One-Dimensional Local Rings

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All the rings in this note will be commutative and Noetherian and have a unit element. Throughout, R will denote a one-dimensional local ring having maximal ideal M. We know that for a M-primary ideal I, the length of the R-module R/I^n is given by $en-r$ for all large values n, where $e=e(I)$ and $r=r(I)$ are integers called the multiplicity and reduction number of I respectively.

Northcott first introduced the notion of neighbourhood rings and studied some important connections between e and r for $I=M$. We quote Northcott[4], [5], Kirby[1] and Matlis[2] as references for these notion and results. In this note we consider certain extensions of these results and give a direct method for their proofs in the case of non-maximal I.

The terminology used in note is in general the same as that of [2] and [3]. We recall some basic definitions. We shall assume that I denote an M-primary ideal unless otherwise stated. An element a in I is called an I-superficial element of degree s, if there is an integer c such that $(I^n : a)\cap I^c=I^{n-s}$ for all large n. The set of those elements forms a multiplicatively closed set S and $R\{I\}$ denotes the set of elements b/a in $S^{-1}R$, where $b\in I^s$ and a is an I-superficial element of degree s (s variable). Then it is easy to see that $R\{I\}$ is a semi-local subring of $S^{-1}R$, and $Ker(R\to R\{I\})$ is the height 0 unmixed part $U(0)$ of the zero ideal. In particular, if R is Cohen-Macaulay, then $R\{I\}$ contains R. Here we note the following.

(a) (1) *Let* $\bar{R}=R/U(0)$. *Then* $e(I)=e(I\bar{R})$.

(2) $e(I)=L(T/I\bar{R})$, $r(I\bar{R})=L(T/\bar{R})$, *where* $T=R\{I\}\supset\bar{R}$.

(3) $r(I)=r(I\bar{R})-L(U(0))$. ($L(N)$ *denotes the length of the* R-module N)

(4) $r(I)=-L(U(0))\Longleftrightarrow I\bar{R}$ *is a principal ideal.*

(5) *R is Cohen-Macaulay* $\Longleftrightarrow r(I)\geq 0$ *for all* I.

Proof. The proofs of (1) and (3) are easy, and (4) follows from (2). In fact, $\bar{R}$ is Cohen-Macaulay, and $r(I\bar{R})=0$ if and only if $T=\bar{R}$. This holds if and only if $IT=I\bar{R}$ is a regular principal ideal. As for (2), since $R\{I\}=\bar{R}\{I\bar{R}\}$, we may assume that R is Cohen-Macaulay. In this case, the proof is almost analogous to that of those assertions in the case of $I=M$ (cf. [2]) and may be omitted. The assertion (5) follows from (2), (3) and (4).

We also note the following elementary fact.

(b) *For every parameter element a in M, we have*

(1) $a^nR/a^{n+1}R\cong R/(aR+(0:a^n))$.

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(2) $L(a^lR/a^{l+1}R)=e(aR) \Longleftrightarrow 0 : a^l=U(0)$.

In particular, R is Buchsbaum if and only if $L(aR/a^2R)=e(aR)$ for every parameter a.

Proof. The kernel of $R \to a^nR/a^{n+1}R\,(x \to a^nx)$ is $a^{n+1}R : a^nR=aR+(0 : a^n)$, which implies (1). Suppose $L(a^lR/a^{l+1}R)=e(aR)$. Then $aR+U(0)=aR+(0 : a^l)$ by (1) and (a), (1). Let $x=ay+z$ be any element in $U(0)$, where $y \in R$ and $z \in 0 : a^l \subset U(0)$. Then $x-z=ay \in U(0)$, hence $y \in U(0) : aR=U(0)$. Therefore $U(0)=aU(0)+(0 : a^l)$ and $U(0)=(0 : a^l)$ by Nakayama. The converse of (2) is obvious by (1) and (a), (1). The last assertion is immediate from the definition.

We shall treat the Cohen-Macaulay case from now on. Then $R\{I\}$ is a finite ring extension of R in the total quotient ring $Q(R)$.

(c) *R is analytically unramified if and only if there is an integer l such that $r(I) \leq l$ for all I. If R is not necessarily Macaulay, then $r(I)$ is bounded if and only if R/P is analytically unramified for every height 0 prime ideal P of R.*

Proof. In fact, for any finite ring extension S in $Q(R)$, there is a regular element a in R such that $I=Sa$ is an M-primary ideal with $I^2=Ia$ and hence a is an I-superficial element of degree 1. Since $r(I)=L(S/R)$, the first assertion is obvious by (a). If R is general, the similar argument show that $\bar{R}=R/U(0)$ is analytically unramified if and only if $r(I)=r(I\bar{R})-L(U(0))$ is bounded for all I. From this fact the last assertion follows immediately.

We also note that $R\{I\}=R[I^sa^{-1}]$ for any I-superficial element a of degree s, and hence $R\{I\}=I^{ns}a^{-n}$ for all large n, which can be proved similarly as in [2].

