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Introduction

In this note we shall consider the homeomorphism type of the compact aspherical homogeneous space of dimension 4. Let G be a connected, simply connnected Lie group and H a closed subgroup of G such that G/H is a compact aspherical homogeneous space of dimension 4 and G acts on G/H irreducibly, i. e. no proper subgroup of G does not act transitively. When G is a solvable group, the homeomorphism type of the solvmanifold is uniquely determined by its fundamental group ([5], [6]). It is proved that if M and N are both compact connected negatively curved Riemannian manifolds with isomorphic fundamental groups, then M and N are homeomorphic provided $\dim M \neq 3$ and 4 ([2]). Recently it is shown that the homeomorphism type of the compact homogeneous space is determined up to a finite covering by its fundamental group ([7]). We shall prove the following

THEOREM. *Let G/H be a 4 dimensional compact homogeneous space, where G is a connected, simply connected Lie group and H is a closed subgroup of G. If G acts on G/H irreducibly, the fundamental group of G/H is solvable.*

From the result that compact aspherical homogeneous spaces with isomorphic solvable fundamental groups are homeomorphic ([4]), we have the following

COROLLARY. *Let G_i be a connected, simply connected Lie group and H_i its closed subgroup such that G_i/H_i is a closed 4 dimensional aspherical manifold on which G_i acts irreducibly for $i=1, 2$. If the fundamental groups of G_1/H_1 and G_2/H_2 are isomorphic, then G_1/H_1 and G_2/H_2 are homeomorphic.*

1. Preliminaries

In this note we use the following notations.

(1) **Z, R** and **N** denote the ring of integers, the field of real numbers and the set of natural numbers, respectively.

(2) For a Lie group G, G^o denotes its identity component.

(3) $[G, G]$ denotes the derived subgroup of group G.

(4) Let G be a group and H its subgroup. $N_G(H)$ denotes the normalizer of H in G.

(5) For a Lie group G, $L(G)$ denotes its Lie algebra.

(6) For a group G, $z(G)$ denotes the center of G.

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Then we have the following definitions.

Definition. A manifold M is said aspherical or of type $K(\pi, 1)$ if all its homotopy groups, except possibly its fundamental group, are equal to zero.

Definition. A transitive action of a Lie group G on manifold M is said irreducible if the following conditions are satisfied ;

(1) G acts on M effectively

(2) G does not contain proper subgroups which are transitive on M.

Definition. Let G be a Lie group. A subgroup T of G is said to be triangular if its image $ad(T)$ by the adjoint representation of G is a triangular group of transformations of $L(G)$.

Let G be a connected Lie group and $G = S \cdot R$ be a Levi decomposition of G, where S is a semisimple subgroup of G and R is the radical of G. There are well known following facts.

THEOREM 1 ([3]). *Let $M = G/H$ be compact and aspherical, and let G be simply connected and locally effective on M. Then the following assertions are true ;*

(1) G is *diffeomorphic* to $\mathbf{R}^n$

(2) $H^o \subset T \cdot R$, *where T is some maximal connected triangular subgroup of S*

(3) *if G is irreducible on M, then $H^o \subset [T, T] \cdot R$.*

It follows, from this Theorem 1, that if $G = S \cdot R$ is a Levi decomposition of G which is difeomorphic to $\mathbf{R}^n$, then S is isomorphic to $A_1 \times \cdots \times A_p$, where A_i is a simple Lie group which is isomorphic to the universal covering group A of $SL(2, \mathbf{R})$ and diffeomorphic to $\mathbf{R}^3$.

THEOREM 2 ([4]). *Let M be a compact homogeneous space of type $K(\pi, 1)$ with π solvable. Then the following assertions are true;*

(1) *M is homeomorphic to solvmaifold*

(2) *if M_1 is another compact homogeneous space of type $K(\pi, 1)$, then M_1 is homeomorphic to M.*

This Theorem 2 means that the homeomorphism type of the compact aspherical homogeneous space is determined by its fundamental group, if it is solvable.

