A note on the unit index in a CM-field

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Abstract. Let K/k be a totally imaginary quadratic extension of a totally real algebraic number field. Such a field K is called a CM-field. Let E_K and E_k be the unit group of K and k, respectively. Let μ_K be the group of every power roots of unity contained in K. An index $|E_K : E_k\mu_K|$ takes a value in 1 or 2, and is called the unit index of the CM-field K. We study relations between the value of the unit index and the cohomology group $H^1(Gal(K/k), \mu_K)$.

1. Introduction. Let k be a totally real algebraic number field and K be its totally imaginary quadratic extension. The ratio of the class number of K by that of k is a positive integer called the relative class number of K. As reviewing briefly in the next section, this number is described with residues of Dedekind's zeta functions at $s = 1$ and special values of an L-function at $s = 0, 1$. In the formula of the relative class number, a constant taking its value in 1 or 2 appears. This constant is called the unit index of K/k. The problem of determining the unit index was originated by [1] , and was pointed out the incorrectness of Satz 29 by showing counterexamples in [2]. They studied the case of imagiary quadratic fields of type (2,2,2) in [3]. The problem was studied in CM-fields generally in [4]. We shall reformulate Lemmermeyer's theorem (Theorem 1 in [4]) from the aspect of classical Galois cohomology in §3. In the present paper, we shall show that a special 1-cocycle generating $H^1(Gal(K/k), \mu_K)$ has an important role to determine the unit index, where μ_K is the torsion subgroup of the unit group E_K of K. The unit index equals 2 if and only if the generating cocycle becomes a coboundary by regarding it as a 1-cycle of $G(K/k)$ with coefficients in E_K. We shall study this vanishing in connection with capitulation of ideal classes of k.

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2. The relative class number formula. Let J be the Galois group of K/k generated by an element τ. The complex valued characters of J are consisted of the unit character ε and a non-trivial character χ. We regard χ a Dirichlet character of k with a conductor $\mathfrak{f}$. The L-functions relative to these characters are functions obtained from functions

$$L(s,\varepsilon)=\prod_{\mathfrak{p}}\left(1-\frac{1}{N\mathfrak{p}^s}\right),$$
$$L(s,\chi)=\prod_{\mathfrak{p}\nmid\mathfrak{f}}\left(1-\frac{\chi(\mathfrak{p})}{N\mathfrak{p}^s}\right)$$

defined in the region satisfying $\Re s>1$ by continuing analytically to meromorphic functions in the whole complex plane,respectively, where $\mathfrak{p}$'s in products runs through the set of prime ideals of k. We notice that $L(s,\varepsilon)$ is the Dedekind zeta function $\zeta_k(s)$ of k. We may consider these two functions as Artin's L-functions relative to one dimensional representations of the group J. Since the character of the regular representation of the group ring $\boldsymbol{C}J$ by left multiplication is $\varepsilon+\chi$, the corresponding Artin's L-function relative to the regular representation satisfies

(1) $$L(s,\varepsilon+\chi)=L(s,\varepsilon)L(s,\chi).$$

By dividing the infinite product in the definition of $L(s,\chi)$ into two parts at values of $\chi(\mathfrak{p})$, we obtain

(2) $$L(s,\varepsilon+\chi)=\prod_{\chi(\mathfrak{p})=1}\left(1-\frac{1}{N\mathfrak{p}^s}\right)^2\prod_{\chi(\mathfrak{p})=-1}\left(1-\frac{1}{N\mathfrak{p}^{2s}}\right)\times\prod_{\mathfrak{p}|\mathfrak{f}}\left(1-\frac{1}{N\mathfrak{p}^s}\right).$$

