回転する弾性体の動的接触問題の変分法に基づく数値的取り扱いについて

学位論文要旨

回転する弾性体の動的接触問題の変分法に基づく数値的取り扱いについて

(Numerical scheme for a dynamical rolling elastic contact problem based on a variational method)

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要約

We investigate a rolling contact problem in elastodynamics. Contact problems in elasticity appear in various fields such as manufacturing and earthquake engineering. In particular, we have in mind the application to printers, where paper sheets are driven through the printer by rollers. A typical problem for such printers is that the roller may produce a squeaking sound. As a step towards preventing such a sound, we study a simplified model in which the roller is modeled as an elastic body driven by a rotation. The paper sheets are modeled as a rigid obstacle. For simplicity, we assume no frictional forces between the roller and the obstacle. The resulting equations of motion are of hyperbolic type with a free boundary.

The aim of the paper is to develop a numerical scheme to solve these equations of motion. The scheme is based on a variational method called the discrete Morse flow. The novelty is that this scheme has not been applied to a hyperbolic system with a free boundary where the unknown function is vector-valued.

The paper is organised as follows. First we derive the set of equations (P) for the rolling contact problem. Next we apply the discrete Morse flow to develop a numerical scheme $(\mathrm{P_k})$ for (P). Lastly we solve $(\mathrm{P_k})$ numerically and discuss the application to the rolling contact problem.

Governing equations

Geometry Let $\Omega \subset \mathbb{R}^2$ be a bounded domain representing the area occupied by an elastic body. The closure $\bar{\Omega}$ of the set Ω is called the reference configuration. We denote by $\boldsymbol{\varphi} : \bar{\Omega} \to \mathbb{R}^2$ the displacement of the reference configuration $\bar{\Omega}$, and refer to $\boldsymbol{\varphi}(\bar{\Omega})$ as the deformed configuration. We call the components of $\boldsymbol{x}$ the Lagrangian coordinates, and the components of $\boldsymbol{X} = \boldsymbol{\varphi}(\boldsymbol{x})$ the Eulerian coordinates (see Figure 1) in the deformed configuration.

At each point $\boldsymbol{x} \in \Omega$, the deformation gradient is given by

$$\boldsymbol{F}(\boldsymbol{x}) := \nabla\boldsymbol{\varphi}(\boldsymbol{x}) = \begin{pmatrix} \dfrac{\partial \varphi_1}{\partial x_1}(\boldsymbol{x}) & \dfrac{\partial \varphi_1}{\partial x_2}(\boldsymbol{x}) \\ \dfrac{\partial \varphi_2}{\partial x_1}(\boldsymbol{x}) & \dfrac{\partial \varphi_2}{\partial x_2}(\boldsymbol{x}) \end{pmatrix}. \tag{1}$$

We require that the determinant of the deformation gradient is positive at all points of the reference configuration, that is

$$J(\boldsymbol{x}) := \det\boldsymbol{F}(\boldsymbol{x}) > 0, \tag{2}$$

for all $\boldsymbol{x} \in \Omega$. As a consequence, the matrix $\boldsymbol{F}(\boldsymbol{x})$ is invertible.

Before linearizing, we describe the equations for mechanical equilibrium in terms of non-linear elasticity. The Cauchy stress tensor $\boldsymbol{T} = (T_{ij})$ is defined in the deformed configuration

$$\boldsymbol{T}(\boldsymbol{X}) := \frac{1}{J(\boldsymbol{x})}\left\{\mu\left(\boldsymbol{F}(\boldsymbol{x})\boldsymbol{F}^T(\boldsymbol{x}) - \boldsymbol{I}\right) + \frac{\lambda}{2}\left(J(\boldsymbol{x})^2 - 1\right)\boldsymbol{I}\right\}, \tag{3}$$

for all $\boldsymbol{x} \in \Omega$, where $\boldsymbol{X} = \boldsymbol{\varphi}(\boldsymbol{x})$, μ and λ are the Lamé constants ($\lambda + \mu \geq 0$, $\mu > 0$), $\boldsymbol{F}^T(\boldsymbol{x})$ is the transpose matrix of $\boldsymbol{F}(\boldsymbol{x})$, and $\boldsymbol{I}$ is the identity matrix.

In our model for the roller, the displacement naturally decomposes as

$$\boldsymbol{\varphi} = \boldsymbol{R}(\mathbf{id} + \boldsymbol{\xi}) \quad \text{in } \Omega, \tag{4}$$

where the matrix $\boldsymbol{R} = (R_{ij})$ describes the counter-clockwise rotation by angle θ (see Figure 1), $\mathbf{id} : \mathbb{R}^2 \to \mathbb{R}^2$ denotes the identity map, and $\boldsymbol{\xi} : \Omega \to \mathbb{R}^2$ is assumed to have small derivatives. More precisely, we assume that

$$\left|\frac{\partial \xi_i}{\partial x_j}(\boldsymbol{x})\right| < \varepsilon, \qquad \left|\frac{\partial^2 \xi_i}{\partial x_j \partial \xi_k}(\boldsymbol{x})\right| < \varepsilon, \tag{5}$$

for some $\varepsilon > 0$ small enough, uniformly for $\boldsymbol{x} \in \Omega$ and $1 \leq i, j, k \leq 2$.

