WEIGHTED ESTIMATES FOR MAXIMAL FUNCTIONS ASSOCIATED WITH FOURIER MULTIPLIERS

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Abstract. We prove some weighted estimates for maximal functions associated with certain Fourier multipliers of Bochner-Riesz type.

1. Introduction

Let $\gamma(t,\xi)$ be a continuous function on $(0,\infty)\times\mathbb{R}^n$ such that $\gamma(t,0)=0$ and $\gamma(t,\xi)>0$ for all $\xi\neq 0$ and $t>0$. Also, we assume the following:

(1.1)
$$\lim_{t\to\infty}\gamma(t,\xi)=0 \quad \text{for all } \xi\in\mathbb{R}^n, \qquad \lim_{|\xi|\to\infty}\gamma(t,\xi)=\infty \quad \text{for all } t>0;$$

$$\{\xi\in\mathbb{R}^n : 1/2\le\gamma(t,\xi)\le 1\}\subset\{\xi\in\mathbb{R}^n : c_1t<|\xi|<c_2t\} \tag{1.2}$$

for all $t>0$ with some constants $0<c_1<c_2$;

$$|\{\xi\in\mathbb{R}^n : \gamma(t,t\xi)\in[1-\delta,1]\}|\le c\delta \tag{1.3}$$

for all $\delta\in(0,1/2]$ and $t>0$, where $|E|$ denotes the Lebesgue measure of a measurable set E.

Let $\hat{f}(\xi)=\int f(x)e^{-2\pi i\langle x,\xi\rangle}\,dx$ be the Fourier transform, where $\langle x,\xi\rangle$ denotes the inner product in $\mathbb{R}^n$. We also write $\hat{f}=\mathcal{F}(f)$. Throughout this note we assume that $n\ge 2$. We consider the Bochner-Riesz mean of order λ with respect to γ defined by

$$S_t^\lambda(f)(x)=\int_{\mathbb{R}^n}(1-\gamma(t,\xi))_+^\lambda\hat{f}(\xi)e^{2\pi i\langle x,\xi\rangle}\,d\xi,$$

where $s_+^\lambda=s^\lambda$ if $s>0$, $s_+^\lambda=0$ if $s\le 0$. When $\gamma(t,\xi)=(|\xi|/t)^2$, this is the ordinary Bochner-Riesz mean. Define the maximal function

$$S_*^\lambda(f)(x)=\sup_{t>0}|S_t^\lambda(f)(x)|.$$

In this note we generalize some known results on weighted estimates for the maximal functions associated with the ordinary Bochner-Riesz means by considering the generalized Bochner-Riesz means $S_t^\lambda(f)$. In

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particular, we shall prove some weighted inequalities for S_*^λ in the cases when $\gamma(t,\xi) = t^{-1}|\Phi(\xi)|$ and $\gamma(t,\xi) = |\Phi(t^{-1}\xi)|$, where Φ is a mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$ satisfying certain regularity conditions. It will be shown that if h is a positive homogeneous function of degree 1 which is infinitely differentiable away from the origin, we can find a suitable Φ such that $|\Phi(\xi)| = h(\xi)$.

Now, we further assume that $\gamma(t,\cdot) \in C^\infty(\mathbb{R}^n \setminus \{0\})$ for all $t > 0$ and that there exists $\epsilon_0 > 0$ such that

$$|(\partial\xi)^\alpha \gamma(t, t\xi)| \le C_\alpha |\xi|^{\epsilon_0 - |\alpha|} \quad \text{in } U_{c_2} \setminus \{0\} \tag{1.4}$$

for all $t > 0$ and multi-indices $\alpha = (\alpha_1, \dots, \alpha_n)$, where $(\partial\xi)^\alpha = (\partial/\partial\xi_1)^{\alpha_1} \dots (\partial/\partial\xi_n)^{\alpha_n}$, $|\alpha| = \alpha_1 + \dots + \alpha_n$ and $U_r = \{\xi \in \mathbb{R}^n : |\xi| < r\}$ (c_2 is as in (1.2)). Then we have the following:

Theorem 1. *Suppose that γ satisfies the conditions* (1.1)–(1.4)*. Let $\lambda > (n-1)/2$* (*the critical index*)*. Then*

$$\left\| S_*^\lambda(f) \right\|_{L^2(w)} \le C_{\lambda,w} \|f\|_{L^2(w)} \qquad (f \in \mathcal{S}(\mathbb{R}^n))$$

for $w \in A_1(\mathbb{R}^n)$ (*the Muckenhoupt class*)*, where $\mathcal{S}(\mathbb{R}^n)$ denotes the Schwartz space on $\mathbb{R}^n$ and $\|f\|_{L^r(w)} = (\int |f(x)|^r w(x)\, dx)^{1/r}$.*

This is a particular case of the following result.