We see the right hand side of this equation is equal to the Dedekind zeta-function on K. Thus, by (1) and (2), we have

$$L(s,\chi)=\frac{\zeta_K(s)}{\zeta_k(s)}.$$

Here, since the zeta functions have single poles at $s=1$, the value of $L(1,\chi)$ equals the ratio of residue of the poles:

$$L(1,\chi)=\frac{\frac{(2\pi)^n}{|\mu_K|\sqrt{|d_K|}}h_KR_K}{\frac{2^n}{2d_k}h_kR_k},$$

where n is the degree of k over $\boldsymbol{Q}$, d_k and d_K denote the discriminants of k and K, respectively, and where h_k (*resp.* R_k) and h_K (*resp.* R_K) are the class numbers (*resp.* the regulators) of k (*resp.* K). We note that formulas $|d_k|=d_k^2N\mathfrak{f}$ and $R_K=|E_K:E_k\mu_K|2^{n-1}R_k$ hold, *c.f.* Chap. 3 of [6]. Therefore,

$$\frac{h_K}{h_k}=\frac{|\mu_K||E_K:E_k\mu_K|(N\mathfrak{f})^{\frac{1}{2}}}{(2\pi)^n}L(1,\chi)$$

Denote by h_K^- the relative class number. We use the functional equation of $L(s,\chi)$ and transform the above formula to a familiar one. Put

$$\Lambda(s)=d_k(N\mathfrak{f})^{\frac{s}{2}}\pi^{-\frac{1+s}{2}}\Gamma(\tfrac{1+s}{2})^nL(s,\chi).$$

The functional equation is an equation

(3) $$\Lambda(1-s)=W(\chi)\Lambda(s),$$

which holds on the whole complex plane, where $W(\chi)$ is a complex number

$$W(\chi) = \sqrt{-1}^{-n}\tau(\chi)(N\mathfrak{f})^{\frac{1}{2}}$$

containing the Gauss sum $\tau(\chi)$ with respect to the Dirichlet character χ, *c.f.* Chap. 0 of [5]. Set $s = 0$, we obtain

$$L(1,\chi) = \frac{\pi^n W(\chi)}{(N\mathfrak{f})^{\frac{1}{2}}} L(0,\chi),$$

and furthermore,

$$h_K^- = \frac{W(\chi)\cdot|E_K : E_k\mu|\cdot|\mu_K|}{2^n} L(0,\chi).$$

We note $W(\chi)^2 = 1$ is obtained if we set $s = 1$ in (3) and compare with the result obtained by setting $s = 0$. The factor $|E_K : E_k\mu_K|$ in this formula is called the unit index of K. We denote by q_K the unit index in the sequal.

3. Some exact sequences. Let $\mathfrak{f}$ be an integral ideal of K. Denote by $E_K(\mathfrak{f})$ a subgroup of the unit group E_K defined by

$$E_K(\mathfrak{f}) = \{x \in E_K : x \equiv 1 \mod \mathfrak{f}\}.$$

Let 2^l be the highest power of 2 dividing $|\mu_K|$. Put $\zeta = e^{2\pi\sqrt{-1}/2^l}$. We suppose $\mathfrak{f}$ is chosen so that $E_K(\mathfrak{f})$ contains ζ. It is of a finite index in E_K and the quotient group $E_K(\mathfrak{f})/\mu_K$ is isomorphic to $\boldsymbol{Z}^{n-1}$. When $x \in E_K(\mathfrak{f})$, $x^{\tau-1}$ is an element of μ_K, because K is a CM-field. This means the group J acts trivially on the quotient group. We can determine the Tate cohomology groups of J with coefficients in $E_K(\mathfrak{f})/\mu_K$:

$$(4)\quad H^n(J, E_K(\mathfrak{f})/\mu_K) = \begin{cases} (\boldsymbol{Z}/2\boldsymbol{Z})^{n-1} & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

By a short exact sequence of $\boldsymbol{Z}J$-modules

$$1 \to \mu_K \to E_K(\mathfrak{f}) \to E_K(\mathfrak{f})/\mu_K \to 1,$$

we have the long exact sequence of the Tate cohomology groups:

$$\cdots \to H^{-1}(J, E_K(\mathfrak{f})/\mu_K) \to H^0(J,\mu_K) \to H^0(J, E_K(\mathfrak{f})) \to H^0(J, E_K(\mathfrak{f})/\mu_K) \to H^1(J,\mu_K) \to H^1(J, E_K(\mathfrak{f})) \to H^1(J, E_K(\mathfrak{f})/\mu_K) \to \cdots.$$

