

# Brauer's class number relation for the S-ideal class number of an algebraic number field

メタデータ	言語: English 出版者: 公開日: 2017-10-03 キーワード (Ja): キーワード (En): 作成者: Yamashita, Hiroshi メールアドレス: 所属:
URL	<a href="http://hdl.handle.net/2297/23749">http://hdl.handle.net/2297/23749</a>

# Brauer's class number relation for the $S$ -ideal class number of an algebraic number field

Hiroshi YAMASITA

**Abstract.** Let  $K/k$  be a Galois extension of algebraic number fields with a Galois group  $G$ . Let  $\Gamma$  be the set of all the subgroups of  $G$ . Let  $S$  be a finite set of prime ideals of  $k$ . Denote by  $h_S(H)$  the  $S$ -class number of  $K^H$ . Let  $\psi_H$  be the induced character from the trivial character of  $H$ . If a  $\mathbf{Z}$ -linear combination  $\sum_{H \in \Gamma} n_H \psi_H$  equals 0, we shall show a formula giving the value of  $\delta(h_S) = \prod_{H \in \Gamma} h_S(H)^{n_H}$  (Brauer's class number relation) and shall study its applications when  $G$  is an abelian  $p$ -group for a prime number  $p$ .

**1. Introduction.** The trivial character  $1_H$  of the subgroup of a finite group  $G$  induces a character of  $G$ , which is called an induced character. We denote this character by  $\psi_H$ . More concretely, it is the character afforded with a  $\mathbf{Q}[G]$ -module  $\mathbf{Q}[G/H] = \mathbf{Q}[G] \otimes_{\mathbf{Q}[H]} \mathbf{Q}$ . If a  $\mathbf{Z}$ -linear combination of  $\psi_H$  is equal to 0 as a function on  $G$ , we call a relation

$$(1) \quad \sum_{H \in \Gamma} n_H \psi_H = 0,$$

a character relation, where  $\Gamma$  is the set consisting of every subgroup of  $G$ . We are interested in this relation if it is non-trivial. Let  $\psi_+$  (resp.  $\psi_-$ ) be the partial sum of  $n_H \psi_H$  such that  $n_H > 0$  (resp.  $n_H < 0$ ). We have  $\psi_+ = \psi_-$ .

We suppose  $G$  is the Galois group of a Galois extension  $K/k$  of algebraic number

fields. Let  $h(H)$  denote the class number of the intermediate field corresponding to a subgroup  $H$ . Then, associated with the character relation (1), we define  $\delta(h)$  by

$$\delta(h) = \prod_{H \in \Gamma} h(H)^{n_H}.$$

The class number relation with respect to (1) is a formula describing the value  $\delta(h)$ , coming from Artin's  $L$ -functions  $L(s, \psi_H)$ 's. Namely, it is well-known that  $L(s, \psi_H)$  coincides with the Dedekind's zeta function  $\zeta_{K^H}(s)$  and has a multiplicative property

$$L(s, \psi_H + \psi'_H) = L(\psi_H)L(\psi'_H),$$

c.f. [8, Chapter 0]. Therefore, we obtain a relation of Artin's  $L$ -function, and further obtained that of zeta functions. The class number relation yields by taking residue at

$s = 1$ , *c.f.* [5, §1]. This class number relation contains a term concerning regulators of subfields  $K^H$ 's. This term can be removed. For instance, when  $k = \mathbf{Q}$ , there is a unit  $\epsilon$  of  $K$  such that  $\mathbf{Z}[G]\epsilon \cong \mathbf{Z}[G]/\mathbf{Z}s_G$  holds for  $s_G = \sum_{\sigma \in G} \sigma$ . Thus, the unit group  $E$  of  $K$  contains a subgroup  $M$  which is isomorphic to  $\mathbf{Z}[G]/\mathbf{Z}s_G$ . In this case, the following formula of  $\delta(h)$  was obtained:

$$(2) \quad \delta(h) = \prod_{H \in \Gamma} ([E^H : M^H][G : H])^{n_H}$$

*c.f.* [9, Theorem 4.1]. It was proved in [9] that this formula is also valid for an arbitrary Galois extension  $K/k$  in [9].

On the other hand, another generalization was showed in [5]. Let  $S$  be a finite set of prime ideals of  $k$ . Denote by  $S(H)$  the set of every primes of  $K^H$  lying above every primes contained in  $S$ . The  $S$ -ideal class group of  $K^H$  is the quotient group of the ideal class group of  $K^H$  by a subgroup generated by every prime ideals contained in  $S(H)$ . Denote by  $h_S(H)$  the order of the  $S$ -ideal class group. Then, it was shown in [5, Theorem 2.7] that a class number relation holds for the  $S$ -class number. However, it contains terms concerning  $S$ -regulators. The aim of the present paper is transform this class number relation to the similar formula describing  $\delta(h_S)$  as (2) by applying the theory of hermitian  $\mathbf{Z}[G]$ -modules developed in [5]. The formula is given in Theorem 3 in §4 below. In §5, we obtain a special character relation for abelian  $p$ -group

$G$ , where  $p$  is a prime number. This relation is a generalization of the character relation for  $G = (\mathbf{Z}/p\mathbf{Z})^m$  studied in [10]. In §6, we give two examples of class number relations deducing from this character relation.

**2. A symmetric  $\mathbf{Z}[G]$ -module.** Let  $\mathbf{Z}[G]$  be the group ring of a finite group  $G$  over the ring  $\mathbf{Z}$  of integers. A finitely generated torsion free  $\mathbf{Z}[G]$ -module is called a  $\mathbf{Z}[G]$ -lattice. The contragredient module of a  $\mathbf{Z}[G]$ -lattice  $M$  is a  $\mathbf{Z}[G]$ -lattice  $\text{Hom}(M, \mathbf{Z})$ . We identify  $M^{**}$  to  $M$  canonically, *c.f.* [2, §10.D]. Let  $V$  be the  $\mathbf{R}[G]$ -module obtained by extension of coefficients to the field  $\mathbf{R}$  of real numbers:  $V = M \otimes \mathbf{R}$ . The  $\mathbf{R}$ -contragredient  $V^*$  is the dual space as an  $\mathbf{R}$ -linear space. An  $\mathbf{R}[G]$ -homomorphism  $h_V : V \rightarrow V^*$  defines a  $G$ -invariant bilinear  $\mathbf{R}$ -form on  $V \times V$ , which is given by

$$(3) \quad \langle u, v \rangle = h_V(u)(v), \quad u, v \in V.$$

This form is non-degenerate if and only if  $h_V$  is an  $\mathbf{R}$ -isomorphism. Conversely, if  $V$  has a  $G$ -invariant form, an  $\mathbf{R}[G]$ -homomorphism  $h_V$  is defined by (3). This notion was generalized to  $\mathbf{Z}[G]$ -modules in [5]. Let  $M$  be a finitely generated  $\mathbf{Z}[G]$ -module. We denote by  $M_{\text{tor}}$  the maximal torsion submodule. We see the quotient module  $\bar{M} = M/M_{\text{tor}}$  is a  $\mathbf{Z}[G]$ -lattice. So we obtain an  $\mathbf{R}[G]$ -module  $V = \bar{M} \otimes \mathbf{R}$ , which contains  $\bar{M}$  as a full sublattice. Since an isomorphism of  $M \otimes \mathbf{R}$  onto  $V$  is induced from the canonical map  $\bar{i} : M \rightarrow \bar{M}$ ,