By $c(I)$ we denote the least number c such that $L(R/I^n)=e(I)n-r(I)$ for all n with $n \geq c$. We also set $R(I)=R[It]$ and $G(I)=R(I)/IR(I)$. Let $K=\Sigma K_nt^n$ be the height one unmixed part of $IR(I)$. Then we have the following, which contains certain extensions of Theorem 12.10 and 12.11 in [2].

(d) (1) $K_n=I^{n+1}R\{I\} \cap I^n$, $n \geq 0$.

(2) $K_n=I^{n+1}$, $n \geq c(I)$.

(3) *Suppose* $c(I) \geq 1$. *Then*

$$L(I^{n-1}/K_{n-1})+1 \leq L(I^n/K_n) \leq L(I^n/I^{n+1}) \leq e(I)-1,\ n \leq c(I)-1.$$

Proof. An element $a \in I^s$ is I-superficial of degree s if and only if $at^s \in R(I)-K$ (cf. [3], 22.). Let W be the multiplicatively closed set consisting of homogeneous elements in $R(I)-K$ and let $A=W^{-1}R(I)$. Then $K=IA \cap R(I)$. Comparing the degree n homogeneous part, the assertion (1) is immediate from the definition of $R\{I\}$. As for (2), considering $R(X)=R[X]_{M[X]}$ if necessary (cf. [3], 6., 22.), we may assume that there is an I-superficial element of degree 1. In fact, since the theorem of transition holds for rings R and $R(X)$ (cf. [2], p.108), the results of (2) and (3) for $R(X)$ and $IR(X)$ immediately yields those for R and I, and hence it is sufficient to prove (2) and (3) under the above assumption. Let b be an I-superficial element of degree 1. Since $R\{I\}$ is finite over R, there is an integer d such that

$$R \subset Ib^{-1} \subset I^2b^{-2} \subset \cdots\cdots \subset I^db^{-d}=R\{I\}.$$

Then $I^db=b^{d+1}R\{I\}=Ib^dR\{I\}=I^{d+1}$ since $IR\{I\}=bR\{I\}$. For every $k \geq 0$, we have

$$L(R/I^{d+k})=L(R/I^db^k)=L(R/b^kR)+L(b^kR/b^kI^d)=e(I)k+L(R/I^d).$$

This implies $d \geq c(I)$. On the other hand, letting $c = c(I)$, we have

$$L(R/I^{c+1}) = e(I) + e(I)c - r(I) = L(R/bR) + L(bR/bI^c) = L(R/I^c b),$$

which implies $I^c b = I^{c+1}$, hence $I^c b^{-c} = I^{c+1} b^{-(c+1)}$. Therefore we see that $I^c b^{-c} = R\{I\}$ and hence $c \geq d$. Thus we have $c(I) = d$. Suppose $n \geq c$. Then $I^n b^{-n} = R\{I\}$ and $I^n = b^n R\{I\} = I^n R\{I\}$. By virtue of (1), we have

$$I^n/K_n = I^n/I^n \cap I^{n+1}R\{I\} \cong I^n + I^{n+1}R\{I\}/I^{n+1}R\{I\} = I^n R\{I\}/I^{n+1}R\{R\}$$
$$= b^n R\{I\}/b^{n+1}R\{I\} \cong R\{I\}/IR\{I\}.$$

Since $n \geq c$, $L(I^n/I^{n+1}) = e(I) = L(R\{I\}/IR\{I\}) = L(I^n/K_n)$. Thus $K_n = I^{n+1}$ for all $n \geq c(I)$, which proves the assertion (2).

By what was proved above, $c(I)$ is the least integer d such that $I^d b^{-d} = R\{I\}$, and in particular, $I^c b^{-c} = R\{I\}$ with $c = c(I)$. Now we proceed with the proof of the assertion (3). By the above remark we may assume that there is an I-superficial element b of degree 1. Set $U = Ib^{-1} + IR\{I\}/IR\{I\}$ and $U^0 = R + IR\{I\}/IR\{I\}$. Then, $U$is an submodule of $R\{I\}/IR\{I\}$, and $U^n = I^n b^{-n} + IR\{I\}/IR\{I\}$, $n \geq 0$. This yields the following ascending chain:

$$U^0 \subset U^1 \subset U^2 \subset \cdots\cdots \subset U^c = U^{c+1} = \cdots\cdots = R\{I\}/IR\{I\},$$

where $U^n \cong I^n + b^n IR\{I\}/b^n IR\{I\} = I^n + I^{n+1}R\{I\}/I^{n+1}R\{I\} \cong I^n/K_n$, $n \geq 0$.

