Theoretical and numerical studies of the shallow water equations with a transmission boundary condition

Dissertation Abstract

Theoretical and numerical studies of the shallow water equations with a transmission boundary condition

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Date of Submission (Revised version)	: September 12, 2019

Abstract

In this work, the stability of the shallow water equations (SWEs) with a transmission boundary condition is studied theoretically and numerically using a suitable energy. In the theoretical part, using a suitable energy, we begin with deriving an equality which implies an energy estimate of the SWEs with the Dirichlet and the slip boundary conditions. For the SWEs with a transmission boundary condition, an inequality for the energy estimate is proved under some assumptions to be satisfied in practical computation. In the numerical part, based on the theoretical results, the energy estimate of the SWEs with a transmission boundary condition is confirmed numerically by a finite difference method (FDM) and Lagrange–Galerkin method (LGM). The choice of a positive constant c_0 used in the transmission boundary condition is investigated additionally. Furthermore, we present numerical results by a LGM, which are similar to those by the FDM. The computation of the SWEs with the transmission boundary condition are also made for the Bay of Bengal by a LGM with the triangular mesh. To see the performance of the LGM we have investigated the experimental order of convergence for the LGM with a suitable choice of exact solutions for five different cases of boundary setting for the norms several norms. The results are satisfactory. In order to see whether the transmission boundary condition is independent of its position or not, simulations are made in the Bay of Bengal, setting the transmission boundary condition in two different places. We have computed the mass and L^2-norm of η and the results shows that the transmission boundary condition works well numerically it is almost independent of its position.

1 Introduction

The shallow water equations (SWEs) can be considered as a coupled system of a pure convection equation for the function ϕ of total wave height and a simplified Navier–Stokes equation for the velocity $u = (u_1, u_2)^T$ obtained by averaging function values in x_3-direction, which are often used for the simulation of tsunami/storm surge in the bay.

In such simulation there are some boundaries in the open sea, see Figure 1. In a real situation, if wave propagates towards such boundaries in the open sea, then there should not be any reflection on these boundaries.

In this study, following [?], we employ a transmission boundary condition on the boundaries in the open sea which is capable to remove this kind of artificial reflection.

It is to be noted here that our final goal is to develop a storm surge prediction model for the Bay of Bengal. For such models researchers, usually employ a radiation type boundary condition on the boundaries in the open sea, see, e.g., [?, ?, ?, ?], which is very similar to the transmission boundary condition used in [?].

The transmission boundary condition of the form

$$u(x,t) = c(x)\frac{\eta(x,t)}{\phi(x,t)}n(x) \tag{1}$$

is often used on Γ_T, where $c(x)$ is a given positive function and $\eta(x,t) = \phi(x,t) - \zeta(x)$ is the elevation from the reference height for a given depth function ζ.

Figure 1: The Bay of Bengal

In this paper, in order to understand the transmission boundary condition mathematically, we study the stability of the SWEs in terms of a suitable energy, and confirm the stability numerically by a finite difference method (FDM) and a finite element method (FEM).

It is to be noted here that we can show a (successful) energy estimate of the SWEs, when only the Dirichlet and the slip boundary conditions are employed, cf. Corollary **??**-(ii), where such discussions have been done under the periodic boundary condition, e.g., [?, ?]. As far as we know, however, there is no mathematical results on the energy estimate of the SWEs with the transmission boundary condition.

Although, at present, the mathematical results do not derive the stability estimate of the SWEs with the transmission boundary condition directly, we have good information and can study the stability numerically by using the theoretical results.

It is known that the FEM is more suitable than FDM for a domain of irregular shape. As the shape of the Bay of Bengal is very irregular, the simulation of the SWEs with the transmission boundary condition are also made by a Lagrange–Galerkin method (LGM) for this domain. The LGM is a FEM based on the time discretization of the material derivative,

$$\frac{\phi^{k+1}(x) - \phi^k(x - u^k(x)\Delta t)}{\Delta t}.$$

2 Statement of the problem

In this section, we state the mathematical problem to be considered in this paper. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain and T a positive constant. We consider the problem : find $(\phi, u) : \overline{\Omega} \times [0,T] \to \mathbb{R} \times \mathbb{R}^2$ such that

$$\begin{cases} \dfrac{\partial \phi}{\partial t} + \nabla \cdot (\phi u) = 0 & \text{in } \Omega \times (0,T), \\ \rho\phi\Big[\dfrac{\partial u}{\partial t} + (u \cdot \nabla)u\Big] - 2\mu\nabla \cdot (\phi D(u)) + \rho g \phi \nabla \eta = 0 & \text{in } \Omega \times (0,T), \\ \phi = \eta + \zeta & \text{in } \Omega \times (0,T), \end{cases} \tag{2}$$

with boundary conditions

$$u = 0 \qquad \text{on } \Gamma_D \times (0,T), \tag{3}$$

$$(D(u)n) \times n = 0, \quad u \cdot n = 0 \qquad \text{on } \Gamma_S \times (0,T), \tag{4}$$

$$u = c\frac{\eta}{\phi} n \qquad \text{on } \Gamma_T \times (0,T), \tag{5}$$

and initial conditions

$$u = u^0, \quad \eta = \eta^0 \qquad \text{in } \Omega, \text{ at } t = 0, \tag{6}$$