we identify  $M \otimes \mathbf{R}$  with  $V$  by this isomorphism. If there is a  $\mathbf{Z}[G]$ -homomorphism  $h : M \rightarrow V^*$ ,  $\bar{i}$  factors  $h$ . Take the map  $\bar{h} : \bar{M} \rightarrow V^*$  so that  $h = \bar{h} \circ \bar{i}$  holds. Thus, an  $\mathbf{R}[G]$ -homomorphism  $\bar{h}_V : V \rightarrow V^*$  is yielded. A bilinear  $\mathbf{R}$ -form is obtained by means of (3) from  $\mathbf{Z}[G]$ -lattice  $\bar{M}$ . We abuse notation and denote by  $h(u, v)$  this  $G$ -invariant bilinear  $\mathbf{R}$ -form on  $V$ . According to [5, Definition 2.1], the pair  $(M, h)$  is called an  $\mathbf{R}$ -valued hermitian  $\mathbf{Z}[G]$ -module. However, we say  $(M, h)$  is an  $\mathbf{R}$ -valued symmetric  $\mathbf{Z}[G]$ -module or a symmetric  $\mathbf{Z}[G]$ -module in short, because we study the case that the  $\mathbf{R}$ -form  $h(u, v)$  is a symmetric form. We note that this form is non-degenerate if and only if  $\bar{h}$  is injective. Let  $r$  be the rank of  $\bar{M}$ . If  $\bar{h}$  is injective, the Gram matrix

$$(h(m_i, m_j))_{1 \leq i, j \leq r}$$

is defined for an arbitrary  $\mathbf{Z}$ -basis  $\{m_i : 1 \leq i \leq r\}$  of  $\bar{M}$ . The discriminant of the symmetric  $\mathbf{Z}[G]$ -module  $(\bar{M}, \bar{h})$  is defined to be the absolute value of the determinant of the Gram matrix, and is denoted by  $\text{disc}(\bar{M}, \bar{h})$ . In general, the discriminant of a symmetric  $\mathbf{Z}[G]$ -module  $(M, h)$  is defined to be

$$(4) \quad \text{disc}(M, h) = \frac{\text{disc}(\bar{M}, \bar{h})}{|M_{\text{tor}}|},$$

*c.f.* [5, Definition 2.3]. A morphism of a symmetric  $\mathbf{Z}[G]$ -module  $(M_1, h_1)$  into  $(M_2, h_2)$  is a  $\mathbf{Z}[G]$ -homomorphism  $\sigma : M_1 \rightarrow M_2$  satisfying the relation  $h_1(u, v) = h_2(\sigma u, \sigma v)$  for  $u, v \in V_1^*$ , where  $V_1 = M_1 \otimes$

$\mathbf{R}$ . If  $\sigma$  is an isomorphism, we call it an isometry and say that  $(M_1, h_1)$  is isometric to  $(M_2, h_2)$ . A direct sum and a tensor product of two modules are defined by means of functorial isomorphisms:

$$\begin{aligned} (V_1 \oplus V_2)^* &\cong V_1^* \oplus V_2^* \\ (V_1 \otimes_{\mathbf{R}} V_2)^* &\cong \text{Hom}_{\mathbf{R}}(V_2, V_1^*) \\ &\cong V_2^* \otimes_{\mathbf{R}} V_1^* \cong V_1^* \otimes_{\mathbf{R}} V_2^* \end{aligned}$$

*c.f.* [2, Proposition 10.30]. We identify  $(V_1 \oplus V_2)^*$  (*resp.*  $(V_1 \otimes_{\mathbf{R}} V_2)^*$ ) with  $V_1^* \oplus V_2^*$  (*resp.*  $V_1^* \otimes_{\mathbf{R}} V_2^*$ ) by these isomorphisms.  $h_1$  and  $h_2$  induce  $\mathbf{Z}[G]$ -homomorphisms

$$\begin{aligned} h_1 \oplus h_2 &: M_1 \oplus M_2 \rightarrow V_1^* \oplus V_2^* \\ h_1 \otimes h_2 &: M_1 \otimes M_2 \rightarrow V_1^* \otimes_{\mathbf{R}} V_2^*. \end{aligned}$$

These symmetric  $\mathbf{Z}[G]$ -modules is denoted by  $(M_1 \oplus M_2, h_1 \oplus h_2)$  and  $(M_1 \otimes M_2, h_1 \otimes h_2)$ , respectively. Let  $H$  be a subgroup of  $G$ .  $\bar{h}_V$  maps the submodule  $V^H$  of  $H$ -invariant elements into  $V^{*H}$ . We have  $(V^H)^* = (V^*)^H$  by [2, Proposition 10.28]. Since  $\bar{M}^H$  is a submodule  $(\bar{M})^H$  of finite index, we have  $\overline{M^H} \otimes \mathbf{R} = (\bar{M})^H \otimes \mathbf{R} = V^H$ . Thus, if we define a homomorphism  $h^H$  by

$$\frac{1}{|H|} h : M^H \rightarrow V^{*H},$$

the pair  $(M^H, h^H)$  is a symmetric  $\mathbf{Z}[\{1\}]$ -module. Denote this symmetric  $\mathbf{Z}$ -module by  $(M, h)^H$  in short, *c.f.* [5, Notation 4.8].

The group ring is provided with involution

$$\left( \sum_{\sigma \in G} a_{\sigma} \sigma \right)^* = \sum_{\sigma \in G} a_{\sigma} \sigma^{-1}.$$

We have  $(xy)^* = y^*x^*$  for product of two elements  $x$  and  $y$  of the group ring. A non-degenerate  $G$ -invariant symmetric bilinear  $\mathbf{R}$ -form  $\langle x, y \rangle$  on  $\mathbf{R}[G]$  is defined from the trivial character  $1_G$  of  $G$ :

$$(5) \quad \langle x, y \rangle = 1_G(y^*x)$$

Hereafter, we denote by the symbol  $V$  this symmetric  $\mathbf{R}$ -space  $\mathbf{R}[G]$ .  $V$  is self-dual, that is  $V^* = V$ . We denote a sum of every element contained in a subset  $A$  of  $G$  by  $s_A$  or  $s(A)$ . An idempotent element associated to the subgroup  $H$  in the group ring  $\mathbf{R}[G]$  is defined to be

$$(6) \quad e_H = \frac{1}{|H|} s_H.$$

Denote by  $\mathbf{Z}[G]e_H$  a  $\mathbf{Z}[G]$ -submodule generated by  $e_H$ . Put  $V_H = \mathbf{R}[G]e_H$ . We define a non-degenerate  $G$ -invariant symmetric bilinear  $\mathbf{R}$ -form  $h_H$  by

$$h_H(u, v) = |H| \langle u, v \rangle$$

on  $V_H$ . We see  $h_H(\sigma e_H, \sigma e_H) = 1$  and  $h_H(\sigma e_H, \tau e_H) = 0$  if  $\sigma e_H \neq \tau e_H$ . Thus, we have  $V_H$  is self-dual with respect to  $h_H$ . The form  $h_H$  is considered it is induced from inclusion  $\mathbf{Z}[G]e_H \rightarrow V_H$ . The inclusion map gives a symmetric  $\mathbf{Z}[G]$ -module structure. We also denote this structure by  $h_H$ :

$$(7) \quad h_H : \mathbf{Z}[G]e_H \rightarrow V_H = (V_H)^*.$$

Moreover, since  $\{\sigma e_H\}$  is a  $\mathbf{Z}$ -basis of  $\mathbf{Z}[G]e_H$ , the symmetric  $\mathbf{Z}[G]$ -module

$(\mathbf{Z}[G]e_H, h_H)$  is unimodular, c.f. [5, Notation 5.14].

The following lemma is a consequence from Corollary 4.14 in [5]. We shall give an elementary proof following to the proof of Proposition 10.31 in [2].