Equations of motion To derive the equations of motion, we change variables in (3) by writing it in terms of $\boldsymbol{\xi}$ on Ω, and expand it in terms of ε by relying on (5). Since $\nabla\boldsymbol{\xi}$ plays the role of the deformation in linearized elasticity, we introduce the strain tensor

$$\boldsymbol{\epsilon}[\boldsymbol{\xi}] := \frac{1}{2}\left(\nabla\boldsymbol{\xi} + \nabla\boldsymbol{\xi}^T\right), \tag{6}$$

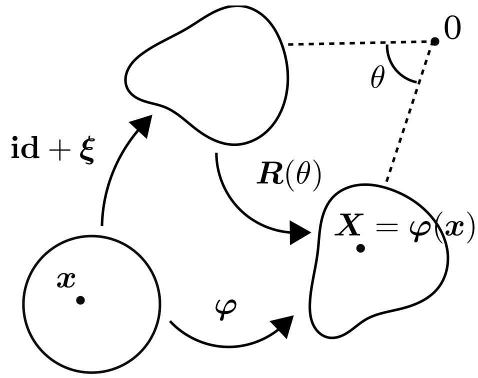

Figure 1: Sketch of the reference domain $\bar{\Omega}$ and the deformed configuration $\varphi(\bar{\Omega})$.

and the stress tensor

$$\boldsymbol{\sigma}[\boldsymbol{\xi}] := 2\mu\boldsymbol{\epsilon}[\boldsymbol{\xi}] + \lambda(\mathrm{div}\boldsymbol{\xi})\boldsymbol{I}, \tag{7}$$

in the reference configuration.

The divergence of the Cauchy tensor $\boldsymbol{T}$ is then

$$(\mathbf{div}_{\boldsymbol{X}}\boldsymbol{T})_i := \frac{\partial T_{ij}}{\partial X_j} = R_{ik}\frac{\partial}{\partial x_\ell}\sigma_{k\ell}[\boldsymbol{\xi}] + O(\varepsilon^2). \tag{8}$$

Next we derive the equations of motion. We encode the forced rotation of the elastic body by a given smooth function $\theta : [0, T) \to \mathbb{R}$ that corresponds to the rotation angle of $\boldsymbol{R}$. Then, by (4), the now time-dependent fields $\boldsymbol{\varphi} : \bar{\Omega} \times [0, T) \to \mathbb{R}^2$ and $\boldsymbol{\xi} : \bar{\Omega} \times [0, T) \to \mathbb{R}^2$ satisfy

$$\boldsymbol{\varphi}(\boldsymbol{x}, t) = \boldsymbol{R}(\theta(t))(\boldsymbol{x} + \boldsymbol{\xi}(\boldsymbol{x}, t)), \tag{9}$$

for all $\boldsymbol{x} \in \bar{\Omega}$ and all $t \geq 0$. After neglecting higher order terms of ε in (8), the conservation of linear momentum yields the equation of elastodynamics,

$$\rho\ddot{\boldsymbol{\varphi}} = \mathbf{div}_{\boldsymbol{X}}\,\boldsymbol{T} \approx \boldsymbol{R}(\theta)\,\mathbf{div}\,\boldsymbol{\sigma}[\boldsymbol{\xi}] \quad \text{in } \Omega \times (0, T), \tag{10}$$

where $\rho > 0$ is the density, superposed dots denote partial differentiation with respect to time (i.e., $\ddot{\boldsymbol{\varphi}} := \partial^2\boldsymbol{\varphi}/\partial t^2$), and

$$\mathbf{div}\,\boldsymbol{\sigma} := \begin{pmatrix} \dfrac{\partial\sigma_{11}}{\partial x_1} + \dfrac{\partial\sigma_{12}}{\partial x_2} \\ \dfrac{\partial\sigma_{21}}{\partial x_1} + \dfrac{\partial\sigma_{22}}{\partial x_2} \end{pmatrix}. \tag{11}$$

Calculating (10), we obtain

$$\rho\ddot{\boldsymbol{\xi}} = \mathbf{div}\boldsymbol{\sigma}[\boldsymbol{\xi}] + \rho\left(\ddot{\theta}\boldsymbol{R}(-\pi/2)(\mathbf{id} + \boldsymbol{\xi}) + \dot{\theta}^2(\mathbf{id} + \boldsymbol{\xi}) + 2\dot{\theta}\boldsymbol{R}(-\pi/2)\dot{\boldsymbol{\xi}}\right) \quad \text{in } \Omega \times (0, T). \tag{12}$$