where ϕ is the total height of wave, $u = (u_1, u_2)^T$ is the velocity, $\eta : \overline{\Omega} \times [0,T] \to \mathbb{R}$ is the water level from the reference height, $\zeta(x) > 0$ $(x \in \overline{\Omega})$ is the depth of water from the reference height, see Figure 2, $D(u) := \big(\nabla u + (\nabla u)^T\big)/2$ is the strain-rate tensor, n is the unit outward normal vector to the boundary of Ω, $\Gamma := \partial\Omega$ is the boundary of Ω, we assume that Γ consists of non-overlapped three parts, Γ_D, Γ_S and Γ_T, i.e., $\overline{\Gamma} = \overline{\Gamma}_D \cup \overline{\Gamma}_S \cup \overline{\Gamma}_T$, $\Gamma_D \cap \Gamma_S = \varnothing$, $\Gamma_S \cap \Gamma_T = \varnothing$, $\Gamma_T \cap \Gamma_D = \varnothing$, the subscripts "$D$", "$S$", and "$T$" mean Dirichlet, slip, and transmission boundaries, respectively, $\rho > 0$ is a constant which represents the density of water, $\mu > 0$ is a constant which represents the viscosity, $g > 0$ is the acceleration due to gravity, and $c(x) := c_0\sqrt{g\zeta(x)}$ with a positive constant c_0. In the rest of paper, we assume $\zeta \in C^1(\overline{\Omega})$. It is important to note here that the equations in (**??**) are derived in [**?**] by considering one-layer viscous SWEs. It is of interest to note here that [**?**] studied about the existence, uniqueness and [**?**] studied about the convergence of a finite element scheme for linearized SWEs but there is no theoretical results, as far we know, for the existence, uniqueness or regularity for the model (**??**)–(**??**) yet. Also it is pertinent to point out here that $\phi(x,t) > 0$ for all $x \in \Omega$ and $t \in [0,T]$ can not be shown theoretically for (**??**)–(**??**), but for this problem with $\Gamma_T = \emptyset$, we have the following Remark.

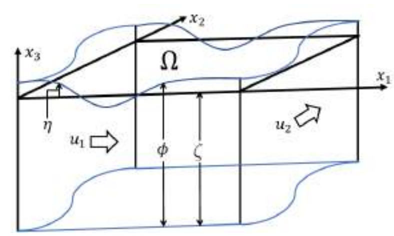


Figure 2: Model domain

Remark 2.1. (**Remark 3.1.1. in thesis**) If $\Gamma \in C^2$, $u \cdot n \geq 0$ on Γ, $\phi(x,0) > 0$ for all $x \in \overline{\Omega}$, then by the characterstic method it can be shown that $\phi(x,t) > 0$ for all $x \in \overline{\Omega}$ and $t \in [0,T]$.

3 Energy estimate

In this section, we define the total energy and study the stability of solutions to the problem stated in Section **??** in terms of the energy. For a solution of (**??**) the total energy $E(t)$ at time $t \in [0,T]$ is defined by

$$E(t) := E_1(t) + E_2(t), \tag{7}$$

where $E_1(t)$ and $E_2(t)$ are the kinetic and the potential energies defined by

$$E_1(t) := \int_\Omega \frac{\rho}{2}\phi|u|^2 dx, \quad E_2(t) := \int_\Omega \frac{\rho g|\eta|^2}{2} dx.$$

Let symbols $I_i(t;\Gamma)$, $i = 1, \ldots, 3$, and $I_4(t;\Omega)$, $t \in [0,T]$, be integrals defined by

$$I_1(t;\Gamma) := -\frac{\rho}{2}\int_\Gamma \phi|u|^2 u \cdot n\, ds, \qquad I_2(t;\Gamma) := -\rho g \int_\Gamma \phi\eta u \cdot n\, ds,$$

$$I_3(t;\Gamma) := 2\mu\int_\Gamma \phi\big[D(u)n\big] \cdot u\, ds, \qquad I_4(t;\Omega) := -2\mu\int_\Omega \phi|D(u)|^2\, dx.$$

These are used in the rest of this paper. Let us assume

$$\phi \in C^1(\overline{\Omega} \times [0,T] : \mathbb{R}), \quad u \in C^1(\overline{\Omega} \times [0,T] : \mathbb{R}^2), \tag{8}$$

and

$$\partial_i\partial_j u \in C^1(\overline{\Omega} \times [0,T] : \mathbb{R}^2) \text{ for } i, j = 1, 2. \tag{9}$$

Theorem 3.1. (**Theorem 3.2.1. in thesis**) *Suppose that a pair of functions* $(\phi, u) : \overline{\Omega} \times [0,T] \to \mathbb{R} \times \mathbb{R}^2$ *satisfies* (**??**) *with* (**??**) *and* (**??**). *Then, we have*

$$\frac{d}{dt}E(t) = \sum_{i=1}^{3} I_i(t;\Gamma) + I_4(t;\Omega). \tag{10}$$

We prove Theorem **??** using the following lemma.

