model has not been studied and they didn't mention a suitable numerical method to solve that model numerically. Hence, the aims of this study are :

1. Prove the L^2-stability estimates of that model.

2. Propose a suitable numerical method to solve the model based on the Lagrange–Galerkin scheme and Adams-Bashforth time discretization.

3. Investigate the experimental order of convergence of the scheme.

4. Apply the numerical scheme to simulate some fluid flow in the non-homogeneous porous media

1.3 Overview of the Dissertation

This dissertation consists of six chapters: In Chapter 1 the motivation of our study, the objective of our study, and the overview of the dissertation, are introduced. The mathematical formulation is presented in Chapter 2, which includes averaging technique and how to apply averaging techniques to get macroscopic continuity and energy equation for fluid flow in porous media, and a statement of the problem that we will work on. In Chapter 3 the stability estimates of the problem is presented. The basic idea to extending the method of characteristics and Lagrange-Galerkin scheme is considered in Chapter 4. Chapter 5 presents the experimental order of convergence related to our scheme and the numerical results related to the fluid flow in simple and complex structures of porosity. Finally, the conclusion of our study is presented in Chapter 6.

2

Chapter 2

Mathematical Formulation

Summary

In this chapter, we present all of the assumptions that C.T. Hsu and P. Cheng used in the derivation of their model, the volume average technique proposed by S. Whitaker, the derivation of macroscopic continuity and momentum equation and a derivation of the macroscopic energy equation. In this chapter, we rewrite the derivation which has been done by C.T. Hsu and P. Cheng in [13].

2.1 Governing equations

2.1.1 Assumptions

In this study we classify the assumption in two parts. The first part is the assumption for porous media and second is the assumption for the derivation of the model (following the assumption coming from C.T. Hsu and P. Cheng [13]).

A porous medium is a material with a solid matrix structure and void spaces. The void spaces permit the fluids to pass through the media. Some example of porous media in nature are soil, sand, sponge, and fractured rock. Porous media also can be found in material engineering such as metal, ceramic, and filter. In this study, we will specify the porous media that we interest. The porous media is assumed to be non-homogeneous and isotropic. The solid matrix is assumed to be incompressible and motionless. We employ multiple length scales in the modeling of porous media; they are macroscopic length scale (L) and microscopic length scale (d_p). The macroscopic length scale is defined over the physical domain. The microscopic length scale (d_p) represents the detail of the morphology in the microscopic scale (i.e., a diameter of each particle). The macroscopic length scale is sufficiently large than the microscopic scale. A representative elementary volume (REV) defined as a volume with size (l_{REV}), which is larger than the microscopic length scale, therefore smaller than the macroscopic scale ($d_p << l_{REV} << L$) [31]. The macroscopic variables defined by the volume average of the microscopic variables over REV. It is assumed that the value of the macroscopic variables do not change when the average volume is larger then REV [31] see Fig 2.1. The porosity in the porous media is defined as the fraction of the volume occupied in the fluid phase in a REV

$$\phi = \frac{V_\alpha}{V_\alpha + V_\beta} \tag{2.1}$$

In the derivation of fluid flow in the non-homogeneous porous media done by C.T. Hsu and P. Cheng, the following assumptions hold.

1. The porosity is define by a continuous function ϕ.

2. $V \in R^3, v'(x, x', t) \in R^3, p_\alpha(x, x', t), x \in \Omega, x' \in V_\alpha(x), t \in R$

3. Only rigid porous media are considered ($v_s = 0$).

4. The physical properties inside the porous media are taken to be constant.

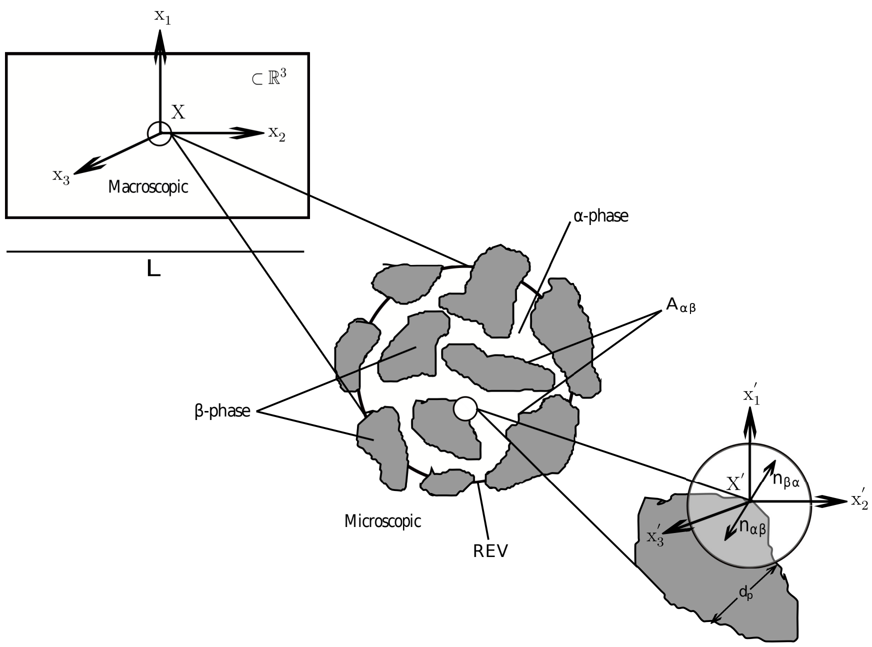


Figure 2.1: Representative elementary volume (REV)

5. The porous media are isotropic (their properties do not depend on the orientation in space).

6. The pore sizes of the porous media are very small.

7. The averaging technique are applied for any physical quantities such as velocity, pressure, and temperature.

8. Fluid are in-compressible, i.e., ρ is constant.

9. Each phase of fluids is separated from the others.

10. $v'(x, x', t) \cdot \mathbf{n}_{\beta\alpha} = 0$ on $A_{\alpha\beta}(x)$ for $x \in \Omega, x' \in A_{\alpha\beta}(x), t \in R$

11. The macroscopic quantities in a representative volume in the porous medium V are well behaved, that is, they are very smoothly and slowly on a microscopic scale, so this condition implies $\langle \hat{\mathbf{v}} \rangle = 0$ and $\langle\langle \mathbf{v} \rangle\rangle = \langle \mathbf{v} \rangle$

12. v' is a continuous defferentiable function.

13. The velocity $v' = 0$ in $x' \in A_{\alpha\beta}$ at the pore surface $A_{\alpha\beta}$ is zero due to the no-slip condition.

2.1.2 The Averaging Technique

There are three definitions of the average of some quantity $\mathbf{W}$, which will be useful. These are the spatial average, the phase average, and the intrinsic phase average. The spatial phase average is defined by

$$\langle \mathbf{W} \rangle^{sp} = \frac{1}{|V|} \int_V \mathbf{W} dx, \tag{2.2}$$

where $\langle \mathbf{W} \rangle^{sp}$ represents the value of $\mathbf{W}$ averaged over both the α-phase and the β-phase, and dx is the volumetric integration.

Second, the phase average is defined as

$$\langle \mathbf{W}_\alpha \rangle^{av} = \frac{1}{|V|} \int_{V_\alpha(x)} \mathbf{W}_\alpha dx. \tag{2.3}$$

Here we are taking the average of $\mathbf{W}_\alpha$ over the space contained in the averaging volume V. $\mathbf{W}_\alpha$ represents the value of $\mathbf{W}$ in the α-phase. $\mathbf{W}_\alpha$ has zero value in the β-phase. Because of this fact, the integral only needs to be evaluated over the volume of the α-phase in V. It means that $\langle \mathbf{W}_\alpha \rangle$ is defined throughout in space and takes non-zero values on the β-phase.

For the analysis of mass transfer and chemical reaction it is more convenient to work with the intrinsic phase average define by

$$\langle \mathbf{W}_\alpha \rangle = \frac{1}{|V_\alpha|} \int_{V_\alpha} \mathbf{W}_\alpha dx. \tag{2.4}$$

This average represents a function evaluated at the point with which we associate the averaging volume. We assume throughout the derivation that the averages are continuously differentiable functions with respect to time and space.

2.1.3 Macroscopic Continuity Equations

C.T. Hsu and P. Cheng [13] reported the macroscopic continuity of mass and momentum equations for fluid flow through the porous media based on the average of the microscopic continuity of mass and momentum over the representative elementary volume (REV). In this technique, the average theorems proposed by S. Whitaker and J.C. Slattery are needed to relate the average of the derivative to the derivative average [9,11].

Let us consider the porous media composed of the α and β phases which represent fluid and solid, respectively. Let $\Omega \subset \mathbb{R}^3$ be a bounded (macroscopic) domain. For $x \in \Omega$, let $V_\alpha(x)$ and $V_\beta(x)$ be microscopic volumes of α and β phases, respectively, and let $V(x) := V_\alpha(x) \cup V_\beta(x) \subset \mathbb{R}^3$ be an REV satisfying $|V(x)| = |V_\alpha(x)| + |V_\beta(x)| < \infty$, where $|V_\alpha(x)|$ represents the measure of $V_\alpha(x)$. We assume that $|V(x)|$ is constant. We denote it by $|V|$. The porosity is given by $\phi(x) = \frac{|V_\alpha(x)|}{|V|} \in (0, 1]$. We denote by $v' = v'(x', x) \in \mathbb{R}^3$ the microscopic velocity at $x' \in V_\alpha(x)$, where x' denotes the coordinates of $V_\alpha(x)$. Then the macroscopic intrinsic phase average for the velocity $\langle v' \rangle$ is define by:

$$\langle v' \rangle = \frac{1}{|V_\alpha(x)|} \int_{V_\alpha(x)} v'(x', x) dx'.$$

The averaging technique assumes that the total macroscopic source of the system at a point x is equal to the total microscopic source to the system at a point x', and total flux through the surface $A_{\alpha\beta}$, see Fig. 2.1. Then this assumption yields

$$\nabla \cdot \left[\frac{1}{|V|} \int_{V_\alpha} v' dx' \right] = \frac{1}{|V|} \int_{V_\alpha} \nabla' \cdot v' dx' + \frac{1}{|V|} \int_{A_{\alpha\beta}} v' \cdot n_{\beta\alpha} \, ds, \tag{2.5}$$

where $n_{\beta\alpha}$ is the unit normal vector from the β-phase to the α-phase and ds is the arc-length on the interface $A_{\alpha\beta}$. In other words, we assume

$$\nabla \cdot (\phi \langle v' \rangle) = \phi \langle \nabla' \cdot v' \rangle + \frac{1}{|V|} \int_{A_{\alpha\beta}} v' \cdot n_{\beta\alpha} \, ds.$$

For the time-dependent case, S. Whitaker and J.C. Slattery assumed that the microscopic velocity $v'(x', x, t)$ and pressure $p(x', x, t)$ are governed by the Navier–Stokes equations in $V_\alpha(x)$, and derived its macroscopic equations in porous media by taking the average in REV. To do this here, we will split the Navier-Stokes equations into two parts. The first part is the microscopic continuity equation and the second part the is microscopic momentum equation. The microscopic continuity equation for in-compressible flow is given by

$$\nabla' \cdot v'(x', x, t) = 0. \tag{2.6}$$

Integrating the equation with respect to representative volume in the porous media, then dividing the result expression by $|V|$ and with aid averaging theorem, we get

$$\frac{1}{|V|} \int_{V_\alpha(x)} (\nabla' \cdot v') dx' = 0, \tag{2.7a}$$

$$\nabla \cdot \left[\frac{1}{|V|} \int_{V_\alpha(x)} v' dx' \right] + \frac{1}{|V|} \int_{A_{\alpha\beta}} v' \cdot n_{\beta\alpha} ds = 0, \tag{2.7b}$$

$$\nabla \cdot \left[\frac{|V_\alpha|}{|V|} \frac{1}{|V_\alpha|} \int_{V_\alpha(x)} v' dx' \right] + \frac{1}{|V|} \int_{A_{\alpha\beta}} v' \cdot n_{\beta\alpha} ds = 0. \tag{2.7c}$$

By following the assumption that there is no flux in the interface of α and β phases, we obtain

$$\nabla \cdot (\phi(x)\langle v'(x', x, t)\rangle) = 0. \tag{2.8}$$

We remark that these superficial quantities are represented by their macroscopic average $\langle v'\rangle$ and $\langle p'\rangle$ as follows:

$$u(x,t) = \phi(x)\langle v'(\cdot, x, t)\rangle, \qquad p(x,t) = \phi(x)\langle p'(\cdot, x, t)\rangle. \tag{2.9}$$

The equation above can be written as

$$\nabla \cdot u(x', x, t) = 0. \tag{2.10}$$

The superficial velocity u is called the Darcy velocity.

