On the value at s=-1 of partial zeta functions for ray classes of real quadratic field

On the value at s=−1 of partial zeta functions for ray classes of real quadratic field

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Abstract. Let k be a real quadratic field. Let α be an integer of k and m be a positive rational integer. Denote by $\zeta((\alpha),(m),s)$ be a partial zeta function associated to a ray class containing the principal ideal (α) which is defined with a conductor (m). We give a formula of the value of $\zeta((\alpha),(m),-1)$ by means of the modification of the sum defined by L. Carlitz, *c.f.* [2]. It is an analogue to the generalized Dedekind sum defined in [6].

1. Introduction. We studied the value at zero of a partial zeta function on a real quadratic field in [8, 9, 10], based on the Shintai formula in [7]. A formula of the value at $s=0$ for a ray class containing a principal ideal in a real quadratic field was given and it was shown that the value is able to be computed when the conductor of the ray class group is a positive integer. In the present paper, we shall study that the value at $s=-1$ with the analogous method. To this end, we use a sum

$$c_{r,s}(h,k)=\sum_{(\mu)\mod k} B_r\left(\frac{\mu}{k}\right)B_s\left(\left\{\frac{h\mu}{k}\right\}\right)$$

defined from the transformation of Lambert series in [2], where $\{x\}$ is the fractional part of a real number x. This sum is extension of the generalized Dedekind sum in [1]. We call it the Carlitz sum in this paper.

2. Review of the Shintai formula. Let α be an integer of a real quadratic field k. Denote by (α) an ideal generated by α. We assume (α) is prime to the conductor (m) of the ray class group. The partial zeta function associated to a ray class containing (α) is defined to be

$$\zeta((\alpha),(m),s)=\sum_{\mathfrak{b}}\frac{1}{N\mathfrak{b}^s},\qquad s>1,$$

which is continued meromorphically to the complex plain, where $\mathfrak{b}$ runs the set of every integral ideals contained in the ray class. Let σ_1 and σ_2 be embeddings of k

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into the field of real numbers. We identify x with $\sigma_1 x$ and write $\sigma_2 x$ as x'. The quadratic field k is mapped into a subset of a two dimensional real vector space by $x \longrightarrow (\sigma_1 x, \sigma_2 x)$. Let C be the set of every vectors whose coordinates are positive. Let $E_+(m)$ be a subgroup of units which are congruent to 1 with modulo m and totally positive. We see $E_+(m)$ is a free abelian group of rank one. Let ε be a generator of $E_+(m)$ such that $\varepsilon > \varepsilon'$. The group $E_+(m)$ acts on the set C by multiplication. Let C_1 be a subset of C generated by a vector $v_1 = (m, m)$:

$$C_1 = \{xv : x > 0\}.$$

Set $v_2 = (m\varepsilon, m\varepsilon')$. Let C_2 be a subset C consisting of every vectors $x_1 v_1 + x_2 v_2$, $x_1 > 0$, $x_2 > 0$. The union of these two sets is a fundamental domain with respect to action of $E_+(m)$. Let S_1 (*resp.* S_2) be a bounded subset of C_1 (*resp.* C_2) of every element such that $0 < x \leq 1$ (*resp.* $0 < x_1, x_2 \leq 1$). Since the fractional ideal $(m^{-1}\alpha)$ is mapped into a discrete set in the two dimensional vector space, the intersection $(1 + (m^{-1}\alpha)) \cap S_i$ is a finite set, where $1 + (m^{-1}\alpha)$ is a set of elements $1 + y$ for every $y \in (m^{-1}\alpha)$. We need the sets of coordinates of these finite sets to formulate the value of the partial zeta function. Namely, put

$$R(1) = \{x : xv_1 \in S_1\},$$
$$R(2) = \{(x_1, x_2) : x_1 v_1 + x_2 v_2 \in S_2\}.$$

These sets $R(i)$ are determined concretely in [8]. Let $1, w$ be the standard basis of the ring of integers of k, such that $w > 0$. We write α as a Z-linear combination concerning this basis and decompose it as

$$\alpha = c(f_1 + f_2 w), \quad c > 0, \quad (f_1, f_2) = 1.$$

Let a and b be positive integers such that

$$\varepsilon = a + bw.$$

We see $a \equiv 1 \mod m$ and $b \equiv 0 \mod m$. We set $\lambda = 0$ if $f_2 = 0$. When $f_2 \neq 0$, we solve a congruent equation

$$(1) \qquad f_2 \lambda \equiv f_1 + f_2 Tr(w) \mod |N(\beta)|$$

and set the solution to λ.