We note that if θ is linear in time, then in the right-hand side the second term vanishes, the third term is the centrifugal force, and the last term is the Coriolis force. To abbreviate the term in parentheses, we define the function $\boldsymbol{f}$ by

$$\boldsymbol{f}(t, \boldsymbol{x}, \boldsymbol{\xi}, \dot{\boldsymbol{\xi}}) := \ddot{\theta}(t)\boldsymbol{R}(-\pi/2)(\boldsymbol{x} + \boldsymbol{\xi}) + \dot{\theta}(t)^2(\boldsymbol{x} + \boldsymbol{\xi}) + 2\dot{\theta}(t)\boldsymbol{R}(-\pi/2)\dot{\boldsymbol{\xi}}. \tag{13}$$

We remark that the time dependence of the rotation angle is not covered by the setting in [21, 16].

Boundary conditions We subdivide the boundary $\partial\Omega$ into Γ_D and Γ_C (see Figure 2), where

$$\Gamma_D \cup \Gamma_C = \partial\Omega, \quad \Gamma_D \cap \Gamma_C = \emptyset, \quad \Gamma_D \neq \emptyset. \tag{14}$$

On the boundary Γ_D we model the forced rotation of Ω by imposing the Dirichlet boundary condition

$$\boldsymbol{\varphi}(\boldsymbol{x}) = \boldsymbol{R}\boldsymbol{x} \qquad \text{for } \boldsymbol{x} \in \Gamma_D, \tag{15}$$

which is equivalent to

$$\boldsymbol{\xi} = \boldsymbol{0} \qquad \text{on } \Gamma_D. \tag{16}$$

We describe the height of the obstacle by a smooth function $g : [0, T) \to \mathbb{R}$. The condition that the deformed configuration remains above the obstacle is given by

$$\varphi_2(\boldsymbol{x}) = (\boldsymbol{x} + \boldsymbol{\xi}(\boldsymbol{x})) \cdot \left(\boldsymbol{R}^T \boldsymbol{e}_2\right) \geq g \qquad \text{for all } \boldsymbol{x} \in \Gamma_C, \tag{17}$$

where $\boldsymbol{e}_i \in \mathbb{R}^2$ are the unit vectors of the canonical basis in the Lagrangian frame. We call

$$\{\boldsymbol{\varphi}(\boldsymbol{x}) : \boldsymbol{x} \in \Gamma_C,\ \varphi_2(\boldsymbol{x}) = g\}$$

the contact zone, and note that it is an unknown subset of Γ_C.

Using the contact zone, we describe the boundary conditions on Γ_C. Outside of the contact zone, we impose homogeneous Neumann boundary conditions (i.e., zero traction). At the contact zone, we impose zero traction in tangential direction (i.e., no friction force between the elastic body and the obstacle), and require the normal force of the obstacle on the elastic body to be non-negative. This leads to the following boundary conditions on Γ_C:

$$\left.\begin{aligned} (\mathbf{id} + \boldsymbol{\xi}) \cdot \left(\boldsymbol{R}^T \boldsymbol{e}_2\right) &\geq g \\ (\boldsymbol{\sigma}[\boldsymbol{\xi}]\boldsymbol{n}) \cdot \left(\boldsymbol{R}^T \boldsymbol{e}_1\right) &= 0 \\ (\boldsymbol{\sigma}[\boldsymbol{\xi}]\boldsymbol{n}) \cdot \left(\boldsymbol{R}^T \boldsymbol{e}_2\right) &\geq 0 \\ \left((\mathbf{id} + \boldsymbol{\xi}) \cdot \left(\boldsymbol{R}^T \boldsymbol{e}_2\right) - g\right) (\boldsymbol{\sigma}[\boldsymbol{\xi}]\boldsymbol{n}) \cdot \left(\boldsymbol{R}^T \boldsymbol{e}_2\right) &= 0 \end{aligned}\right\} \quad \text{on } \Gamma_C, \tag{18}$$

where $\boldsymbol{n}$ is the unit outward normal vector to Γ_C.

Full model Summarizing the equations above, and adding initial conditions, we obtain the complete system