In fact, suppose that $U^{k-1} = U^k$ for some $k \leq c$. Then $U^{c-1} = U^{c-k}U^{k-1} = U^c = R\{I\}/IR\{I\}$, and hence $I^{c-1}b^{-(c-1)} + IR\{I\} = R\{\mathrm{I}\}$. Since $R\{I\}$ is a finite R-module and $I \subseteq M$, we have $I^{c-1}b^{-(c-1)} = R\{I\}$ by Nakayama. But this contradicts the definition of $d(=c)$. Since $L(I^n/K_n) = L(U^n)$ and $I^{n+1} \subseteq K_n$, $n \geq 0$, the assertion is proved except for the last inequality in (3). On the other hand, $b \in I$ is I-superficial of degree 1, and hence

$$L(I^n/I^{n+1}) \leq L(I^n/I^n b) = L(R/bR) + L(bR/I^n b) - L(R/I^n) = L(R/bR) = e(I).$$

Therefore, $L(I^n/I^{n+1}) = e(I)$ if only if $I^{n+1} = I^n b$. This holds if and only if $I^n b^{-n} = I^{n+1}b^{-(n+1)} = \cdots\cdots = R\{I\}$. From the definition of $d(=c)$, if $n \leq c-1$, then $L(I^n/I^{n+1}) \leq e(I) - 1$. Thus the assertion (3) is proved completely.

In the proof of (d) we obtain the following.

(e) $R[It]/K = \Sigma(I^n/K_n)t^n \cong U^0 + U^1 t + U^2 t^2 + \cdots\cdots = U^0[Ut]$, $U^c = R\{I\}/IR\{I\}$ *where* $IR\{I\} = bR\{I\}$ *for a suitable* $b \in IR\{I\}$ *and* $U = Ib^{-1} + IR\{I\}/IR\{I\}$.

In fact, $U^n \cong I^n/K_n$, $n \geq 0$ as in the above proof, and there is an I-superficial element a of degree s for some natural number s. Then $I^{ns}a^{-n} = R\{I\}$ for large n, and hence $I^{ns} = a^n R\{I\} = I^{ns}R\{I\} = b^{ns}R\{I\}$. This implies that there is the least integer d with $I^d = b^d R\{I\}$. It is easy to see that $d = c(I)$ and $U^d = R\{I\}/IR\{I\}$.

As a simple application we have the following.

(f) *Let* $K = \Sigma K_n$ *be the height one unmixed part of* $IR[It]$. *Set* $r_0 = e(I) - L(R/I)$, $h = e(I) - L(R/K_0)$ *and* $c = c(I)$.

(1) $Max(c, r_0) \leq r(I) \leq hc - \frac{1}{2}c(c-1)$.

(2) $c \leq h$ *and* $c = h$ *if and only if* $L(I^n/K_n) = e(I) - c + n$ *for* $n = 0, 1, \cdots, c-1$.

Proof. If $c = 0$, then $r(I) = 0$ from the definition and hence $r_0 = 0$. Thus the assertion is true in this case.

Suppose $c \geq 1$. By virtue of (d) we have

$$r_0 \leq \sum_{n=0}^{c-1}(e(I)-L(I^n/I^{n+1})) = r(I) \leq \sum_{n=0}^{c-1}(e(I)-L(I^n/K_n))$$
$$\leq \sum_{n=0}^{c-1}(e(I)-(L(R/K_0)+n) = hc - \frac{1}{2}c(c-1).$$

Since $L(I^n/I^{n+1})+1 \leq e(I)$ for $n=0, 1, \cdots\cdots, c-1$ by (d), we have $r(I) \geq c$. Thus the assertion (1) is proved. On the other hand, by virtue of (d), we have

$$h = e(I)-L(R/K_0) \geq 1 + e(I) - L(I/K_1) \geq 2 + e(I) - L(I^2/K_2) \geq$$
$$\cdots\cdots\cdots\cdots \geq c-1+e(I)-L(I^{c-1}/K_{c-1}) \geq c.$$

Thus $c \leq h$ and the equality $c=h$ hold if and only if $e(I)-L(I^n/K_n)=c-n$ for $n=0, 1, \cdots\cdots$, $c-1$, which prove the assertion (2).

The following is a refinement of Theorem 2 in [1].

(g) *With the same notation as in* (g) *we have the following.*

$$r(I) \leq (e(I)-1)c - \frac{1}{2}c(c-1) \leq \frac{1}{2}e(I)(e(I)-1).$$

In particular, if $G(I)$, *the associated graded ring of* R *with respect I, is Macaulay, then* $r(I) \leq r_0(r_0+1)/2$, *where* $r_0 = e(I)-L(R/I)$.

Proof. Since $h=e(I)-L(R/K_0) \leq e(I)-1$, the first inequality is obvious by virtue of (g), (1). On the other hand , set

$$p(n) = (e(I)-1)n - \frac{1}{2}n(n-1),\ n=0, 1, 2, \cdots\cdots.$$

Then it is easily seen that $p(n)$ has the maximum at $n=e(I)-1$ and $p(e(I)-1)=e(I)(e(I)-1)/2$, which proves the assertion. If $R(I)$ is Macaulay, then $h=r_0$ and the last assertion is immediate from the similar argument above.

References

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