# On the value at s=-1 of partial zeta functions for ray classes of real quadratic field


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# On the value at s=−1 of partial zeta functions for ray classes of real quadratic field

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**Abstract.** Let $k$ be a real quadratic field. Let $\alpha$ be an integer of $k$ and $m$ be a positive rational integer. Denote by $\zeta((\alpha),(m),s)$ be a partial zeta function associated to a ray class containing the principal ideal $(\alpha)$ which is defined with a conductor $(m)$. We give a formula of the value of $\zeta((\alpha),(m),-1)$ by means of the modification of the sum defined by L. Carlitz, *c.f.* [2]. It is an analogue to the generalized Dedekind sum defined in [6].

**1. Introduction.** We studied the value at zero of a partial zeta function on a real quadratic field in [8, 9, 10], based on the Shintai formula in [7]. A formula of the value at $s=0$ for a ray class containing a principal ideal in a real quadratic field was given and it was shown that the value is able to be computed when the conductor of the ray class group is a positive integer. In the present paper, we shall study that the value at $s=-1$ with the analogous method. To this end, we use a sum

$$c_{r,s}(h,k)=\sum_{(\mu) \bmod k} B_r\left(\frac{\mu}{k}\right) B_s\left(\left\{\frac{h\mu}{k}\right\}\right)$$

defined from the transformation of Lambert series in [2], where $\{x\}$ is the fractional part of a real number $x$. This sum is extension of the generalized Dedekind sum in [1]. We call it the Carlitz sum in this paper.

**2. Review of the Shintai formula.** Let $\alpha$ be an integer of a real quadratic field $k$. Denote by $(\alpha)$ an ideal generated by $\alpha$. We assume $(\alpha)$ is prime to the conductor $(m)$ of the ray class group. The partial zeta function associated to a ray class containing $(\alpha)$ is defined to be

$$\zeta((\alpha),(m),s)=\sum_{\mathfrak{b}} \frac{1}{N\mathfrak{b}^s}, \qquad s>1,$$

which is continued meromorphically to the complex plain, where $\mathfrak{b}$ runs the set of every integral ideals contained in the ray class. Let $\sigma_1$ and $\sigma_2$ be embeddings of $k$

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into the field of real numbers. We identify $x$ with $\sigma_1 x$ and write $\sigma_2 x$ as $x'$. The quadratic field $k$ is mapped into a subset of a two dimensional real vector space by $x \longrightarrow (\sigma_1 x, \sigma_2 x)$. Let $C$ be the set of every vectors whose coordinates are positive. Let $E_+(m)$ be a subgroup of units which are congruent to 1 with modulo $m$ and totally positive. We see $E_+(m)$ is a free abelian group of rank one. Let $\varepsilon$ be a generator of $E_+(m)$ such that $\varepsilon > \varepsilon'$. The group $E_+(m)$ acts on the set $C$ by multiplication. Let $C_1$ be a subset of $C$ generated by a vector $v_1 = (m, m)$:

$$C_1 = \{xv : x > 0\}.$$

Set $v_2 = (m\varepsilon, m\varepsilon')$. Let $C_2$ be a subset $C$ consisting of every vectors $x_1 v_1 + x_2 v_2$, $x_1 > 0$, $x_2 > 0$. The union of these two sets is a fundamental domain with respect to action of $E_+(m)$. Let $S_1$ (*resp.* $S_2$) be a bounded subset of $C_1$ (*resp.* $C_2$) of every element such that $0 < x \leq 1$ (*resp.* $0 < x_1, x_2 \leq 1$). Since the fractional ideal $(m^{-1}\alpha)$ is mapped into a discrete set in the two dimensional vector space, the intersection $(1 + (m^{-1}\alpha)) \cap S_i$ is a finite set, where $1 + (m^{-1}\alpha)$ is a set of elements $1 + y$ for every $y \in (m^{-1}\alpha)$. We need the sets of coordinates of these finite sets to formulate the value of the partial zeta function. Namely, put

$$R(1) = \{x : xv_1 \in S_1\},$$
$$R(2) = \{(x_1, x_2) : x_1 v_1 + x_2 v_2 \in S_2\}.$$