$$(\mathrm{P}) \begin{cases} \rho\ddot{\boldsymbol{\xi}} - \mathbf{div}\boldsymbol{\sigma}[\boldsymbol{\xi}] = \rho\boldsymbol{f}(\cdot, \cdot, \boldsymbol{\xi}, \dot{\boldsymbol{\xi}}) & \text{in} \quad \Omega \times (0, T) \\ \boldsymbol{\xi} = \boldsymbol{0} & \text{on} \quad \Gamma_D \times [0, T) \\ (\mathbf{id} + \boldsymbol{\xi}) \cdot \left(\boldsymbol{R}^T(\theta) \boldsymbol{e}_2\right) \geq g & \text{on} \quad \Gamma_C \times [0, T) \\ (\boldsymbol{\sigma}[\boldsymbol{\xi}]\boldsymbol{n}) \cdot \left(\boldsymbol{R}^T(\theta) \boldsymbol{e}_1\right) = 0 & \text{on} \quad \Gamma_C \times [0, T) \\ (\boldsymbol{\sigma}[\boldsymbol{\xi}]\boldsymbol{n}) \cdot \left(\boldsymbol{R}^T(\theta) \boldsymbol{e}_2\right) \geq 0 & \text{on} \quad \Gamma_C \times [0, T) \\ \left((\mathbf{id} + \boldsymbol{\xi}) \cdot \left(\boldsymbol{R}^T(\theta) \boldsymbol{e}_2\right) - g\right) (\boldsymbol{\sigma}[\boldsymbol{\xi}]\boldsymbol{n}) \cdot \left(\boldsymbol{R}^T(\theta) \boldsymbol{e}_2\right) = 0 & \text{on} \quad \Gamma_C \times [0, T) \\ \boldsymbol{\xi}(\cdot, 0) = \boldsymbol{\xi}^0 & \text{in} \quad \Omega, \\ \dot{\boldsymbol{\xi}}(\cdot, 0) = \boldsymbol{\eta}^0 & \text{in} \quad \Omega. \end{cases}$$

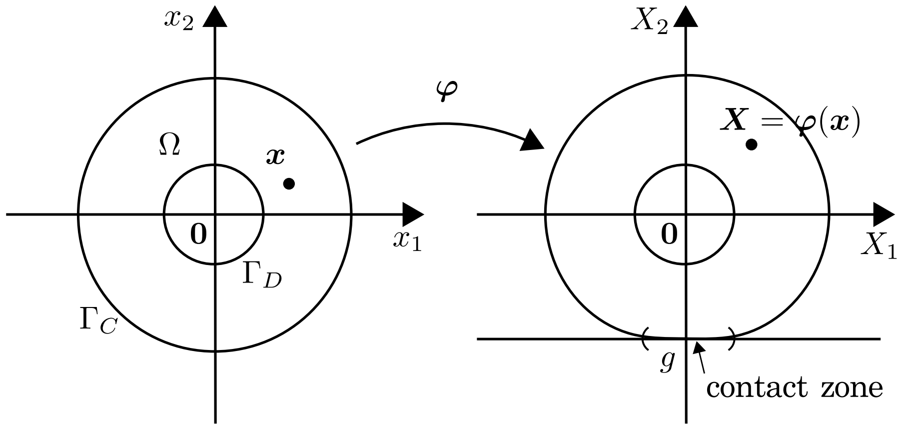

Figure 2: Sketch of the boundary components Γ_D, Γ_C and the contact zone.

where $\theta, g : [0, T) \to \mathbb{R}$ and $\boldsymbol{\xi}^0, \boldsymbol{\eta}^0$ are given functions, and $\boldsymbol{f}$ is defined in (13). (P) describes the complete set of equations of motion for $\boldsymbol{\xi}$ which we solve numerically in the remainder of this paper.

Numerical method

Time-discretized problem For the discretization in time, let $T > 0$ be the end time, $M \in \mathbb{N}$ be the number of time steps, and $\Delta t := T/M$ the time step size. For each time step $k = 0, 1, \cdots, M$, we set

$$\theta^k := \theta(k\Delta t), \quad g^k := g(k\Delta t),$$

and denote by $\boldsymbol{\xi}^k : \Omega \to \mathbb{R}^2$ the time-discretized approximation of the solution $\boldsymbol{\xi}$ of (P) at time $k\Delta t$. For convenience, we set $\boldsymbol{\xi}^k|_{k=-1} := \boldsymbol{\xi}^0 - \Delta t \boldsymbol{\eta}^0$.

Using the Crank-Nicholson scheme, we discretize the elastodynamics equation in time as

$$\rho \frac{\boldsymbol{\xi}^k - 2\boldsymbol{\xi}^{k-1} + \boldsymbol{\xi}^{k-2}}{(\Delta t)^2} = \mathbf{div}\boldsymbol{\sigma}\left[\frac{\boldsymbol{\xi}^k + \boldsymbol{\xi}^{k-2}}{2}\right] + \rho \boldsymbol{f}^{k-1} \quad \text{in } \Omega, \tag{19}$$

where we define

$$\boldsymbol{f}^{k-1}(\boldsymbol{x}) := \boldsymbol{f}((k-1)\Delta t, \boldsymbol{x}, \boldsymbol{\xi}^{k-1}, (\boldsymbol{\xi}^{k-1} - \boldsymbol{\xi}^{k-2})/\Delta t). \tag{20}$$

Using the definition of $\boldsymbol{f}$, (20) reads

$$\boldsymbol{f}^{k-1}(\boldsymbol{x}) = \ddot{\theta}^{k-1}\boldsymbol{R}(-\pi/2)(\boldsymbol{x} + \boldsymbol{\xi}^{k-1}) + (\dot{\theta}^{k-1})^2(\boldsymbol{x} + \boldsymbol{\xi}^{k-1}) + 2\dot{\theta}^{k-1}\boldsymbol{R}(-\pi/2)\frac{\boldsymbol{\xi}^{k-1} - \boldsymbol{\xi}^{k-2}}{\Delta t}. \tag{21}$$