LEMMA 1. *Let  $(M, h)$  be an arbitrary non-degenerate symmetric  $\mathbf{Z}[G]$ -module. Then, we have an isometry*

$$(\mathbf{Z}[G]e_H \otimes M, h_H \otimes h)^G \rightarrow (M, h)^H.$$

*Proof.* Let  $[G/H]$  be the complete set of representatives of right cosets. The set  $\{\sigma e_H : \sigma \in [G/H]\}$  is a  $\mathbf{Z}$ -basis of the free  $\mathbf{Z}$ -module  $\mathbf{Z}[G]e_H$ . Thus, each element  $x$  is written uniquely as a sum

$$x = \sum_{\sigma \in [G/H]} \sigma e_H \otimes m_{\bar{\sigma}}, \quad m_{\bar{\sigma}} \in M,$$

where  $\bar{\sigma}$  denotes the right coset  $\sigma H$ . If  $x$  is  $G$ -invariant, we see

$$gx = \sum_{\sigma} g\sigma e_H \otimes gm_{\bar{\sigma}} = x$$

for every  $g \in G$ . Since the coefficient  $m_{\bar{\sigma}}$  of each  $\sigma$  is uniquely determined for  $x$ , we have  $m_{\bar{g}} = gm_{\bar{1}}$ . In particular, if we set  $g \in H$ , we have  $m_{\bar{1}} = gm_{\bar{1}}$ . Thus, by sending  $x \in (\mathbf{Z}[G]e_H \otimes M)^G$  to  $m_{\bar{1}} \in M^H$ , an injective mapping is defined. It is easy to verify this mapping is a surjective homomorphism. Therefore,  $(\mathbf{Z}[G]e_H \otimes M)^G \cong M^H$  as  $\mathbf{Z}$ -modules. We shall show this isomorphism is an isometry. Let  $x$  and  $y$  be two

elements of  $(\mathbf{Z}[G]e_H \otimes M)^G$ :

$$\begin{aligned} x &= \sum_{\sigma \in G/H} \sigma e_H \otimes \sigma m, \\ y &= \sum_{\sigma \in G/H} \sigma e_H \otimes \sigma n \end{aligned}$$

for  $m, n \in M^H$ . Denote by  $\bar{m}$  and  $\bar{n}$  the images into  $\bar{M} \otimes \mathbf{R}$ . We have

$$\begin{aligned} \frac{1}{|G|} \cdot h_H \otimes h(x)(y) &= \frac{1}{|G|} \sum_{\sigma, \tau \in G/H} h_H(\sigma e_H)(\tau e_H) \cdot h(\sigma \bar{m})(\tau \bar{n}). \\ &= \frac{1}{|G|} \sum_{\sigma \in G/H} h(\sigma \bar{m})(\sigma \bar{n}) \\ &= \frac{1}{|H|} h(\bar{m})(\bar{n}). \end{aligned}$$

This shows the isomorphism is an isometry.  $\square$

The subset consisting of every  $H$  such that  $n_H > 0$  (resp.  $n_H < 0$ ) is denoted by  $\Gamma_+$  (resp.  $\Gamma_-$ ). Associated to these subsets, we define  $\mathbf{Z}[G]$ -modules  $M_{\pm}$  to be  $M_{\pm} = \bigoplus_{H \in \Gamma_{\pm}} (\mathbf{Z}[G]e_H)^{|n_H|}$ . The non-degenerate symmetric  $\mathbf{Z}[G]$ -module structures are defined on  $M_{\pm}$  by

$$(8) \quad (M_{\pm}, h_{\pm}) = \bigoplus_{H \in \Gamma_{\pm}} (\mathbf{Z}[G]e_H, h_H)^{n_H}$$

from (7). Since the character relation (1) asserts there is an  $\mathbf{Q}[G]$ -isomorphism

$$M_+ \otimes \mathbf{Q} \cong M_- \otimes \mathbf{Q},$$

there is an injective  $\mathbf{Z}[G]$ -homomorphism of  $M_-$  into  $M_+ \otimes \mathbf{Q}$ . Put  $V_{\pm} = M_{\pm} \otimes \mathbf{R}$ . Let  $j$  be an  $\mathbf{R}[G]$ -isomorphism of  $V_-$  into  $V_+$  obtained from this  $\mathbf{Z}[G]$ -homomorphism. Let

$j^*$  be the adjoint of  $j$  with respect to  $\mathbf{R}$ -forms on  $V_{\pm}$ . Namely,  $j^*$  is defined by

$$(9) \quad h_+(j(u), v) = h_-(u, j^*(v)).$$

$j^*$  is an  $\mathbf{R}[G]$ -isomorphism of  $V_+$  onto  $V_-$ .

By [5, §5.4], the fundamental invariant  $\delta(M_+, M_-; M)$  is defined for an arbitrary non-degenerate symmetric  $\mathbf{Z}[G]$ -module  $(M, h)$ . We write it as  $\delta(M, h)$  or  $\delta(M)$  in short. The following discriminant relation holds from [5, Theorem 6.1]:

$$(10) \quad \delta(M, h) = \frac{\text{disc}((M_+ \otimes M, h_+ \otimes h)^G)}{\text{disc}((M_- \otimes M, h_- \otimes h)^G)}.$$

If we define a function  $f$  on  $\Gamma$  by  $f(H) = \text{disc}((M, h)^H)$ , we have by virtue of Lemma 1 a formula

$$\delta(M) = \prod_{H \in \Gamma} f(H)^{n_H}.$$

We generalize this notion to an arbitrary function  $f$  taking values in non-zero real numbers. We define a functional  $\delta$  on such  $f$ 's to be

$$\delta(f) = \prod_{H \in \Gamma} f(H)^{n_H}.$$

This functional is multiplicative. When  $f$  is a constant function, we see  $\delta(f) = f(1)^{\sum n_H}$ . However, this value  $\delta(f)$  equals to 1, because we have

$$\begin{aligned} \langle 1_G, \sum_{H \in \Gamma} n_H \psi_H \rangle_G &= \sum_{H \in \Gamma} n_H \langle 1_G, \psi_H \rangle_G \\ &= \sum_H n_H \langle 1_H, 1_H \rangle_H \end{aligned}$$

from the Frobenius reciprocity law, *c.f.* [1, (4)].

REMARK 1. If  $M$  is a  $\mathbf{Z}[G]$ -module of finite order, it has a trivial symmetric  $\mathbf{Z}[G]$ -module structure, because of  $M \otimes \mathbf{R} = 0$ . We denote this structure by  $(M, *)$ . Note

$$\delta(M, *) = \prod_{H \in \Gamma} \frac{1}{|M^H|^{n_H}}$$

from the definition (4).