To derive the macroscopic momentum equation we define the microscopic momentum equation for incompressible flow from the Navier-Stokes equation by:

$$\rho_\alpha \left[\frac{\partial v'}{\partial t} + \nabla' \cdot (v' \otimes v')\right] = -\nabla' p_\alpha + \mu_\alpha \nabla'^2 v', \tag{2.11}$$

where ρ_α and μ_α are the density and the viscosity of the fluids, respectively, p_α is the pressure of the fluids, and $v' \otimes v'$ is the dyadic product, which is a particular case of the tensor product, whose resulting second rank tensor. The divergence of second rank tensors is a vector (first-rank tensor). Integrating equation 2.11 concerning a representative volume in the porous media, and then dividing the resulting expression by $|V|$ we have by the averaging technique that,

$$\frac{1}{|V|}\int_{V_\alpha} \rho_\alpha \frac{\partial v'}{\partial t} dx' + \frac{1}{|V|}\int_{V_\alpha} \rho_\alpha \nabla' \cdot (v' \otimes v') dx' = \frac{1}{|V|}\int_{V_\alpha} \left[-\nabla' p_\alpha + \mu_\alpha \nabla'^2 v'\right] dx'. \tag{2.12}$$

We evaluate each terms in equation 2.12 as follows :

For the first term, by applying Leibniz integral rule we can interchange differentiation and integration in the first term (as we assumed above that all of the averages are continuous differentiable function), and referring to the definition of the intrinsic phase average (2.4), we have,

$$\begin{aligned}\frac{1}{|V|}\int_{V_\alpha}\rho_\alpha\frac{\partial v'}{\partial t}dx' &= \rho_\alpha\frac{\partial}{\partial t}\left[\frac{1}{|V|}\int_{V_\alpha}v'dx'\right],\\ &= \rho_\alpha\frac{\partial}{\partial t}\left[\frac{|V_\alpha|}{|V|}\frac{1}{|V_\alpha|}\int_{V_\alpha}v'dx'\right],\\ &= \rho_\alpha\frac{\partial}{\partial t}\left(\phi(x)\langle v(x)\rangle\right).\end{aligned}$$

For the second term, we get

$$\begin{aligned}\frac{1}{|V|}\int_{V_\alpha}\rho_\alpha\nabla'\cdot(v'\otimes v')dx' &= \rho_\alpha\frac{1}{|V|}\int_{V_\alpha}\nabla'\cdot(v'\otimes v')dx',\\ &= \rho_\alpha\nabla\cdot\left[\frac{1}{|V|}\int_{V_\alpha}(v'\otimes v')dx'\right]+\rho_\alpha\frac{1}{|V|}\int_{A_{\alpha\beta}}(v'\otimes v')\cdot n_{\beta\alpha}ds,\\ &= \rho_\alpha\nabla\cdot\left[\frac{|V_\alpha|}{|V|}\frac{1}{|V_\alpha|}\int_{V_\alpha}(v'\otimes v')dx'\right],\\ &= \rho_\alpha\nabla\cdot(\phi(x)\langle v'\otimes v'\rangle),\end{aligned}$$

where $\langle v'\otimes v'\rangle = \frac{1}{|V_\alpha|}\int_{V_\alpha}(v'\otimes v')dx'$. For the third term, we obtain

$$\begin{aligned}-\frac{1}{|V|}\int_{V_\alpha}\nabla' p_\alpha dx' &= -\nabla\left[\frac{1}{|V|}\int_{V_\alpha}p_\alpha dx'\right]-\frac{1}{|V|}\int_{A_{\alpha\beta}}p_\alpha n_{\beta\alpha}ds,\\ &= -\nabla\left[\frac{|V_\alpha|}{|V|}\frac{1}{|V_\alpha|}\int_{V_\alpha}p_\alpha dx'\right]-\frac{1}{|V|}\int_{A_{\alpha\beta}}p_\alpha n_{\beta\alpha}ds,\\ &= -\nabla(\phi(x)\langle p_\alpha\rangle)-\frac{1}{|V|}\int_{A_{\alpha\beta}}p_\alpha n_{\beta\alpha}ds,\end{aligned}$$

where $\langle p_\alpha\rangle = \frac{1}{|V_\alpha|}\int_{V_\alpha}p_\alpha dx'$ is the average (macroscopic) pressure.

For the last term, we get

$$\begin{aligned}\frac{1}{|V|}\int_{V_\alpha}\mu_\alpha\nabla'^2 v' dx' &= \mu_\alpha\frac{1}{|V|}\int_{V_\alpha}\nabla'\cdot(\nabla' v')dx',\\ &= \mu_\alpha\nabla\cdot\left[\frac{1}{|V|}\int_{V_\alpha}(\nabla' v')dx'\right] - \frac{\mu_\alpha}{|V|}\int_{A_{\beta\alpha}}(\nabla' v')\cdot n_{\beta\alpha}ds.\end{aligned}$$

To avoid the difficulty coming from the first term in the right hand side, we use the following identity in geometric calculus (the gradient of a vector field is the sum of a scalar field and a bi-vector field):

$$\nabla\mathbf{A} = \nabla\cdot\mathbf{A} + \nabla\wedge\mathbf{A}. \tag{2.13}$$

By assuming that the bi-vector term is equal to zero, we get

$$\begin{aligned}\frac{1}{|V|}\int_{V_\alpha}\mu_\alpha\nabla'^2 v' dx' &= \mu_\alpha\nabla\cdot\left[\frac{1}{|V|}\int_{V_\alpha}\nabla\cdot v' dx'\right] + \frac{\mu_\alpha}{|V|}\int_{A_{\alpha\beta}}(\nabla' v')\cdot n_{\alpha\beta}ds,\\ &= \mu_\alpha\nabla\cdot\left[\nabla\cdot\left[\frac{1}{|V|}\int_{V_\alpha}v' dx'\right] - \frac{\mu_\alpha}{|V|}\int_{A_{\alpha\beta}}v'\cdot n_{\alpha\beta}ds\right]\\ &+ \frac{\mu_\alpha}{|V|}\int_{A_{\alpha\beta}}(\nabla' v')\cdot n_{\alpha\beta}ds,\\ &= \mu_\alpha\nabla^2(\phi(x)\langle v(x)\rangle) + \frac{\mu_\alpha}{|V|}\int_{A_{\alpha\beta}}\frac{\partial v'}{\partial n}ds.\end{aligned}$$

We collecting these results together, we have

$$\rho_\alpha\left[\frac{\partial}{\partial t}(\phi(x)\langle v\rangle) + \nabla\cdot(\phi(x)\langle v'\otimes v'\rangle\right] = -\nabla(\phi(x)\langle p_\alpha\rangle) + \mu_\alpha\nabla^2(\phi(x)\langle v\rangle) + B, \tag{2.14}$$

where

$$B = -\frac{1}{|V|}\int_{A_{\alpha\beta}}p_\alpha n_{\beta\alpha}ds + \frac{\mu_\alpha}{|V|}\int_{A_{\alpha\beta}}\frac{\partial v'}{\partial n}ds, \tag{2.15}$$

which is the total drag force per unit volume (body force) due to the presence of solid particles.

We need to represent the term $\langle v' \otimes v' \rangle$ into $\langle v \rangle$. To do that we need to relate microscopic and macroscopic quantities through perturbation variables [10] defined as:

$$\hat{v} = v' - \langle v \rangle. \tag{2.16}$$

Then, the average of the dot product becomes

$$\begin{aligned}\langle v' \otimes v' \rangle &= \langle ((\langle v \rangle + \hat{v}) \otimes (\langle v \rangle + \hat{v}) \rangle, \\ &= \langle \langle v \rangle \otimes \langle v \rangle \rangle + \langle \langle v \rangle \otimes \hat{v} \rangle + \langle \hat{v} \otimes \langle v \rangle \rangle + \langle \hat{v} \otimes \hat{v} \rangle, \\ &= \langle \langle v \rangle \rangle \otimes \langle \langle v \rangle \rangle + \langle v \rangle \otimes \langle \hat{v} \rangle + \langle \hat{v} \rangle \otimes \langle v \rangle + \langle \hat{v} \otimes \hat{v} \rangle.\end{aligned}$$

To simplify the equation above we require that the macroscopic quantities in the representative volume in porous media are "well behaved", that is, they are very smoothly and slowly on a microscopic scale. This condition implies that $\langle \hat{v} \rangle$ is equal to zero and the $\langle \langle v \rangle \rangle = \langle v \rangle$. This definition leads to a dispersion term that is non-zero even when v' is uniform. Then, we have,

$$\langle v' \otimes v' \rangle = \langle v \rangle \otimes \langle v \rangle + \langle \hat{v} \otimes \hat{v} \rangle, \tag{2.17}$$

where we neglected the high order term $(\langle \hat{v} \otimes \hat{v} \rangle)$. Then, substituting equation (2.17) into (2.14) yields

$$\rho_\alpha \left[\frac{\partial}{\partial t} \left(\phi(x) \langle v \rangle \right) + \nabla \cdot \left(\phi(x) \langle v \rangle \otimes \langle v \rangle \right) \right] = -\nabla(\phi(x)\langle p_\alpha(x) \rangle) + \mu_\alpha \nabla^2 (\phi(x) \langle v \rangle) + B. \tag{2.18}$$

By definition of the Darcy velocity and the pressure $P_\alpha(x) = \phi(x)\langle p_\alpha \rangle$, we have

$$\rho_\alpha \left[\frac{\partial}{\partial t} \left(u(x', x, t) \right) + \nabla \cdot \left(\frac{u(x', x, t)}{\phi(x)} \otimes u(x', x, t) \right) \right] = -\nabla P_\alpha(x) + \mu_\alpha \nabla^2 u(x', x, t) + B. \tag{2.19}$$

By using the following tensor product identity and equation (2.10), we can rewrite equation (2.19) as

$$\nabla \cdot (v \otimes v) = \left(v \cdot \nabla v + v(\nabla \cdot v) \right), \tag{2.20}$$

$$\rho_\alpha \left[\frac{\partial \left(u(x', x, t) \right)}{\partial t} + \left(u(x', x, t) \cdot \nabla \right) \frac{u(x', x, t)}{\phi} \right] = -\nabla P_\alpha(x) + \mu_\alpha \nabla^2 u(x', x, t) + B. \tag{2.21}$$

When the buoyancy force is considered, the momentum equation is,

$$\begin{aligned} \rho_\alpha \left[\frac{\partial u(x', x, t)}{\partial t} + \left(u(x', x, t) \cdot \nabla \right) \frac{u(x', x, t)}{\phi} \right] &= -\nabla P_\alpha(x) + \mu_\alpha \nabla^2 u(x', x, t) \\ &\quad + B - \phi \rho_\alpha g(\theta_f - \theta_i). \end{aligned} \tag{2.22}$$

2.1.4 Drag Force Model

To approximate the drag force per unit volume B, we start with the definition of drag force coefficient. For an arbitrary microscopic geometry which has microscopic length scale d, then the drag coefficient can be expressed as [6; 16; 20]

$$C_d = c_{do} + c_{d1} Re_d^{-1} + c_{d2} Re_d^{-1/2} + O(Re_d^{-3/2}), \tag{2.23}$$

where

$$Re_d = \frac{|\bar{u}| d}{u}, \tag{2.24}$$

C_d is the drag coefficient, $\bar{u}$ is the microscopic average velocity vector, u is the macroscopic velocity vector, Re_d is the microscopic Reynold number, and c_{do}, c_{d1}, c_{d2} are the microscopic drag coefficient constants. The zeroth order, -1 order, $-1/2$ order, and the $-3/2$ order terms have correlation with the inertial effect, the Stokes drag, the skin friction, and negligible higher-order term, respectively. Hence, the drag force per unit volume of the porous media B can be defined as

$$B = \frac{F_{D_{fs}}}{V_s + V_f}, \tag{2.25}$$

where $F_{D_{fs}}$ is a drag force, V_s is the volume of the solid and V_f is the volume of the fluid.