Theorem 2. *Let γ be as in Theorem* 1*. Suppose that $\lambda > (n-1)/2$, $(n-1)/\lambda < p \le 2$, $1 < p$ and $1 < r \le p$. Then*

$$\left\| S_*^\lambda(f) \right\|_{L^p(w)} \le C_{\lambda,w} \|f\|_{\dot{F}_p^{0,r}(w)} \qquad (f \in \mathcal{S}(\mathbb{R}^n))$$

for all $w \in A_1(\mathbb{R}^n)$, where $\dot{F}_p^{0,r}(w)$ is the weighted (*homogeneous*) *Triebel-Lizorkin space.*

See [4] for the Triebel-Lizorkin space $\dot{F}_p^{s,r}$ (see also [14]). The definition of the norm for the weighted Triebel-Lizorkin space $\dot{F}_p^{s,r}(w)$ is the same as that for $\dot{F}_p^{s,r}$ except that the weighted measure $w(x)\,dx$ is used in place of the Lebesgue measure (see [1]). Note that, if $1 < r \le p \le 2$, $w \in A_p$ and $f \in \mathcal{S}(\mathbb{R}^n)$,

$$\|f\|_{L^p(w)} \approx \|f\|_{\dot{F}_p^{0,2}(w)} \le c\|f\|_{\dot{F}_p^{0,p}(w)} \le c\|f\|_{\dot{F}_p^{0,r}(w)}. \tag{1.5}$$

Thus Theorem 1 follows from Theorem 2 with $p = r = 2$.

Let $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ be a bijection. We define a space BL to be the space of all those bijections Φ which satisfy $\Phi(0) = 0$ and

$$c|\xi - \eta| \le |\Phi(\xi) - \Phi(\eta)| \le C|\xi - \eta| \quad \text{for all } \xi, \eta \in \mathbb{R}^n$$

with some constants $0 < c < C$. Note that if $\Phi \in BL$, $|\Phi(\xi)| \approx |\xi|$ and $|\Phi(E)| \approx |E|$ for a measurable set E.

Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a mapping with the components $F_1, F_2 \dots, F_n$. We define a subspace D of BL. Let $F \in BL$. We say $F \in D$ if $F_j \in C^\infty(\mathbb{R}^n \setminus \{0\})$ $(j = 1, 2, \dots, n)$ and there exists a neighborhood U of the origin such that

$$\max_{1 \le j \le n} |(\partial \xi)^\alpha F_j(\xi)| \le C_\alpha |\xi|^{1-|\alpha|} \quad \text{in } U \setminus \{0\}$$

for all multi-indices α.

For a mapping $\Phi : \mathbb{R}^n \to \mathbb{R}^n$, we consider $\gamma(t, \xi)$ defined by either of the following two equations:

$$\gamma(t, \xi) = t^{-1}|\Phi(\xi)|, \quad \gamma(t, \xi) = |\Phi(t^{-1}\xi)|.$$

Then we have the following:

Corollary 1. *Suppose that $\Phi \in D$ and let $\gamma(t, \xi)$ be as above. Suppose that $\lambda > (n-1)/2$. Then*

$$\left\|S_*^\lambda(f)\right\|_{L^2(w)} \le C_{\lambda,w}\|f\|_{L^2(w)} \qquad (f \in \mathcal{S}(\mathbb{R}^n))$$

for $w \in A_1(\mathbb{R}^n)$.

This follows from Theorem 1, since under the hypotheses of Corollary 1 $\gamma(t, \xi)$ satisfies the conditions (1.1)–(1.4) with $\epsilon_0 = 1$ in (1.4).

Let h be a positive homogeneous function of degree 1. By this we mean that $h(t\xi) = th(\xi)$ for all $t > 0$ and $\xi \in \mathbb{R}^n$, $h(0) = 0$ and $h(\xi) > 0$ for $\xi \neq 0$. Then, in fact, Corollary 1 is equivalent to the following:

Corollary 2. *Suppose that $\Phi \in D$ and $h \in C^\infty(\mathbb{R}^n \setminus \{0\})$. Let $\gamma(t, \xi) = t^{-1}(h \circ \Phi)(\xi) = t^{-1}h(\Phi(\xi))$ or $\gamma(t, \xi) = (h \circ \Phi)(t^{-1}\xi)$. Suppose that $\lambda > (n-1)/2$. Then*

$$\left\|S_*^\lambda(f)\right\|_{L^2(w)} \le C_{\lambda,w}\|f\|_{L^2(w)} \qquad (f \in \mathcal{S}(\mathbb{R}^n))$$

for $w \in A_1(\mathbb{R}^n)$.