THEOREM 3 ([6]). *Let G be a connected Lie group and H a closed subgroup of G such that G/H is compact. Then $N_G(H^o)$ contains a maximal connected triangular subgroup of G.*

2. Example

This section is mainly concerned with the manifold which is constructed in [4] . Let G be a Lie group and F, H closed subgroups of G such that $F \supset H$. We consider a fibration

$$F/H \rightarrow G/H \rightarrow G/F,$$

where G/H is the base space and F/H is the fiber space. If $G_1 \subset G$ is a subgroup which is transitive on G/F, then the fibration is homogeneous with respect to G_1. The stationary group of the fiber is $F_1 = F \cap G_1$. We have the following

PROPOSITION 1. *If $F_1 \subset H$, then the action of F_1 on F/H is induced by the action of F_1 by inner automorphisms of F.*

PROOF. Let gH be an element of F/H, Then since $g_1 \in H$, we have

$$g_1 \cdot (gH) = g_1 g g_1^{-1} g_1 H = g_1 g g_1^{-1} H,\ g_1 \in F_1.$$

Thus the assertion is proved.

Let G be $SL(2, \mathbf{R}) \times_{ad} \mathbf{R}^3$, where ad is the adjoint representation of $SL(2, \mathbf{R})$ and let T be the group of the triangular matrices with positive elements on the diagonal. Then $T \times_{ad \mid T} \mathbf{R}^3$ is a solvable group and there exists a subspace $V \subset \mathbf{R}^3$ of codimension 1 which is invarianat with respect to T. Hence $T \times_{ad \mid T} \mathbf{R}^3/V$ is 3 dimensional solvable group and so $[[\mathrm{T}, \mathrm{T}], \mathbf{R}^3] \subset V$. Thus we obtain an abelian Lie group $[T, T] \cdot \mathbf{R}^3/V$. The subgroup

$$D_\alpha = \left\{ \begin{pmatrix} e^{k\alpha} & 0 \\ 0 & e^{-k\alpha} \end{pmatrix} \mid k \in \mathbf{Z} \right\}$$

acts on $[T, T] = \mathbf{R}^1$ by the multiplication by $e^{2\alpha}$. If $\alpha \in \mathbf{R}$ is such that $e^{2\alpha} + e^{-2\alpha} \in \mathbf{N}$, then D_α is conjugate with the subgroups of integral valued matrices in the group of automorphisms of $[T, T] \cdot \mathbf{R}^3/V = \mathbf{R}^2$. Hence D_α preserves some lattice Γ in $[T, T] \cdot \mathbf{R}^3/V$. We set $H_1 = p^{-1}(\Gamma)$, where $p : [T, T] \cdot \mathbf{R}^3 \to [T, T] \cdot \mathbf{R}^3/V$ is a natural projection. Let H be the subgroup of G generated by D_α and H_1. Then it follows that H is a closed in G and $G/H = M_\alpha$ is a compact 4 dimensional homogeneous space. If we set $H' = z(SL(2,\mathbf{R})) \cdot H$ in place of H, then we get another analogous manifold $G/H' = M'_\alpha$.

PROPOSITION 2. *M_α and M'_α are diffeomorphic to solvmanifolds.*

PROOF. Since $H \subset T \cdot \mathbf{R}^3$, we have the fibration $T \cdot \mathbf{R}^3/H \to G/H \to G/T \cdot \mathbf{R}^3$. Here the base space $G/T \cdot \mathbf{R}^3$ is diffeomorphic to S^1. Then a maximal compact subgroup K of $SL(2, \mathbf{R})$ is transitive on $G/T \cdot \mathbf{R}^3$ and so , by Proposition 1, the diffeomorphism of the fiber space is trivial. Hence G/H is diffeomorphic to $T \cdot \mathbf{R}^3/H \times S^1$, because a bundle over S^1 is determined by a diffeomorphism of the fiber space. This manifold is a solvmanifold. For M'_α this argument is also applicable, because $z(SL(2, \mathbf{R}))$ acts trivially on $\mathbf{R}^3$.

3. Proof of Theorem

In this section we shall prove Theorem in Introduction. let G be a connected, simply connected Lie group which is irreducible on $M = G/H$ and $G = S \cdot R$ be a Levi decomposition with the natural projection $q : G \to S$. Then, from the remark following Theorem 1, we have $S = S_1 \times \cdots \times S_p$. Let $p_i : S \to A_i$ be the natural projection for $i = 1, \cdots, p$. From Theorem 1 we have that $\dim p_i \cdot q(H) \leq 1$. Since $\dim M = 4$, we get that $p \leq 2$.