Proposition 1. We have

(i) The set $R(1)$ is equal to

$$\left\{ \frac{1}{m} + \frac{l}{c} : 0 \leq l \leq c - 1 \right\}.$$

(ii) Put $N = |N(\beta)| b/m$ and $\delta = a + b\lambda$. The set $R(2)$ is

$$\{(x_1, x_2) : 1 \leq j \leq c, 1 \leq s \leq mcN\}.$$

where

$$x_1 = \frac{j - \left\{ -\frac{c}{m} + \frac{\delta s}{mN} \right\}}{c},$$
$$x_2 = \frac{\delta}{mcN}.$$

We define Bernoulli polynomials attached to linear forms for matrices

$$A_1 = \begin{pmatrix} m & m \end{pmatrix}, \qquad A_2 = \begin{pmatrix} m & m \\ m\varepsilon & m\varepsilon' \end{pmatrix},$$

which are defined to be

$$L(t) = A_1 \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}, \qquad \begin{pmatrix} L_1(t) \\ L_2(t) \end{pmatrix} = A_2 \begin{pmatrix} t_1 \\ t_2 \end{pmatrix},$$

where we abbreviate (t_1, t_2) to t. A Bernoulli polynomial $B_n^{(i)}(A_1; x)$ with respect to $L(t)$ is defined from a function

$$(2) \qquad \frac{1}{uL(t)} \cdot \frac{uL(t)e^{uL(t)x}}{e^{uL(t)} - 1}\Bigg|_{t_i=1}.$$

We expand this function as a Laurent series and denote the coefficient of $u^{2(n-1)}t_{3-i}^{n-1}$ as

$$\frac{1}{(n!)^2} B_n^{(i)}(A_1; x).$$

By symmetricity of t_1 and t_2, we see $B_n^{(1)}(A_1; x) = B_n^{(2)}(A_1; x)$. The Bernoulli polynomial relative to A_1 is

$$B_n(A_1; x) = B_n^{(1)}(A_1; x).$$

A Bernoulli polynomial $B_n^{(i)}(A_2; x_1, x_2)$ relative to A_2 is defined similarly from a function

$$(3) \qquad \prod_{j=1}^{2} \frac{1}{uL_j(t)} \cdot \frac{uL_j(t)e^{uL_j(t)x_j}}{e^{uL_j(t)} - 1}\Bigg|_{t_i=1}.$$

After expanding this expression, we take the coefficient of $u^{2(n-1)}t_{3-i}^{n-1}$ and express it as

$$\frac{1}{(n!)^2} B_n^{(i)}(A_2; x_1, x_2).$$

Since ε and ε' are conjugate to each other with respect to the action of the Galois group of k, $B_n^{(2)}(A_1; x)$ is conjugate to $B_n^{(1)}(A_1; x)$. The Bernoulli polynomial relative to A_2 is a polynomial of two variables defined to be

$$B_n(A_2; x_1, x_2) = \frac{1}{2} Tr\left(B_n^{(1)}(A_2; x_1, x_2)\right).$$

Then, by the Shintai formula, we obtain

THEOREM 2. For $p = 1, 2, \cdots$, the value of $\zeta((\alpha), (m), 1-p)$ is given by a formula

$$-\frac{|N(\alpha)|^{p-1}}{p^2} \sum_{x \in R(1)} B_p(A_1, x) + \frac{|N(\alpha)|^{p-1}}{p^2} \sum_{(x_1, x_2) \in R(2)} B_p(A_2; x_1, x_2).$$

3. The Carlitz sum. The primary Bernoulli polynomial $B_n(x)$ is defined from a generating function

$$\frac{te^{tx}}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} t^n.$$

We use the following well-known formulae:

(I) $B_n = B_n(0)$.

(II) $B_n(1 - x) = (-1)^n B_n(x)$.

(III) $\sum_{j=1}^{k-1} B_n\left(x + \frac{j}{k}\right) = k^{1-n} B_n(kx)$.