**3. The group of  $S$ -units.** We assume  $G$  is the Galois group of a finite Galois extension  $K/k$  of algebraic number fields. Let  $S$  be a finite set of places of  $k$  containing all the archimedean places. Denote by  $S_0$  the subset of every non-archimedean places. Suppose  $S = \{v_1, \dots, v_s\}$ . We choose a prolongation onto  $K$  of each  $v_i$  and fix it once for all. Denote by  $w_i$  the selected place. Let  $G_i$  be the decomposition group of  $w_i$ . Every place of  $K^H$  lying above  $v_i$  is obtained from the decomposition into two sided cosets:

$$G = \dot{\cup}_{j=1}^{s_i} H\sigma_{ij}G_i.$$

Let  $u_i$  be restriction of  $w_i$  onto  $K^H$ . There are  $s_i$  places  $\sigma_{ij}u_i$ ,  $j = 1, \dots, s_i$  over  $v_i$ . Denote by  $S_0(K^H)$  the union of all such places for every  $v_i \in S_0$ . Let  $\mathcal{P}_k$  be the set of the all places of  $k$ . Denote by  $|\cdot|_v$  be the normalized multiplicative valuation for  $v \in \mathcal{P}_k$  so that the product formula holds. Namely,

$$\prod_{v \in \mathcal{P}_k} |x|_v = 1$$

holds for every  $x \in k^\times$ . Further, we associate a multiplicative valuation  $\|\cdot\|_{w_i}$  to each  $w_i$  so that the value  $\|x\|_{w_i}$  for every

$x \in k$  agrees to the value  $|x|_{v_i}$ . Denote by  $h_S(H)$  the order of the  $S$ -ideal class group of  $K^H$ .  $h_S(H)$  is a function on  $\Gamma$ . An element  $x \in K$  is called an  $S$ -unit if an arbitrary prime divisor of the principal ideal  $(x)$  belongs to the set of valuation ideals of places contained in  $S_0(K)$ . The subgroup of  $K^\times$  generated by every  $S$ -unit of  $K$  is called the group of  $S$ -units of  $K$  and is denoted by  $E_S$ . We shall give two non-degenerate symmetric  $\mathbf{Z}[G]$ -module structures on  $E_S$ . We abbreviate the idempotent  $e_{G_i}$  defined by (6) to  $e_i$ . Put

$$L_S = \bigoplus_{i=1}^s \mathbf{Z}[G]e_i,$$

$$V_S = \bigoplus_{i=1}^s V e_i.$$

A non-degenerate symmetric  $\mathbf{R}$ -form on  $V_S$  is defined by

$$(11) \quad \left\langle \sum_{i=1}^s u_i, \sum_{i=1}^s v_i \right\rangle = \sum_{i=1}^s h_{G_i}(u_i, v_i).$$

The inclusion map of  $L_S$  into  $V_S$  is given by

$$h_S = \bigoplus_{i=1}^s h_{G_i} : L_S \rightarrow V_S = V_S^*,$$

which is a non-degenerate symmetric  $G$ -invariant  $\mathbf{Z}[G]$ -module structure on  $L_S$ . Let  $[G/G_i]$  be a complete set of representatives of  $G/G_i$ . Put  $\alpha_i = s([G/G_i])$ .  $L_S^G$  is a free  $\mathbf{Z}$ -module on a basis  $\{\alpha_i e_i : 1 \leq i \leq s\}$ . Put  $\eta = (\alpha_1 e_1, \dots, \alpha_s e_s) \in L_S^G$ .  $V_S^G$  contains a one-dimensional subspace generated by  $\eta$ . Since  $L_S \cap V_\eta = \mathbf{Z}\eta$ , there is an injective  $\mathbf{Z}[G]$ -homomorphism  $L_S/\mathbf{Z}\eta \rightarrow$

$V_S/V_\eta$ . Moreover,  $V_\eta$  has an orthogonal homomorphism defined to be complement  $V_{S,1}$  in  $V_S$ :

$$V_{S,1} = \{u \in V_S : \langle \eta, u \rangle = 0\}$$

with respect to (11). We observe

$$\left( \sum_{i=1}^s \sum_{\sigma_i \in [G/G_i]} a_{\sigma_i \sigma_i e_i, \eta} \right) = \sum_{i=1}^s \sum_{\sigma_i \in [G/G_i]} a_{\sigma_i}.$$

Thus, if we define  $|u| = \langle u, \eta \rangle$ , we see  $u \in V_{S,1}$  is equivalent to  $|u| = 0$ . We consider  $V_{S,1}$  as a symmetric space by restricting the  $\mathbf{R}$ -form on  $V_S$ . Since this symmetric form is  $G$ -invariant, we have  $V_{S,1}^* = V_{S,1}$  as  $\mathbf{R}[G]$ -modules with respect to the symmetric form. Let  $h_{S,1}$  be the composite map of the canonical map  $V_S/V_\eta \rightarrow V_{S,1}$  induced from the projection onto  $V_{S,1}$  and the homomorphism of  $L_S/\mathbf{Z}\eta$  into  $V_S/V_\eta$ . Put  $L_{S,1} = L_S/\mathbf{Z}\eta$ . The pair  $(L_{S,1}, h_{S,1})$  is a non-degenerate symmetric  $\mathbf{Z}[G]$ -module.

We apply the generalized Dirichlet-Herbrand theorem on  $S$ -units, *c.f.* [4, Theorem I.3.7]. There is a  $\mathbf{Q}[G]$ -isomorphism

$$(12) \quad E_S \otimes \mathbf{Q} \rightarrow L_{S,1} \otimes \mathbf{Q}.$$

Thus,  $E_S \otimes \mathbf{R} \cong V_{S,1}$ . Since  $E_S$  is mapped into  $E_S \otimes \mathbf{R}$  by  $x \rightarrow x \otimes 1$ , there is a  $\mathbf{Z}[G]$ -homomorphism  $h$  of  $E_S$  into  $V_{S,1}$ . This makes  $E_S$  a non-degenerate symmetric  $\mathbf{Z}[G]$ -module.

$E_S$  is provided with another non-degenerate symmetric  $\mathbf{Z}[G]$ -module structure. Let  $l : E_S \rightarrow V_S$  be a  $\mathbf{Z}[G]$ -

$$l(u) = \left( \sum_{\sigma \in G} \log \|\sigma^{-1}u\|_{w_i} \sigma e_i \right)_{1 \leq i \leq s}.$$

We see

$$|l(u)| = \log \left( \prod_{i=1}^s \prod_{\sigma_i \in [G/G_i]} \|\sigma_i^{-1}u\|_{w_i}^{|G_i|} \right).$$

The product formula of the multiplicative valuations normalized to the algebraic number field  $K$  asserts this value is equal to 0. Hence,  $l$  takes values in  $V_{S,1}$ . Since  $S$  contains every archimedean place,  $\text{Ker } l = E_{\text{tor}}$ . Thus,  $\mathbf{Z}[G]$ -module  $(E_S, l)$  is a non-degenerate symmetric. We compute  $\langle l(u), l(v) \rangle$  and obtain

$$(13) \quad \langle l(u), l(v) \rangle = \sum_{i=1}^s \sum_{\sigma_i \in [G/G_i]} \log \|\sigma_i^{-1}u\|_{w_i} \log \|\sigma_i^{-1}v\|_{w_i} |G_i|.$$

This shows  $\langle l(u), l(v) \rangle$  coincides with the form  $\rho_S(u, v)$  defined in [5, (8.1)]. We restate here the following formula obtained in [5, Theorem 2.7]:

**THEOREM 2 (Kani).** *Let  $w$  be the function on  $\Gamma$  defined to be  $w(H) = |E_{S, \text{tor}}^H|$ . Then, we have*

$$\delta(h_S)^2 = \frac{\delta(\mathbf{Z})\delta(w)}{\delta(E_S, l)\delta(L_S)}.$$

**REMARK 2.** If  $H$  is cyclic, we have  $\delta(\mathbf{Z}[G]e_H) = 1$  by [5, Example 2.13. a)]. Therefore,  $\delta(\mathbf{Z}[G]e_i) = 1$  if  $v_i$  is archimedean. We see  $\delta(L_S) = \delta(L_{S_0})$ .

REMARK 3. We define a function  $n_G$  on  $\Gamma$  to be  $n_G(H) = |G : H|$ . We have

$$\delta(\mathbf{Z}) = \prod_{H \in \Gamma} |H|^{-n_H} = \delta(n_G),$$

c.f. [5, (2.7)].

REMARK 4. Let  $w_2$  be the 2-part of  $w$ . We have  $\delta(w) = \delta(w_2)$  from [1, §2.5].