Lemma 3.2. (**Lemma 3.2.2. in thesis**) *For the functions* $\phi : \overline{\Omega} \times [0,T] \to \mathbb{R}$ *and* $u : \overline{\Omega} \times [0,T] \to \mathbb{R}^2$ *satisfying* (**??**), *we have the following.*

$$(i)\ \frac{\partial}{\partial t}(\phi u) + \nabla \cdot [(\phi u) \otimes u] = \left(\frac{\partial \phi}{\partial t} + \nabla \cdot (\phi u)\right) u + \phi \left(\frac{\partial u}{\partial t} + (u \cdot \nabla) u\right),$$
$$(ii)\ \int_\Omega (\nabla \cdot [(\phi u) \otimes u]) \cdot u dx = \frac{1}{2}\int_\Gamma \phi |u|^2 u \cdot n ds + \frac{1}{2}\int_\Omega [\nabla \cdot (\phi u)]|u|^2 dx.$$

Proof of Theorem **??**. Differentiating (**??**) with respect to t, we get

$$\frac{d}{dt}E(t) = \frac{d}{dt}E_1(t) + \frac{d}{dt}E_2(t). \tag{11}$$

We compute $\frac{d}{dt}E_1(t)$ and $\frac{d}{dt}E_2(t)$ separately.

Firstly, $\frac{d}{dt}E_1(t)$ is computed as follows. From Lemma **??**-(i) and the first equation of (**??**), we have

$$\phi\Big[\frac{\partial u}{\partial t} + (u \cdot \nabla) u\Big] = \frac{\partial}{\partial t}(\phi u) + \nabla \cdot [(\phi u) \otimes u],$$

which implies

$$\rho\Big[\frac{\partial}{\partial t}(\phi u) + \nabla \cdot [(\phi u) \otimes u]\Big] - 2\mu \nabla \cdot \big[\phi D(u)\big] + \rho g \phi \nabla \eta = 0. \tag{12}$$

Multiplying (**??**) by u and integrating with respect to x over Ω, we get

$$\begin{aligned} &\rho \int_\Omega \Big[\frac{\partial}{\partial t}(\phi u)\Big] \cdot u\ dx + \rho \int_\Omega \Big[\nabla \cdot [(\phi u) \otimes u]\Big] \cdot u\ dx - 2\mu \int_\Omega \Big[\nabla \cdot (\phi D(u))\Big] \cdot u\ dx \\ &+ \rho g \int_\Omega \phi \nabla \eta \cdot u\ dx = 0. \end{aligned} \tag{13}$$

From the equation (**??**) above and the next two identities:

$$\begin{aligned} &\rho \int_\Omega \Big[\frac{\partial}{\partial t}(\phi u)\Big] \cdot u\, dx + \rho \int_\Omega \Big[\nabla \cdot [(\phi u) \otimes u]\Big] \cdot u\ dx \\ &= \rho \int_\Omega \Big(\frac{\partial \phi}{\partial t}|u|^2 + \phi \frac{\partial u}{\partial t} \cdot u\Big)\, dx + \frac{\rho}{2}\int_\Gamma \phi |u|^2 u \cdot n\ ds + \frac{\rho}{2}\int_\Omega \big[\nabla \cdot (\phi u)\big]|u|^2\, dx \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(from Lemma ??-}(ii)) \\ &= \rho \int_\Omega \Big(\frac{1}{2}\frac{\partial \phi}{\partial t}|u|^2 + \phi u \cdot \frac{\partial u}{\partial t}\Big) dx + \frac{\rho}{2}\int_\Gamma \phi |u|^2 u \cdot n\, ds \quad \text{(from the first eq. of (??))} \\ &= \frac{d}{dt}\Big[\frac{\rho}{2}\int_\Omega \phi |u|^2\ dx\Big] + \frac{\rho}{2}\int_\Gamma \phi |u|^2 u \cdot n\, ds = \frac{d}{dt}E_1(t) - I_1(t;\Gamma), \end{aligned}$$

$$\begin{aligned} -2\mu \int_\Omega \Big[\nabla \cdot (\phi D(u))\Big] \cdot u\ dx &= -2\mu \int_\Gamma \phi\big[D(u)n\big] \cdot u\ ds + 2\mu \int_\Omega \phi |D(u)|^2\ dx \\ &= -I_3(t;\Gamma) - I_4(t;\Omega), \end{aligned}$$

we obtain

$$\frac{d}{dt}E_1(t) = I_1(t;\Gamma) + I_3(t;\Gamma) + I_4(t;\Omega) - \rho g \int_\Omega \nabla \eta \cdot (\phi u)\ dx. \tag{14}$$

Secondly, $\frac{d}{dt}E_2(t)$ is computed as follows:

$$\frac{d}{dt}E_2(t) = \frac{d}{dt}\Big[\frac{\rho g}{2}\int_\Omega |\eta|^2 dx\Big]$$

$$
\begin{aligned}
&= \rho g \int_\Omega \eta \frac{\partial \eta}{\partial t}\, dx \\
&= \rho g \int_\Omega \eta \frac{\partial \phi}{\partial t}\, dx \qquad \text{(from the third eq. of (??))} \\
&= \rho g \int_\Omega \eta \big[-\nabla \cdot (\phi u)\big] dx \qquad \text{(from the first eq. of (??))} \\
&= -\rho g \int_\Omega \nabla \cdot (\eta \phi u) dx + \rho g \int_\Omega \nabla \eta \cdot (\phi u)\, dx \\
&= I_2(t;\Gamma) + \rho g \int_\Omega \nabla \eta \cdot (\phi u)\, dx. \qquad (15)
\end{aligned}
$$