Note that the drag force F_D is a force acting opposite to the relative motion of any object moving with respect to a surrounding fluid. This can exist between two fluid layers (or surfaces) or a fluid and a solid surface. The drag force can be expressed as

$$F_D \propto P_d A_{fs}, \tag{2.26}$$

where P_d is the pressure exerted by fluid in the area A_{fs}. P_d is represents to the dynamic pressure due to the kinetic energy of fluid undergoing relative flow velocity $\bar{u}$, and then the kinetic energy equation is defined by:

$$P_d = \frac{1}{2}\rho\bar{u}^2. \tag{2.27}$$

Then, by inserting (2.27) into (2.26), we get

$$F_D \propto 1/2\rho_f|\bar{u}|\bar{u}A_{fs}. \tag{2.28}$$

By adding the C_d into (2.28), we obtain the expression for drag force:

$$F_D = -1/2\rho_f |\bar{u}|\bar{u} A_{fs} C_d. \tag{2.29}$$

The minus sign appears because of the force acting in the opposite direction with respect to the fluids flow. Then equation (2.25) becomes

$$B = \frac{Drag_{fs}}{V_s + V_f} = -\frac{1/2\rho_f |\bar{u}|\bar{u} A_{fs} C_d}{V_s/(1-\phi)}, \tag{2.30}$$

where

$$\phi = \frac{V_f}{V_s + V_f}. \tag{2.31}$$

By defining the geometry factor

$$\eta = \frac{A_{fs} d}{V_s}, \tag{2.32}$$

and inserting equation (2.23) and (2.24), then the equation (2.30) becomes

$$B = -(1-\phi)\eta \frac{\rho_f u_f^2}{2d^3} \left(c_{do} Re_d^2 + c_{d1} Re_d^1 + c_{d2} Re_d^{3/2} \right) \hat{\mathbf{e}}_f, \tag{2.33}$$

where $\hat{\mathbf{e}}_f$ is the unit vector pointing to the macroscopic velocity. From Eq.(2.33), it can be understood that the geometry factor η and the bulk porosity ϕ relate the macroscopic drag force and the microscopic drag coefficient for arbitrary microscopic geometry.

Darcy, Brinkman and Forchheimer model the drag force B in porous media as

$$\nabla \bar{p}_f = -\left[\frac{\mu_f u_f}{K} + \frac{C_F \rho_f u_f |u_f|}{\sqrt{K}} \right]. \tag{2.34}$$

By multiplying Eq. (2.34) by ϕ, the drag force per unit volume is

$$B = \nabla p = \nabla(\phi \bar{p}_f) = -\phi \left[\frac{\mu_f u_f}{K} + \frac{C_F \rho_f u_f |u_f|}{\sqrt{K}} \right]. \tag{2.35}$$

which can be expressed as

$$B = -\phi \left[\frac{\mu_f u_f}{K} + \frac{C_F \rho_f u_f |u_f|}{\sqrt{K}} \right] \tag{2.36a}$$

$$= -\phi \left[\frac{\mu_f \phi \bar{u}_f}{K} + \frac{C_F \rho_f \phi^2 \bar{u}_f |\bar{u}_f|}{\sqrt{K}} \right] \tag{2.36b}$$

$$= -\left[\frac{\mu_f \phi^2}{K} \left(\frac{u_f}{d} \right) Re_d + \frac{C_F \rho_f \phi^3}{\sqrt{K}} \left(\frac{u_f}{d} \right)^2 Re_d^2 \right] \hat{\mathbf{e}}. \tag{2.36c}$$

Comparing Eqs. (2.11) and (2.15) we obtain

$$\frac{\mu_f \phi^2}{K} \left(\frac{u_f}{d} \right) = (1-\phi)\eta \frac{\rho_f u_f^2}{2d^3} c_{d1} \tag{2.37}$$

and

$$\frac{C_F \rho_f \phi^3}{\sqrt{K}} \left(\frac{u_f}{d} \right)^2 = (1-\phi)\eta \frac{\rho_f u_f^2}{2d^3} c_{do}. \tag{2.38}$$

From Eqs. (2.37) and (2.38), the permeability and the Forchheimer coefficient can be expressed in terms of the drag coefficient and geometry factor for an arbitrary structured porous medium as

$$K = \frac{\phi^2}{(1-\phi)\eta} \frac{2d^2}{c_{d1}}, \qquad C_F = \frac{\sqrt{(1-\phi)\eta}}{\phi^2} \frac{c_{d0}}{\sqrt{2c_{d1}}}. \tag{2.39}$$

Thus the Darcy number can be recast as

$$Da = \frac{K}{L^2} = \frac{\phi^2}{(1-\phi)\eta} \left(\frac{d}{L} \right)^2 \frac{2}{c_{d1}}. \tag{2.40}$$

From Erguns experimental study [8] of a bed of packed spheres, the permeability and the Forchheimer coefficient are related to the porosity by

$$F(\phi) = \frac{b}{\sqrt{a\phi^3}}, \quad K = \frac{d_p^2\phi^3}{a(1-\phi)^2}. \tag{2.41}$$

Thus the Ergun's constants can be expressed as,

$$a = \frac{\phi}{(1-\phi)}\frac{\eta}{2}c_{d1}, \qquad b = \frac{\eta}{2}c_{d0}. \tag{2.42}$$

2.1.5 Macroscopic Energy Equation

Assume $\theta'_\alpha(x', x, t) \in \mathbb{R}^2, v'(x', x) \in \mathbb{R}^2, x' \in V_\alpha, x \in \Omega, t \in \mathbb{R}$. Then, the microscopic energy equations for the fluid and solid phases are

$$(\rho C_p)_\alpha \left[\frac{\partial \theta'_\alpha}{\partial t} + \nabla' \cdot (v'_\alpha \theta'_\alpha)\right] = \nabla' \cdot (k_\alpha \nabla \theta'_\alpha) \tag{2.43}$$

and

$$(\rho C_p)_\beta \frac{\partial \theta'_\beta}{\partial t} = \nabla' \cdot \left(k_\beta \nabla' \theta'_\beta\right), \tag{2.44}$$

where the interface conditions are

$$v_\alpha = 0 \qquad \text{on } A_{\alpha\beta}, \tag{2.45a}$$

$$\theta'_\alpha = \theta'_\beta \qquad \text{on } A_{\alpha\beta}, \tag{2.45b}$$

$$n_{\alpha\beta} \cdot k_\alpha \nabla' \theta'_\alpha = n_{\alpha\beta} \cdot \left(k_\beta \nabla' \theta'_\beta\right) \qquad \text{on } A_{\alpha\beta}, \tag{2.45c}$$

where C_p is the specific heat at constant pressure, θ'_α is the microscopic temperature of the fluid, θ'_β is the microscopic temperature of the solid metric, k_α is the thermal conductivity of the fluid phase, k_β is the thermal conductivity of the solid phase, and $n_{\alpha\beta}$ is the outward unit normal vector from fluid to solid.

The macroscopic energy equation for convective heat transfer in porous media is obtained by the volume average of the microscopic energy equation lies in the fluid and solid phases over the representative elementary volume (REV). To derive the macroscopic energy equation for fluid phase, we integrate the equation (2.43) respect to the representative volume in the porous media, and then divide the resulting expression by $|V|$, we have by using the averaging technique that

$$\frac{1}{|V|}\int_{V_\alpha}(\rho C_p)_\alpha \frac{\partial \theta'_\alpha}{\partial t}dx' + \frac{1}{|V|}\int_{V_\alpha}(\rho C_p)_\alpha \nabla' \cdot (v'_\alpha \theta'_\alpha)\, dx' = \frac{1}{|V|}\int_{V_\alpha} \nabla' \cdot (k_\alpha \nabla' \theta'_\alpha)\, dx'. \tag{2.46}$$

We evaluate each term in (2.46) as follows:

$$\begin{aligned}\frac{1}{|V|}\int_{V_\alpha}(\rho C_p)_\alpha \frac{\partial \theta'_\alpha}{\partial t}dx' &= (\rho C_p)_\alpha \frac{d}{dt}\left[\frac{1}{|V|}\int_{V_\alpha}\theta'_\alpha dx'\right] \\ &= (\rho C_p)_\alpha \frac{d}{dt}\left[\frac{|V_\alpha|}{|V|}\frac{1}{|V_\alpha|}\int_{V_\alpha}\theta'_\alpha dx'\right] \\ &= (\rho C_p)_\alpha \frac{d}{dt}\left(\phi(x)\langle\theta'_\alpha\rangle\right).\end{aligned}$$

$$\begin{aligned}\frac{1}{|V|}\int_{V_\alpha}(\rho C_p)_\alpha \nabla' \cdot (v'_\alpha \theta'_\alpha)\, dx' &= (\rho C_p)_\alpha \frac{1}{|V|}\int_{V_\alpha}\nabla' \cdot (v'_\alpha \theta'_\alpha)\, dx' \\ &= (\rho C_p)_\alpha \nabla \cdot \left[\int_{V_\alpha}\frac{|V_\alpha|}{|V|}\frac{1}{|V_\alpha|}(v'_\alpha \theta'_\alpha)\right] \\ &\quad + (\rho C_p)_\alpha \frac{1}{|V|}\int_{A_{\alpha\beta}} n_{\alpha\beta}\cdot (v'_\alpha \theta'_\alpha)\, ds \\ &= (\rho C_p)_\alpha \nabla \cdot \left[\int_{V_\alpha}\frac{|V_\alpha|}{|V|}\frac{1}{|V_\alpha|}(v'_\alpha \theta'_\alpha)\right] \\ &= (\rho C_p)_\alpha \nabla \cdot (\phi(x)\langle v'_\alpha \theta'_\alpha\rangle).\end{aligned}$$

$$\begin{aligned}\frac{1}{|V|}\int_{V_\alpha}\nabla' \cdot (k_\alpha \nabla' \theta'_\alpha)\, dx' &= \nabla \cdot \left[\frac{1}{|V|}\int_{V_\alpha}(k_\alpha \nabla' \theta'_\alpha)\, dx'\right] + \frac{1}{|V|}\int_{A_{\beta\alpha}} n_{\alpha\beta}\cdot (k_\alpha \nabla' \theta'_\alpha)\, ds \\ &= k_\alpha \nabla \cdot \left[\frac{1}{|V|}\int_{V_\alpha}(\nabla' \theta'_\alpha)\, dx'\right] + \frac{1}{|V|}\int_{A_{\alpha\beta}} n_{\alpha\beta}\cdot (k_\alpha \nabla' \theta'_\alpha)\, ds\end{aligned}$$

$$
\begin{aligned}
&= k_\alpha \nabla \cdot \left[\nabla \cdot \left[\frac{1}{|V|} \int_{V_\alpha} \theta'_\alpha dx' \right] + \frac{1}{|V|} \int_{V_{\alpha\beta}} n_{\alpha\beta} \cdot \theta' ds \right] \\
&\quad + \frac{1}{|V|} \int_{A_{\alpha\beta}} n_{\alpha\beta} \cdot \left(k_\alpha \nabla' \theta'_\alpha \right) ds \\
&= k_\alpha \nabla^2 (\phi(x) \langle \theta'_\alpha \rangle) + k_\alpha \nabla \cdot \left[\frac{1}{|V|} \int_{A_{\alpha\beta}} n_{\alpha\beta} \cdot \theta'_\alpha ds \right] \\
&\quad + \frac{1}{|V|} \int_{A_{\alpha\beta}} n_{\alpha\beta} \cdot \left(k_\alpha \nabla' \theta'_\alpha \right) ds.
\end{aligned}
$$