The advantage of the Crank-Nicholson scheme in contrast to the purely implicit scheme used in previous works [28] is that it conserves the time-discrete energy in the case when $\theta \equiv 0$ with homogeneous Dirichlet boundary conditions:

Theorem 1. *If $\theta \equiv 0$, $\boldsymbol{\xi}^k = \mathbf{0}$ on $\partial\Omega$ for $k = 0, 1, \cdots, M$, and $\boldsymbol{\xi}^k$ satisfies (19) for $k = 1, 2 \cdots, M$, then the time-discrete energy*

$$E^k := \frac{1}{2}\int_\Omega \frac{|\boldsymbol{\xi}^k - \boldsymbol{\xi}^{k-1}|^2}{(\Delta t)^2}\,dx + \frac{1}{2}\int_\Omega \frac{\boldsymbol{\sigma}[\boldsymbol{\xi}^k] : \boldsymbol{\epsilon}[\boldsymbol{\xi}^k] + \boldsymbol{\sigma}[\boldsymbol{\xi}^{k-1}] : \boldsymbol{\epsilon}[\boldsymbol{\xi}^{k-1}]}{2}\,dx \quad \text{for } k = 1, 2, \cdots, M \tag{22}$$

does not depend on k. Here, $\boldsymbol{\sigma} : \boldsymbol{\epsilon} := \sigma_{ij}\epsilon_{ij}$.

The Crank-Nicholson discretization above yields the following time-discretized scheme for (P). The choice of $(\boldsymbol{\xi}^k + \boldsymbol{\xi}^{k-2})/2$ in the boundary conditions is motivated by the variational formula. Let $\boldsymbol{\xi}^0, \boldsymbol{\eta}^0 \in W^{1,2}(\Omega; \mathbb{R}^2)$ be given, and set $\boldsymbol{\xi}^k|_{k=-1} := \boldsymbol{\xi}^0 - \Delta t \boldsymbol{\eta}^0$. For $k = 1, 2, \cdots, M$, find $\boldsymbol{\xi}^k : \Omega \to \mathbb{R}^2$ such that the following equations are satisfied:

$$(\mathrm{P}_k)\begin{cases} \rho\dfrac{\boldsymbol{\xi}^k - 2\boldsymbol{\xi}^{k-1} + \boldsymbol{\xi}^{k-2}}{(\Delta t)^2} - \mathbf{div}\boldsymbol{\sigma}\left[\dfrac{\boldsymbol{\xi}^k + \boldsymbol{\xi}^{k-2}}{2}\right] = \rho \boldsymbol{f}^{k-1} & \text{in} \quad \Omega \\ \boldsymbol{\xi}^k = \mathbf{0} & \text{on} \quad \Gamma_D \\ (\mathbf{id} + \boldsymbol{\xi}^k) \cdot \big(\boldsymbol{R}^T(\theta^k)\boldsymbol{e}_2\big) \geq g^k & \text{on} \quad \Gamma_C \\ \left(\boldsymbol{\sigma}\left[\dfrac{\boldsymbol{\xi}^k + \boldsymbol{\xi}^{k-2}}{2}\right]\boldsymbol{n}\right) \cdot \big(\boldsymbol{R}^T(\theta^k)\boldsymbol{e}_1\big) = 0 & \text{on} \quad \Gamma_C \\ \left(\boldsymbol{\sigma}\left[\dfrac{\boldsymbol{\xi}^k + \boldsymbol{\xi}^{k-2}}{2}\right]\boldsymbol{n}\right) \cdot \big(\boldsymbol{R}^T(\theta^k)\boldsymbol{e}_2\big) \geq 0 & \text{on} \quad \Gamma_C \\ \Big((\mathbf{id} + \boldsymbol{\xi}^k) \cdot \big(\boldsymbol{R}^T(\theta^k)\boldsymbol{e}_2\big) - g^k\Big)\left(\boldsymbol{\sigma}\left[\dfrac{\boldsymbol{\xi}^k + \boldsymbol{\xi}^{k-2}}{2}\right]\boldsymbol{n}\right) \cdot \big(\boldsymbol{R}^T(\theta^k)\boldsymbol{e}_2\big) = 0 & \text{on} \quad \Gamma_C \end{cases}$$