Let Δ be a monoid generated by $X = X^1$. Δ is a set $\{X^0, X^1, X^2, \cdots\}$ on which multiplication is defined to be $X^m X^n = X^{m+n}$. Denote by $\boldsymbol{Q}\Delta$ a linear space over $\boldsymbol{Q}$ generated by Δ. We make this linear space a ring by defining multiplication as extension of that of Δ. When we write

$$f(X)|_{X^n \to B_n}$$

for an element $f(x)$ of $\boldsymbol{Q}\Delta$, it means to replace each X^n contained in $f(X)$ to a Bernoulli number B_n. We also use

$$f(X)|_{X^n \to B_n(x)}$$

to replace each X^n in $f(X)$ to a Bernoulli polynomial $B_n(x)$. These replacement symbol is extensively used in two variables case such as

$$f(X,Y)|_{X^mY^n\to B_mB_n}.$$

We note the element $f(X)$ is needed to be expanded completely before replacement.

Let h and k be integers such that $(h,k)=1$ and $k>0$. In [2], a sum

$$c_{r,s}=\sum_{j=0}^{k-1}B_r\left(\frac{j}{k}\right)B_s\left(\left\{\frac{hj}{k}\right\}\right)$$

is defined and deduces relations from the transformation formula of the Lambert series. This sum is extension of the generalized Dedekind sum defined in [1]. By the property (II) of Bernoulli polynomials, we have

$$\begin{aligned}\sum_{j=0}^{k-1}&B_r\left(\frac{j}{k}\right)B_s\left(\left\{\frac{hj}{k}\right\}\right)\\ &=\sum_{j=1}^{k}B_r\left(\frac{k-j}{k}\right)B_s\left(\left\{\frac{h(k-j)}{k}\right\}\right)\\ &=(-1)^{r+s}\sum_{j=1}^{k}B_r\left(\frac{j}{k}\right)B_s\left(\left\{\frac{hj}{k}\right\}\right),\end{aligned}$$

which implies

$$\begin{aligned}(-1)^{r+s}&c_{r,s}(h,k)=\\ &c_{r,s}(h,k)+B_rB_s-(-1)^rB_rB_s.\end{aligned}$$

Therefore, $c_{r,s}(h,k)=0$ if $r+s$ is odd and $r\neq 1$. If $r=1$, we see $2c_{r,s}(h,k)=B_s$.

In the below, we assume $r+s$ is even and abuse notation and write $c_r(h,k)$ for $c_{2n-r,r}(h,k)$:

$$c_r(h,k)=\sum_{j=0}^{k-1}B_{2n-r}\left(\frac{j}{k}\right)B_r\left(\left\{\frac{hj}{k}\right\}\right)$$

We have

$$c_0(h,k)=c_{2n}(h,k)=\frac{B_n}{k^{2n-1}}$$

by the property (III) of Bernoulli polynomials. A formula describing relations between $c_r(h,k)$'s was given in Theorem 2, [3]:

THEOREM 3 (L. Carlitz). Let h and k be positive integers such that $(h,k)=1$. For $0\le r\le 2n-1$,

$$\begin{aligned}&\binom{2n}{r+1}hk^{2n-1}c_{2n-1-r}(h,k)+\binom{2n}{r}k^{2n-1}c_{2n-r}(h,k)\\ &=\binom{2n}{r+1}h^{2n-1-r}\left(X-kX^0\right)^{r+1}X^{2n-1-r}|_{X^p\to c_p(k,h)}\\ &\qquad+\binom{2n}{r}\left(Xk+Yh\right)^{2n-r}Y^r|_{X^pY^q\to B_pB_q}.\end{aligned}$$

COROLLARY 4. Set $n=2$. Then,

(i) for $r=0$,

$$\begin{aligned}&4hk^3c_3(h,k)=\\ &-4h^3kc_3(k,h)-\frac{h^4}{30}+\frac{h^2k^2}{6}-\frac{k^4}{30}-\frac{1}{10},\end{aligned}$$

(ii) for $r=1$,

$$\begin{aligned}&6hk^3c_2(h,k)+4k^3c_3(h,k)=\\ &\quad-12h^2kc_3(k,h)+6h^2k^2c_2(k,h)\\ &\qquad-\frac{2h^3}{15}+\frac{hk^2}{3}-\frac{1}{5h},\end{aligned}$$

(iii) for $r=2$,

$$\begin{aligned}&4hk^3c_1(h,k)+6k^3c_2(h,k)=\\ &-4hk^3c_1(k,h)+12hk^2c_2(k,h)-12hkc_3(k,h)\\ &\qquad+\frac{k^2}{6}-\frac{2}{15h^2}-\frac{h^2}{5},\end{aligned}$$

(iv) for $r = 3$,

$$4k^3c_1(h,k) = -4kc_3(k,h) + 6k^2c_2(k,h) - 4k^3c_1(k,h) - \frac{1}{30h^3} - \frac{k^4}{30h^3} - \frac{h}{10}.$$

We deduce reciprocity laws from this corollary.