Combining these results together we have

$$
\begin{aligned}
&(\rho C_p)_\alpha \frac{d}{dt} \left(\phi(x) \langle \theta'_\alpha \rangle \right) + (\rho C_p)_\alpha \nabla \cdot \left(\phi(x) \langle v'_\alpha \theta'_\alpha \rangle \right) \\
&\qquad = k_\alpha \nabla^2 (\phi(x) \langle \theta'_\alpha \rangle) + k_\alpha \nabla \cdot \left[\frac{1}{|V|} \int_{A_{\alpha\beta}} n_{\alpha\beta} \cdot \theta'_\alpha ds \right] \\
&\qquad + \frac{1}{|V|} \int_{A_{\alpha\beta}} n_{\alpha\beta} \cdot \left(k_\alpha \nabla' \theta'_\alpha \right) ds.
\end{aligned} \tag{2.47}
$$

To derive the macroscopic energy equation for the solid phase we integrate the equation (2.44) with respect to the representative volume in the porous media, nd then divide the resulting expression by $|V|$, we have by using the averaging technique that

$$
\frac{1}{|V|} \int_{V_\beta} (\rho C_p)_\beta \frac{\partial \theta'_\beta}{\partial t} dx' = \frac{1}{|V|} \int_{V_\beta} \nabla' \cdot \left(k_\beta \nabla' \theta'_\beta \right) dx'. \tag{2.48}
$$

$$
\begin{aligned}
\frac{1}{|V|} \int_{V_\beta} (\rho C_p)_\beta \frac{\partial \theta'_\beta}{\partial t} dx' &= (\rho C_p)_\beta \frac{d}{dt} \left[\int_{V_\beta} \frac{1}{|V|} \theta'_\beta dx' \right] \\
&= (\rho C_p)_\beta \frac{d}{dt} \left[\int_{V_\beta} \frac{V_\beta}{|V|} \frac{1}{|V_\beta|} \theta'_\beta dx' \right] \\
&= (\rho C_p)_\beta \frac{d}{dt} \left((1 - \phi(x) \langle \theta'_\beta \rangle) \right).
\end{aligned}
$$

$$\begin{aligned}\frac{1}{|V|}\int_{V_\beta}\nabla'\cdot\left(k_\beta\nabla'\theta'_\beta\right)dx' &= k_\beta\nabla\cdot\left[\frac{1}{|V|}\int_{V_\beta}\nabla'\theta'_\beta dx'\right]+k_\beta\frac{1}{|V|}\int_{A_{\alpha\beta}}n_{\beta\alpha}\cdot\nabla'\theta'_\beta dx' \\ &= k_\beta\nabla\cdot\left[\nabla\cdot\left[\frac{1}{|V|}\int_{V_\beta}\theta'_\beta dx'+\frac{1}{|V|}\int_{A_{\alpha\beta}}n_{\beta\alpha}\cdot\theta'_\beta ds\right]\right] \\ &\quad + k_\beta\frac{1}{|V|}\int_{A_{\alpha\beta}}n_{\beta\alpha}\cdot\nabla'\theta'_\beta ds \\ &= k_\beta\nabla\cdot\left[\nabla\cdot\left[\frac{V_\beta}{|V|}\frac{1}{|V_\beta|}\int_{V_\beta}\theta'_\beta dx'\right]+\frac{1}{|V|}\int_{A_{\alpha\beta}}n_{\beta\alpha}\cdot\theta'_\beta ds\right] \\ &\quad + K_\beta\frac{1}{|V|}\int_{A_{\alpha\beta}}n_{\beta\alpha}\cdot\nabla'\theta'_\beta ds \\ &= k_\beta\nabla\cdot\left(\nabla\cdot\left[(1-\phi(x))\langle\theta_\beta\rangle\right]\right)+k_\beta\nabla\cdot\left[\frac{1}{|V|}\int_{A_{\alpha\beta}}n_{\beta\alpha}\cdot\theta'_\beta ds\right] \\ &\quad + \frac{1}{|V|}\int_{A_{\alpha\beta}}n_{\beta\alpha}\cdot(k_\beta\nabla'\theta'_\beta)ds.\end{aligned}$$

Combining these results together we have

$$\begin{aligned}(\rho C_p)_\beta\frac{\partial}{\partial t}\left((1-\phi(x))\langle\theta'_\beta\rangle\right) = k_\beta\nabla\cdot\left(\nabla\cdot\left[(1-\phi(x))\langle\theta'_\beta\rangle\right]\right)+k_\beta\nabla\cdot\left[\frac{1}{|V|}\int_{A_{\alpha\beta}}n_{\beta\alpha}\cdot\theta'_\beta ds\right] \\ +\frac{1}{|V|}\int_{A_{\alpha\beta}}n_{\beta\alpha}\cdot(k_\beta\nabla'\theta'_\beta)ds,\end{aligned} \tag{2.49}$$

where $(\rho C_p)_\alpha$ and $(\rho C_p)_\beta$ are the heat capacities of the fluid and solid phases, respectively. Adding equation (2.47) and (2.49), we find form the boundary condition (2.45b) that

$$\frac{d}{dt}\left[(\rho C_p)_\alpha \phi(x)\langle\theta'_\alpha\rangle + (\rho C_p)_\beta(1-\phi(x))\langle\theta'_\beta\rangle\right] + (\rho C_p)_\alpha \nabla\cdot(\phi(x)\langle v'\theta'_\alpha\rangle)$$
$$= \nabla^2\left[k_\alpha(\phi(x)\langle\theta'_\alpha\rangle) + k_\beta((1-\phi(x))\langle\theta'_\beta\rangle)\right]$$
$$+\nabla\cdot\left[\frac{1}{|V|}\int_{A_{\alpha\beta}}(k_\alpha\theta'_\alpha - k_\beta\theta'_\beta)\cdot n_{\alpha\beta}ds\right]. \tag{2.50}$$

Now we decompose θ'_α and θ'_β as

$$\hat{\theta}_\alpha = \theta'_\alpha - \langle\theta_\alpha\rangle, \tag{2.51a}$$

$$\hat{\theta}_\beta = \theta'_\beta - \langle\theta_\beta\rangle. \tag{2.51b}$$

In the equilibrium condition, we assumed

$$\langle\theta'_\alpha\rangle = \langle\theta'_\beta\rangle = \langle\theta'\rangle. \tag{2.52}$$

Substituting equation (2.17) and (2.50) into (2.51) and using (2.52) yields

$$\frac{d}{dt}\left\{\left[(\rho C_p)_\alpha\phi(x) + (\rho C_p)_\beta(1-\phi(x))\right]\langle\theta'\rangle\right\} + (\rho C_p)_\alpha\nabla\cdot\Big(\phi(x)(\langle v\rangle\langle\theta\rangle + \langle\hat{v}\hat{\theta}_\alpha\rangle)\Big)$$
$$= \nabla^2\left\{\left[k_\alpha\phi(x) + k_\beta(1-\phi(x))\right]\langle\theta'\rangle\right\}$$
$$+\nabla\cdot\left[\frac{1}{|V|}\int_{A_{\alpha\beta}}(k_\alpha\theta' - k_\beta\theta')\cdot n_{\alpha\beta}ds\right]. \tag{2.53}$$

Nozad et al. [25] approximate the terms on right-hand side of equation (2.53) by

$$\nabla^2 \left\{ \left[k_\alpha \phi(x) + k_\beta (1 - \phi(x)) \right] \langle \theta' \rangle \right\} + \nabla \cdot \left[\frac{1}{|V|} \int_{A_{\alpha\beta}} (k_\alpha \theta' - k_\beta \theta') \cdot n_{\alpha\beta} ds \right] = \nabla \cdot (k_d \nabla \langle \theta' \rangle), \tag{2.54}$$

where $k_d := k_\alpha \phi + k_\beta (1 - \phi)$ is the stagnan thermal conductivity of the saturated porous medium. Then equation (2.53) becomes

$$\frac{d}{dt} \left\{ \left[(\rho C_p)_\alpha \phi(x) + (\rho C_p)_\beta (1 - \phi(x)) \right] \langle \theta' \rangle \right\} + (\rho C_p)_\alpha \nabla \cdot \Big(\phi(x) (\langle v \rangle \langle \theta' \rangle + \langle \hat{v} \hat{\theta}_\alpha \rangle) \Big) = \nabla \cdot (k_d \nabla \langle \theta' \rangle), \tag{2.55}$$

$$\frac{d}{dt} \left\{ \left[(\rho C_p)_\alpha \phi(x) + (\rho C_p)_\beta (1 - \phi(x)) \right] \langle \theta' \rangle \right\} + (\rho C_p)_\alpha \nabla \cdot (\phi(x) (\langle v \rangle \langle \theta' \rangle) = \nabla \cdot (k_d \nabla \langle \theta' \rangle). \tag{2.56}$$

We define the Darcy temperature and the Darcy velocity as follows

$$\theta := \langle \theta' \rangle \qquad u := \phi(x) \langle v \rangle.$$

Then, we have

$$\frac{d}{dt} \left\{ \left[(\rho C_p)_\alpha \phi(x) + (\rho C_p)_\beta \left(1 - \phi(x) \right) \right] \theta \right\} + (\rho C_p)_\alpha \nabla \cdot (u\theta) = \nabla \cdot (k_d \nabla \theta). \tag{2.57}$$

By defining $\sigma := \sigma_\alpha \phi + \sigma_\beta (1 - \phi)$ as the entropy per unit volume, $\sigma_\alpha = (\rho C_p)_\alpha$ as the entropy per unit volume for α-phase, we have

$$\sigma \frac{d\theta}{dt} + \sigma_\alpha \nabla \cdot (u\theta) = \nabla \cdot (k_d \nabla \theta). \tag{2.58}$$

Chapter 3

Stability estimates

Summary

In this chapter we present the proof of the stability estimate for the model of non-steady flow in porous media proposed by C.T.Hsu and P.Cheng. To prove the stability estimate we start with defining the problem that we will work on, and then find the weak formulation of the problem. The stability estimates are easily derived from the key inequality after we present the theoretical results Theorem (3.2.1) and Corollary (3.2.2).

3.1 Statement of the problem

In this section, we introduce a mathematical framework for the model presented in Section 2.