Variational structure (P_k) For any $k = 1, 2, \cdots, M$, problem (P_k) is an elliptic problem with an obstacle. It is the Euler-Lagrange equation for the minimizer of the functional

$$\mathcal{J}^k(\boldsymbol{\xi}) := \rho\int_\Omega \frac{|\boldsymbol{\xi} - 2\boldsymbol{\xi}^{k-1} + \boldsymbol{\xi}^{k-2}|^2}{2(\Delta t)^2}\,dx + \frac{1}{2}\int_\Omega \left(\frac{1}{2}\boldsymbol{\sigma}[\boldsymbol{\xi}] + \boldsymbol{\sigma}[\boldsymbol{\xi}^{k-2}]\right) : \boldsymbol{\epsilon}[\boldsymbol{\xi}]\,dx - \rho\int_\Omega \boldsymbol{f}^{k-1} \cdot \boldsymbol{\xi}\,dx \tag{23}$$

over to the admissible set

$$\mathcal{K}^k := \left\{\boldsymbol{\xi} \in W^{1,2}(\Omega; \mathbb{R}^2);\ \boldsymbol{\xi} = \mathbf{0} \text{ a.e. on } \Gamma_D,\ (\mathbf{id} + \boldsymbol{\xi}) \cdot \big(\boldsymbol{R}^T(\theta^k)\boldsymbol{e}_2\big) \geq g^k \text{ a.e. on } \Gamma_C\right\}. \tag{24}$$

Indeed, by calculating the first variation of $\mathcal{J}^k$ over $\mathcal{K}^k$ we obtain that any minimizer $\boldsymbol{\xi}^k$ satisfies (P_k). The existence of a unique minimizer follows from the facts that $\mathcal{J}^k$ is weakly lower-semicontinuous on $W^{1,2}(\Omega; \mathbb{R}^2)$, is bounded from below, has bounded sublevel sets, and that $\mathcal{K}^k$ is convex and closed in $W^{1,2}(\Omega; \mathbb{R}^2)$.

Numerical method for solving the minimization problem The aim is to minimize $\mathcal{J}^k$ over $\mathcal{K}^k$ numerically using the finite element method.

Given a space discretization parameter $\Delta x > 0$, the domain Ω is approximated by a triangular mesh giving a numerical domain $\tilde{\Omega}$. We first distribute equispaced nodes of distance approximately Δx on Γ_D and Γ_C and then we generate the interior nodes by applying the

Poisson disk sampling algorithm due to [4] with parameter $r = \frac{2}{3}\Delta x$. The triangular mesh is then given by the Delaunay triangulation [25] on the constructed nodes.

We approximate the minimizer of $\mathcal{J}^k$ by a continuous function on $\widetilde{\Omega}$ that is linear on each element of the mesh. We denote the space of such functions V. Let $N \in \mathbb{N}$ be the number of the nodes, $\{\boldsymbol{x}_n\}_{n=1}^N$ be the nodes, and I_D and I_C be defined

$$I_D := \{n; \boldsymbol{x}_n \in \Gamma_D\}, \quad I_C := \{n; \boldsymbol{x}_n \in \Gamma_C\}. \tag{25}$$

We define the basis functions $\zeta_n : \mathbb{R}^2 \to \mathbb{R}$ as the continuous functions, linear on each element, satisfying

$$\zeta_n(\boldsymbol{x}_m) = \delta_{nm}. \tag{26}$$

For the vector

$$\widetilde{\boldsymbol{\xi}} = (\widetilde{\xi}_{1,1}, \widetilde{\xi}_{1,2}, \cdots, \widetilde{\xi}_{1,N}, \widetilde{\xi}_{2,1}, \cdots, \widetilde{\xi}_{2,N}) \in \mathbb{R}^{2N},$$

we define the operator $P : \mathbb{R}^{2N} \to V$ as

$$P(\widetilde{\boldsymbol{\xi}})(\boldsymbol{x}) := \left(\sum_{n=1}^N \widetilde{\xi}_{1,n}\zeta_n(\boldsymbol{x}), \sum_{n=1}^N \widetilde{\xi}_{2,n}\zeta_n(\boldsymbol{x}) \right). \tag{27}$$

We set $\widetilde{\boldsymbol{\xi}}^0, \widetilde{\boldsymbol{\xi}}^{-1} \in \mathbb{R}^{2N}$ as

$$\widetilde{\xi}^0_{d,n} := \xi^0_d(\boldsymbol{x}_n), \quad \widetilde{\xi}^{-1}_{d,n} := \xi^{-1}_d(\boldsymbol{x}_n), \tag{28}$$

for $d = 1, 2$, $n = 1, 2, \cdots, N$. Then for any $k = 1, 2, \cdots, M$, we seek inductively a minimizer $\widetilde{\boldsymbol{\xi}}^k$ of the discrete functional