We can derive Corollary 2 from Corollary 1 as follows. Define $\Lambda : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\Lambda(\xi) = \begin{cases} |\xi|h(\xi)^{-1}\xi & \text{if } \xi \neq 0, \\ 0 & \text{if } \xi = 0. \end{cases}$$

Note that $\Lambda^{-1}(\eta) = h(\eta)|\eta|^{-1}\eta$ $(\eta \neq 0)$, $\Lambda^{-1}(0) = 0$. We can easily see that $\Lambda \in D$. Define $\Gamma = \Lambda^{-1} \circ \Phi \in D$. Since $|\Gamma| = h \circ \Phi$, by applying Corollary 1 to $\gamma(t, \xi) = t^{-1}|\Gamma(\xi)|$ and $\gamma(t, \xi) = |\Gamma(t^{-1}\xi)|$ we get Corollary 2.

When λ is near 0, we have the following estimates with power weights:

Theorem 3. *Let $\gamma(t,\xi) = t^{-1}|\Phi(\xi)|$, $\Phi \in D$. Suppose that $\lambda > 0$ and $-1 < \alpha \le 0$. Then*

$$\int_{\mathbb{R}^n} \left|S_*^\lambda(f)(x)\right|^2 |x|^\alpha \, dx \le C_{\lambda,\alpha} \int_{\mathbb{R}^n} |f(x)|^2 |x|^\alpha \, dx \qquad (f \in \mathcal{S}(\mathbb{R}^n)).$$

When $\gamma(t,\xi) = (|\xi|/t)^2$, this is due to Carbery-Rubio de Francia-Vega [2]. A complex interpolation between Theorem 3 and Corollary 1 with $w(x) = |x|^\alpha$ $(-n < \alpha \le 0)$ gives the following (see [2], [8]):

Corollary 3. *Let $\gamma(t,\xi)$ be as in Theorem 3. Suppose that $0 < \lambda \le (n-1)/2$ and $-2\lambda - 1 < \alpha \le 0$. Then*

$$\int_{\mathbb{R}^n} \left|S_*^\lambda(f)(x)\right|^2 |x|^\alpha \, dx \le C_{\lambda,\alpha} \int_{\mathbb{R}^n} |f(x)|^2 |x|^\alpha \, dx.$$

This result can be used to get the following:

Corollary 4. *Let $\gamma(t,\xi)$ be as in Theorem 3. Suppose that $0 < \lambda \le (n-1)/2$, $2 \le p < 2n/(n-1-2\lambda)$ and $n(1-2/p) < -\alpha < 1 + 2\lambda$. Put $w_\alpha(x) = \min(1, |x|^\alpha)$. Then*

$$\left\|S_*^\lambda(f)\right\|_{L^2(w_\alpha)} \le c\|f\|_{L^2(w_\alpha)} \le c\|f\|_{L^p}.$$

The second inequality of the conclusion of Corollary 4 follows by Hölder's inequality. As in [2], by Corollary 4 we can see that

$$\lim_{t\to\infty} S_t^\lambda(f)(x) = f(x) \quad \text{a.e.}$$

for $0 < \lambda \le (n-1)/2$ and $f \in L^p(\mathbb{R}^n)$ provided $2 \le p < 2n/(n-1-2\lambda)$.

Remark 1. When $\gamma(t,\xi) = t^{-1}h(\xi)$, where h is a certain positive homogeneous function of degree 1, the $L^2(w)$ boundedness of S_*^λ for $\lambda > (n-1)/2$ and $w \in A_1$ can be derived from the estimates of Seeger for the Littlewood-Paley functions (see [10, 11]). The case where $h \in C^\infty(\mathbb{R}^n \setminus \{0\})$ follows form Corollary 2.

Remark 2. Let a be a non-negative, continuous function on $[0,\infty)$. We assume that $a \in C^\infty((0,\infty))$, $a(0) = 0$, $a(1) = 1$, $a'(s) > 0$ for $s > 0$, $a(s) \to \infty$ as $s \to \infty$ and

$$\left|(d/ds)^\ell a(s)\right| \le cs^{\epsilon_1 - \ell}$$

for all $s \in (0,\gamma)$ and $\ell \ge 0$ with some positive constants γ, ϵ_1. Then Theorem 3 and Corollaries 1–4 stated above still hold with $\gamma(t,\xi) = a(t^{-1}(h \circ \Phi)(\xi))$ and also Corollaries 1, 2 remain true with $\gamma(t,\xi) = a((h \circ \Phi)(t^{-1}\xi))$, where h is a positive homogeneous function of degree 1 in $C^\infty(\mathbb{R}^n \setminus \{0\})$ and $\Phi \in D$. In particular, this remark applies to the function $a(s) = s^m$, $m > 0$. In this case, $\gamma(t,\xi) = t^{-m}(H \circ \Phi)(\xi)$ or $\gamma(t,\xi) = (H \circ \Phi)(t^{-1}\xi)$, where H is a homogeneous function of degree m (see [3], [6], [7], [12] for related results).