**4. Brauer's class number relations.**  $V_{S,1}^* = V_{S,1}$  contains a  $\mathbf{Z}[G]$ -lattice isomorphic to  $L_{S,1} = L_S/\mathbf{Z}\eta$ . The inverse image  $M'$  by  $h : E_S \rightarrow V_{S,1}^*$  of the lattice is a submodule containing  $E_{S,tor}$ . Since  $\text{Ker } h = E_{S,tor}$ ,  $M'^{|\mathbf{Z}[G]|}$  is torsion free and is isomorphic to  $L_{S,1}$ . Hence,  $E_S$  contains a  $\mathbf{Z}[G]$ -submodule isomorphic to  $L_{S,1}$ . Let  $M_S$  be an arbitrary such  $\mathbf{Z}[G]$ -submodule. By restricting the two symmetric  $\mathbf{Z}[G]$ -module structures of  $E_S$ ,  $M_S$  is also provided with two structures. We denote them by  $(M_S, h)$  and  $(M_S, l)$ , respectively. We shall prove the following class number relation holds:

THEOREM 3. We define a function  $i_{E_S, M_S}$  on  $\Gamma$  to be  $i_{E_S, M_S}(H) = [E_S^H : M_S^H]$ . Then, we have

$$\delta(h_S) = \frac{\delta(i_{E_S, M_S})\delta(n_G)}{\delta(L_{S_0})}.$$

This theorem is a generalization to  $S$ -class numbers of Brauer's class number relation proved in [9, Theorem 4.1]. The key of the proof is the following lemma:

LEMMA 4.  $\delta(M_S, h) = \delta(M_S, l)$ .

*Proof.* Let  $\iota$  (resp.  $\iota_{\pm}$ ) be identity map (resp. identity maps) on  $V_{S,1}^*$  (resp.  $V_{\pm}^*$ ). Since the adjoint map  $j^*$  in (9) is an isomorphism,  $j^* \otimes \iota$  is an isomorphism of  $V_+^* \otimes_{\mathbf{R}} V_{S,1}^*$  onto  $V_-^* \otimes_{\mathbf{R}} V_{S,1}^*$ . Denote by  $(j^* \otimes \iota)(G)$  restriction of  $(j^* \otimes \iota)$  on the  $G$ -invariant submodules. Let  $\alpha$  be an automorphism on  $V_{S,1}^*$  which is induced from an isomorphism  $l \circ h^{-1} : h(M_S) \rightarrow l(M_S)$  of sublattices. We abbreviate  $\iota_{\pm} \otimes \alpha$  to  $\alpha_{\pm}$  in short and denote by  $\alpha_{\pm}(G)$  restriction onto  $(V_{\pm}^* \otimes_{\mathbf{R}} V_{S,1}^*)^G$ . We have

$$\alpha_+(G) = (j^* \otimes \iota)(G)^{-1} \circ \alpha_-(G) \circ (j^* \otimes \iota)(G).$$

Thus,  $\det \alpha_+(G) = \det \alpha_-(G)$ . Concerning two symmetric  $\mathbf{Z}[G]$ -module structures  $(h_{\pm} \otimes h)^G$  and  $(h_{\pm} \otimes l)^G$  on  $(M_{\pm} \otimes M_S)^G$ , we have the following commutative diagram:

$$\begin{array}{ccc} (M_{\pm} \otimes M_S)^G & \xrightarrow{(h_{\pm} \otimes h)^G} & (V_{\pm}^* \otimes V_{S,1}^*)^G \\ \downarrow id & & \downarrow \alpha_{\pm}(G) \\ (M_{\pm} \otimes M_S)^G & \xrightarrow{(h_{\pm} \otimes l)^G} & (V_{\pm}^* \otimes V_{S,1}^*)^G \end{array}$$

Thus, a relation between the Gram matrices

$$\begin{aligned} \text{disc}((M_{\pm} \otimes M_S, h_{\pm} \otimes l)^G) &= \\ (\det \alpha_{\pm}(G))^2 \text{disc}((M_{\pm} \otimes M_S, h_{\pm} \otimes h)^G). \end{aligned}$$

is obtained. Hence, it follows  $\delta(M_S, h) = \delta(M_S, l)$  from (10).  $\square$

*Proof of Theorem 3.* The quotient module  $E_S/M_S$  is of finite order. It is a trivial non-degenerate symmetric  $\mathbf{Z}[G]$ -module  $(E_S/M_S, *)$ . Thus, we have an exact se-

quence in the category of symmetric  $\mathbf{Z}[G]$ -modules: Therefore, we obtain

$$(14) \quad 1 \rightarrow (M_S, h) \rightarrow (E_S, h) \rightarrow (E_S/M_S, *) \rightarrow 1.$$

$$(16) \quad \frac{1}{[E_S^H : M_S^H]} = \frac{|\mathrm{Im} \delta_H|}{|(E_S/M_S)^H|}.$$

and an auxiliary formula

$$(17) \quad \delta(E_S/M_S)\psi = \delta(i_{E_S, M_S})^{-1}$$

For each  $H \in \Gamma$ , the following sequence is exact:

$$1 \rightarrow \mathbf{Z}[G]e_H \otimes M_S \rightarrow \mathbf{Z}[G]e_H \otimes E_S \rightarrow \mathbf{Z}[G]e_H \otimes E_S/M_S \rightarrow 1.$$

$$(18) \quad \delta(L_S) = \delta(n_G)\delta(M_S, h)$$

Moreover, since  $L_S/\mathbf{Z}\eta \cong M_S$ , we also have

We have a cohomology long exact sequence

$$1 \rightarrow (\mathbf{Z}[G]e_H \otimes M_S)^G \rightarrow (\mathbf{Z}[G]e_H \otimes E_S)^G \rightarrow (\mathbf{Z}[G]e_H \otimes E_S/M_S)^G \xrightarrow{\delta_H} H^1(G, \mathbf{Z}[G]e_H \otimes M_S)$$

from "exact sequence formula". Combining (15), (17) and (18), we have

$$\frac{\delta(E_S, h)}{\delta(w)} = \frac{\delta(L_S)}{\delta(n_G)\delta(i_{E_S, M_S})^2}$$

from this sequence. We can apply "exact sequence formula", c.f. [5, Theorem 6.21]. We have

because of  $\delta(E_{S, \mathrm{tor}}) = \delta(w)^{-1}$ . Moreover, in account of Lemma 4, we can substitute  $\delta(E_S, h)$  for  $\delta(E_S, l)$  in the formula of Theorem 2. In consequence, we have a formula

$$(15) \quad \delta(E_S, h)\delta(E_{S, \mathrm{tor}}) = \delta(M_S, h)\delta(E_S/M_S)^2\psi^2,$$

$$\delta(h_S)^2 = \frac{\delta(n_G)^2\delta(i_{E_S, M_S})^2}{\delta(L_S)^2}.$$

where

$$\psi = \prod_{H \in \Gamma} |\mathrm{Im} \delta_H|^{n_H}.$$

This proves the theorem.  $\square$

We notice that the first three terms in a cohomology long exact sequence

$$1 \rightarrow M_S^H \rightarrow E_S^H \rightarrow (E_S/M_S)^H \xrightarrow{f_H} H^1(H, M_S)$$

We shall give two applications of Theorem 3.