THEOREM 5. We have the following formulae:

(i)

$$k^2c_3(h,k)+h^2c_3(k,h) = -\frac{k^3}{120h} - \frac{h^3}{120k} + \frac{hk}{24} - \frac{1}{40hk}.$$

(ii)

$$kc_2(h,k)-hc_2(k,h) = +\frac{2}{3}\left(\frac{k}{h}c_3(h,k) - \frac{h}{k}c_3(k,h)\right) + \frac{1}{90}\left(\frac{k^2}{h^2} - \frac{h^2}{k^2}\right).$$

(iii)

$$c_1(h,k)+c_1(k,h) = \frac{3}{4}\left(\frac{c_2(h,k)}{h} + \frac{c_2(k,h)}{k}\right) - \frac{1}{2}\left(\frac{c_3(h,k)}{h^2} + \frac{c_3(k,h)}{k^2}\right) - \frac{1}{60}\left(\frac{h}{h^3} + \frac{k}{h^3}\right) - \frac{1}{120h^3k^3}.$$

Proof. The formula (i) follows from Corollary 4, immediately. We also have

$$6hk^3c_2(h,k) - 6h^2k^2c_2(k,h) = -4(k^3c_3(h,k) + 3h^2kc_3(k,h)) - \frac{1}{5h} + \frac{hk^2}{3} - \frac{2h^3}{15}$$

Hence,

$$kc_2(h,k) - hc_2(k,h) = -\frac{2}{3hk^2}\left(k^3c_3(h,k) + 3h^2kc_3(k,h)\right) - \frac{1}{30h^2k^2} + \frac{1}{18} - \frac{h^2}{45k^2}$$

Applying the formula (i), we have

$$kc_2(j,k) - hc_2(k,h) = -\frac{4h}{3k}c_3(k,h) \quad \frac{1}{180}\left(\frac{h^2}{k^2} + \frac{k^2}{h^2}\right) - \frac{1}{60h^2k^2} - \frac{h^2}{45k^2} + \frac{1}{36}.$$

Let $\sigma = (h,k)$ be a transposition of symbols h and k. Let ι be the identity transposition. We apply the alternizer $\mathcal{A} = \frac{1}{2}(\iota - \sigma)$ to the expression and obtain

$$kc_2(h,k) - hc_2(k,h) = \frac{2}{3}\left(\frac{k}{h}(c_3(h,k) - \frac{h}{k}c_3(k,h)\right) + \frac{1}{90}\left(\frac{k^2}{h^2} - \frac{h^2}{k^2}\right)$$

Similarly, applying the symmetrizer $\mathcal{S} = \frac{1}{2}(\iota + \sigma)$ to the expression of (iii) in Corollary 4, the formula (iv) is deduced. □

We note $c_3(h,k) = c_1(h',k)$ holds for an integer h' such that $h'h \equiv 1 \mod k$, because of an equality

$$\sum_{j=0}^{k-1} B_1\left(\frac{j}{k}\right) B_3\left(\left\{\frac{hj}{k}\right\}\right) = \sum_{j=0}^{k-1} B_1\left(\left\{\frac{h'j}{k}\right\}\right) B_3\left(\frac{j}{k}\right).$$

4. The formula of $\zeta((\alpha),(m),-1)$. We calculate explicit expression of $B_2(A_1,x)$ and $B_2(A_2;x_1,x_2)$. The coefficient of $u^{2(n-1)}$ in the expansion of (2) is

$$\frac{B_{2n-1}(x)}{(2n-1)!}m^{2(n-1)}(1+t_2)^{2(n-1)}.$$

Thus,

$$\frac{B_n^{(1)}(A_1;x)}{(n!)^2} = \binom{2n-2}{n-1}\frac{m^{2n-2}}{(2n-1)!}B_{2n-1}(x).$$

Setting $n=2$, we have

$$B_2(A_1;x)=\frac{4m^2}{3}B_3(x).$$

Hence,

$$\sum_{l=0}^{c-1}B_2\left(A_1;\frac{1}{m}+\frac{l}{c}\right)=\frac{4m^2}{3c^2}B_3\left(\frac{c}{m}\right). \tag{4}$$