In Section 2, we shall prove Theorem 2. Suppose that h is a positive homogeneous function of degree 1 such that $h \in C^\infty(\mathbb{R}^n \setminus \{0\})$, $\nabla h(\xi) \neq 0$ for $\xi \in \mathbb{R}^n \setminus \{0\}$. Put $\Sigma_h = \{\xi \in \mathbb{R}^n : h(\xi) = 1\}$. If the hypersurface Σ_h has non-vanishing Gaussian curvature and if $\lambda > (n-1)/2$, then

$$\left|\mathcal{F}\left((1-h)_+^\lambda\right)(x)\right| \leq c(1+|x|)^{-n-\epsilon} \qquad \text{for some } \epsilon > 0$$

(see Sogge [13]). Therefore, if $\gamma(t,\xi) = t^{-1}h(\xi)$, we have $S_*^\lambda(f) \leq c\mathcal{M}(f)$, where $\mathcal{M}$ denotes the Hardy-Littlewood maximal operator, and hence S_*^λ is bounded on $L^p(w)$ for $1 < p < \infty$ and $w \in A_p$. Although pointwise estimates similar to those given above are not available in the present situation, we have the weighted L^q estimates for the kernels arising from a decomposition of the operator S_t^λ defined by the general functions $\gamma(t,\xi)$ (Lemma 2), which can be applied to prove Theorem 2.

In Section 3, we shall prove Theorem 3. The proof is based on the weighted L^2 estimates of [2] and [8] for certain Littlewood-Paley functions.

2. Proof of Theorem 2

To handle the singularity of $\gamma(t,\xi)$ at $\xi = 0$, we need the following pointwise estimates for Fourier transform.

Lemma 1. *Let $g : \mathbb{R}^n \to \mathbb{R}$ be continuous and $g(0) = 0$. Let $\varphi \in C_0^\infty(\mathbb{R})$. Suppose that $g^{-1}(\mathrm{supp}(\varphi)) \subset U_\epsilon$ for some $\epsilon > 0$, where $U_\epsilon = \{x \in \mathbb{R}^n : |x| < \epsilon\}$. We further assume that $g \in C^{n+1}(U_\epsilon \setminus \{0\})$ and there exists $m > 0$ such that*

$$|(\partial\xi)^\alpha g(\xi)| \leq c|\xi|^{m-|\alpha|} \qquad \text{in } U_\epsilon \setminus \{0\} \text{ for } |\alpha| \leq n+1.$$

Then

$$|\mathcal{F}(\varphi \circ g)(x)| \leq c(1+|x|)^{-n-\delta} \qquad \text{for some } \delta > 0.$$

Proof. Take $\tilde{\varphi}(\xi) \in C_0^\infty(\mathbb{R}^n)$ such that $\varphi(g(\xi)) = \tilde{\varphi}(\xi)\varphi(g(\xi))$. Write $\varphi(g(\xi)) = \tilde{\varphi}(\xi)(\varphi(g(\xi)) - \varphi(0)) + \varphi(0)\tilde{\varphi}(\xi)$. Then it suffices to estimate the Fourier transform of $\Psi(\xi) := \tilde{\varphi}(\xi)(\varphi(g(\xi)) - \varphi(0))$. We have

$$|(\partial\xi)^\alpha \Psi(\xi)| \leq c|\xi|^{m-|\alpha|} \qquad \text{in } \mathbb{R}^n \setminus \{0\} \text{ for } |\alpha| \leq n+1. \tag{2.1}$$

Let $\psi \in C_0^\infty(\mathbb{R}^n)$ be such that $\mathrm{supp}(\psi) \subset \{1/2 \leq |\xi| \leq 2\}$, $\sum_{j\in\mathbb{Z}} \psi(2^{-j}\xi) = 1$ for $\xi \neq 0$, where $\mathbb{Z}$ denotes the set of all integers. Write

$$\hat{\Psi}(x) = \sum_{j \leq M} \int \psi(2^{-j}\xi)\Psi(\xi)e^{-2\pi i\langle x,\xi\rangle}\,d\xi$$

for some $M \geq 0$. We split the sum on the right hand side into two pieces: $\hat{\Psi}(x) = I + II$, where

$$I = \sum_{j \leq N} \int \psi(2^{-j}\xi)\Psi(\xi)e^{-2\pi i\langle x,\xi\rangle}\, d\xi, \qquad II = \sum_{N<j\leq M} \int \psi(2^{-j}\xi)\Psi(\xi)e^{-2\pi i\langle x,\xi\rangle}\, d\xi,$$

for $N \leq 0$, which will be specified below. We may assume $|x| > 2$. Applying integration by parts k times ($1 \leq k \leq n+1$) and using (2.1), we have

$$\left| \int \psi(2^{-j}\xi)\Psi(\xi)e^{-2\pi i\langle x,\xi\rangle}\, d\xi \right| \leq c|x|^{-k}2^{jn}2^{j(m-k)}. \tag{2.2}$$

To estimate I we use (2.2) with $k = n$ and to estimate II with $k = n+1$. Finally, choosing $N = \log_2(|x|^{-1})$, we can get the conclusion. □