This sequence yields the following short exact sequence

$$(5)\quad 1 \to E_K(\mathfrak{f})/E_K(\mathfrak{f})^J\mu_K \to H^1(J,\mu_K) \to H^1(J, E_K(\mathfrak{f})) \to 1.$$

Since μ_K is of finite order, the Herbrand quotient is one:

$$\frac{|H^0(J,\mu_K)|}{|H^1(J,\mu_K)|} = 1.$$

We see $H^0(J,\mu_K) = \{\pm 1\}$. Hence, $H^1(J,\mu_K) \cong \boldsymbol{Z}/2\boldsymbol{Z}$. Define

$$q_\mathfrak{f} = |E_K(\mathfrak{f}) : E_K(\mathfrak{f})^J\mu_K|.$$

Note $q_\mathfrak{f}$ coincides with q_K when $\mathfrak{f} = 1$. By the exact sequence (5), we obtain the following lemma:

LEMMA 1. $q_\mathfrak{f}\cdot|H^1(J, E_K(\mathfrak{f}))| = 2$. *In particular,* $q_K = 2$ *if and only if the cohomology group* $H^1(J, E_K)$ *vanishes.*

Let $K_\mathfrak{P}$ be the completion of K with topology defining from the $\mathfrak{P}$-adic valuation. Let Π be a prime element in $K_\mathfrak{P}$ and

$U_{\mathfrak{P}}$ be the unit group of the valuation ring. Let $U_{\mathfrak{P}}^{(i)}$ be a subgroup of $U_{\mathfrak{P}}$ defined to be

$$U_{\mathfrak{P}}^{(i)} = \{x \in U_{\mathfrak{P}} : x \equiv 1 \mod \Pi^i\}.$$

$U_{\mathfrak{P}}^{(i)}$ is a subgroup of a finite index in $U_{\mathfrak{P}}$. We define a subgroup $U_K(\mathfrak{f})$ in the idele group of K to be

$$U_K(\mathfrak{f}) = \prod_{i=1}^{r} U_{\mathfrak{P}_i}^{(e_i)} \times \prod_{\mathfrak{P}\nmid\mathfrak{f}} U_{\mathfrak{P}} \times (C^\times)^n$$

for the factorization $\mathfrak{f} = \prod_{i=1}^{r} \mathfrak{P}_i^{(e_i)}$, where $\mathfrak{P}$ in a product runs through the set of ideals of K not dividing $\mathfrak{f}$. When $\mathfrak{f} = 1$, this group coincides with U_K: the group of unit ideles. We choose an embedding of $K^\times$ into the idele group J_K of K and fix it once for all. By virtue of this embedding, we regard $E_K(\mathfrak{f})$ as a subgroup of $U_K(\mathfrak{f})$ and obtain an exact sequence

$$1 \to E_K(\mathfrak{f}) \to U_K(\mathfrak{f}) \to U_K(\mathfrak{f})/E_K(\mathfrak{f}) \to 1$$

of $\boldsymbol{Z}J$-modules. We take a cohomology long exact sequence from this short exact sequence:

$$1 \to E_K(\mathfrak{f})^J \to U_K(\mathfrak{f})^J \xrightarrow{\iota} (U_K(\mathfrak{f})/E_K(\mathfrak{f}))^J$$
$$\xrightarrow{\delta} H^1(J, E_K(\mathfrak{f})) \to H^1(J, U_K(\mathfrak{f})) \to \cdots$$