$$\begin{aligned}
\widetilde{\mathcal{J}}^k(\widetilde{\boldsymbol{\xi}}) := {}& \rho \int_{\widetilde{\Omega}} \frac{\left| P(\widetilde{\boldsymbol{\xi}}) - 2P(\widetilde{\boldsymbol{\xi}}^{k-1}) + P(\widetilde{\boldsymbol{\xi}}^{k-2}) \right|^2}{2(\Delta t)^2}\, dx \\
&+ \frac{1}{2} \int_{\widetilde{\Omega}} \left(\frac{1}{2}\boldsymbol{\sigma}[P(\widetilde{\boldsymbol{\xi}})] + \boldsymbol{\sigma}[P(\widetilde{\boldsymbol{\xi}}^{k-2})] \right) : \boldsymbol{\epsilon}[P(\widetilde{\boldsymbol{\xi}})]\, dx \\
&- \rho \int_{\widetilde{\Omega}} \boldsymbol{f}\left((k-1)\Delta t, \cdot, P(\widetilde{\boldsymbol{\xi}}^{k-1}), (P(\widetilde{\boldsymbol{\xi}}^{k-1}) - P(\widetilde{\boldsymbol{\xi}}^{k-2}))/\Delta t \right) \cdot P(\widetilde{\boldsymbol{\xi}})\, dx,
\end{aligned} \tag{29}$$

over the admissible set

$$\widetilde{\mathcal{K}}^k := \left\{ \widetilde{\boldsymbol{\xi}} \in \mathbb{R}^{2N}; \widetilde{\xi}_{1,n} = \widetilde{\xi}_{2,n} = 0 \text{ for } n \in I_D, (\boldsymbol{x}_n + (\widetilde{\xi}_{1,n}, \widetilde{\xi}_{2,n})) \cdot \left(\boldsymbol{R}^T(\theta^k)\boldsymbol{e}_2 \right) \geq g^k \text{ for } n \in I_C \right\}. \tag{30}$$

For fixed $k \geq 1$, we approximate the minimizer of the functional $\widetilde{\mathcal{J}}^k$ in the admissible set $\widetilde{\mathcal{K}}^k$ using a variant of the nonlinear conjugate gradient method with a projection given by the following steps ($\varepsilon > 0$ is a given stopping tolerance):

(1) initial guess $\widetilde{\boldsymbol{\xi}}_0 \in \widetilde{\mathcal{K}}^k$ (for example, $\widetilde{\boldsymbol{\xi}}_0 = \mathrm{Proj}_{\widetilde{\mathcal{K}}^k}(\widetilde{\boldsymbol{\xi}}^{k-1})$)

(2) $\boldsymbol{g}_1 = -\nabla \widetilde{\mathcal{J}}^k(\widetilde{\boldsymbol{\xi}}_0)$

(3) $\boldsymbol{p}_1 = T^k_{\widetilde{\boldsymbol{\xi}}_0}(\boldsymbol{g}_1)$

(4) $e = \|\boldsymbol{p}_1\|$; if $e \le \varepsilon$ then set $\widetilde{\boldsymbol{\xi}}^k = \widetilde{\boldsymbol{\xi}}_0$ and proceed to next time step $k+1$

(5) For $m = 1, 2, \ldots$:

(i) $\alpha_m = \operatorname{argmin}_{\alpha>0} \widetilde{\mathcal{J}}^k(\widetilde{\boldsymbol{\xi}}_{m-1} + \alpha \boldsymbol{p}_m)$ (Exact solution as the function is quadratic.)

(ii) $\widetilde{\boldsymbol{\xi}}_m = \mathrm{Proj}_{\widetilde{\mathcal{K}}^k}(\widetilde{\boldsymbol{\xi}}_{m-1} + \alpha_m \boldsymbol{p}_m)$

(iii) $\boldsymbol{g}_{m+1} = -\nabla \widetilde{\mathcal{J}}^k(\widetilde{\boldsymbol{\xi}}_m)$

(iv) $\beta_m = \max\left\{0, \frac{(\boldsymbol{g}_{m+1} - \boldsymbol{g}_m)\cdot \boldsymbol{g}_{m+1}}{\|\boldsymbol{g}_m\|^2}\right\}$

(v) $\boldsymbol{p}_{m+1} = T^k_{\widetilde{\boldsymbol{\xi}}_m}(\boldsymbol{g}_{m+1} + \beta_m \boldsymbol{p}_m)$

(vi) $e = \|T^k_{\widetilde{\boldsymbol{\xi}}_m}(\boldsymbol{g}_{m+1})\|$; if $e \le \varepsilon$ then set $\widetilde{\boldsymbol{\xi}}^k = \widetilde{\boldsymbol{\xi}}_m$ and proceed to next time step $k+1$

where

$$\left(\mathrm{Proj}_{\widetilde{\mathcal{K}}^k}(\widetilde{\boldsymbol{\xi}})\right)_{(n,n+N)} := \begin{cases} (\widetilde{\xi}_{1,n}, \widetilde{\xi}_{2,n}) - \min\left\{0, g^k - (\boldsymbol{x}_n + (\widetilde{\xi}_{1,n}, \widetilde{\xi}_{2,n}))\cdot(\boldsymbol{R}^T(\theta^k)\boldsymbol{e}_2)\right\}(\boldsymbol{R}^T(\theta^k)\boldsymbol{e}_2) & \text{if } n \in I_C, \\ (\widetilde{\xi}_{1,n}, \widetilde{\xi}_{2,n}) & \text{otherwise} \end{cases}$$