The result (??) follows by adding (??) and (??) and recalling (??). □

Corollary 3.3. (*i*) (**Corollary 3.2.3. in thesis**) *Suppose that a pair of functions* $(\phi, u) : \overline{\Omega} \times [0,T] \to \mathbb{R} \times \mathbb{R}^2$ *satisfies* (??) *with* (??)-(??), (??) *and* (??) . *Then, we have*

$$\frac{d}{dt} E(t) = \sum_{i=1}^{3} I_i(t;\Gamma_T) + I_4(t;\Omega). \qquad (16)$$

(*ii*) *Furthermore, if* $\Gamma = \Gamma_D \cup \Gamma_S$ *and* $\phi(x,t) > 0$ $((x,t) \in \overline{\Omega} \times [0,T])$, *we have*

$$\frac{d}{dt} E(t) = I_4(t;\Omega) \le 0. \qquad (17)$$

Proof. On Γ_S, from the first equation of (??), there exists a scalar function $w : \overline{\Omega} \times [0,T] \to \mathbb{R}$ such that $D(u)n = w(x,t)n$, which implies

$$\big[D(u)n\big] \cdot u = (wn) \cdot u = w(u \cdot n) = 0.$$

Hence, the result (??) is established from Theorem **??** with (??) and (??).

When $\Gamma = \Gamma_D \cup \Gamma_S$, i.e., $\Gamma_T = \varnothing$, the identity (??) implies (??). □

It is to be noted here that the definition (??), Lemma **??**-(ii) and Corollary **??**-(ii) can also be found in [?], where $u \cdot n = 0$ is assumed.

Theorem 3.4. (**Theorem 3.2.4. in thesis**) *Suppose that a pair of functions* $(\phi, u) : \overline{\Omega} \times [0,T] \to \mathbb{R} \times \mathbb{R}^2$ *satisfies* (??) *with* (??)-(??), (??), (??) *and an inequality*

$$\phi(x,t) > 0, \quad (x,t) \in \overline{\Gamma}_T \times [0,T], \qquad (18)$$

and that there exists $\alpha \in (0,1)$ *such that*

$$\eta(x,t) \ge -\alpha \zeta(x), \quad x \in \overline{\Gamma}_T,\ t \in [0,T], \qquad (19)$$

$$0 < c_0 \le \sqrt{\frac{2}{\alpha}}\,(1-\alpha). \qquad (20)$$

Then, we have the following estimates:

$$I_1(t;\Gamma_T) + I_2(t;\Gamma_T) \le 0, \qquad (21)$$

in particular,

$$\frac{d}{dt} E(t) \le I_3(t;\Gamma_T). \qquad (22)$$

Proof. We prove (??), then (??) and (??) imply (??), since $I_4(t;\Omega)$ is always non-positive. We have

$$
\begin{aligned}
\sum_{i=1}^{2} I_i(t;\Gamma_T) &= -\rho \int_{\Gamma_T} \phi (u \cdot n) \Big[g\eta + \frac{1}{2}|u|^2 \Big]\, ds \\
&= -\rho \int_{\Gamma_T} \phi c \frac{\eta}{\phi} \Big[g\eta + \frac{1}{2} c_0^2 g \zeta \frac{\eta^2}{\phi^2} \Big]\, ds \\
&= -\rho g \int_{\Gamma_T} c \eta^2 \Big[1 + \frac{c_0^2}{2} \frac{\zeta \eta}{(\zeta+\eta)^2} \Big]\, ds.
\end{aligned}
$$

Let $f(r) := r/(1+r)^2$. From $f'(r) = (1-r)/(1+r)^3$, it holds that $f(r_1) \le f(r_2)$ for $-1<r_1 \le r_2 \le 1$. If $\eta < 0$, then since $-1 \le -\alpha \le \eta/\zeta \le 0$, we obtain $f(-\alpha) \le f(\eta/\zeta)$. Again if $\eta \ge 0$ then we also have $f(-\alpha) < 0 < f(\eta/\zeta)$. In both cases we obtain $f(-\alpha) \le f(\eta/\zeta)$ i.e.,

$$-\frac{\alpha}{(1-\alpha)^2} \le \frac{\eta\zeta}{(\zeta+\eta)^2},$$

which implies that

$$\sum_{i=1}^{2} I_i(t;\Gamma_T) \le -\rho g \int_{\Gamma_T} c\eta^2 \left\{ 1 - \frac{c_0^2 \alpha}{2(1-\alpha)^2} \right\} ds \le 0$$

from the condition (**??**). $\square$

Remark 3.5. (**Remark 3.2.5. in thesis**) We observe numerically that $I_2(t;\Gamma)$ is dominant and $\sum_{i=1}^{3} I_i(t;\Gamma)$ is negative, while $I_1(t;\Gamma)$ and $I_3(t;\Gamma)$ may be positive, cf. Subsection **??**. Although the sign of $\frac{d}{dt}E(t)$ is as yet unknown due to $I_3(t;\Gamma_T)$, from the numerical results we can say that the transmission boundary condition (**??**) is reasonable under the conditions (**??**)-(**??**) to be satisfied in practical computation.