The notation to be used in this paper is as follows. For $d = 2, 3$, let $\Omega \subset \mathbb{R}^d$ be a bounded domain, Γ the boundary of Ω, and T a positive constant. Γ is divided into three parts, Γ_i, $i = 0, 1, 2$, which satisfy $\bar{\Gamma} = \bar{\Gamma}_0 \cup \bar{\Gamma}_1 \cup \bar{\Gamma}_2$ and $\Gamma_i \cap \Gamma_j = \emptyset$ for all $i \neq j$. We suppose that Γ is a Lipschitz boundary, and that, for each $i \in \{0, 1, 2\}$,

Γ_i is piecewise smooth, where the total number of the smooth boundaries of Γ_i is finite. The Lebesgue space on Ω for $p \in [1, \infty]$ is denoted by $L^p(\Omega)$ and the Sobolev space $W^{1,2}(\Omega)$ is denoted by $H^1(\Omega)$ with the norm

$$\|u\|_{H^1(\Omega)} \equiv \left(\|u\|_{L^2(\Omega)}^2 + \|\nabla u\|_{L^2(\Omega)}^2\right)^{1/2}.$$

The vector- and matrix-valued function spaces corresponding to, e.g., $L^2(\Omega)$ are denoted by $L^2(\Omega)^d$ and $L^2(\Omega)^{d\times d}$, respectively. The inner products in $L^2(\Omega)$, $L^2(\Omega)^d$, and $L^2(\Omega)^{d\times d}$ are all represented by $(\cdot, \cdot)$.

We consider the following problem governed by the Navier–Stokes equations with non-homogeneous porosity [13]; find $(u, p) : \overline{\Omega} \times [0, T] \to \mathbb{R}^d \times \mathbb{R}$ such that

$$\rho\Big[\frac{\partial u}{\partial t} + (u \cdot \nabla)\frac{u}{\phi}\Big] - \nabla \cdot [2\mu D(u)] + \nabla p = f + B(u, \phi) \quad \text{in } \Omega \times (0, T), \tag{3.1a}$$

$$\nabla \cdot u = 0 \quad \text{in } \Omega \times (0, T), \tag{3.1b}$$

$$u = g \quad \text{on } \Gamma_0 \times (0, T), \tag{3.1c}$$

$$2\mu D(u)n - pn = 0 \quad \text{on } \Gamma_1 \times (0, T), \tag{3.1d}$$

$$[2\mu D(u)n - pn] \times n = 0 \quad \text{on } \Gamma_2 \times (0, T), \tag{3.1e}$$

$$u \cdot n = 0 \quad \text{on } \Gamma_2 \times (0, T), \tag{3.1f}$$

$$u = u^0 \quad \text{in } \Omega, \text{ at } t = 0, \tag{3.1g}$$

where u is the Darcy velocity, p is the pressure, $\mu > 0$ is a dynamic viscosity, $u^0 : \Omega \to \mathbb{R}^d$ is a given initial velocity, $f : \Omega \times (0, T) \to \mathbb{R}^d$ is a given external force, $g : \Gamma_0 \times (0, T) \to \mathbb{R}^d$ is a given boundary velocity, $\phi : \Omega \to (0, 1]$ is a given porosity, $D(u) : \Omega \times (0, T) \to \mathbb{R}^{d\times d}_{\text{sym}}$ is the strain-rate tensor defined by

$$D(u) \equiv \frac{1}{2}\Big[\nabla u + (\nabla u)^T\Big],$$

$B(u, \phi) = B(u, \phi; \mu, \rho, d_p) : \Omega \times (0, T) \to \mathbb{R}^d$ is the total drag force defined in (2.35) with (2.41), and $n : \Gamma \to \mathbb{R}^d$ is the outward unit normal vector. On the boundary, we impose the Dirichlet boundary condition on Γ_0, the stress free boundary condition on Γ_1, and the slip boundary condition on Γ_2.

Throughout this paper, the following two hypotheses are assumed to hold.

Hypothesis 3.1.1. We suppose that $\mathrm{meas}(\Gamma_0) > 0$, $f \in C([0, T]; L^2(\Omega)^d)$, $g \in C([0, T]; H^1(\Omega)^d)$, and $u^0 \in L^2(\Omega)^d$.

Hypothesis 3.1.2. The porosity satisfies the following.

(*i*) $\phi \in W^{1,\infty}(\Omega), \quad \phi_0 \equiv \operatorname*{ess.inf}_{x\in\Omega} \phi(x) > 0.$

(*ii*) $|\nabla\phi| \leq \dfrac{2b}{d_p}(1 - \phi)$ a.e. in Ω.

Let us introduce constants ϕ_1 and α defined by

$$\phi_1 \equiv \operatorname*{ess.sup}_{x\in\Omega} \phi(x) \leq 1, \qquad \alpha \equiv \frac{a(1 - \phi_1)^2}{d_p^2 \phi_1^2} \geq 0.$$

We note that

$$\operatorname*{ess.inf}_{x\in\Omega} \frac{\phi(x)}{K(\phi(x))} \geq \alpha \geq 0. \tag{3.2}$$

Remark 3.1.3. From Hypothesis 3.1.1 and the Trace Theorem [10], it holds that $g(\cdot, t)_{|\Gamma_0} \in H^{1/2}(\Gamma_0)^d$ for any $t \in [0, T]$.

Remark 3.1.4. An example the value of $|\nabla\phi|$ in Lavrans field, Halten Terrace, Norway [7] is 4.336×10^{-5} cm^{-1}. In the real situation, the value of $d_p \leq 0.02$ cm, and from the empirical study, S. Ergun [8] suggested the value of $b = 1.75$. Then if we calculate the right hand side in Hypothesis 3.1.2-(*ii*), it results in 157.5 cm^{-1}. Obviously, the spatial derivative of the real porosity $\nabla\phi(x)$ satisfies $|\nabla\phi| \ll 157.5$ cm^{-1}. By this fact, Hypothesis 3.1.2-(*ii*) is not restrictive.

For a function $g_0 \in H^{1/2}(\Gamma_0)^d$, let us introduce function spaces $V(g_0)$, V and Q defined by

$$V(g_0) \equiv \big\{ v \in H^1(\Omega)^d;\ v = g_0 \text{ on } \Gamma_0,\ v \cdot n = 0 \text{ on } \Gamma_2 \big\},\ \ V \equiv V(0),\ \ Q \equiv L^2(\Omega),$$

respectively. When $\Gamma = \Gamma_0$, we replace the definition of Q above with $Q \equiv L^2_0(\Omega) \equiv \big\{ q \in L^2(\Omega);\ (q, 1) = 0 \big\}$ in a conventional way, cf. [10]. We define bilinear forms a_0, b, and c_0, and trilinear forms a_1 and c_1 by

$$a_0(u,v) \equiv 2\mu\big(D(u), D(v)\big), \quad b(v,q) \equiv -\,(\nabla \cdot v, q), \quad c_0(u,v) \equiv \mu\Big(\frac{\phi}{K(\phi)}u, v\Big),$$
$$a_1(u,w,v) \equiv \rho\big((u \cdot \nabla)w, v\big), \quad c_1(\theta,u,v) \equiv \rho\bigg(\frac{F(\phi)\phi\,\theta u}{\sqrt{K(\phi)}}, v\bigg).$$

The weak formulation for problem (3.1) is to find $\{(u,p)(t) \in V(g(t)) \times Q;\ t \in (0,T)\}$ such that, for $t \in (0,T)$,

$$\rho\Big(\frac{\partial u}{\partial t}, v\Big) + a_0(u,v) + a_1\Big(u, \frac{u}{\phi}, v\Big) + b(v,p) + b(u,q) + c_0(u,v) + c_1\big(|u|, u, v\big)$$
$$= (f(t), v)\,, \quad \forall (v,q) \in V \times Q, \tag{3.3a}$$

$$u(0) = u^0 \quad \text{in } L^2(\Omega)^d. \tag{3.3b}$$

3.2 Estimates

In this section, we present the theoretical results Theorem 3.2.1 and Corollary 3.2.2, which provide a key inequality and stability estimates, respectively. The stability estimates are easily derived from the key inequality.

Theorem 3.2.1. *Suppose that Hypotheses 3.1.1 and 3.1.2 hold true. Assume $g = 0$. Suppose that $(u,p) \in (C^1([0,T]; L^2(\Omega)^d) \cap L^2(0,T;V)) \times L^2(0,T;L^2(\Omega))$*

satisfies (3.3). *Then, it holds that*

$$\frac{d}{dt}\Big(\frac{\rho}{2}\|u(t)\|^2_{L^2(\Omega)}\Big) + \frac{\rho}{2}\int_{\Gamma_1} \frac{|u(t)|^2}{\phi} u(t)\cdot n\, ds + \mu\beta_0^2\|u(t)\|^2_{H^1(\Omega)} + \mu\alpha\|u(t)\|^2_{L^2(\Omega)} \leq \frac{1}{4\mu\beta_0^2}\|f(t)\|^2_{L^2(\Omega)}, \tag{3.4}$$

where $\beta_0 > 0$ *is a positive constant to be defined in* (3.7) *below.*

Corollary 3.2.2 (Stability estimates)**.** In addition to the same assumptions in Theorem 3.2.1, suppose that $u \cdot n \geq 0$ on $\Gamma_1 \times [0, T]$. Then, we have the following:

(i) It holds that

$$\sqrt{\rho}\|u\|_{L^\infty(0,T;L^2(\Omega))} + \sqrt{\mu}\beta_0\|u\|_{L^2(0,T;H^1(\Omega))} \leq 2\Big(\sqrt{\rho}\|u^0\|_{L^2(\Omega)} + \frac{1}{\sqrt{\mu}\beta_0}\|f\|_{L^2(0,T;L^2(\Omega))}\Big). \tag{3.5}$$

(ii) It holds that, for any $t \in [0, T]$,

$$\|u(t)\|_{L^2(\Omega)} \leq \exp\Big(-\frac{\mu\alpha}{\rho}\,t\Big)\|u^0\|_{L^2(\Omega)} + \frac{1}{\sqrt{2\rho\mu}\beta_0}\|f\|_{L^2(0,t;L^2(\Omega))}. \tag{3.6}$$

The proofs of Theorem 3.2.1 and Corollary 3.2.2 are given after stating two lemmas.

Lemma 3.2.3 (Korn's inequality, [4; 17])**.** *Let* Ω *be a bounded domain with a Lipschitz-continuous boundary* $\partial\Omega$*, and let* Γ_0 *be a part of* $\partial\Omega$*. Assume* $\mathrm{meas}(\Gamma_0) > 0$. *Then, there exists a positive constant* β_0 *such that*

$$\beta_0\|u\|_{H^1(\Omega)} \leq \|D(u)\|_{L^2(\Omega)} \qquad \forall u \in \{v \in H^1(\Omega)^d;\ v = 0 \ \text{on}\ \Gamma_0\}. \tag{3.7}$$

Lemma 3.2.4. *Suppose Hypothesis 3.1.2-(i) holds true. Assume* $u \in H^1(\Omega)^d$ *and* $\nabla \cdot u = 0$ *in* Ω*. Then, it holds that*

$$\Big((u \cdot \nabla)\Big(\frac{u}{\phi}\Big), u\Big) = \frac{1}{2}\int_\Gamma \frac{|u|^2}{\phi} u \cdot n \, ds + \frac{1}{2}\Big(|u|^2, (u \cdot \nabla)\frac{1}{\phi}\Big). \tag{3.8}$$

Proof. Let $I \equiv ((u \cdot \nabla)(u/\phi), u)$. From the integration by parts, and the assumption, $\nabla \cdot u = 0$, we have:

$$\begin{aligned} I &= \int_\Gamma u_j \left(\frac{u_i}{\phi}\right) u_i n_j ds - \int_\Omega \left(\frac{u_i}{\phi}\right) (u_i u_j)_{,j} \, dx \\ &= \int_\Gamma \left(\frac{u_i}{\phi}\right) u_i u_j n_j ds - \int_\Omega \left(\frac{u_i}{\phi}\right) (u_{i,j} u_j + u_i u_{j,j}) dx \\ &= \int_\Gamma \left(\frac{u_i}{\phi}\right) u_i u \cdot n ds - \int_\Omega \left(\frac{u_i}{\phi}\right) (u \cdot \nabla) u_i dx \\ &= \int_\Gamma \frac{|u|^2}{\phi} u \cdot n \, ds - \Big(\nabla \cdot (u \otimes u), \frac{u}{\phi}\Big) \\ &= \int_\Gamma \frac{|u|^2}{\phi} u \cdot n \, ds - \Big((u \cdot \nabla) u, \frac{u}{\phi}\Big). \end{aligned} \tag{3.9}$$