where δ is the connecting homomorphism. Let x be an element of $U_K(\mathfrak{f})$ such that $x^{\tau-1} \in E_K(\mathfrak{f})$. Let φ be a 1-cocycle in coefficients $E_K(\mathfrak{f})$ which sends $\sigma \in J$ to $x^{\sigma-1}$. The connecting homomorphism δ maps $xE_K(\mathfrak{f})$ to the cohomology class containing φ. Let J_k be the idele group of k. The quotient group $Cl_k(\mathfrak{f}) = J_k/k^\times U_K(\mathfrak{f})^J$ is a finite abelian group. Let $Cl_K(\mathfrak{f})$ be the ray class group of K defined with modulo $\mathfrak{f}$: $Cl_K(\mathfrak{f}) = J_K/K^\times U_K(\mathfrak{f})$. When $\mathfrak{f} = 1$, $Cl_k(\mathfrak{f})$ and $Cl_K(\mathfrak{f})$ are the ideal class groups of k and K, respectively. Let $j_{K/k} : Cl_k(\mathfrak{f}) \longrightarrow Cl_K(\mathfrak{f})$ be the homomorphism induced from an inclusion $J_k \to J_K$. Now, by virtue of Hilbert's Theorem 90, there is $\alpha \in K$ such that $\alpha^{\tau-1} = x^{\tau-1}$ holds. Then, we see $x_0 = x\alpha^{-1}$ is J-invariant. Since $j_{K/k}(x_0(k^\times U_K(\mathfrak{f})^J)) = K^\times U_K(\mathfrak{f})$, we obtain a mapping of $(U_K(\mathfrak{f})/E_K(\mathfrak{f}))^J$ to $\mathrm{Ker}\, j_{K/k}$. This mapping is a homomorphism of groups. Let $x_0(k^\times U(\mathfrak{f})^J)$ be an element of $\mathrm{Ker}\, j_{K/k}$. There are $\alpha \in K^\times$ and $x \in U_K(\mathfrak{f})$ such that $x_0 = \alpha x$. We ovserve $\alpha \in E_K(\mathfrak{f})$ and $x^{\tau-1} = \alpha^{1-\tau} \in E_K(\mathfrak{f})$. Since $x\alpha^{-1} \in U_K(\mathfrak{f})^J$, we obtain $xE_K(\mathfrak{f})U_K(\mathfrak{f}^J) \in \mathrm{Coker}\, \iota$. We notice that $xE_K(\mathfrak{f})$ belongs to the kernel of the mapping if $x \in U_K(\mathfrak{f})^J$. Conversely, let $xE_K(\mathfrak{f}) \in (U_K(\mathfrak{f})/E_K(\mathfrak{f}))^J$ be an element which becomes trivial in $\mathrm{Ker}\, j_{K/k}$. There are $\alpha \in K^\times$ and $x_0 \in k^\times U_K(\mathfrak{f})^J$ such that $x^{\tau-1} = \alpha^{1-\tau}$ and $x_0 = \alpha x$. Since x_0 is a product of elements $a_0 \in k^\times$ and $x_1 \in U_K(\mathfrak{f})^J$, we see $\alpha x = a_0 x_1$. This implies $a_0\alpha^{-1} \in E_K(\mathfrak{f})$ and $x \in E_K(\mathfrak{f})U_K(\mathfrak{f})^J$. Namely, $xE_K(\mathfrak{f}) \in \mathrm{Im}\, \iota$. The kernel of the mapping coincides with $\mathrm{Im}\, \iota$. Therefore, $\mathrm{Coker}\, \iota \cong \mathrm{Ker}\, j_{K/k}$.

LEMMA 2. *We have an exact sequence*

$$(6) \quad 1 \to \mathrm{Ker}\, j_{K/k} \to H^1(J, E_K(\mathfrak{f})) \to H^1(J, U_K(\mathfrak{f})).$$

The kernel of $j_{K/k}$ for $\mathfrak{f} = 1$ is the capitulation kernel of Cl_k relative to the extension K/k. We denote by κ the capitulation kernel.