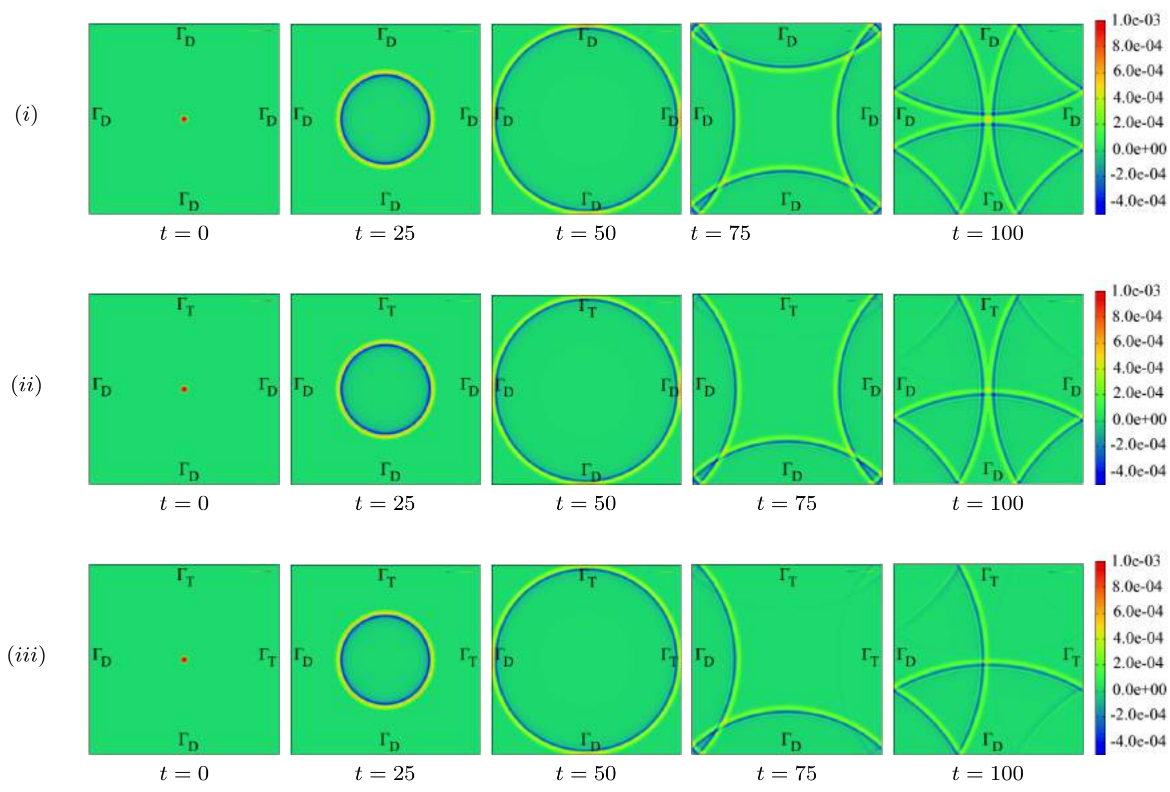
Remark 3.6. (**Remark 3.2.6. in thesis**) The condition (**??**) is not strict in the practical computation, where α and c_0 are chosen typically as, e.g., $\alpha = 0.01$ and $c_0 = 0.9$ [**?**]. These satisfy (**??**), since $\sqrt{2/\alpha}\,(1-\alpha) \approx 14$.

4 Numerical results by a finite difference scheme

In this section, we present numerical results by a finite difference scheme for problem (**??**)–(**??**) with $\Omega = (0, L)^2$ for a positive constant L, $T = 100$, $\zeta = a > 0$, $\mu = 1$, $g = 9.8 \times 10^{-3}$, $\rho = 10^{12}$, $\eta^0 = c_1 \exp(-100|x-p|^2)$ $(c_1 > 0, p \in \Omega)$. These values are in km (length), kg (mass) and s (time). We set $\Gamma_S = \emptyset$ for simplicity. We consider five cases of Γ_T:

$$(i)\ \Gamma_T = \emptyset, \qquad (ii)\ \Gamma_T = \Gamma_{\text{top}}, \qquad (iii)\ \Gamma_T = \Gamma_{\text{top}} \cup \Gamma_{\text{right}} \cup \{(L, L)\},$$
$$(iv)\ \Gamma_T = \Gamma_{\text{top}} \cup \Gamma_{\text{right}} \cup \Gamma_{\text{left}} \cup \{(L, L)\} \cup \{(0, L)\}, \qquad (v)\ \Gamma_T = \Gamma,$$

for $\Gamma_{\text{top}} := \{(x_1, L); 0 < x_1 < L\}$, $\Gamma_{\text{right}} := \{(L, x_2); 0 < x_2 < L\}$, $\Gamma_{\text{left}} := \{(0, x_2); 0 < x_2 < L\}$, and set $\Gamma_D = \Gamma \setminus \Gamma_T$. For the above cases (ii)-(v), $c_0 = 0.9$ is taken following [**?**].