for any $\widetilde{\boldsymbol{\xi}} \in \mathbb{R}^{2N}$ and

$$\left(T^k_{\widetilde{\boldsymbol{\xi}}}(\boldsymbol{p})\right)_{(n,n+N)} := \begin{cases} (p_{1,n}, p_{2,n}) - \min\left\{0, (p_{1,n}, p_{2,n})\cdot(\boldsymbol{R}^T(\theta^k)\boldsymbol{e}_2)\right\}(\boldsymbol{R}^T(\theta^k)\boldsymbol{e}_2) \\ \qquad \text{if } n \in I_C,\ (\boldsymbol{x}_n + (\widetilde{\xi}_{1,n}, \widetilde{\xi}_{2,n}))\cdot(\boldsymbol{R}^T(\theta^k)\boldsymbol{e}_2) \le g^k, \\ (p_{1,n}, p_{2,n}) \qquad \text{otherwise,} \end{cases}$$

for any $\boldsymbol{p} \in \mathbb{R}^{2N}$. The operator $\mathrm{Proj}_{\widetilde{\mathcal{K}}^k}$ is the orthogonal projection onto the set $\widetilde{\mathcal{K}}^k$. The operator $T^k_{\widetilde{\boldsymbol{\xi}}}(\boldsymbol{p})$ restricts the search direction $(\boldsymbol{p})_{(n,n+N)}$ for $n \in I_C$ so as not to jump over the obstacle g^k.

We choose the domain Ω as the annulus

$$\Omega := \{\boldsymbol{x} \in \mathbb{R}^2; r_D < |\boldsymbol{x}| < r_C\}, \quad \Gamma_D := \{\boldsymbol{x} \in \mathbb{R}^2; |\boldsymbol{x}| = r_D\}, \quad \Gamma_C := \{\boldsymbol{x} \in \mathbb{R}^2; |\boldsymbol{x}| = r_C\},$$

where $r_D = 0.25$ and $r_C = 0.5$. We further set the initial data as $\boldsymbol{\xi}^0 = \boldsymbol{0}$ and $\boldsymbol{\eta}^0 = \boldsymbol{0}$.

Numerical results We simulate two cases. In the first case we remove the obstacle, and study the sensitivity of the roller's dynamics with respect to the parameters. In particular, we are interested in the vibrations in the radial and tangential displacements, because the understanding of these vibrations might help in removing the squeaking sound of printer rollers. As feedback on these simulations, we add a vibration to the given rotation $\theta(t)$ to investigate the occurrence of resonance.

In the second case we add the obstacle. We are interested in the shape of the deformed domain and the size of the stress tensor $\boldsymbol{\sigma}[\boldsymbol{\xi}]$ as a function on the deformed domain, especially in the region close to the contact zone.

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学位論文審査報告書（甲）

1．学位論文題目（外国語の場合は和訳を付けること。）

回転する弾性体の動的接触問題の変分法に基づく数値的取り扱いについて

Numerical scheme for a dynamical rolling elastic contact problem based on a variational method

2．論文提出者 (1) 所　属　数物科学　　専攻

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3．審査結果の要旨（600～650 字）

赤川佳穂君は、2014 年 4 月に自然科学研究科数物科学専攻博士後期課程に入学した。それ以降、自由境界を持つ連続体（弾性体）の数値解析に取り組んできた。特に、ローラー形状で回転する弾性体と障害物との動的接触問題について、モデリング・数値計算法構築に取り組んできた。彼の方法は変分に基づく方法で波動型方程式に分類される問題に対して、エネルギー最小化問題を用いた方法論を開発している。物理的モデルとしては、プリンターの紙送り機構（シートフィーダ）を意識しており、この問題の最終的な目標は、フィーダーが紙を取り込む際に起きる「鳴き」と呼ばれる振動現象を解決することであった。弾性体の回転変形は非線形双曲型であり、弾性体と紙の接触部分での自由境界を生ずるなど難問である。最終的な目標は、粘着力・摩擦力などをヒステリシス付きで考察する必要がある非常に困難な問題となり得る。赤川君は、「変形が回転と微小変位に分解できる」という合理的仮定を用いて、この問題を主要項が線形となる弾性体の運動方程式で記述した。これは変分的取り扱いに適した有用なモデリングとなっている。

数値計算方法としては、特に、接触面を自動的に決定する変分的スキームで保存量も的確に取り扱う方法を提案しており、新規性の高い方法論といえる。これは、この問題に関する理論および数値解析に対して、新たな興味深い手法をもたらす結果である。赤川君は、この結果を原著論文 1 本にまとめた。以上により本論文は、博士(理学)を授与するに値すると判断した。

4．審査結果 (1) 判　定（いずれかに○印）　合　格　・　不合格

(2) 授与学位　博　士（　理学　）