These sets $R(i)$ are determined concretely in [8]. Let 1, $w$ be the standard basis of the ring of integers of $k$, such that $w > 0$. We write $\alpha$ as a $Z$-linear combination concerning this basis and decompose it as

$$\alpha = c(f_1 + f_2 w), \quad c > 0, \quad (f_1, f_2) = 1.$$

Let $a$ and $b$ be positive integers such that

$$\varepsilon = a + bw.$$

We see $a \equiv 1 \mod m$ and $b \equiv 0 \mod m$. We set $\lambda = 0$ if $f_2 = 0$. When $f_2 \neq 0$, we solve a congruent equation

$$(1) \qquad f_2 \lambda \equiv f_1 + f_2 Tr(w) \mod |N(\beta)|$$

and set the solution to $\lambda$.

PROPOSITION 1. We have

(i) The set $R(1)$ is equal to

$$\left\{\frac{1}{m} + \frac{l}{c} : 0 \leq l \leq c-1\right\}.$$

(ii) Put $N = |N(\beta)|b/m$ and $\delta = a + b\lambda$. The set $R(2)$ is

$$\{(x_1, x_2) : 1 \leq j \leq c, 1 \leq s \leq mcN\}.$$

where

$$x_1 = \frac{j - \left\{-\frac{c}{m} + \frac{\delta s}{mN}\right\}}{c},$$
$$x_2 = \frac{\delta}{mcN}.$$

We define Bernoulli polynomials attached to linear forms for matrices

$$A_1 = \begin{pmatrix} m & m \end{pmatrix}, \qquad A_2 = \begin{pmatrix} m & m \\ m\varepsilon & m\varepsilon' \end{pmatrix},$$

which are defined to be

$$L(t) = A_1 \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}, \qquad \begin{pmatrix} L_1(t) \\ L_2(t) \end{pmatrix} = A_2 \begin{pmatrix} t_1 \\ t_2 \end{pmatrix},$$

where we abbreviate $(t_1, t_2)$ to $t$. A Bernoulli polynomial $B_n^{(i)}(A_1; x)$ with respect to $L(t)$ is defined from a function

$$(2) \qquad \frac{1}{uL(t)} \cdot \frac{uL(t)e^{uL(t)x}}{e^{uL(t)} - 1}\Bigg|_{t_i=1}.$$

We expand this function as a Laurent series and denote the coefficient of $u^{2(n-1)}t_{3-i}^{n-1}$ as

$$\frac{1}{(n!)^2} B_n^{(i)}(A_1; x).$$

By symmetricity of $t_1$ and $t_2$, we see $B_n^{(1)}(A_1; x) = B_n^{(2)}(A_1; x)$. The Bernoulli polynomial relative to $A_1$ is

$$B_n(A_1; x) = B_n^{(1)}(A_1; x).$$

A Bernoulli polynomial $B_n^{(i)}(A_2; x_1, x_2)$ relative to $A_2$ is defined similarly from a function

$$(3) \qquad \prod_{j=1}^{2} \frac{1}{uL_j(t)} \cdot \frac{uL_j(t)e^{uL_j(t)x_j}}{e^{uL_j(t)} - 1}\Bigg|_{t_i=1}.$$

After expanding this expression, we take the coefficient of $u^{2(n-1)}t_{3-i}^{n-1}$ and express it as

$$\frac{1}{(n!)^2} B_n^{(i)}(A_2; x_1, x_2).$$

Since $\varepsilon$ and $\varepsilon'$ are conjugate to each other with respect to the action of the Galois group of $k$, $B_n^{(2)}(A_1; x)$ is conjugate to $B_n^{(1)}(A_1; x)$. The Bernoulli polynomial relative to $A_2$ is a polynomial of two variables defined to be

$$B_n(A_2; x_1, x_2) = \frac{1}{2} Tr\left(B_n^{(1)}(A_2; x_1, x_2)\right).$$

Then, by the Shintai formula, we obtain

THEOREM 2. For $p = 1, 2, \cdots$, the value of $\zeta((\alpha), (m), 1-p)$ is given by a formula

$$-\frac{|N(\alpha)|^{p-1}}{p^2} \sum_{x \in R(1)} B_p(A_1, x) + \frac{|N(\alpha)|^{p-1}}{p^2} \sum_{(x_1, x_2) \in R(2)} B_p(A_2; x_1, x_2).$$

**3. The Carlitz sum.** The primary Bernoulli polynomial $B_n(x)$ is defined from a generating function

$$\frac{te^{tx}}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} t^n.$$

We use the following well-known formulae:

(I) $B_n = B_n(0)$.