On the other hand, from the product rule, we have:

$$\begin{aligned} I &= \int_\Omega u_j \left(u_i \frac{1}{\phi}\right)_{,j} u_i dx \\ &= \int_\Omega u_j \left(u_{i,j} \frac{1}{\phi} + u_i \left(\frac{1}{\phi}\right)_{,j}\right) u_i dx \\ &= \int_\Omega \left\{\frac{1}{\phi} \left[(u \cdot \nabla) u_i\right] u_i + |u|^2 u_j \left(\frac{1}{\phi}\right)_{,j}\right\} dx \\ &= \left(\left[(u \cdot \nabla)\right] \frac{1}{\phi}, u\right) + \left(|u|^2, (u \cdot \nabla)\left(\frac{1}{\phi}\right)\right) \\ &= \Big((u \cdot \nabla) u, \frac{u}{\phi}\Big) + \Big(|u|^2, (u \cdot \nabla)\frac{1}{\phi}\Big). \end{aligned} \tag{3.10}$$

Adding the two equations (3.9) and (3.10) and dividing by 2, we obtain (3.8). □

Proof of Theorem 3.2.1. Substituting $(u, -p) \in V \times Q$ into (v, q) in (3.3), we have

$$\rho\Big(\frac{\partial u}{\partial t}, u\Big) + a_0(u, u) + a_1\Big(u, \frac{u}{\phi}, u\Big) + c_0(u, u) + c_1(|u|, u, u) = (f, u). \tag{3.11}$$

We evaluate each term in (3.11) as follows:

$$\begin{aligned}
\rho\Big(\frac{\partial u}{\partial t}, u\Big) &= \rho \int_\Omega \frac{\partial}{\partial t}\left(\frac{1}{2}u_i u_i\right) dx \\
&= \frac{d}{dt}\left(\frac{\rho}{2}\int_\Omega |u|^2\right) dx \\
&= \frac{d}{dt}\Big(\frac{\rho}{2}\|u\|^2_{L^2(\Omega)}\Big), && (3.12\text{a}) \\
a_0(u, u) &= 2\mu \int_\Omega D(u) : D(u) dx \\
&= 2\mu \|D(u)\|^2_{L^2(\Omega)} \geq 2\mu\beta_0^2 \|u\|^2_{H^1(\Omega)} \qquad \text{(by Lem. 3.2.3)}, && (3.12\text{b}) \\
a_1\Big(u, \frac{u}{\phi}, u\Big) &= \rho((u\cdot\nabla)u, u) \\
&= \frac{\rho}{2}\int_{\Gamma_1} \frac{|u|^2}{\phi} u\cdot n\, ds + \frac{\rho}{2}\Big(|u|^2, (u\cdot\nabla)\frac{1}{\phi}\Big) \quad \text{(by Lem. 3.2.4)} \\
&\geq \frac{\rho}{2}\int_{\Gamma_1} \frac{|u|^2}{\phi} u\cdot n\, ds - \Big(|u|^2, \frac{\rho|u|}{2}\Big|\nabla\frac{1}{\phi}\Big|\Big), && (3.12\text{c}) \\
c_0(u, u) &= \mu\Big(\frac{\phi}{K(\phi)}, |u|^2\Big) \geq \mu\alpha\|u\|^2_{L^2(\Omega)} \qquad \text{(by (3.2))}, && (3.12\text{d}) \\
c_1(|u|, u, u) &= \rho\left(\frac{F(\phi)\phi|u|u}{\sqrt{K(\phi)}}, u\right) \\
&= \left(|u|^2, \rho|u|\frac{F(\phi)\phi}{\sqrt{K(\phi)}}\right), && (3.12\text{e}) \\
(f, u) &\leq \|f\|_{L^2(\Omega)}\, \|u\|_{L^2(\Omega)} \\
&\leq \mu\beta_0^2\|u\|^2_{L^2(\Omega)} + \frac{1}{4\mu\beta_0^2}\|f\|^2_{L^2(\Omega)} \\
&\leq \mu\beta_0^2\|u\|^2_{H^1(\Omega)} + \frac{1}{4\mu\beta_0^2}\|f\|^2_{L^2(\Omega)}. && (3.12\text{f})
\end{aligned}$$

Here, we note the fact that Hypothesis 3.1.2 yields

$$G_\phi := \frac{1}{2}\left|\nabla \frac{1}{\phi}\right| - \frac{F(\phi)\phi}{\sqrt{K(\phi)}} = \frac{1}{2\phi^2}\Big[|\nabla \phi| - \frac{2b}{d_p}(1-\phi)\Big] \le 0 \quad \text{a.e. in } \Omega. \tag{3.13}$$

Combining (3.12) with (3.11) and using (3.13), we obtain

$$\begin{aligned}\frac{d}{dt}\Big(\frac{\rho}{2}\|u(t)\|^2_{L^2(\Omega)}\Big) + \frac{\rho}{2}\int_{\Gamma_1} \frac{|u(t)|^2}{\phi} u(t)\cdot n\, ds + \mu\beta_0^2\|u(t)\|^2_{H^1(\Omega)} + \mu\alpha\|u(t)\|^2_{L^2(\Omega)} \\ \le \frac{1}{4\mu\beta_0^2}\|f(t)\|^2_{L^2(\Omega)} + \big(|u(t)|^2, \rho|u(t)|G_\phi\big) \le \frac{1}{4\mu\beta_0^2}\|f(t)\|^2_{L^2(\Omega)}.\end{aligned}$$

Thus, we obtain (3.4). □

Proof of Corollary 3.2.2. Firstly, we prove (i). Dropping the non-negative second and fourth terms in (3.4), we have

$$\frac{d}{dt}\Big(\frac{\rho}{2}\|u(t)\|^2_{L^2(\Omega)}\Big) + \mu\beta_0^2\|u(t)\|^2_{H^1(\Omega)} \le \frac{1}{4\mu\beta_0^2}\|f(t)\|^2_{L^2(\Omega)},$$

which implies (3.5). Here, we have used the fact that, for non-negative functions $\eta \in C^1([0,T];\mathbb{R})$ and $\phi, \psi \in L^1([0,T];\mathbb{R})$, the inequality $\eta'(t) + \phi(t) \le \psi(t)$ $(t \in [0,T])$ yields $\|\eta\|_{L^\infty(0,T)} + \|\phi\|_{L^1(0,T)} \le 2[\eta(0) + \|\psi\|_{L^1(0,T)}]$, we also used the inequality $(a+b)/\sqrt{2} \le \sqrt{a^2+b^2}$ $(a, b \in \mathbb{R})$.

Secondly, we prove (ii). Dropping the non-negative second and third terms in (3.4), we get

$$\frac{d}{dt}\Big(\frac{\rho}{2}\|u(t)\|^2_{L^2(\Omega)}\Big) + \mu\alpha\|u(t)\|^2_{L^2(\Omega)} \le \frac{1}{4\mu\beta_0^2}\|f(t)\|^2_{L^2(\Omega)},$$

which implies (3.6) from Gronwall's inequality. □

Chapter 4

Lagrange–Galerkin Scheme

Summary

In this chapter we discuss the basic idea to derived the Lagrange–Galerkin with Adams–Bashforth time discretization by extending the method of characteristic. The idea is to define the macroscopic average velocity w, and then compute the material derivative respect to w. To find our numerical scheme, we approximate the material derivative using Adams–Bashforth method.

4.1 Basic idea of the Scheme

In this section, we present a Lagrange–Galerkin scheme of second-order in time for problem (3.1).

For the Darcy velocity u and the porosity ϕ in problem (3.1), we introduce the macroscopic average velocity $w : \overline{\Omega} \times [0, T] \to \mathbb{R}^d$ and the material derivative D/Dt with respect to w defined by

$$w \equiv \frac{u}{\phi}, \qquad \qquad \frac{D}{Dt} \equiv \frac{\partial}{\partial t} + w \cdot \nabla.$$

Then, we can rewrite $\partial u/\partial t + (u \cdot \nabla)(u/\phi)$ by

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)\frac{u}{\phi} = \phi\Big[\frac{\partial w}{\partial t} + (w \cdot \nabla)w\Big] = \phi\frac{Dw}{Dt}. \tag{4.1}$$

The equation (4.1) is a fundamental relation to the development of our new numerical scheme to be presented.

Let τ be a time increment, $N_T \equiv \lfloor T/\tau \rfloor$ the total number of time steps, and $t^k \equiv k\tau$ for $k \in \{0, 1, \dots, N_T\}$. For a function ψ defined in $\overline{\Omega} \times [0, T]$ or $\Gamma_0 \times [0, T]$, we denote $\psi(\cdot, t^k)$ simply by ψ^k. Let $X : [0, T] \to \mathbb{R}^d$ be a solution of the following ordinary differential equation :

$$X'(t) = w(X(t), t), \quad t \in [0, T], \tag{4.2}$$

subject to an initial condition $X(t^k) = x$. Physically, $X(t)$ represents the position of a fluid particle with respect to the macroscopic average velocity w at time t. For a given velocity $v : \Omega \to \mathbb{R}^d$, let $X_1(v, \tau) : \Omega \to \mathbb{R}^d$ be the mapping defined by

$$X_1(v, \tau)(x) \equiv x - v(x)\tau, \tag{4.3}$$

which is an upwind point of x with respect to the velocity v and a time increment τ. Now, we derive the second-order approximation of $\partial u/\partial t + (u \cdot \nabla)(u/\phi)$ at (x, t^k) by the Adams–Bashforth method as follows:

$$\begin{aligned}
&\Big[\frac{\partial u}{\partial t} + (u \cdot \nabla)\frac{u}{\phi}\Big](x, t^k) = \phi(x)\frac{Dw}{Dt}(x, t^k) = \phi(x)\frac{d}{dt}\left(w(X(t), t)\right)_{|t=t^k} \\
&= \frac{\phi(x)}{2\tau}\Big[3w^k - 4w^{k-1} \circ X_1\big(w^k, \tau\big) + w^{k-2} \circ X_1\big(w^k, 2\tau\big)\Big](x) + O(\tau^2) \qquad (4.4)\\
&= \frac{\phi(x)}{2\tau}\Big[3w^k - 4w^{k-1} \circ X_1\big(w^{(k-1)*}, \tau\big) + w^{k-2} \circ X_1\big(w^{(k-1)*}, 2\tau\big)\Big](x) + O(\tau^2) \\
&= \frac{1}{2\tau}\Big[3u^k - \phi\big[4w^{k-1} \circ X_1(w^{(k-1)*}, \tau) - w^{k-2} \circ X_1(w^{(k-1)*}, 2\tau)\big]\Big](x) + O(\tau^2),
\end{aligned}$$

where the symbol "$\circ$" denotes the composition of functions,

$$[v \circ X_1(v,\tau)](x) = v(X_1(v,\tau)(x)),$$

and $w^{(k-1)*}$ is a second-order approximation of w^k defined by

$$w^{(k-1)*} \equiv 2w^{k-1} - w^{k-2}.$$

The idea of (4.4) has been proposed and employed in [3; 9; 21; 22].