(1) Case of $p=2$

In this case we have that $G = (A_1 \times A_2) \cdot R$ and $\dim q(H) \leq 2$. If $R \neq \{e\}$, then $R \subset H$. So by the reason of irreducibility we have $G = A_1 \times A_2$. Then $H^o = [T_1, T_1] \times [T_2, T_2]$, where T_i is a maximal triangular subgroup of A_i for $i = 1, 2$, and $N_G(H^o) = z(A_1)T_1 \times z(A_2) T_2$. Since $\pi_1(G/H) = H/H^o$ and $H \subset N_G(H^o)$, $H/H^o \subset N_G(H^o)/H^o = z(A_1)\mathbf{R} \times z(A_2)\mathbf{R}$ and so $\pi_1(G/H) = \mathbf{Z}^4$.

Let $H_1 = H \cdot N_G(H^o)^o$. Since $H \subset H_1 \subset N_G(H^o)$ and $H_1{}^o = N_G(H^o)^o = T_1 \times T_2$, it follows that $H_1{}^o$ is a closed subgroup of G, while by the construction $H_1{}^o$ is transitive on H_1/H. We consider the fibration and the corresponding exact sequence of the fundamental group ;

$$H_1/H \to G/H \to G/H_1$$

$$1 \to \pi_1(H_1/H) \to \pi_1(G/H) \to \pi_1(G/H_1) \to 1.$$

Let K_i be a subgroup of A_i whose Lie algebra is maximal and compact. Since $K = K_1 \times K_2$ is an abelian subgroup of G and is transitive on the compact base space G/H_1, G/H_1 is a 2 dimensional torus. With respect to the fiber H_1/H,

$$H_1/H = H_1{}^o/H_1{}^o \cap H = S^1 \times S^1.$$

However $K \subset G$ is transitive on G/H_1 and $K \cap H_1 \subset z(G) \cap H$, because $K \cap H_1 \subset N_G(H^o)$. So the diffeomorphism of the fiber ,which is induced by an inner automorphism by the element of $K \cap H_1$, is trivial and hence, applying Proposition 1, this fibration is trivial over the loops S^1 of the base. Thus G/H is 4 dimensional torus([8]).

(2) Case of $p = 1$

In this case we have $G = A \cdot R$ and further the following two subcases (i) and (ii) because dim $q(H) \leq 1$.

(i) dim $q(H) = 0$ and $H^o \subset R$, codim ${}_R H^o = 1$.

Claim 1. *R is abelian and if we write $G = A \times_\varphi \mathbf{R}^n$, then φ is an irreducible representation.*

If R is nonabelian, then either $[R, R] \cdot H = R$ or $[R, R] \subset H$. *If* $[R, R] \cdot H = R$, then $G' = A \times_\varphi [R, R]$ is transitive on M, which contradicts an irreducibility of G on M. If $[R, R] \subset H$, then this contradicts a local effectiveness of the action of G on M. Thus R is abelian.

Assume φ is reducible. Let $R = V_1 + V_2$ be a direct sum decomposition of two φ invariant subspaces of R, $V_i \neq \{0\}$ for $i = 1, 2$. Then either $V_1 + H^o = R$ or $V_2 + H^o = R$ and so $A \times_\varphi V_1$ or $A \times_\varphi V_2$ is transitive on M. This contradicts an irreducibility of G on M. Thus φ is irreducible.

Claim 2. *φ is unimodular.*

We put $F = N_G(H^o)^o$. By Theorem 3 there exists a maximal connected triangular subgroup T of A such that $T \cdot R \subset F$. But this induces $T \cdot R = F$ and F/H^o is a 3 dimensional solvable Lie group with a lattice $H \cap F/H^o$. This solvable Lie group has the following splitting

$$\begin{array}{ccccccccc} 1 & \longrightarrow & [T, T] \cdot R/H^o & \longrightarrow & F/H^o & \longrightarrow & T/[T, T] & \longrightarrow & 1. \\ & & \| & & & & \| & & \\ & & \mathbf{R}^2 & & & & \mathbf{R}^1 & & \end{array}$$

We get that $F/H^o = \mathbf{R}^1 \times_\psi \mathbf{R}^2$, where $\psi(t) = (Ad(t), \varphi(t))$, Ad is an adjoint map of T on $[T, T]$, for $t \in T/[T, T]$. By the reason of the unimodularity of ψ and ad, φ is unimodular.