The third exact sequence is obtained from a short exact sequence

$$1 \to E_K(\mathfrak{f}) \to E_K \to E_K/E_K(\mathfrak{f}) \to 1.$$

We take a cohomology long exact sequence:

$$1 \to E_K(\mathfrak{f})^J \to E_k \to E_K/E_K(\mathfrak{f}) \to H^1(J, E_K(\mathfrak{f})) \to H^1(J, E_K).$$

By shortening this sequence, we obtain an exact sequence

$$(7) \quad 1 \to E_K/E_K(\mathfrak{f})E_k \to H^1(J, E_K(\mathfrak{f})) \to H^1(J, E_K).$$

Combining the exact sequences (5), (6) and (7), we have the following commutative diagram:

$$\begin{array}{ccccccc}
 & & & & 1 & & \\
 & & & & \downarrow & & \\
 & & & & E_K/\mu_K E_k & & \\
 & & & & \downarrow & & \\
 & & & & H^1(K, \mu_K) & & \\
 & & & & \downarrow & & \\
1 \to & \kappa & \to & & H^1(J, E_K) & \to & H^1(J, U_K) \\
 & \uparrow & & & \uparrow & & \uparrow \\
1 \to & \mathrm{Ker}\, j_{K/k} & \to & & H^1(J, E_K(\mathfrak{f})) & \to & H^1(J, U_K(\mathfrak{f})) \\
 & & & & \uparrow & & \\
 & & & & E_K/E_K(\mathfrak{f})E_k & & \\
 & & & & \uparrow & & \\
 & & & & 1 & &
\end{array}$$

REMARK 1. Since $E_K(\mathfrak{f})$ contains μ_K, we see $q_k = 2$ if $E_K/E_K(\mathfrak{f})E_k$ is not trivial. We also have $q_K = 1$ if the image of Ker $j_{K/k}$ into κ is not trivial.

4. The value of q_K. Put $\eta_l = \zeta + \zeta^{-1} = 2\cos\frac{\pi}{2^{l-1}}$. Note $\eta_1 = -2$, $\eta_2 = 0$ and $\eta_3 = \sqrt{2}$. We see

$$(\zeta - 1)^\tau = -\zeta^{-1}(\zeta - 1).$$

For $l \geq 2$, we define a function φ_l of J to be $\varphi_l(\sigma) = (\zeta - 1)^{\sigma - 1}$ for $\sigma \in J$. This function is a 1-cocycle of J with taking values in μ_K, which generates the cohomology group $H^1(J, \mu_K)$. We define φ_1 to be $\varphi_1(\sigma) = \sqrt{-m}^{\sigma-1}$ for $\sigma \in J$, where m is a totally positive element of k satisfying $K = k(\sqrt{-m})$. When $l = 1$, φ_1 generates $H^1(J, \mu_K)$.

PROPOSITION 3. Let L_0 be a totally real extension of k. Put $L = L_0 K$ and suppose $|\mu_L : \mu_K|$ is odd. Then, $q_L = 2$ if $q_K = 2$.

Proof. We notice that φ_l generates $H^1(J, \mu_K)$ and $H^1(J, \mu_L)$ under the assumption. Suppose $q_K = 2$. The image of φ_l into $H^1(J, E_K)$ is trivial by Lemma 1. Hence, the image into $H^1(J, E_L)$ through $H^1(J, E_K)$ is also trivial. This proves $q_L = 2$ by virtue of Lemma 1. □

REMARK 2. We note that $q_K = 1$ dose not implies $q_L = 1$ even if $|\mu_L : \mu_K|$ is odd. See Remark 4 in the bellow.

The cohomology group $H^1(J, U_K)$ is easily determined from the ramification of prime ideals in K/k. Let $\{\mathfrak{P}_1, \cdots, \mathfrak{P}_r\}$ be a complete set of prime ideals of K ramified over k. Let Π_i be a prime element of $K_{\mathfrak{P}_i}$. Let φ_i be a 1-cocycle of J defined by $\varphi_i(\sigma) = \Pi_i^{\sigma-1}$, $(\sigma \in J)$. The cohomology group $H^1(J, U_{\mathfrak{P}_i})$ is a cyclic group of order 2 which is generated by the cocycle φ_i. We have an isomorphism