Let $\mathcal{T}_h \equiv \{e\}$ be a triangulation of $\overline{\Omega}\,(= \cup_{e\in\mathcal{T}_h})$, h_e be the diameter of $e \in \mathcal{T}_h$, and $h \equiv \max_{e\in\mathcal{T}_h} h_e$ be the maximum element size. We define the function spaces X_h, M_h, V_h and Q_h by

$$X_h \equiv \Big\{ v_h \in C(\overline{\Omega})^d;\ v_{h|e} \in P_2(e)^d,\ \forall e \in \mathcal{T}_h \Big\},$$
$$M_h \equiv \Big\{ q_h \in C(\overline{\Omega});\ q_{h|e} \in P_1(e),\ \forall e \in \mathcal{T}_h \Big\},$$

$V_h \equiv X_h \cap V$ and $Q_h \equiv M_h \cap Q = M_h$, where $P_k(e)$ is the (scalar-valued) polynomial space of degree $k \in \mathbb{N}$ on e.

Let $u_h^0 \in X_h$ and $\{g_h^k\}_{k=1}^{N_T} \subset X_h$ be given approximations of u^0 and g. Our new Lagrange–Galerkin scheme of second-order in time for solving problem (3.1) is to find $\big\{(u_h^k, p_h^k)\big\}_{k=1}^{N_T} \subset V_h(g_h^k) \times Q_h$ such that, for all $(v_h, q_h) \in V_h \times Q_h$,

(initial step)

$$\left(\frac{u_h^1 - \phi[w_h^0 \circ X_1(w_h^0,\tau)]}{\tau}, v_h \right) + a_0(u_h^1, v_h) + b(v_h, p_h^1) + b(u_h^1, q_h)$$
$$+ c_0(u_h^1, v_h) + c_1(|u_h^0|, u_h^1, v_h) = (f^1, v_h), \quad \text{(4.5a)}$$

(general step)

$$\left(\frac{1}{2\tau}\Big[3u_h^k - \phi\big[4w_h^{k-1}\circ X_1(w_h^{(k-1)*},\tau) - w_h^{k-2}\circ X_1(w_h^{(k-1)*},2\tau)\big]\Big], v_h\right)$$
$$+a_0(u_h^k, v_h) + b(v_h, p_h^k) + b(u_h^k, q_h) + c_0(u_h^k, v_h) + c_1(|u_h^{(k-1)*}|, u_h^k, v_h,)$$
$$= (f^k, v_h), \quad k = 2, \dots, N_T, \quad (4.5b)$$

where w_h^k and $w_h^{(k-1)*}$ are defined by

$$w_h^k \equiv \frac{u_h^k}{\phi}, \qquad w_h^{(k-1)*} \equiv 2w_h^{k-1} - w_h^{k-2}.$$

We compute (u_h^1, p_h^1) by (4.5a) and $\{(u_h^k, p_h^k)\}_{k=2}^{N_T}$ by (4.5b). This idea on the initial step treatment has been proposed for the Navier–Stokes equations, cf. [22], where the second-order convergence in time in $L^2(\Omega)$-norm has been proved. Here, we apply it to problem (3.1).

Chapter 5

Numerical Results

Summary

In this section, we confirm the experimental order of convergence of scheme (4.5) and perform some numerical simulation for fluid flow in non-homogeneous porous media. All of the computations in this section are computed on a Intel(R) Core (TM) i7-2600 CPU @ 3.40 GHz with 4GB RAM.

5.1 Experimental Order of Convergence

In this subsection, a two-dimensional test problem is computed by scheme (4.5) to check the order of convergence of the scheme. In problem (3.1) we set $\Omega = (0,\pi)^2$ cm, $T = 1$ s, $\mu = 8.89 \times 10^{-3}$ dyn·s/cm^2, $d_p = 5 \times 10^{-2}$ cm, $\rho = 9.951 \times 10^{-1}$ gr/cm^3, and $\phi = [2 + \sin(2x_2/5)]/3$. The functions g and u^0 are given so that the exact solution is

$$u(x,t) = \Big(-\frac{\partial\psi}{\partial x_2}, \frac{\partial\psi}{\partial x_1}\Big)(x,t), \quad p = \sin(x_1)\sin(x_2)e^{-2t}, \quad \psi = \sin^3(x_1)\sin^3(x_2)e^{-2t}.$$

The problem is solved by scheme (4.5) with $h = \pi/N$ for $N = 4, 8, 16, 32, 128$, and $\tau = h$. For the computation we employed FreeFem++ [12] with P2/P1-element. For the solution (u_h, p_h) of scheme (4.5) we define errors $Er1$ and $Er2$ by

$$Er1 := \max_{n=0,\dots,N_T} \| u_h^n - u^n \|_{H^1(\Omega)}, \quad Er2 := \max_{n=0,\dots,N_T} \| p_h^n - p^n \|_{L^2(\Omega)} .$$

Figure 5.1 shows the graphs of $Er1$ and $Er2$ versus h $(= \tau)$ in logarithmic scale. The values of $Er1$, $Er2$ and slopes are represented in Table 5.1. We can see that both $Er1$ and $Er2$ are almost of second order in h $(= \tau)$.

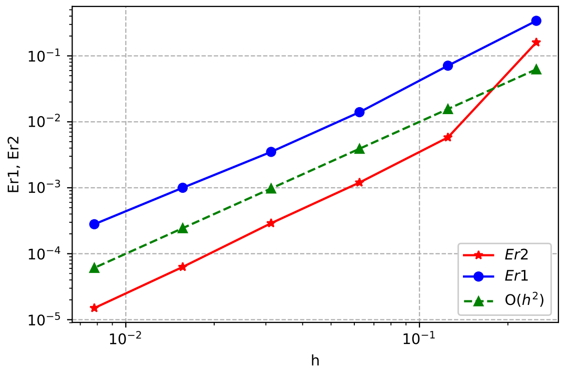


Figure 5.1: The order of convergence for scheme (4.5).

Table 5.1: Values of $Er1$ and $Er2$ and their slopes for the Problem 3 by scheme (4.5).

N	$Er1$	$Er2$	Slope of $Er1$	Slope of $Er2$	CPU times [s]
4	3.4×10^{-1}	1.6×10^{-1}	–	–	1.9
8	7.1×10^{-2}	5.8×10^{-3}	2.26	4.76	16.4
16	1.4×10^{-2}	1.2×10^{-3}	2.34	2.30	174.8
32	3.5×10^{-3}	2.9×10^{-4}	2.00	2.05	577.4
64	1.0×10^{-3}	6.3×10^{-5}	1.81	2.20	5,953.9
128	2.8×10^{-4}	1.5×10^{-5}	1.84	2.07	58,150.9

5.2 Simulation with Non-homogeneous Porosity

In this subsection, we present two cases of numerical simulation for the fluid flow through the non-homogeneous porous media.

5.2.1 Simulation of Flow in Two Layers of Porosity

The purpose of the simulation in the first case is to understand the fluid flow in the two layers of porosity. This simulation is motivated by the real condition of the geothermal reservoir which has a porosity function of the depth. At the top of the reservoir, the value of porosity is large, while at the bottom, the value of porosity is small due to the existence of pressure which comes from the mass of the soils and rocks.

We set $\Omega = (0,3) \times (0,1)$ cm, $\Gamma_1 = \{(x_1, x_2);\ x_1 = 3,\ 0 < x_2 < 1\}$, $\Gamma_0 = \partial\Omega \setminus \overline{\Gamma}_1$, $f = 0$, $g = u^0$ on Γ_0, $\Gamma_2 = \emptyset$, $T = 5$ s, $\rho = 9.951 \times 10^{-1}$ gr/cm^3, and $\mu = 8.89 \times 10^{-3}$ dyn·s/cm^2. We define the initial condition as

$$u^0 = \eta(x_1) \begin{pmatrix} \frac{1}{4} - (x_2 - \frac{1}{2})^2 \\ 0 \end{pmatrix},$$

where η is defined by

$$\eta(x_1) := \begin{cases} \cos(\pi x_1) & (0 \le x_1 \le 0.5), \\ 0 & (0.5 < x_1). \end{cases} \tag{5.1}$$

For the porosity ϕ we set

$$\phi(x) = 0.4 + 0.4 H_\epsilon(x_2 - 0.5),$$

where $\epsilon = \frac{1}{360}$ and H_ϵ is an approximated Heaviside function defined by

$$H_\epsilon(s) = \begin{cases} 1 & (s \geq \epsilon), \\ \dfrac{1}{2} + \dfrac{1}{2}\left(\dfrac{s}{\epsilon} + \dfrac{1}{\pi}\sin\dfrac{\pi s}{\epsilon}\right) & (|s| < \epsilon), \\ 0 & (s \leq -\epsilon). \end{cases} \tag{5.2}$$

For this case we run the simulation with division number $N = 120$, $h = 3/N$, $\tau = h$. Since we have a layer of ϕ on $x_2 = 1/2$, we employ a mesh whose mesh size near $x_2 = 1/2$ is chosen to be around $1/720$. To aid the understanding of the problem setting in this simulation, the boundary conditions and the porosity are illustrated together with the finite element mesh on Ω in Figure 5.2.

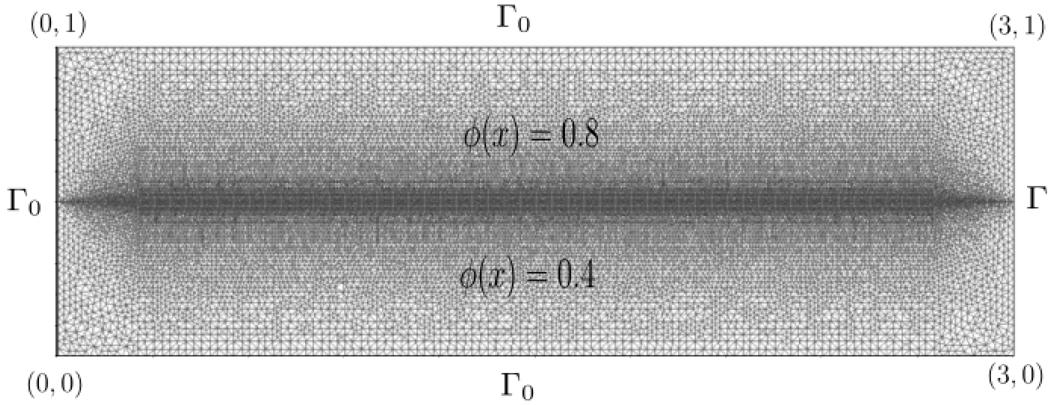
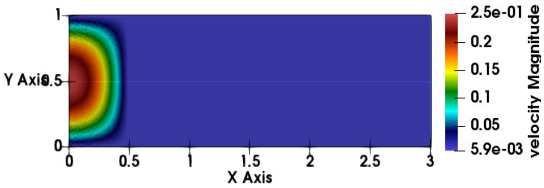
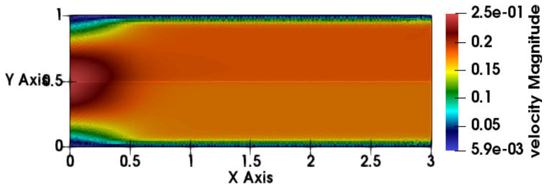


Figure 5.2: The boundary conditions and the finite element mesh.