LEMMA 5.  $\delta(h_S)$  is a unit in the ring  $\mathbf{Z}_p$  of  $p$ -adic integers for every prime number  $p$  not dividing  $|G|$ .

are equal to those in the above cohomology exact sequence by virtue of Lemma 1. Hence,  $|\mathrm{Im} \delta_H| = |\mathrm{Im} f_H|$  and

$$\frac{|\mathrm{Im} \delta_H|}{|(E_S/M_S)^H|} = \frac{|\mathrm{Im} f_H|}{|(E_S/M_S)^H|}.$$

*Proof.* Let  $p$  be a prime not dividing  $|G|$ . We see  $\delta(n_G) \in \mathbf{Z}_p^\times$ . By the formula of  $\delta(\mathbf{Z}[G/H])$  in [5, Example 2.1, b)], we have  $\delta(\mathbf{Z}[G]e_i) \in \mathbf{Z}_p^\times$  for  $i = 1, \dots, s$ . Let  $f_0$  be a function on  $\Gamma$  defined by  $f_0(H) =$

$|(E_S/M_S)^H|$ . Since  $p \nmid |H^1(H, M_S)|$ , we have from (16) that  $\delta(i_{E_S, M_S})$  is a  $p$ -adic integer if and only if  $\delta(f_0)$  is also. Let  $Y$  be the  $p$ -primary submodule of  $E_S/M_S$ . Let  $p^m$  be the exponent of  $Y$ . Put  $Y_n = Y^{p^n}$  for  $n = 0, \dots, m$ .  $Y_{n-1}/Y_n$  is an  $\mathbf{F}_p[G]$ -module. Let  $\chi_n$  be the character of  $G$  afforded with an  $\mathbf{F}_p[G]$ -module  $Y_{n-1}/Y_n$ . Let  $\zeta^{(1)}, \dots, \zeta^{(r)}$  be a basic set of irreducible  $\mathbf{Q}_p$ -characters of  $G$ , where  $\mathbf{Q}_p$  denotes the field of  $p$ -adic numbers. Since  $p \nmid |G|$ , an  $\mathbf{F}_p$ -irreducible character is obtained from each  $\zeta^{(i)}$  by reduction with respect to mod  $p$ . Denote by  $\bar{\zeta}^{(i)}$  the  $\mathbf{F}_p$ -irreducible character.  $\chi_n$  is a linear combination of  $\bar{\zeta}^{(i)}$ 's with non-negative integral coefficients:

$$\chi_n = \sum_{i=1}^r c_i \bar{\zeta}^{(i)}.$$

The dimension of  $(Y_{n-1}/Y_n)^H$  over  $\mathbf{F}_p$  is given by the value of

$$\sum_{i=1}^r c_i \langle \zeta^{(i)}, \psi_H \rangle_G \dim_{\mathbf{F}_p} U_i,$$

where  $U_i$  are simple  $\mathbf{F}_p[G]$ -modules affording the characters  $\zeta^{(i)}$ 's. We see

$$\sum_{H \in \Gamma} n_H \dim_{\mathbf{F}_p} (Y_{n-1}/Y_n)^H = 0$$

from (1). Thus, if we define a function  $f_n$  on  $\Gamma$  by  $f_n(H) = |(Y_{n-1}/Y_n)^H|$ , we have  $\delta(f_n) = 1$ . Since  $f_0(H) = \prod_{n=1}^m f_n(H)$ , we see  $p \nmid \delta(i_{E_S, M_S})$ .  $\square$

**COROLLARY 6.** *Let  $h_S^{(p)}(H)$  be the highest power of  $h_S(H)$  with respect to a prime  $p$ . If  $p \nmid |G|$ , we have  $\delta(h_S^{(p)}) = 1$ .*

We assume  $K$  is a CM-field and  $k$  is a totally real subfield. The Galois group  $G$  contains the complex conjugation map  $\tau$ . By Lemma 8 in the next section, the character relation (1) holds if and only if

$$\sum_{H \in \Gamma} n_H \bar{e}_H = 0$$

holds. Denote by  $H^+$  a subgroup generated by  $H$  and  $\tau$ . Put  $e^+ = \frac{1}{2}(1 + \tau)$ . Since

$$e_{H^+} = e_H e^+,$$

we have  $\sum_{H \in \Gamma} n_H \bar{e}_{H^+} = 0$  from the above idempotent relation. Thus, by Lemma 8, a character relation  $0 = \sum_{H \in \Gamma} n_H \psi_{H^+}$  is yielded. Hence,

$$(19) \quad 0 = \sum_{H \in \Gamma} n_H (\psi_H - \psi_{H^+}).$$

Let  $\Gamma_1$  be a subset of  $\Gamma$  consisting of  $H$  such that  $H \neq H^+$ . We define functions  $f^\pm$  from an arbitrary function  $f$  on  $\Gamma$  to be

$$f^-(H) = \frac{f(H)}{f(H^+)} \quad \text{and} \quad f^+(H) = f(H^+).$$

Then, the functional  $\delta'$  defined from (19) satisfies  $\delta'(f^+) = 1$ ,  $\delta'(f) = \delta'(f^-)$  and

$$\delta(f^-) = \delta'(f) = \prod_{H \in \Gamma_1} f^-(H)^{n_H}.$$

Suppose  $S$  is the set of all the archimedean places. We put  $E_S^+ = E_S^{\leq \tau}$ . Since  $E_S^{2\omega(1)} \subset E_S^+$ , we can choose  $M_S$  from a subgroup of  $E_S^+$ . We observe an index relation

$$i_{E_S, M_S}(H) = [E_S^H : \mu_K^H E_S^{+H}] [\mu_K^H E_S^{+H} : M_S]$$

holds, where  $\mu_K$  is the subgroup of  $E_S$  consisting of every root of unity. Let  $Q$  be a function on  $\Gamma$  whose value is equal to the unit index of  $K^H$  if  $\tau \notin H$  and which takes 1 when  $H = H^+$ . We have

$$i_{E_S, M_S}(H) = \frac{Q(H)w(H)i_{E_S, M_S}(H^+)}{2}.$$

Therefore, we obtain

**COROLLARY 7.** *Let  $K$  be a CM-field which is a Galois extension on a totally real subfield  $k$ . If  $S$  is the set of all the archimedean places, then we have*

$$\delta(h_S) = \delta(Q^-)\delta(w^-)\delta(n_{\bar{G}})$$

with respect to the character relation (1).

**REMARK 5.** Each of  $\delta(Q^-)$ ,  $\delta(w^-)$  and  $\delta(n_{\bar{G}})$  takes a value of an integral power of 2.

**5. Character relations.** The induced character  $\psi_H$  is defined to be

$$\psi_H(\sigma) = \frac{1}{|H|} \sum_{g \in G} \dot{1}_H(g^{-1}\sigma g)$$

where  $\dot{1}_H$  is a function on  $G$  taking value 1 for every element of  $H$  and taking 0 for elements in  $G \setminus H$ , c.f. [2, (10.3)]. Let  $\chi$  be a linear combination of  $\psi_H$  with integral coefficients  $n_H$ . We have

$$\begin{aligned} \sum_{\sigma \in G} \chi(\sigma)\sigma^{-1} &= \sum_{\sigma} \left( \sum_{H \in \Gamma} n_H \psi_H(\sigma) \right) \sigma^{-1} \\ &= \sum_H \frac{n_H}{|H|} \sum_{\sigma} \sum_{g \in G} \dot{1}_H(g^{-1}\sigma g) \sigma^{-1} \\ &= \sum_H \frac{n_H}{|H|} \sum_g \sum_{\sigma \in gHg^{-1}} \sigma^{-1} \\ &= \sum_H n_H \sum_g g e_H g^{-1} \end{aligned}$$

Put  $\tilde{e}_H = \frac{1}{|H|} \sum_{g \in G} e_{gHg^{-1}}$ . We have

$$\sum_{\sigma \in G} \chi(\sigma)\sigma^{-1} = \sum_{H \in \Gamma} n_H \tilde{e}_H.$$