$$H^1(J, U_K) \cong \prod_{i=1}^{r} H^1(J, U_{\mathfrak{P}_i}).$$

We notice an isomorphism $H^1(J, U_{\mathfrak{P}_i}) \cong (K_{\mathfrak{P}_i}^{\times}/U_{\mathfrak{P}}) \otimes \boldsymbol{Z}/2\boldsymbol{Z}$ holds. Let $w_{\mathfrak{P}_i}$ be a normalized additive valuation of $K_{\mathfrak{P}_i}$. We observe that $q_K = 1$ holds if there is a ramified prime ideal $\mathfrak{P}_i$ satisfying $w_{\mathfrak{P}_i}(\zeta - 1) \equiv 1 \mod 2$, because φ_l dose not vanish in $H^1(J, U_{\mathfrak{P}_i})$.

Assume $l \geq 2$. $\zeta - 1$ is a prime element in $\boldsymbol{Q}(\zeta)$ and $2 - \eta_l$ is that in its maximal real subfield. Let

$$(8) \qquad (2 - \eta_l) = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_r^{e_r}$$

be the primary decomposition of the principal ideal $(2 - \eta_l)$ in k. Suppose a prime factor $\mathfrak{p}_i$ in (8) is restriction of a prime ideal $\mathfrak{P}_i$ onto k. Let $w_{\mathfrak{p}_i}$ be a normalized additive valuation of $k_{\mathfrak{p}_i}$. Let t be the ramification index of $\mathfrak{p}_i$ in K/k. We see

$$2w_{\mathfrak{P}_i}(\zeta - 1) = t w_{\mathfrak{p}_i}(2 - \eta_l).$$

Hence, we obtain $2w_{\mathfrak{P}_i}(\zeta - 1) = te_i$ Therefore, if e_i is odd, we have $t = 2$ and $w_{\mathfrak{P}_i}(\zeta - 1) = e_i$. In this case, the prime ideal $\mathfrak{P}_i$ is ramified over k and $q_K = 1$. *c.f.* Theorem 1 in [4].

We also consider the ideal (m) generated by a totally positive element m of k such that $K = k(\sqrt{-m})$ holds. Let

$$(9) \qquad (m) = \mathfrak{p}_i^{e_1} \cdots \mathfrak{p}_r^{e_r}$$

be a primary decomposition in k. Since values of the cocycle φ_1 are contained in $\{\pm 1\}$, the cocycle is a coboundary if $l \geq 2$. We consider it in the case of $l = 1$. The cohomology group $H^1(J, \mu_K)$ is generated by φ_1. We observe $q_K = 1$ holds if and only if φ_1 is not a coboundary in E_K. If e_i is odd in (9), the prime ideal $\mathfrak{p}_i$ is ramified in K. Let $\mathfrak{P}_i$ be a prime ideal of K lying above $\mathfrak{p}_i$. We have $w_{\mathfrak{P}_i}(\sqrt{-m})$ is odd. Thus, $q_K = 1$. *c.f.* Kap.III of [1].

THEOREM 4. If an odd exponent appears in (8) or (9), the unit index q_K equals one.

COROLLARY 5. *If a ramified prime ideal of K over k is tamely ramified, we have* $q_K = 1$

The unit index is not determined when K/k satisfies the following condition:

(*) The principal ideal generated by a totally real element m of k satisfying $K = k(\sqrt{-m})$ is a square of an ideal $\mathfrak{m}$ of k, and the ideal $(2-\eta_l)$ is a square of an ideal $\mathfrak{a}$ of k if $l \geq 2$.

THEOREM 6. Under the condition (*), the necessary and sufficient condition for $q_K = 2$ is that the ideal $\mathfrak{a}$ and $\mathfrak{m}$ are principal ideals of k.