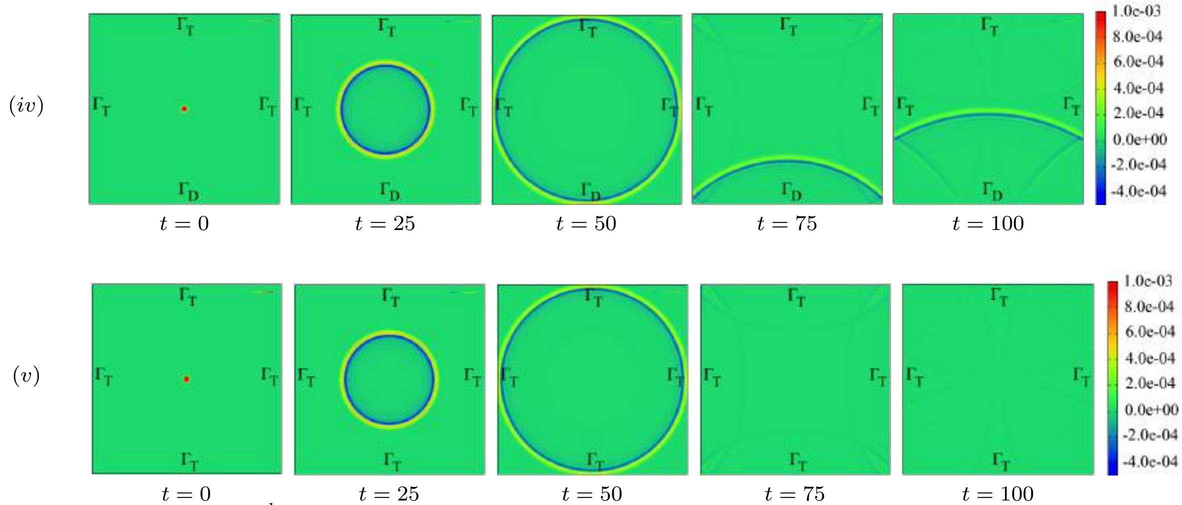


Figure: 3 Color contours of η_h^k by finite difference scheme for the five cases (i)-(v) discussed in Subsection **??**.

4.1 Numerical results for five cases of boundary settings

Numerical simulations are carried out by FDM for $L = 10$, $a = 1$, $u^0 = 0$, $c_1 = 0.01$, $p = (5,5)^T$, $N = 1,000$ and $\Delta t = 0.05$ $(N_T = 2,000)$. Figure 3 shows color contours of η_h^k for $k = 0$, 500, 1,000, 1,500 and 2,000, which correspond to times $t = 0, 25, 50, 75$ and 100, respectively, where (i)-(v) represent simulated results for the cases (i)-(v) stated at the beginning of this section. It can be clearly found that the artificial reflection is almost removed on the transmission boundaries for the cases (ii)-(v) (see Figure 3).

4.2 Numerical study of energy estimate

We study the stability of solutions to the problem (**??**)-(**??**) numerically by FDM in terms of the energy $E(t)$ defined in (**??**). Using solution $\{(u_h^k, \phi_h^k)\}_{k=1}^{N_T}$ with $\{\eta_h^k\}_{k=1}^{N_T}$ the values of $E(t^k) \approx E_h^k$ and $I_i(t^k; \Gamma) \approx I_h^k$, $i = 1, 2, 3$, $I_4(t^k; \Omega)$ are computed. The results are presented in Figure 4 and 5.

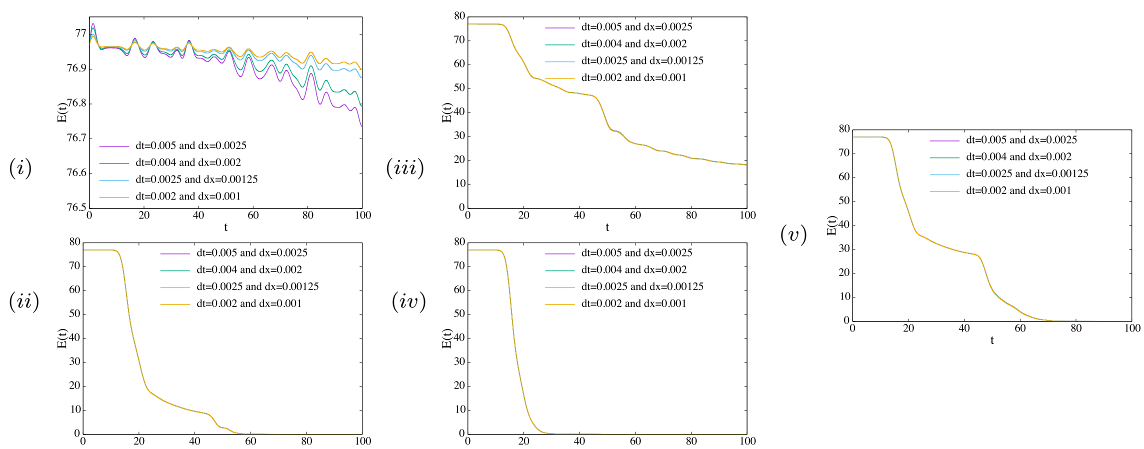


Figure: 4 Graphs of E_h^k versus $t = t^k$ $(\geq 0, k \in \mathbb{Z})$ for the five cases (i)-(v).