(II) $B_n(1-x) = (-1)^n B_n(x)$.

(III) $\sum_{j=1}^{k-1} B_n\left(x + \frac{j}{k}\right) = k^{1-n} B_n(kx)$.

Let $\Delta$ be a monoid generated by $X = X^1$. $\Delta$ is a set $\{X^0, X^1, X^2, \cdots\}$ on which multiplication is defined to be $X^m X^n = X^{m+n}$. Denote by $\boldsymbol{Q}\Delta$ a linear space over $\boldsymbol{Q}$ generated by $\Delta$. We make this linear space a ring by defining multiplication as extension of that of $\Delta$. When we write

$$f(X)|_{X^n \to B_n}$$

for an element $f(x)$ of $\boldsymbol{Q}\Delta$, it means to replace each $X^n$ contained in $f(X)$ to a Bernoulli number $B_n$. We also use

$$f(X)|_{X^n \to B_n(x)}$$

to replace each $X^n$ in $f(X)$ to a Bernoulli polynomial $B_n(x)$. These replacement symbol

is extensively used in two variables case such as

$$f(X,Y)|_{X^mY^n\to B_mB_n}.$$

We note the element $f(X)$ is needed to be expanded completely before replacement.

Let $h$ and $k$ be integers such that $(h,k)=1$ and $k>0$. In [2], a sum

$$c_{r,s}=\sum_{j=0}^{k-1}B_r\left(\frac{j}{k}\right)B_s\left(\left\{\frac{hj}{k}\right\}\right)$$

is defined and deduces relations from the transformation formula of the Lambert series. This sum is extension of the generalized Dedekind sum defined in [1]. By the property (II) of Bernoulli polynomials, we have

$$\begin{aligned}&\sum_{j=0}^{k-1}B_r\left(\frac{j}{k}\right)B_s\left(\left\{\frac{hj}{k}\right\}\right)\\&=\sum_{j=1}^{k}B_r\left(\frac{k-j}{k}\right)B_s\left(\left\{\frac{h(k-j)}{k}\right\}\right)\\&=(-1)^{r+s}\sum_{j=1}^{k}B_r\left(\frac{j}{k}\right)B_s\left(\left\{\frac{hj}{k}\right\}\right),\end{aligned}$$

which implies

$$\begin{aligned}&(-1)^{r+s}c_{r,s}(h,k)=\\&\qquad c_{r,s}(h,k)+B_rB_s-(-1)^rB_rB_s.\end{aligned}$$

Therefore, $c_{r,s}(h,k)=0$ if $r+s$ is odd and $r\neq 1$. If $r=1$, we see $2c_{r,s}(h,k)=B_s$.

In the below, we assume $r+s$ is even and abuse notation and write $c_r(h,k)$ for $c_{2n-r,r}(h,k)$:

$$c_r(h,k)=\sum_{j=0}^{k-1}B_{2n-r}\left(\frac{j}{k}\right)B_r\left(\left\{\frac{hj}{k}\right\}\right)$$

We have

$$c_0(h,k)=c_{2n}(h,k)=\frac{B_n}{k^{2n-1}}$$

by the property (III) of Bernoulli polynomials. A formula describing relations between $c_r(h,k)$'s was given in Theorem 2, [3]:

THEOREM 3 (L. Carlitz). Let $h$ and $k$ be positive integers such that $(h,k)=1$. For $0\le r\le 2n-1$,

$$\begin{aligned}&\binom{2n}{r+1}hk^{2n-1}c_{2n-1-r}(h,k)+\binom{2n}{r}k^{2n-1}c_{2n-r}(h,k)\\&=\binom{2n}{r+1}h^{2n-1-r}\,(X-kX^0)^{r+1}X^{2n-1-r}|_{X^p\to c_p(k,h)}\\&\qquad+\binom{2n}{r}(Xk+Yh)^{2n-r}Y^r|_{X^pY^q\to B_pB_q}.\end{aligned}$$