Now, we give a proof of Theorem 2. Decompose

$$(1 - \gamma(t,\xi))_+^\lambda = \sum_{j=0}^{\infty} 2^{-j\lambda} n_j(\gamma(t,\xi)),$$

where $n_j \in C_0^\infty(\mathbb{R})$ $(j \geq 0)$, $\mathrm{supp}(n_j) \subset [1-2^{-j}, 1]$ $(j \geq 1)$, $\mathrm{supp}(n_0) \subset (-1,1)$ and $|(d/dr)^\ell n_j(r)| \leq c_\ell 2^{j\ell}$ for $\ell \geq 0$. Let $L_{j,t}^\lambda(x) = \mathcal{F}^{-1}(2^{-j\lambda}n_j(\gamma(t,\cdot)))(x)$ for $j \geq 0$ and $K_t^\lambda(x) = \mathcal{F}^{-1}((1-\gamma(t,\cdot))_+^\lambda)(x) - L_{0,t}^\lambda(x)$, where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform. Put $G_t(\xi) = \gamma(t, t\xi)$. Note that $G_t^{-1}(\mathrm{supp}(n_0)) \subset U_{c_2}$ for all $t > 0$, where U_{c_2} is as in (1.4). This can be seen by using the second condition of (1.1), (1.2) and the intermediate value theorem. By (1.4) and Lemma 1 with $g = G_t$ and $\varphi = n_0$, we have $\sup_{t>0} |L_{0,t}^\lambda * f| \leq c\mathcal{M}f$. Since

$$\left\| \sup_{t>0} |L_{0,t}^\lambda * f| \right\|_{L^p(w)} \leq c\|\mathcal{M}f\|_{L^p(w)} \leq c\|f\|_{L^p(w)} \leq c\|f\|_{\dot{F}_p^{0,p}(w)}$$

(see (1.5)) to prove Theorem 2, it suffices to show

$$\int \sup_{t>0} |K_t^\lambda * f(x)|^p w(x)\, dx \leq c\left(\|f\|_{\dot{F}_p^{0,p}(w)}\right)^p. \tag{2.3}$$

Decompose $K_t^\lambda(x) = \sum_{j=1}^\infty L_{j,t}^\lambda(x)$. Then, by Hölder's inequality we have

$$|K_t^\lambda * f(x)|^p \leq \left(\sum_{j=1}^\infty c_j^{-q/p}\right)^{p/q} \left(\sum_{j=1}^\infty c_j \left|L_{j,t}^\lambda * f(x)\right|^p\right),$$

where $1/p + 1/q = 1$ and $\{c_j\}$ is a sequence of positive numbers such that $\sum_{j=1}^{\infty} c_j^{-q/p} < \infty$. Thus we have

(2.4)
$$\int \sup_{t>0} |K_t^\lambda * f(x)|^p w(x)\,dx \le c \sum_{j=1}^{\infty} c_j \int \sup_{t>0} \left|L_{j,t}^\lambda * f(x)\right|^p w(x)\,dx$$
$$\le c \sum_{j=1}^{\infty} c_j \sum_{k\in\mathbb{Z}} \int \sup_{2^k\le t\le 2^{k+1}} \left|L_{j,t}^\lambda * f(x)\right|^p w(x)\,dx.$$

Note that (1.2) implies

$$\sup_{2^k\le t\le 2^{k+1}} \left|L_{j,t}^\lambda * f(x)\right| = \sup_{2^k\le t\le 2^{k+1}} \left|L_{j,t}^\lambda * \Delta_k f(x)\right|, \tag{2.5}$$

where

$$(\Delta_k f)\hat{}\,(\xi) = \Psi(2^{-k}\xi)\hat{f}(\xi)$$

with $\Psi \in C_0^\infty(\mathbb{R}^n)$ satisfying

$$\operatorname{supp}(\Psi) \subset \{b_1 \le |\xi| \le b_2\}, \quad \Psi(\xi) = 1 \quad \text{if } a_1 \le |\xi| \le a_2$$

for some suitable numbers a_1, a_2, b_1, b_2 such that $0 < b_1 < a_1 < a_2 < b_2$. By (2.4) and (2.5), to prove (2.3) it suffices to show that there exists $\epsilon > 0$ such that

$$\int \sup_{2^k\le t\le 2^{k+1}} \left|L_{j,t}^\lambda * f(x)\right|^p w(x)\,dx \le c2^{-j\epsilon} \int |f(x)|^p w(x)\,dx, \tag{2.6}$$

where the constant c is independent of k and j. Indeed, by (2.6) we have

(2.7)
$$\sum_{k\in\mathbb{Z}} \int \sup_{2^k\le t\le 2^{k+1}} \left|L_{j,t}^\lambda * \Delta_k f(x)\right|^p w(x)\,dx$$
$$\le c2^{-j\epsilon} \sum_{k\in\mathbb{Z}} \int |\Delta_k f(x)|^p w(x)\,dx \le c2^{-j\epsilon} \left(\|f\|_{\dot{F}_p^{0,p}(w)}\right)^p,$$

where the last inequality follows by a standard argument (see [1]); thus, using (2.5) and (2.7) in (2.4) and choosing $\{c_j\}$ suitably, we get (2.3).