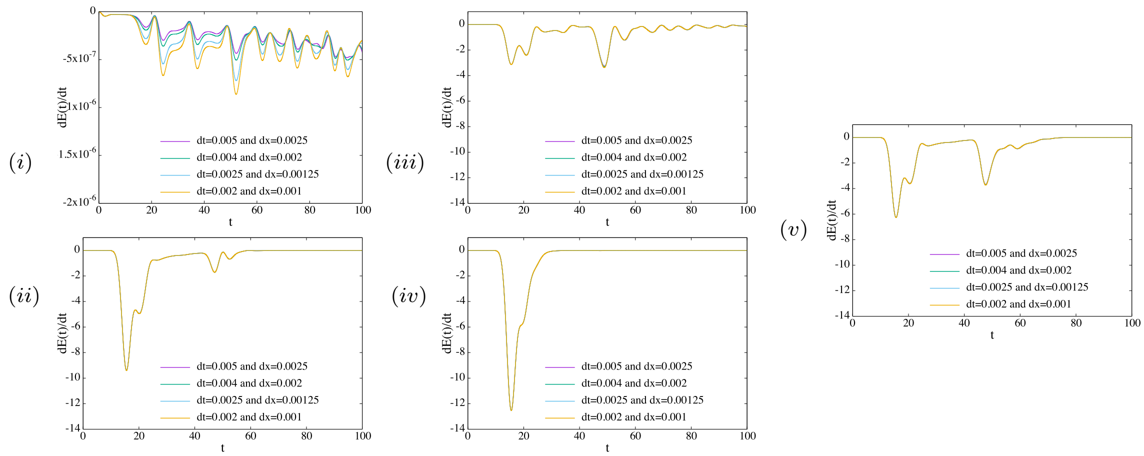


Figure: 5 Graphs of $\sum_{i=1}^{4} I_{hi}^k \approx \frac{d}{dt}E(t)$ versus $t = t^k$ ($\geq 0, k \in \mathbb{Z}$) for the five cases (i)-(v).

5 Numerical results by an LG scheme

In this section, we present an LG scheme for the problem described in Section **??**.

Let $\mathcal{T}_h = \{K\}$ be a triangulation of Ω, and M_h the so-called P1 (piecewise linear) finite element space. We set $\Psi_h := M_h$ for the water level η, and

$$V_h(\psi_h) := \left\{ v_h \in M_h^2;\ \begin{array}{ll} v_h(P) = c(P)\dfrac{\psi_h(P) - \zeta(P)}{\psi_h(P)} n(P), & \forall P : \text{node on } \Gamma_T, \\ v_h(Q) = 0, & \forall Q : \text{node on } \Gamma_D \end{array} \right\}$$

for the velocity u. The LG scheme is to find $\{(\phi_h^k, u_h^k)\}_{k=1}^{N_T} \subset \Psi_h \times V_h$ such that, for $k = 1, \ldots, N_T$,

$$\left\{ \begin{array}{ll} \displaystyle\int_\Omega \frac{\phi_h^k - \widetilde{\phi}_h^{k-1} \circ X_{1h}^{k-1} \gamma_h^{k-1}}{\Delta t} \psi_h \, dx = 0, & \forall \psi_h \in \Psi_h, \\ \displaystyle\rho \int_\Omega \phi_h^k \frac{u_h^k - \widetilde{u}_h^{k-1} \circ X_{1h}^{k-1}}{\Delta t} \cdot v_h \, dx + 2\mu \int_\Omega \phi_h^k D(u_h^k) : D(v_h) \, dx & \\ \displaystyle\qquad + \rho g \int_\Omega \phi_h^k \nabla \eta_h^k \cdot v_h \, dx = 0, & \forall v_h \in V_h, \\ \qquad \phi_h^k = \eta_h^k + \Pi_h^{\mathrm{FEM}} \zeta, & \end{array} \right. \tag{23}$$

where $X_{1h}^k(x) := x - u_h^k(x)\Delta t$, $\gamma_h^k : \Omega \to \mathbb{R}$ is defined by

$$\gamma_h^k(x) := \det\Big(\frac{\partial X_{1h}^k(x)}{\partial x}\Big),$$

the symbol "$\circ$" represents the composition of functions, i.e., $[v_h \circ X_{1h}^k](x) := v_h(X_{1h}^k(x))$, $\Pi_h^{\mathrm{FEM}} : C(\overline{\Omega}) \to M_h$ is the Lagrange interpolation operator, and

$$\widetilde{\psi}_h(x) = \begin{cases} \psi_h(x), & x \in \overline{\Omega}, \\ \psi_h(P_x), & x \in \mathbb{R}^2 \setminus \overline{\Omega}, \end{cases}$$

where $P_x \in \Gamma$ is the "nearest" nodal point from x. In each step, firstly, $\phi_h^k \in \Psi_h$ is obtained from the first equation of scheme (**??**). Secondly, $u_h^k \in V_h$ is obtained by using ϕ_h^k from the second equation. In the first equation of (**??**), the idea of mass conservative Lagrange–Galerkin scheme [**?**] is employed.

A numerical simulation is carried out by LG scheme (**??**) for the Bay of Bengal (see Figure 5).