COROLLARY 4. Set $n=2$. Then,

(i) for $r=0$,

$$\begin{aligned}&4hk^3c_3(h,k)=\\&-4h^3kc_3(k,h)-\frac{h^4}{30}+\frac{h^2k^2}{6}-\frac{k^4}{30}-\frac{1}{10},\end{aligned}$$

(ii) for $r=1$,

$$\begin{aligned}&6hk^3c_2(h,k)+4k^3c_3(h,k)=\\&-12h^2kc_3(k,h)+6h^2k^2c_2(k,h)\\&\qquad-\frac{2h^3}{15}+\frac{hk^2}{3}-\frac{1}{5h},\end{aligned}$$

(iii) for $r=2$,

$$\begin{aligned}&4hk^3c_1(h,k)+6k^3c_2(h,k)=\\&-4hk^3c_1(k,h)+12hk^2c_2(k,h)-12hkc_3(k,h)\\&\qquad+\frac{k^2}{6}-\frac{2}{15h^2}-\frac{h^2}{5},\end{aligned}$$

(iv) for $r = 3$,

$$4k^3c_1(h,k) = -4kc_3(k,h) + 6k^2c_2(k,h) - 4k^3c_1(k,h) - \frac{1}{30h^3} - \frac{k^4}{30h^3} - \frac{h}{10}.$$

We deduce reciprocity laws from this corollary.

THEOREM 5. We have the following formulae:

(i)

$$k^2c_3(h,k)+h^2c_3(k,h) = -\frac{k^3}{120h} - \frac{h^3}{120k} + \frac{hk}{24} - \frac{1}{40hk}.$$

(ii)

$$kc_2(h,k)-hc_2(k,h) = +\frac{2}{3}\left(\frac{k}{h}c_3(h,k) - \frac{h}{k}c_3(k,h)\right) + \frac{1}{90}\left(\frac{k^2}{h^2} - \frac{h^2}{k^2}\right).$$

(iii)

$$c_1(h,k)+c_1(k,h) = \frac{3}{4}\left(\frac{c_2(h,k)}{h} + \frac{c_2(k,h)}{k}\right) - \frac{1}{2}\left(\frac{c_3(h,k)}{h^2} + \frac{c_3(k,h)}{k^2}\right) - \frac{1}{60}\left(\frac{h}{h^3} + \frac{k}{h^3}\right) - \frac{1}{120h^3k^3}.$$

*Proof.* The formula (i) follows from Corollary 4, immediately. We also have

$$6hk^3c_2(h,k) - 6h^2k^2c_2(k,h) = -4(k^3c_3(h,k) + 3h^2kc_3(k,h)) - \frac{1}{5h} + \frac{hk^2}{3} - \frac{2h^3}{15}$$

Hence,

$$kc_2(h,k) - hc_2(k,h) = -\frac{2}{3hk^2}\left(k^3c_3(h,k) + 3h^2kc_3(k,h)\right) - \frac{1}{30h^2k^2} + \frac{1}{18} - \frac{h^2}{45k^2}$$

Applying the formula (i), we have

$$kc_2(j,k) - hc_2(k,h) = -\frac{4h}{3k}c_3(k,h) \quad \frac{1}{180}\left(\frac{h^2}{k^2} + \frac{k^2}{h^2}\right) - \frac{1}{60h^2k^2} - \frac{h^2}{45k^2} + \frac{1}{36}.$$

Let $\sigma = (h,k)$ be a transposition of symbols $h$ and $k$. Let $\iota$ be the identity transposition. We apply the alternizer $\mathcal{A} = \frac{1}{2}(\iota - \sigma)$ to the expression and obtain

$$kc_2(h,k) - hc_2(k,h) = \frac{2}{3}\left(\frac{k}{h}(c_3(h,k) - \frac{h}{k}c_3(k,h)\right) + \frac{1}{90}\left(\frac{k^2}{h^2} - \frac{h^2}{k^2}\right)$$

Similarly, applying the symmetrizer $\mathcal{S} = \frac{1}{2}(\iota + \sigma)$ to the expression of (iii) in Corollary 4, the formula (iv) is deduced. □