Let  $\{\zeta^{(1)}, \dots, \zeta^{(r)}\}$  be the basic set of irreducible  $\mathbf{C}$ -characters of  $G$ . Put  $z = \sum_{H \in \Gamma} n_H \tilde{e}_H$ . We have

$$\begin{aligned} \langle \chi, \zeta^{(i)} \rangle_G &= \frac{1}{|G|} \sum_{\sigma \in G} \chi(\sigma) \zeta^{(i)}(\sigma^{-1}) \\ &= \frac{1}{|G|} \zeta^{(i)}\left(\sum_{\sigma} \chi(\sigma)\sigma^{-1}\right) \\ &= \frac{1}{|G|} \zeta^{(i)}(z). \end{aligned}$$

By [2, Proposition 9.23], we have every class function on  $\mathbf{C}[G]$  takes value 0 at  $z$  if  $\langle \chi, \zeta^{(i)} \rangle_G = 0$  for  $i = 1, \dots, r$ . Furthermore, this condition implies  $z = 0$ , because  $z$  is an element of the center of  $\mathbf{C}[G]$ . Conversely, if  $z = 0$ , we also have  $\chi = 0$ . Thus, we have

**LEMMA 8.** *The character relation (1) holds if and only if  $\sum_{H \in \Gamma} n_H \tilde{e}_H = 0$ .*

**REMARK 6 (norm relations).** Let  $U(G)$  be a subset of  $\mathbf{Z}^{|\Gamma|}$  consisting of  $\alpha = (\alpha_H)$  such that  $\alpha_H s_H = 0$ . This subset is a submodule and is called the module of norm relations in [7]. Let  $\Delta_H$  be the subset of  $\Gamma$  consisting of every cyclic subgroup of  $G$ . Denote by  $\Delta_{H,U}$  for each cyclic subgroup  $U$  the subset  $\{N \in \Delta_H : N \geq U\}$ . In [7, Satz 1], an element  $\gamma^H$  of  $U(G)$  was defined by

$$\gamma_U^H = \begin{cases} 0 & \text{if } U \notin \Delta_G \setminus \{H\}, \\ 1 & \text{if } U = H, \\ -\sum_{N \in \Delta_{H,U}} \mu(|N:U|) & \text{if } U \in \Delta_G \setminus \{H\}, \end{cases}$$

and it was proved the set  $\{\gamma^H : H \notin \Delta_G\}$  is a  $\mathbf{Z}$ -basis of  $U(G)$ .

REMARK 7 (the formula of Kani-Rosen). Let  $R(G)$  be a  $\mathbf{Q}$ -linear subspace of  $\mathbf{Q}^{|\Gamma|}$  consisting of  $\beta = (\beta_H)$  such that  $\sum_{H \in \Gamma} \beta_H \psi_H = 0$ . Let  $\Delta / \sim$  be the set of conjugacy classes of cyclic subgroups. Then, the dimension of  $R(G)$  is given by

$$\dim R(G) = |\Gamma| - |\Delta / \sim|$$

c.f. [6, the formula (6)].

Hereafter, we restrict our concern onto abelian  $p$ -groups for a fixed prime  $p$ . Suppose

$$(20) \quad G \cong \mathbf{Z}/p^{m_1}\mathbf{Z} \times \cdots \times \mathbf{Z}/p^{m_n}\mathbf{Z},$$

for integers  $m_1 \geq \cdots \geq m_n \geq 1$ . Let  $\hat{G}$  be the group of all the characters, that is  $\hat{G} = \text{Hom}(G, \mathbf{C}^\times)$ . Denote by  $H^\perp$  (resp.  $X^\perp$ ) for a subgroup  $H$  (resp.  $X$ ) of  $G$  (resp.  $\hat{G}$ ) the annihilator of  $H$  (resp.  $X$ ). We have  $(X^\perp)^\perp = X$  and  $H_1^\perp \cap H_2^\perp = (H_1 H_2)^\perp$  for subgroups  $H_1$  and  $H_2$  of  $G$ . Put  $G^* = (\hat{G}^p)^\perp$ . We denote by  $H^*$  the subgroup  $H G^*$  for  $H$ . Put  $Z = \hat{G} \setminus \hat{G}^p$ . Let  $\Gamma_0$  be the subset of  $\Gamma$  consisting of  $\text{Ker } \chi$  for every  $\chi \in Z$ . We have

$$\hat{G} = \bigcup_{H \in \Gamma_0} H^\perp, \quad H^\perp \cap G^{*\perp} = H^{*\perp}.$$

Thus, if  $H = \langle \zeta \rangle^\perp$  for  $\zeta \in Z$ , we have  $H^{*\perp} = \langle \zeta^p \rangle$ .

LEMMA 9. If  $H_1, H_2 \in \Gamma_0$  are distinct, we have  $H_1^\perp \cap H_2^\perp \leq H_1^{*\perp} \cap H_2^{*\perp}$ .

*Proof.* There are  $\zeta_i \in Z$  such that  $\langle \zeta_i \rangle = H_i^\perp$ . If  $\zeta_1 \notin \langle \zeta_2 \rangle$ , we see  $\langle \zeta_1 \rangle \cap \langle \zeta_2 \rangle \leq \langle \zeta_2^p \rangle$ . Thus, we have  $H_1^\perp \cap H_2^\perp \leq H_2^{*\perp}$ . Similarly, we have  $H_1^\perp \cap H_2^\perp \leq H_1^{*\perp}$ .  $\square$

THEOREM 10. We have a character relation

$$\psi_1 - \psi_{G^*} = \sum_{H \in \Gamma_0} (\psi_H - \psi_{H^*}).$$

*Proof.* By Lemma 9, we have

$$(H_1^\perp \setminus H_1^{*\perp}) \cap (H_2^\perp \setminus H_2^{*\perp}) = \emptyset$$

for every pair  $(H_1, H_2) \in \Gamma_0 \times \Gamma_0$  such that  $H_1 \neq H_2$ . Since  $H^{*\perp} = G^{*\perp} \cap H^\perp$ ,  $Z$  is a disjoint union of  $H^\perp \setminus H^{*\perp}$ :

$$(21) \quad Z = \hat{G} \setminus G^{*\perp} = \bigcup_{H \in \Gamma_0} H^\perp \setminus H^{*\perp}.$$

Every induced character  $\psi_H$  is a linear combination of  $\zeta \in \hat{G}$  with non-negative integral coefficients  $m_\zeta$ . The value of  $m_\zeta$  is computed by the Frobenius reciprocity law:

$$m_\zeta = \langle \psi_H, \zeta \rangle_G = \langle 1_H, \zeta \downarrow_H \rangle_H,$$

where  $\zeta \downarrow_H$  denotes restriction onto  $H$ . Clearly,  $m_\zeta = 1$  if  $\zeta \in H^\perp$ , and  $m_\zeta = 0$  if  $\zeta \downarrow_H$  is not trivial. Therefore,  $\psi_H = \sum_{\zeta \in H^\perp} \zeta$ . By (21), we have

$$\begin{aligned} \psi_1 - \psi_{G^*} &= \sum_{\zeta \in \hat{G}} \zeta - \sum_{\zeta^* \in G^{*\perp}} \zeta^* \\ &= \sum_{H \in \Gamma_0} \left( \sum_{\zeta \in H^\perp} \zeta - \sum_{\zeta^* \in H^{*\perp}} \zeta^* \right) \\ &= \sum_{H \in \Gamma_0} (\psi_H - \psi_{H^*}). \end{aligned}$$