Proof. We observe $\mathfrak{a} = (\zeta - 1)$ or $\mathfrak{m} = (\sqrt{m})$ holds in K. If $\mathfrak{a}$ or $\mathfrak{m}$ is not principal, the capitulation kernel κ is not trivial. This implies $q_K = 1$. Suppose these ideals are principal. When $l \geq 2$, there is $a \in k$ such that $\mathfrak{a} = (a)$, and hence, there is $x \in E_K$ such that $\zeta - 1 = ax$. This means the 1-cocycle φ_l is a coboundary and the homomorphism $H^1(J, \mu_K) \to H^1(J, E_K)$ is null. We have, $q_K = 2$. When $l = 1$, the 1-cocycle φ_1 generates $H^1(J, \mu_K)$. By the similar argument in the case of $l \geq 2$, $q_K = 2$ follows from the assumption that $\mathfrak{m}$ is principal. □

REMARK 3. Let $l \geq 2$. Let ζ' be a 2^{l+1}th root of unity such that $\zeta'^2 = \zeta$. Suppose K/k satisfies (*) and $K_{\mathfrak{P}} = \boldsymbol{Q}_2(\zeta')$ for a prime ideal $\mathfrak{P}$ of K dividing 2. The cocycle φ_l is coboundary in $U_{\mathfrak{P}}$, because of $\varphi_l(\tau) = ((\zeta' - 1)^{\tau})^{2^l - 2}$. By applying Theorem 6, we have the unit index equals one if and only if $\mathfrak{a}$ is principal. Put $L = K(\zeta')$. $\zeta' - 1$ is a prime element of $K_{\mathfrak{P}}(\zeta')$. Hence, $q_L = 1$. Note $|\mu_L : \mu_K|$ is even.

5. The capitulation kernel. Let E_k^+ be the group of totally positive units of k, E_k^+ is isomorphic to $\boldsymbol{Z}^{n-1}$, where the group J acts trivially. Since E_K/E_k^+ is of finite order, we have

$$\frac{|H^0(J, E_K)|}{|H^1(J, E_K)|} = \frac{|H^0(J, E_k^+)|}{|H^1(J, E_k^+)|} = 2^{n-1}$$

from Herbrand's lemma. By setting $\mathfrak{f} = 1$ in Lemma 1, we obtain

$$|H^0(J, E_K)| = 2^{n-1}|H^1(J, E_K)| = \frac{2^n}{q_K}.$$

Let $\mathfrak{P}_1, \cdots, \mathfrak{P}_r$ be ideals of K ramified in K/k. The subgroup I^J of the ideal class group I of K is generated by these ramified ideals and natural extension of ideals of k. Denote by B the subgroup $I^J P/P$ of the ideal class group Cl_K, where P is the group of principal ideals contained in K. Since J is cyclic, the quotient group Cl_K^J/B is isomorphic to the first unit knot $\omega_{K/k} := E_k \cap N_{K/k}K^\times/N_{K/k}E_K$. Thus,

$$|\omega_{K/k}| = \frac{|Cl_K^J|}{|B|}.$$

On the other hand, the value of $|Cl_K^J|$ is given by the genus number formula

$$|Cl_K^J| = \frac{h_k 2^{t+n}|\omega_{K/k}|}{2|H^0(J, E_K)|} = h_k 2^{t-1} q_K |\omega_{K/k}|.$$

From these two formulas, we have the formula of the order of B:

$$|B| = 2^{t-1} h_k q_K.$$

Since the quotient group $B/j_{K/k}(Cl_k)$ is generated by t residue classes containing the prime ideal $\mathfrak{P}_i$'s and since $\mathfrak{P}_i^2$'s are natural extension of ideals of k, there is an integer $\delta \geq 0$ such that $B/j_{K/k}(Cl_k) \cong (\boldsymbol{Z}/2\boldsymbol{Z})^{t-\delta}$ holds. Therefore, a formula

$$|B| = \frac{2^{t-\delta} h_k}{|\kappa|}$$

follows. Equating these two formulas concerning the order of B, we have

$$q_K|\kappa| = 2^{1-\delta}.$$

We observe $\delta = 0$ or $\delta = 1$ holds in this equation. The value of δ is one if and only if there is a non-empty subset $\{\mathfrak{P}_{i_1}, \cdots, \mathfrak{P}_{i_s}\}$ of the set of ramified prime ideals $\{\mathfrak{P}_1, \cdots, \mathfrak{P}_t\}$, an element α of K

and an ideal $\mathfrak{b}$ of k satisfying a relation $\mathfrak{P}_{i_1} \cdots \mathfrak{P}_{i_s} = (\alpha)\mathfrak{b}$.