To prove (2.6), we use the following estimates:

Lemma 2. *Let $t \in [2^k, 2^{k+1}]$, $k \in \mathbb{Z}$. For any λ, p and δ satisfying $\lambda > (n-1)/2$, $(n-1)/\lambda < p \le 2$, $1 < p$ and $0 < \delta < p\lambda + 1 - n$, there exists $\epsilon > 0$ such that*

$$\int_{\mathbb{R}^n} \left|(L_{j,t}^\lambda)_{2^k}(x)\right|^q (1 + |x|)^{(n+\delta)q/p}\,dx \le c2^{-j\epsilon},$$

where $1/p + 1/q = 1$, $(L_{j,t}^\lambda)_r(x) = r^{-n} L_{j,t}^\lambda(x/r)$ $(r > 0)$ and the constant c is independent of t, k and j.

Proof. Fix $t \in [2^k, 2^{k+1}]$. Since (1.4) holds, integration by parts gives

$$|(L^\lambda_{j,t})_{2^k}(x)| \le C_M 2^{-j\lambda} 2^{-j}(1+2^{-j}|x|)^{-M} \qquad \text{for all } M > 0, \tag{2.8}$$

where C_M is independent of t, k and j. Also, by (1.3) we have

$$|\{\xi \in \mathbb{R}^n : \gamma(t, 2^k\xi) \in [1-\delta, 1]\}| \le c\delta, \tag{2.9}$$

where c is independent of $\delta \in (0, 1/2]$, t and k.

By (2.9) and the Hausdorff-Young inequality we have

(2.10)

$$\begin{aligned}
\int_{\mathbb{R}^n} \left|(L^\lambda_{j,t})_{2^k}(x)\right|^q \, dx &\le \left(\int \left|\mathcal{F}\left((L^\lambda_{j,t})_{2^k}\right)(\xi)\right|^p \, d\xi\right)^{q/p} \\
&\le c\left(2^{-pj\lambda}|\{\xi \in \mathbb{R}^n : \gamma(t, 2^k\xi) \in [1-2^{-j}, 1]\}|\right)^{q/p} \\
&\le c 2^{-j(q\lambda + q/p)}.
\end{aligned}$$

This implies

$$\begin{aligned}
&\int_{|x|\le 2^j} \left|(L^\lambda_{j,t})_{2^k}(x)\right|^q (1+|x|)^{(n+\delta)q/p} \, dx \\
&\qquad \le c 2^{j(n+\delta)q/p} 2^{-j(q\lambda+q/p)} = c 2^{-j(q-1)(p\lambda+1-n-\delta)}.
\end{aligned} \tag{2.11}$$

Let $0 < \tau < 1/2$. Then, by Hölder's inequality we have

$$\begin{aligned}
&\int_{|x|>2^j} \left|(L^\lambda_{j,t})_{2^k}(x)\right|^q (1+|x|)^{(n+\delta)q/p} \, dx \\
&\le \left(\int_{|x|>2^j} \left|(L^\lambda_{j,t})_{2^k}(x)\right|^q (1+|x|)^{(n+\delta)q/(p\tau)} \, dx\right)^\tau \left(\int_{\mathbb{R}^n} \left|(L^\lambda_{j,t})_{2^k}(x)\right|^q \, dx\right)^{1-\tau}.
\end{aligned}$$

By (2.8) we see that

$$\begin{aligned}
&\left(\int_{|x|>2^j} \left|(L^\lambda_{j,t})_{2^k}(x)\right|^q (1+|x|)^{(n+\delta)q/(p\tau)} \, dx\right)^\tau \\
&\le c(2^{-j\lambda}2^{-j})^{q\tau} \left(\int_{|x|>2^j} (2^{-j}|x|)^{-qM} |x|^{(n+\delta)q/(p\tau)} \, dx\right)^\tau \\
&\le c(2^{-j\lambda}2^{-j})^{q\tau} 2^{j(n+\delta)q/p} 2^{jn\tau} \left(\int_{|x|>1} |x|^{-qM+(n+\delta)q/(p\tau)} \, dx\right)^\tau \\
&\le c(2^{-j\lambda}2^{-j})^{q\tau} 2^{j(n+\delta)q/p} 2^{jn\tau},
\end{aligned}$$

where M and τ are chosen so that $n + (n+\delta)q/(p\tau) < qM$. By this and (2.10) we have

$$\begin{aligned}(2.12)\qquad &\int_{|x|>2^j} \left|(L^\lambda_{j,t})_{2^k}(x)\right|^q (1+|x|)^{(n+\delta)q/p}\,dx \\ &\le c(2^{-j\lambda}2^{-j})^{q\tau}2^{j(n+\delta)q/p}2^{jn\tau}2^{-j(q\lambda+q/p)(1-\tau)} \\ &= c2^{-j(q-1)(p\lambda+1-n-\delta-\tau(n-1)/(q-1))},\end{aligned}$$

where we further assume that $p\lambda + 1 - n - \delta - \tau(n-1)/(q-1) > 0$. Combining (2.11) and (2.12), we get the conclusion. This completes the proof of Lemma 2. □