The Bernoulli polynomial $B_2(A_2;x_1,x_2)$ is yielded by expanding (3). Put $u'=mu$. The coefficient of $u'^{2(n-1)}$ in the expansion is

$$\frac{1}{(2n)!}\sum_{j=0}^{2n}\binom{2n}{j}B_{2n-j}(x_1)B_j(x_2)\times(1+t_2)^{2n-j-1}(\varepsilon+\varepsilon' t_2)^{j-1}.$$

Since $\varepsilon\varepsilon'=1$, we replace $\varepsilon+\varepsilon' t_2$ to $\varepsilon(1+\varepsilon^{-2}t_2)$. Let P_j be the coefficient of t_2^{n-1} in $(1+t_2)^{2n-j-1}(1+\varepsilon^{-2}t_2)^{j-1}$. We compute

$$L_j=\binom{2n}{j}C_nB_{2n-j}(x_1)B_j(x_2)\varepsilon^{j-1}P_j$$

where $C_n=\dfrac{m^{2n-2}}{\binom{2n}{n}}$.

(i) $j=0$

$$P_0=\sum_{p+q=n-1}\binom{2n-1}{p}(-1)^q(\varepsilon^{-2})^q.$$
$$L_0=C_nB_{2n}(x_1)\varepsilon^{-1}\sum_{q=0}^{2n-1}\binom{2n-1}{n-1-q}(-\varepsilon^{-2})^q$$

(ii) $j=2n$

$$P_{2n}=\sum_{p+q=n-1}\binom{2n-1}{q}\varepsilon^{-2q}(-1)^p.$$
$$L_{2n}=C_nB_{2n}(x_2)\varepsilon\sum_{p=0}^{n-1}\binom{2n-1}{n-1-p}(-\varepsilon^2)^p$$

(iii) $0<j<n$

$$P_j=\sum_{q=0}^{j-1}\binom{2n-j-1}{n-1-q}\binom{j-1}{q}\varepsilon^{-2q}.$$
$$L_j=\binom{2n}{j}C_nB_{2n-j}(x_1)B_j(x_2)\varepsilon^{j-1}\times\sum_{q=0}^{j-1}\binom{2n-j-1}{n-1-q}\binom{j-1}{q}\varepsilon^{-2q}.$$

(iv) $n\le j<2n$

$$P_j=\sum_{q=j-n}^{n-1}\binom{2n-j-1}{n-1-q}\binom{j-1}{q}\varepsilon^{-2q}.$$
$$L_j=\binom{2n}{j}C_nB_{2n-j}(x_1)B_j(x_2)\varepsilon^{j-1}\times\sum_{q=j-n}^{n-1}\binom{2n-j-1}{n-1-q}\binom{j-1}{q}\varepsilon^{-2q}.$$

Set $n=2$. We see $C_n=\frac{m^2}{6}$ and

$$L_0=\frac{m^2(3\varepsilon^{-1}-\varepsilon^{-3})}{6}B_4(x_1)$$
$$L_4=\frac{m^2(3\varepsilon-\varepsilon^{3})}{6}B_4(x_2)$$
$$L_1=\frac{4m^2}{3}B_3(x_1)B_1(x_2)$$
$$L_2=m^2Tr(\varepsilon)B_2(x_1)B_2(x_2)$$
$$L_3=\frac{4m^2}{3}B_1(x_1)B_3(x_2)$$

Therefore, the value of $\zeta((\beta c),(m),-1)$ equals

$$-\frac{|N(\beta)|c^2}{4}\cdot\frac{4m^2}{3c^2}B_3\left(\frac{c}{m}\right)+\frac{|N(\beta)|c^2}{4}\sum_{x_1,x_2}\sum_{i=0}^{4}L_i$$

We introduce a modified Carlitz sum

$$c_r(h,k;c')=\sum_{j=0}^{k-1}B_{2n-r}\left(\frac{j}{k}\right)B_r\left(\left\{\frac{hj+c'}{k}\right\}\right).$$

Put

$$\tau(s) = \left\{-\frac{c}{m} + \frac{\delta s}{mN}\right\},$$

We have

$$x_1 = \frac{j - \tau(s)}{c}.$$

Hence, $c_r(\delta, mcN; c)$ equals

$$S_r = \sum_{j=1}^{c} \sum_{s=1}^{mcN} B_{4-r}\left(\frac{s}{mcN}\right) B_r\left(\frac{j-\tau(s)}{c}\right).$$