THEOREM 7. We have $q_K|\kappa| = 2$ if $\delta = 0$, and have $q_K = |\kappa| = 1$ if $\delta = 1$.

This theorem is a reformulation of Theorem 1 in [4] from the aspect of genus theory in class field theory.

COROLLARY 8. If K/k is unramified in the narrow sense, we have $q_K|\kappa| = 2$.

REMARK 4. Let p be an odd prime number such that $p \equiv 1 \mod 8$. We show how the unit index is determined for $\boldsymbol{Q}(\sqrt{-2}, \sqrt{2p})$ and $\boldsymbol{Q}(\sqrt{-1}, \sqrt{2p})$ from our theorems. We note this problem had settled in Proposition 1 of [3] completely. Set $k = \boldsymbol{Q}(\sqrt{2p})$ and put $K_1 = k(\sqrt{-2})$, $K_2 = k(\sqrt{-1})$. We see $|\mu_{K_i}|$ equals 2^i and the prime ideal (2) of $\boldsymbol{Q}$ is ramified in K_i. Let $\mathfrak{p} = (2, \sqrt{2p})$ be the prime ideal of k dividing 2. We observe every exponents in (7) and (8) are even and the ideal $\mathfrak{m}$ or $\mathfrak{a}$ in (*) is coincided with $\mathfrak{p}$. Put $\alpha_1 = \sqrt{-2}$ and $\alpha_2 = \sqrt{-1} - 1$. We see $\mathfrak{p} = (\alpha_i)$ in K_i. Let ζ' be an eighth root of unity. It follows from Theorem 6 that $q_{K_i} = 2$ if and only if $\mathfrak{p}$ is principal, because of $\boldsymbol{Q}_2(\sqrt{-2}, \sqrt{2p}) = \boldsymbol{Q}_2(\sqrt{-1}, \sqrt{2p}) = \boldsymbol{Q}_2(\zeta')$.

The genus number of $k/\boldsymbol{Q}$ is

$$\frac{2^2}{2|E_{\boldsymbol{Q}} : E_{\boldsymbol{Q}} \cap N_{k/\boldsymbol{Q}} U_k|}.$$

Since $p \equiv 1 \mod 8$, -1 is locally norm and the genus number is two. We study the unit knot $\omega_{k/\boldsymbol{Q}} = E_{\boldsymbol{Q}} \cap N_{k/\boldsymbol{Q}} k^\times / N_{k/\boldsymbol{Q}} E_k$ with respect to the extension $k/\boldsymbol{Q}$. Let $\mathfrak{q}$ be the another prime ideal of k ramifiyng in $k/\boldsymbol{Q}$. Note $\mathfrak{q} = (p, \sqrt{2p})$ and $\mathfrak{p}\mathfrak{q} = (\sqrt{2p})$. Denote by $B_{k/\boldsymbol{Q}}$ the subgroup of the ideal class group of k generated by $cl(\mathfrak{p})$ and $cl(\mathfrak{q})$. We have $\omega_{k/\boldsymbol{Q}} = Cl_k^G / B_{k/\boldsymbol{Q}}$, where G is the Galois group of $k/\boldsymbol{Q}$. Since the order of Cl_k^G is two by the genus number formula, we observe the order of $B_{k/\boldsymbol{Q}}$ is one if and only if $\omega_{k/\boldsymbol{Q}} \neq \{1\}$. Our proof is concluded by the observation that an eqality $|\omega_{k/\boldsymbol{Q}}| = (3 - N_{k/\boldsymbol{Q}}\varepsilon_k)/2$ for the foundamental unit ε_k holds.

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