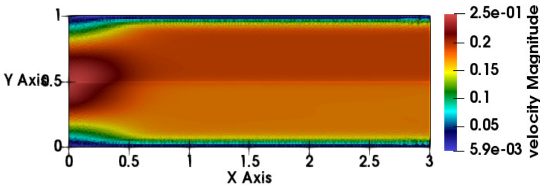
The results of the first case simulation are presented in Figure 5.3. Figure 5.3-(a) is the initial condition of the simulation. From this figure, we can see that the profile distribution of the velocity is symmetric. As time increases, the profile distribution becomes asymmetric; this happens because of the difference of values of the porosity. From equation (2.41), it can be understood that high porosity implies high permeability. High permeability means that the resistance of fluids to flow is small so that the fluid can flow faster rather than the area with small



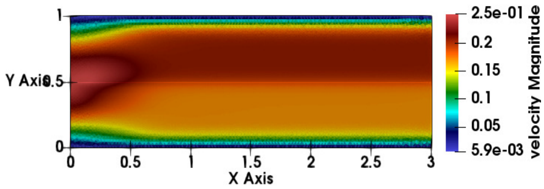
(a) $t = 0.0$ [s]



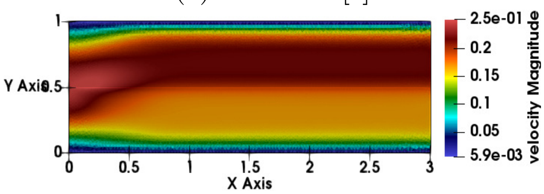
(b) $t = 0.083$ [s]



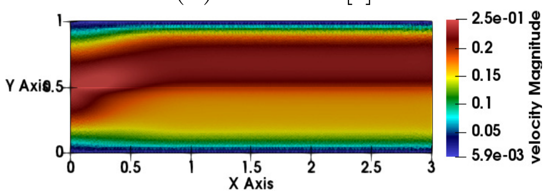
(c) $t = 0.16$ [s]



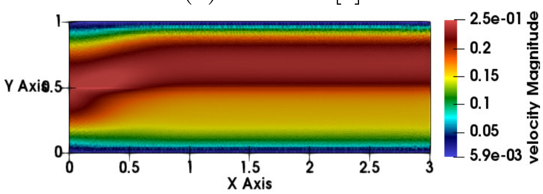
(d) $t = 0.33$ [s]



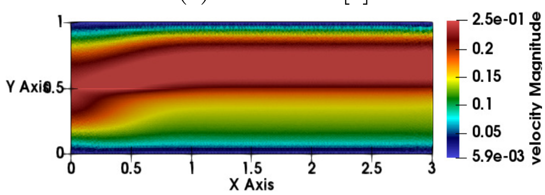
(e) $t = 0.5$ [s]



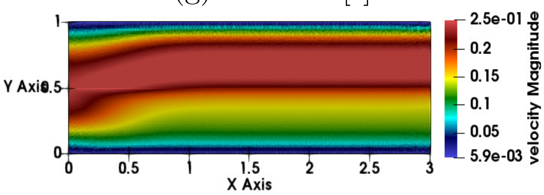
(f) $t = 0.66$ [s]



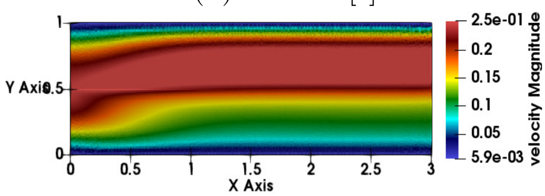
(g) $t = 0.83$ [s]



(h) $t = 1.6$ [s]



(i) $t = 3.3$ [s]



(j) $t = 5.0$ [s]

Figure 5.3: Time evolution of velocity magnitude.

porosity. It clearly can be seen in (c)-(j) in Figure 5.3 that the flow in the top layer with $\phi = 0.8$ is faster than that in the bottom layer with $\phi = 0.4$. This behavior of our numerical results has a good qualitative agreement with the natural flow in the simple case of the porous media.

5.2.2 Simulation of Flow in Complex Porosity

The purpose of the second simulation is to understand the fluid flow for the complex value of porosity. This simulation is motivated by the real condition of the porosity distribution in the rock structure, such as in carbonate rock, where the value of porosity is irregular. For this simulation, we set $\Omega = (0, 3\pi) \times (0, \pi)$ cm, $T = 5$ s, $\rho = 9.951 \times 10^{-1}$ gr/cm^3, $\mu = 8.89 \times 10^{-3}$ dyn.s/cm^2, $f = 0$, and

$$u^0 = \eta(x_1) \begin{pmatrix} 0.01 \left(\frac{\pi^2}{4} - \left(x_2 - \frac{\pi}{2} \right)^2 \right) \\ 0 \end{pmatrix},$$

where η is the function defined in (5.1). For the porosity ϕ we set

$$\phi(x) = \frac{\gamma_1 - \gamma_0}{2} \sin(2x_2) \cos(2x_1) + \frac{\gamma_1 + \gamma_0}{2},$$

where $\gamma_0 = 0.15$ and $\gamma_1 = 0.65$. For this case we run the simulation with division number $N = 300$, $h = 3\pi/N$, $\tau = h$. To aid the understanding of the problem setting in this simulation, we plotted the distribution function of porosity in the computational domain in Figure 5.4.

The results of the second case simulation are presented in Figure 5.5. Figure 5.5-(a) illustrates the initial velocity magnitude of the simulation. From Figure 5.5, we can see that the fluid is flowing faster in the area which has a large porosity; for the area which has small porosity, the fluid is flowing slowly. In the area with small porosity, we can see the gradation motion of the fluid clearly; this

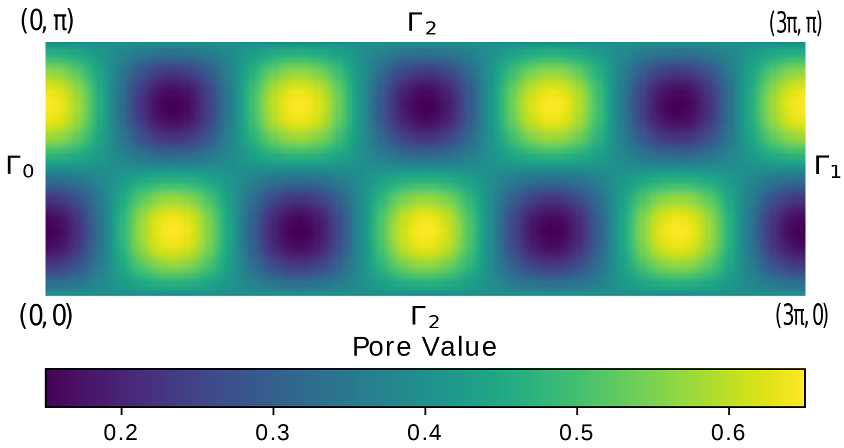
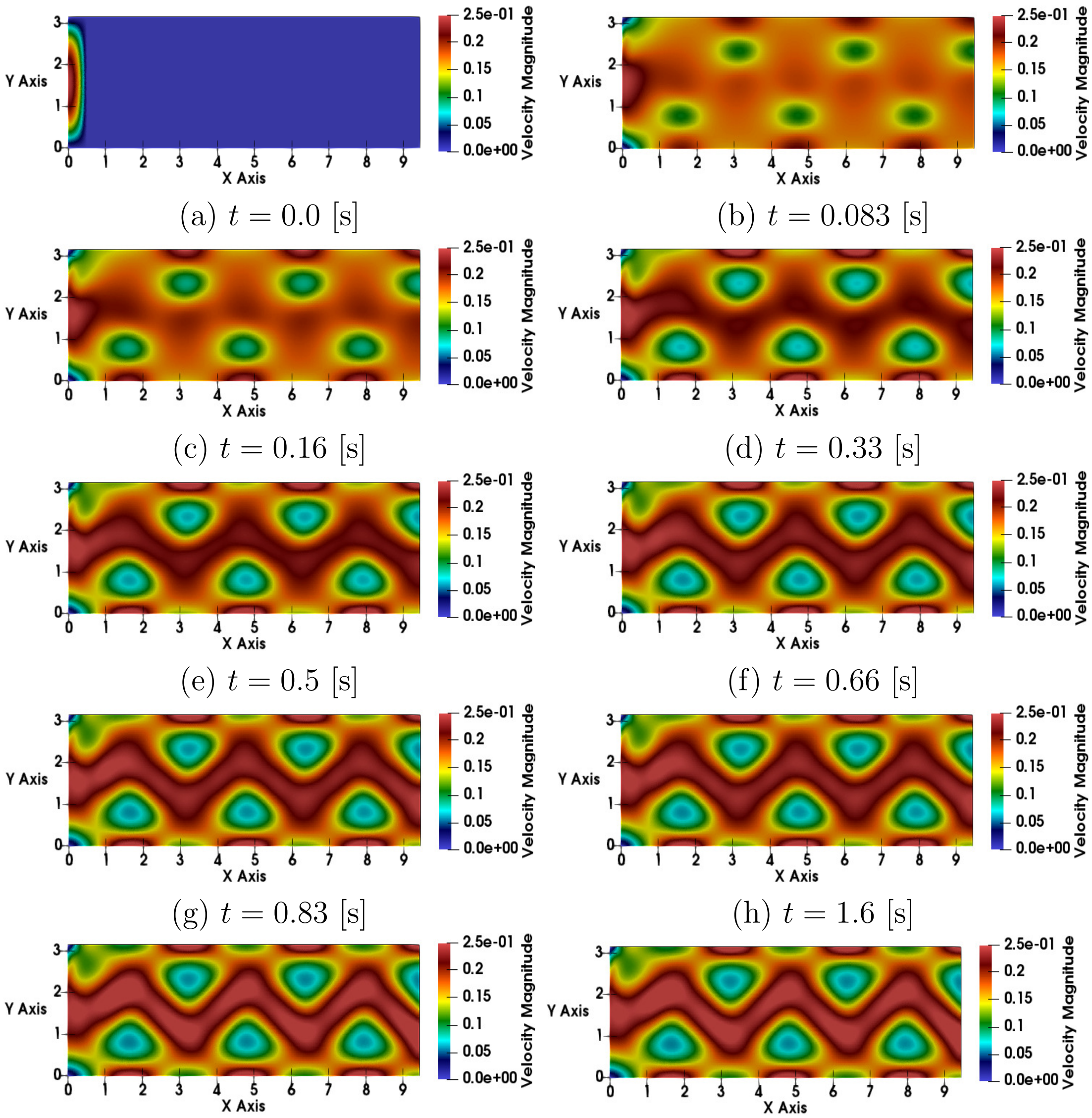


Figure 5.4: Computation domain and porosity value distribution

fact emphasizes that scheme (4.5) can deal with the irregular pattern of porosity. Figure 5.5 has a good qualitative agreement with the natural flow in the irregular design of porous media.



(a) $t = 0.0$ [s] (b) $t = 0.083$ [s]

(c) $t = 0.16$ [s] (d) $t = 0.33$ [s]

(e) $t = 0.5$ [s] (f) $t = 0.66$ [s]

(g) $t = 0.83$ [s] (h) $t = 1.6$ [s]

(i) $t = 3.3$ [s] (j) $t = 5.0$ [s]

Figure 5.5: Time evolution of magnitude velocity.

Chapter 6

Conclusions

To approach the phenomena in the geothermal reservoir, we dealt with the equations of non-steady flow in the non-homogeneous porous media proposed by C.T. Hsu and P. Cheng. In this work, we succeeded to prove the L^2-stability estimates of the model by establishing Lemma (3.2.4) to extract the influence of the non-homogeneity of the porosity. To established the energy stability estimates, we control this term with the Forchheimer term coming from the Darcy-Brinkmann-Forchheimer model. As a numerical scheme, we proposed a characteristic finite element method (Lagrange-Galerkin scheme). We extended the idea of the characteristics method and introduced the macroscopic average velocity w to overcome the difficulty which comes from the convection term with the non-homogeneous porosity ϕ. To check the convergence order of the scheme, we compared a simple problem with the analytical solution and showed that our scheme has second-order accuracy both in space and in time. From the numerical simulation presented in Subsection 5.2 and 5.3, our results have a good qualitative agreement with the natural flow in the simple and complex structures of porosity.

In this work, we succeeded to propose the Lagrange-Galerkin scheme for solving the model. However the theoretical convergence of this scheme has not been proved yet. Another challenge to improve the stability estimates of our results is to extend the Hypothesis 3.1.2.(i) to allow for ϕ to have a jump.

For the next work, we plan to couple our system with the thermal energy to simulate the fluid flow and heat transfer in the geothermal reservoir in 3D to predict the electrical generating capacity and the life time of the reservoir.

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