We compute S_4 as follows:

$$\begin{aligned}
&\sum_{j=1}^{c}\sum_{s=1}^{mcN} B_4\left(\frac{s}{mcN}\right) \\
&= c\sum_{j=0}^{c-1}\sum_{t=1}^{mN} B_4\left(\frac{t+jmN}{mcN}\right) \\
&= c^{-2}\sum_{t=1}^{mN} B_4\left(\frac{t}{mN}\right) \\
&= c^{-2}\left(\sum_{t=0}^{mN-1} B_4\left(\frac{t}{mN}\right) - B_4 + B_4(1)\right) \\
&= -\frac{1}{30m^3N^3c^2}.
\end{aligned}$$

Since

$$(5) \quad \sum_{j=1}^{c} B_r\left(\frac{j-\tau(s)}{c}\right) = \sum_{j=0}^{c-1} B_r\left(\frac{j+1-\tau(s)}{c}\right) = c^{1-r}B_r(1-\tau(s)),$$

we also have

$$S_0 = -\frac{1}{30m^3N^3c^2}.$$

The deformation (5) is also available to S_r when $1 \le r \le 3$. We obtain

$$\begin{aligned}
S_r &= c^{1-r}\sum_{s=1}^{mcN} B_{4-r}\left(\frac{s}{mcN}\right) B_r(1-\tau(s)) \\
&= c^{-2}\sum_{s=1}^{mN} B_{4-r}\left(\frac{s}{mN}\right) B_r(1-\tau(s)).
\end{aligned}$$

Here, by changing s to $mN - s$, the sum

$$\sum_{s=1}^{mN} B_{4-r}\left(\frac{s}{mN}\right) B_r(1-\tau(s))$$

is transformed to

$$\sum_{s=0}^{mN-1} B_{4-r}\left(\frac{mN-s}{mN}\right) B_r\left(1 - \left\{-\frac{c}{m} + \frac{\delta(mN-s)}{mN}\right\}\right),$$

Hence,

$$\begin{aligned}
S_r &= (-1)^r c^{-2} \sum_{s=0}^{mN-1} B_{4-r}\left(\frac{s}{mN}\right) B_r\left(\left\{\frac{c}{m} + \frac{\delta s}{mN}\right\}\right) \\
&= (-1)^r c^{-2} c_r(\delta, mN; cN).
\end{aligned}$$

The formula in Theorem 2 is described explicitly with $c_r(\delta, mcN; c')$.

THEOREM 6. The quotient of the value of $\zeta((\beta c), (m), -1)$ by $|N(\beta)|$ is equal to the value of an expression

$$\begin{aligned}
&-\frac{m^2}{3}B_3\left(\frac{c}{m}\right) + \frac{Tr(\varepsilon^3) - 3\varepsilon}{720mN^3} \\
&-\frac{m^2}{3}\left(c_1(\delta, mN; cN) + c_3(\delta, mN; cN)\right) \\
&\qquad + \frac{m^2 Tr(\varepsilon)}{4} c_2(\delta, mN; cN).
\end{aligned}$$

We needed to obtain the value of the generalized Dedekind sum $s(h, k; c)$ in [6] to compute the value $\zeta((\alpha), (m), 0)$. Since the reciprocity formula of $s(h, k; c) + s(k, h; c)$ is known by [4] and its elementary proof was given in [5], the value of $s(h, k; c)$ can be computed rapidly by using this reciprocity law and the Euclidean algorithm. The analogous method is expected to the modified Carlitz sum. Denote by $\Delta_r(h, k; c)$ the difference of $c_r(h, k; c)$ from $c_r(h, k)$:

$$c_r(h, k; c) = c_r(h, k) + \Delta_r(h, k; c).$$

The formula of Theorem 6 is separated into two factors. We note $N = |N(\beta)|b/m$ and

δ is obtained from a solution of congruence equation (1). Hence, a part

$$\frac{Tr(\varepsilon^3-3\varepsilon)}{720mN^3}-\frac{m^2}{3}\left(c_1(\delta,mN)+c_3(\delta,mN)\right)+\frac{m^2Tr(\varepsilon)}{4}c_2(\delta,mN),$$

is dependent only on β and b. This sum is computable by means of Theorem 5. The remainder sum

$$-\frac{m^2}{3}(\Delta_1(\delta,mN;cN)+\Delta_3(\delta,mN;cN))+\frac{m^2Tr(\varepsilon)}{4}\Delta_2(\delta,mN;cN)-\frac{m^2}{3}B_3\left(\frac{c}{m}\right)$$

is determined with depending on c.

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