By the above claims, φ coincides with the adjoint representation used in Proposition 2. Hence $G = A \times_{ad} \mathbf{R}^3$ and $G/H = M_\alpha$.

(ii) dim $q(H) = 1$ and codim ${}_R R \cap H = 2$.

Claim 3. *If $G = A \cdot R$, then $R = N$, where N is the nilradical of R.*

Assume $R \neq N$. Let $H_2 = H \cap R$ and then $H_2 \cdot N \subset R$. Since both cases $N \subset H_2$ and $H_2 \cdot N = R$ are impossible by the reason of the irreducibility of G on M, we have codim ${}_R H_2 \cdot N = 1$. Let $p : R \to R/N$ be the natural projection. Since $p(H_2 \cdot N) = H_2/H_2 \cap H$ has codimension 1 in the abelian Lie group R/N, there exists a 1 dimensional subgroup V such that $V \cdot p(H_2 \cdot N) = R/N$ and $V \cap p(H_2 \cdot N) = \{e\}$. Since A acts on R/N trivially, $p^{-1}(V) \subset R$ is invariand with respect to A and hence $G' = A \cdot p^{-1}(V)$ is a proper subgroup of G. Since $L(G') + L(H) = L(G)$, G' is transitive on M, which contradicts the irrducibility of G on M. Thus $R=N$.

Claim 4. *N is abelian.*

Since $T \cdot N$ is a maximal connected triangular subgroup of G, we have $N_G(H^o)^o = T \cdot N$ by Theorem 3. Let $C \subset T$ be a Cartan subgroup such that $C \cdot [T, T] = T$ and $C \cap [T, T] = \{e\}$. Since dim $q(H) = 1$, $q(H^o) = [T, T]$. We have $H \subset N_G(H^o) = z(A)T \cdot N$ and so $H^o \subset T \cdot N$. Then 3 dimensional solvable Lie group $C \cdot N/H^o \cap C \cdot N$ is transitive on the compact solvmanifold $T \cdot N/H^o \cap C \cdot N$ of dimension 3. $C \cdot N/H^o \cap C \cdot N$ contains N/H_2 as a normal subgroup of dimension 2. From the description of 3 dimensional solvable Lie groups having lattices ([1]), N/H_2 is abelian and hence $[N, N] \subset H_2$. But assume $[N, N] \neq \{e\}$, this contradicts the local effectiveness of the action of G on M. Thus N is abelian.

Claim 5. *If we write $G = A \times_\varphi \mathbf{R}^n$, φ is trivial and so $G = A \times \mathbf{R}^n$.*

Since T normalizes H^o, φ induces a representation $\tilde{\varphi} : T \to GL(\mathbf{R}^n/H_2{}^o)$, which is trivial. In fact $\mathbf{R}^n$ is abelian and $T \cdot \mathbf{R}^n/H^o$ is 3 dimensional solvable group and so $[T, \mathbf{R}^n] \subset H_2$. Let $V_1 \subset \mathbf{R}^n$ be the space of the fixed elements of φ and V_2 its invariant complement. From the triviality of $\tilde{\varphi}$ it follows that $V_2 \subset H_2$. Assume $V_2 \neq \{0\}$. Then the subgroup $G' = A \times_\varphi V_1$ is a proper subgroup of G and transitive on M, which contradicts the irreducibility of G on M. Thus $V_1 = \mathbf{R}^n$ and so φ is trivial.

By the same argument as in case (1), $M = G/H$ is diffeomorphic to a 4 dimensional torus.

(3) Case of $p=0$

In this case we have $G=R$ and so G/H is a solvmanifold.

In all cases we proved that the fundamental group of G/H is solvable since the fundamental group of solvmanifold is solvable. This completed the proof of Theorem.

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