We can find in Sogge [13, pp. 70–71] an argument similar to the one used in the proof of Lemma 2. Now we can prove (2.6). By Hölder's inequality and Lemma 2

$$\begin{aligned}(2.13)\qquad &\sup_{t\in[2^k,2^{k+1}]} \left|(L^\lambda_{j,t})_{2^k} * 2^{kn}f_{2^k}(x)\right|^p \\ &\le \sup_{t\in[2^k,2^{k+1}]} \left(\int \left|(L^\lambda_{j,t})_{2^k}(x-y)\right|^q (1+|x-y|)^{(n+\delta)q/p}\,dy\right)^{p/q} \\ &\times \int |f(2^{-k}y)|^p(1+|x-y|)^{-n-\delta}\,dy \\ &\le c2^{-(\epsilon p/q)j}\int |f(2^{-k}y)|^p(1+|x-y|)^{-n-\delta}\,dy.\end{aligned}$$

Since $w \in A_1$, we have

(2.14)

$$\int (1+|x-y|)^{-\delta-n}w(2^{-k}x)\,dx \le C_w w(2^{-k}y) \quad \text{for a.e. } y\in\mathbb{R}^n.$$

By (2.13) and (2.14) we see that

$$\begin{aligned}&\int \sup_{t\in[2^k,2^{k+1}]} \left|L^\lambda_{j,t} * f(x)\right|^p w(x)\,dx \\ &= \int \sup_{t\in[2^k,2^{k+1}]} \left|(L^\lambda_{j,t})_{2^k} * 2^{kn}f_{2^k}(x)\right|^p 2^{-kn}w(2^{-k}x)\,dx \\ &\le c2^{-(\epsilon p/q)j}\int 2^{-kn}|f(2^{-k}y)|^p\left(\int (1+|x-y|)^{-n-\delta}w(2^{-k}x)\,dx\right)dy \\ &\le c2^{-(\epsilon p/q)j}\int |f(y)|^p w(y)\,dy,\end{aligned}$$

which proves (2.6). This completes the proof of Theorem 2.

3. Proof of Theorem 3

To prove Theorem 3, we use the following.

Proposition 1. *Let* $\gamma(t,\xi) = t^{-1}|\Phi(\xi)|$, $\Phi \in BL$. *Let* $0 < \delta < 7/8$ *and let* $m_\delta(r)$ *be a continuously differentiable function supported in the interval* $[1-\delta, 1]$. *Suppose that* $\|(d/dr)m_\delta\|_{L^1(\mathbb{R})} \le 1$. *Define*

$$\left(U_t^\delta f\right)\hat{}\,(\xi) = \hat{f}(\xi)m_\delta(\gamma(t,\xi)).$$

Then

$$\int_{\mathbb{R}^n}\int_0^\infty \left|U_t^\delta f(x)\right|^2 |x|^\alpha \,\frac{dt}{t}\,dx \le c\delta\int_{\mathbb{R}^n}|f(x)|^2|x|^\alpha\,dx,$$

where $-1 < \alpha \le 0$ *and the constant* c *is independent of* δ.

When $\gamma(t,\xi) = |\xi|/t$, this was proved in Carbery-Rubio de Francia-Vega [2] and Rubio de Francia [8] (see [9] for a related result).

To prove Proposition 1, we use the following result, which can be found in [5, 8].

Lemma 3. *Let* $0 < \beta < 2$. *Then*

$$\int\limits_{\mathbb{R}^n}\int_0^\infty |g(t,x)|^2|x|^\beta\,\frac{dt}{t}\,dx = c_\beta \iint\limits_{\mathbb{R}^n\times\mathbb{R}^n}\int_0^\infty |\hat{g}(t,\xi)-\hat{g}(t,\eta)|^2|\xi-\eta|^{-n-\beta}\,\frac{dt}{t}\,d\xi\,d\eta,$$

where $\hat{g}(t,\xi) = \mathcal{F}(g(t,\cdot))(\xi)$. *We also have*

$$\int_{\mathbb{R}^n}|g(x)|^2|x|^\beta\,dx = c_\beta\iint_{\mathbb{R}^n\times\mathbb{R}^n}|\hat{g}(\xi)-\hat{g}(\eta)|^2|\xi-\eta|^{-n-\beta}\,d\xi\,d\eta.$$

Now, we give a proof of Proposition 1. By duality, to prove Proposition 1 it suffices to show