$\square$

**6. Examples.**

EXAMPLE 1. Suppose  $n \geq 2$  and  $m_1 = \dots = m_n = m \geq 2$  in (20). We have  $G^* = G^{p^{m-1}}$  and  $\Gamma_0 = \{H : G/H \cong \mathbf{Z}/p^m\mathbf{Z}\}$ . Put  $K^* = K^{G^*}$ . If  $H \in \Gamma_0$  is the kernel of a character  $\chi$ , we have  $H^{*\perp} = \langle \chi^p \rangle$ . Thus, we see  $H_1^* = H_2^*$  is equivalent to  $\langle \chi_1^p \rangle = \langle \chi_2^p \rangle$ . If we choose  $H \in \Gamma_0$  arbitrarily, the number of subgroups  $H' \in \Gamma_0$  such that  $H^* = H'^*$  is equal to  $p^{n-1}$ . By Theorem 10, we obtain a character relation

$$0 = (\psi_1 - \psi_{G^*}) - \sum_{H \in \Gamma_0} (\psi_H - \psi_{H^*}).$$

Put  $\Gamma_0^* = \{H^* : H \in \Gamma_0\}$ . Since  $\delta(n_G) = p^{n-|\Gamma_0|}$ , the corresponding class number relation is

$$\frac{h_S(K)}{h_S(K^*)} = \frac{\prod_{H \in \Gamma_0} h_S(K^H)}{\left( \prod_{H^* \in \Gamma_0^*} h_S(K^{H^*}) \right)^{p^{n-1}}} \cdot \frac{\delta(i_{E_S, M_S})}{\delta(L_{S_0})} \cdot p^{n-|\Gamma_0|}.$$

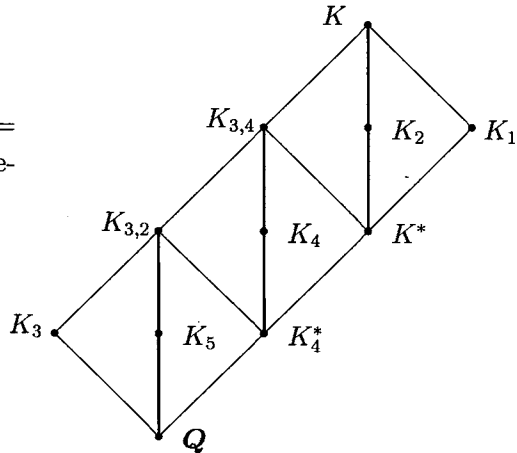
EXAMPLE 2. Set  $p = 2$ . Let  $q$  be a positive integer such that  $8 \mid \phi(q)$ . Let  $m$  be a square free integer prime to  $q$ . Let  $K_1$  be the cyclic extension of degree 8 and of conductor  $q$  over  $\mathbf{Q}$ . Put  $K = K_1(\sqrt{m})$  and  $k = \mathbf{Q}$ . We see  $n = 2$ ,  $m_1 = 8$  and  $m_2 = 2$ . Put  $K_3 = \mathbf{Q}(\sqrt{m})$ . We define  $\chi_1 \sim \chi_2$  for  $\chi_i \in \hat{G}$  to be  $\langle \chi_1 \rangle = \langle \chi_2 \rangle$ . Let  $\rho$  (resp.  $\chi$ ) be a Dirichlet character corresponding to  $K_1$  (resp.  $K_3$ ) of order 8 (resp. 2). A set

of complete representatives of  $Z/\sim$  is given by

$$\{\rho, \rho\chi, \rho^2\chi, \rho^4\chi, \chi\}.$$

For each character  $\zeta$ , we associate a subgroup  $H = \text{Ker } \zeta$  and a subfield  $L = K^H$  as follows.

$\zeta$	$\rho$	$\rho\chi$	$\rho^2\chi$	$\rho^4\chi$	$\chi$
$H$	$H_1$	$H_2$	$H_4$	$H_5$	$H_3$
$L$	$K_1$	$K_2$	$K_4$	$K_5$	$K_3$
$ \zeta $	8	8	4	2	2



We observe  $G^* = \langle \rho^2 \rangle^\perp$  and

$$H_1^* = H_2^* = G^*, H_3^* = H_5^* = G, H_4^* = \langle \rho^4 \rangle^\perp.$$

Write  $K_4^*$  for  $K^{H_4^*}$ . Theorem 10 yields a character relation

$$(22) \quad 0 = \psi_1 - \sum_{i=1}^5 \psi_{H_i} + \psi_{G^*} + \psi_{H_4^*} + 2\psi_G.$$

Since  $\delta(n_G) = 8^{-1}$ , the corresponding class number relation is

$$(23) \quad \frac{h_S(K)h_S(K_4^*)h_S(K^*)}{\prod_{i=1}^5 h_S(K_i)} = \frac{\delta(i_{E_S, M_S})}{8\delta(L_{S_0})}.$$

We compute the norm relation  $\gamma^G$  and obtain

$$0 = s_G - \sum_{i=1}^5 s_{H_i} + s_{H_{3,2}} + s_{H_{3,4}} + 2s_1,$$

where  $H_{3,2}$  and  $H_{3,4}$  are terms of the composition series

$$H_3 > H_{3,2} > H_{3,4} > \{1\}.$$

Denote by  $K_{3,2}$  and  $K_{3,4}$  the fixed field by  $H_{3,2}$  and  $H_{3,4}$ , respectively. Since  $\tilde{e}_H = |G|e_H$ , the character relation

$$0 = \psi_1 - \psi_{H_1} - \psi_{H_2} - 4\psi_{H_3} \\ - 2\psi_{H_4} - 4\psi_{H_5} + 2\psi_{H_{3,2}} + \psi_{H_{3,4}} + 8\psi_G$$

yields from Lemma 8. If we apply Theorem 10 to Galois extension  $K_{3,4}/K_4^*$  and  $K_{3,2}/Q$  and lift the obtained character relations onto those of  $G$ , we can transform the above relation to (22).

We assume  $K_1$  is real abelian and  $K_3$  is imaginary quadratic and  $S_0 = \emptyset$ . We see  $K$  and  $K_2, K_3, K_4, K_5$  are imaginary abelian field. By Corollary 7, we have

$$\frac{h^-(K)}{\prod_{i=2}^5 h^-(K_i)} = \frac{\delta(Q^-)\delta(w^-)}{8}.$$

## References

- [1] R. BRAUER; Beziehungen zwischen Klassenzahlen von Teilkörpern eines galoisschen Körpers. *Math. Nachr.* **4**(1951), 158-174
- [2] C. W. CURTIS and I. REINER; *Methods of Representation Theory with Applications to Finite Groups and Orders I.* John Willy & Sons, 1981.
- [3] C. W. CURTIS and I. REINER; *Methods of Representation Theory with Applications to Finite Groups and Orders II.* John Willy & Sons, 1987.
- [4] G. GRAS; *Class Field Theory: From Theory to Practice.* Springer, 2004.
- [5] E. KANI; Discriminants of hermitian  $R[G]$ -modules and Brauer's class number relation. *Algebra and number theory* (Essen, 1992), 43-135, de Guyter, Berlin, 1994.
- [6] E. KANI and M. ROSEN; Idempotent relations and factors of Jacobians. *Math. Ann.* **45**(1989), 307-327.
- [7] H. P. REHM; Über die gruppentheoretische Struktur der Relationen zwischen Relativnormabbildungen in endlichen Galoisschen Körpererweiterungen. *J. Number Theory* **7**(1975), 49-70.
- [8] J. TATE; *Les Conjecture de Stark sur les Fonctions  $L$  d'Artin.* Birkhäuser Boston (1984).
- [9] C. WALTER; Brauer's class number relation. *Acta Arith.* **35**(1979), 33-40.
- [10] C. WALTER; Kuroda's class number relation. *Acta Arith.* **35**(1979), 41-51.