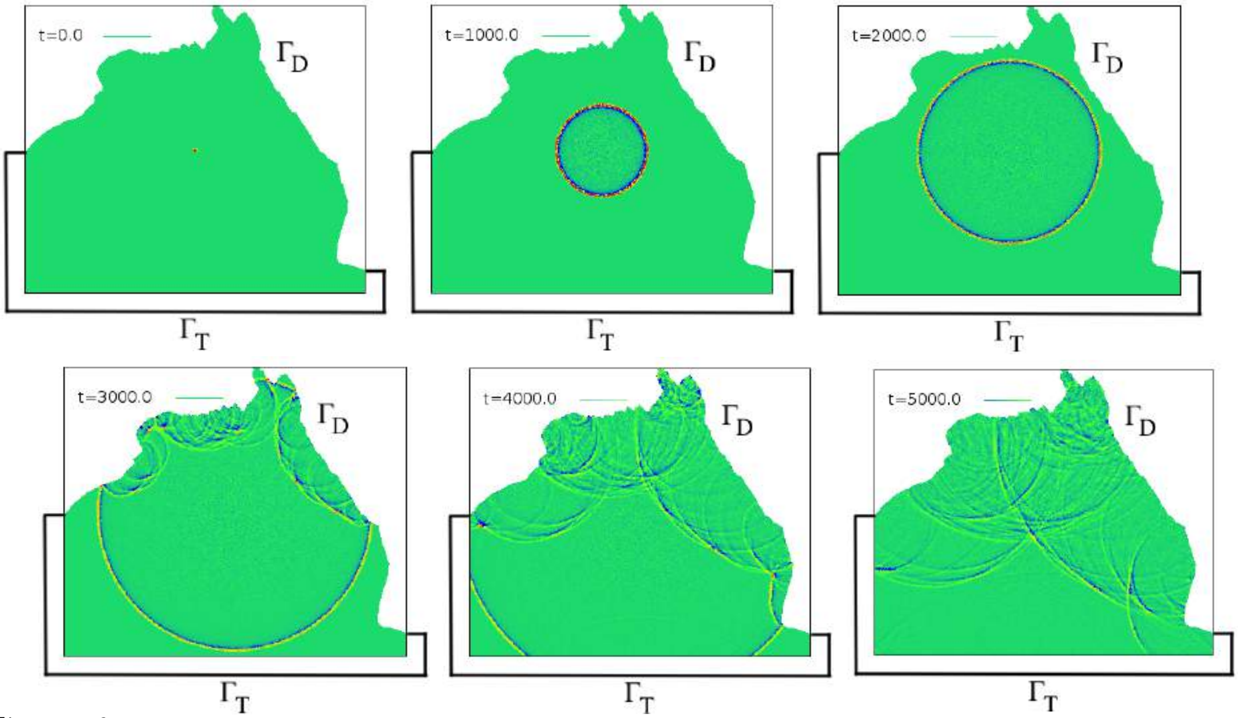


Figure: 6 Simulation of SWEs by LGM in the Bay of Bengal with transmission and Dirichlet boundary conditions, here Γ_T and Γ_D represent the transmission and the Dirichlet boundaries, respectively

6 Conclusions

Energy estimates of the SWEs with a transmission boundary condition have been studied mathematically and numerically. For a suitable energy, we have obtained an equality that the time-derivative of the energy is equal to a sum of three line integrals and a domain integral in Theorem **??**. The theorem implies a (successful) energy estimate of the SWEs with the Dirichlet and the slip boundary conditions, cf. Corollary **??**-(ii). After that, an inequality for the energy estimate of the SWEs with the transmission boundary condition has been proved in Theorem **??**. In the proof, it has been shown that a sum of two line integrals over the transmission boundary is non-positive under some conditions to be satisfied in practical computation. Based on the theoretical results, the energy estimate of SWEs with the transmission boundary condition has been confirmed numerically. It is found that the transmission boundary condition works well numerically and that the transmission boundary condition reduces the energy drastically via the term I^k_{h2}. Furthermore, we have presented simulated results for the Bay of Bengal by an LGM (see Figure 6), which shows that the transmission boundary condition works well. As far as we know, there is not a single model using LGM, for the prediction of storm surge in the Bay of Bengal, therefore we strongly believe that our results will be helpful to develop an appropriate storm surge prediction model using LGM for the Bay of Bengal in the near future.

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令和　元年　8月　6日

学位論文審査報告書（甲）

1．学位論文題目（外国語の場合は和訳を付けること。）

Theoretical and numerical studies of the shallow water equations with a transmission boundary condition（透過境界条件付き浅水波方程式に対する理論的および数値的研究）

2．論文提出者　(1) 所　　属　　数物科学　　　　専攻

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3．審査結果の要旨（600～650字）

Md. Masum Murshed さんは2016年10月に自然科学研究科博士後期課程数物科学専攻に入学し、浅水波方程式に対する数値解析の研究を行ってきた。非圧縮浅水波方程式は津波や高波のシミュレーションに工学で広く使用されているが、数学的に未解決の事柄が多くある。Masum さんは、出身国であるバングラデシュの沿岸部およびベンガル湾において毎年多くの高波の被害があることから、信頼性の高い高波シミュレーションを目標として本研究に取り組んだ。

本研究では、海上に設定される人工境界上で必要とされる透過境界条件に着目し、その理論的数学解析を行った。また数値計算手法として、風上型差分法による数値解法に加え、入り組んだ地形を表現するのに適した有限要素法によるラグランジュ・ガラーキン型スキームを提案し数値実験を行った。透過型境界条件は人工境界での不自然な波の反射などを抑えるために考案されたもので、工学では広く使用されているがこれまで数学的な理論解析はほとんどされて来なかった。Masum さんは浅水波方程式の解が満たすエネルギー等式を導くことで、その透過型境界条件が波を透過させることでエネルギーを逃す性質があることを数学的に明らかにした。また、そのエネルギー等式に現れる各項を差分法および有限要素法を用いて数値計算し、理論的な解析を裏付けた。

これらの結果は原著論文1報にまとめられている。以上により本論文は、博士（理学）を授与するに値すると判断した。

4．審査結果　　(1) 判　　定（いずれかに○印）　合　格　・　不合格

(2) 授与学位　　博　士（　理学　）