(3.1)
$$\int_{\mathbb{R}^n}\left|\int_0^\infty U_t^\delta f_t(x)\,\frac{dt}{t}\right|^2 |x|^{-\alpha}\,dx \le c\delta\int_{\mathbb{R}^n}\int_0^\infty |f(t,x)|^2|x|^{-\alpha}\,\frac{dt}{t}\,dx,$$

where we write $f_t(x) = f(t,x)$ for a function f in $C_0^\infty((0,\infty)\times\mathbb{R}^n)$. Define an operator L_F by

$$(L_F f)\hat{}\,(\xi) = \hat{f}(F\xi),$$

where F is a mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$. Let

$$\left(V_t^\delta f\right)\hat{}\,(\xi) = \hat{f}(\xi)m_\delta(|\xi|/t).$$

Then, by taking Fourier transform, we can see that

$$U_t^\delta f(x) = L_\Phi V_t^\delta L_{\Phi^{-1}} f(x). \tag{3.2}$$

Since $\Phi \in BL$, by the second part of Lemma 3, (3.2) and a change of variables we have

(3.3)
$$\int_{\mathbb{R}^n} \left| \int_0^\infty U_t^\delta f_t(x) \, \frac{dt}{t} \right|^2 |x|^{-\alpha} \, dx \approx \int_{\mathbb{R}^n} \left| \int_0^\infty V_t^\delta L_{\Phi^{-1}} f_t(x) \, \frac{dt}{t} \right|^2 |x|^{-\alpha} \, dx$$

(see [8] for this argument). Since the estimates in (3.1) are known when $\gamma(t,\xi) = |\xi|/t$, by (3.3) we have

(3.4)
$$\int_{\mathbb{R}^n} \left| \int_0^\infty U_t^\delta f_t(x) \, \frac{dt}{t} \right|^2 |x|^{-\alpha} \, dx \le c\delta \int_{\mathbb{R}^n} \int_0^\infty |L_{\Phi^{-1}} f_t(x)|^2 |x|^{-\alpha} \, \frac{dt}{t} \, dx.$$

The first part of Lemma 3 implies

$$\text{(3.5)} \quad \int_{\mathbb{R}^n} \int_0^\infty |L_{\Phi^{-1}} f_t(x)|^2 |x|^{-\alpha} \, \frac{dt}{t} \, dx \approx \int_{\mathbb{R}^n} \int_0^\infty |f(t,x)|^2 |x|^{-\alpha} \, \frac{dt}{t} \, dx.$$

By (3.4) and (3.5) we get (3.1). This completes the proof of Proposition 1.

Put

$$\tilde{S}_t^\lambda(f)(x) = \int_{\mathbb{R}^n} \eta(\gamma(t,\xi)) \, (1-\gamma(t,\xi))_+^\lambda \, \hat{f}(\xi) e^{2\pi i \langle x,\xi \rangle} \, d\xi,$$

where $\eta \in C_0^\infty(\mathbb{R})$ is such that $\eta(s) = 1$ if $3/4 \le s \le 2$ and $\eta(s) = 0$ if $s \le 1/2$. Define

$$\tilde{S}_*^\lambda(f)(x) = \sup_{t>0} |\tilde{S}_t^\lambda(f)(x)|.$$

Then, by applying Proposition 1 we can prove the following result as in [2].

Proposition 2. *Let $\gamma(t,\xi)$ be as in Proposition 1. Let $\lambda > 0$ and $-1 < \alpha \le 0$. Then*

$$\int_{\mathbb{R}^n} \left| \tilde{S}_*^\lambda(f)(x) \right|^2 |x|^\alpha \, dx \le C_\lambda \int_{\mathbb{R}^n} |f(x)|^2 |x|^\alpha \, dx.$$

Now, we can finish the proof of Theorem 3. Let $\zeta \in C_0^\infty(\mathbb{R})$ be such that $\zeta(s) + \eta(s) = 1$ for $s \in [0,2]$ and $\operatorname{supp}(\zeta) \subset [-1, 3/4]$, where η is as in the definition of $\tilde{S}_t^\lambda$. Put

$$M_t^\lambda f(x) = \int_{\mathbb{R}^n} \zeta\left(\gamma(t,\xi)\right) (1-\gamma(t,\xi))_+^\lambda \, \hat{f}(\xi) e^{2\pi i \langle x,\xi \rangle} \, d\xi.$$

Then $S_*^\lambda(f)(x) \le \sup_{t>0} |M_t^\lambda f(x)| + \tilde{S}_*^\lambda(f)(x)$. As in the first part of the proof of Theorem 2, by Lemma 1 we have $\sup_{t>0} |M_t^\lambda f(x)| \le c\mathcal{M}f(x)$. Thus the conclusion follows from Proposition 2 and the $L^2(w)$ boundedness of $\mathcal{M}$ for $w \in A_2(\mathbb{R}^n)$.

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