We note $c_3(h,k) = c_1(h',k)$ holds for an integer $h'$ such that $h'h \equiv 1 \mod k$, because of an equality

$$\sum_{j=0}^{k-1} B_1\left(\frac{j}{k}\right) B_3\left(\left\{\frac{hj}{k}\right\}\right) = \sum_{j=0}^{k-1} B_1\left(\left\{\frac{h'j}{k}\right\}\right) B_3\left(\frac{j}{k}\right).$$

**4. The formula of $\zeta((\alpha),(m),-1)$.** We calculate explicit expression of $B_2(A_1,x)$ and $B_2(A_2;x_1,x_2)$. The coefficient of $u^{2(n-1)}$ in the expansion of (2) is

$$\frac{B_{2n-1}(x)}{(2n-1)!}m^{2(n-1)}(1+t_2)^{2(n-1)}.$$

Thus,

$$\frac{B_n^{(1)}(A_1;x)}{(n!)^2} = \binom{2n-2}{n-1}\frac{m^{2n-2}}{(2n-1)!}B_{2n-1}(x).$$

Setting $n=2$, we have

$$B_2(A_1;x)=\frac{4m^2}{3}B_3(x).$$

Hence,

$$(4)\quad \sum_{l=0}^{c-1}B_2\left(A_1;\frac{1}{m}+\frac{l}{c}\right)=\frac{4m^2}{3c^2}B_3\left(\frac{c}{m}\right).$$

The Bernoulli polynomial $B_2(A_2;x_1,x_2)$ is yielded by expanding (3). Put $u'=mu$. The coefficient of $u'^{2(n-1)}$ in the expansion is

$$\frac{1}{(2n)!}\sum_{j=0}^{2n}\binom{2n}{j}B_{2n-j}(x_1)B_j(x_2)\times(1+t_2)^{2n-j-1}(\varepsilon+\varepsilon' t_2)^{j-1}.$$

Since $\varepsilon\varepsilon'=1$, we replace $\varepsilon+\varepsilon' t_2$ to $\varepsilon(1+\varepsilon^{-2}t_2)$. Let $P_j$ be the coefficient of $t_2^{n-1}$ in $(1+t_2)^{2n-j-1}(1+\varepsilon^{-2}t_2)^{j-1}$. We compute

$$L_j=\binom{2n}{j}C_nB_{2n-j}(x_1)B_j(x_2)\varepsilon^{j-1}P_j$$

where $C_n=\dfrac{m^{2n-2}}{\binom{2n}{n}}$.

(i) $j=0$

$$P_0=\sum_{p+q=n-1}\binom{2n-1}{p}(-1)^q(\varepsilon^{-2})^q.$$
$$L_0=C_nB_{2n}(x_1)\varepsilon^{-1}\sum_{q=0}^{2n-1}\binom{2n-1}{n-1-q}(-\varepsilon^{-2})^q$$

(ii) $j=2n$

$$P_{2n}=\sum_{p+q=n-1}\binom{2n-1}{q}\varepsilon^{-2q}(-1)^p.$$
$$L_{2n}=C_nB_{2n}(x_2)\varepsilon\sum_{p=0}^{n-1}\binom{2n-1}{n-1-p}(-\varepsilon^2)^p$$

(iii) $0<j<n$

$$P_j=\sum_{q=0}^{j-1}\binom{2n-j-1}{n-1-q}\binom{j-1}{q}\varepsilon^{-2q}.$$
$$L_j=\binom{2n}{j}C_nB_{2n-j}(x_1)B_j(x_2)\varepsilon^{j-1}\times\sum_{q=0}^{j-1}\binom{2n-j-1}{n-1-q}\binom{j-1}{q}\varepsilon^{-2q}.$$

(iv) $n\le j<2n$

$$P_j=\sum_{q=j-n}^{n-1}\binom{2n-j-1}{n-1-q}\binom{j-1}{q}\varepsilon^{-2q}.$$
$$L_j=\binom{2n}{j}C_nB_{2n-j}(x_1)B_j(x_2)\varepsilon^{j-1}\times\sum_{q=j-n}^{n-1}\binom{2n-j-1}{n-1-q}\binom{j-1}{q}\varepsilon^{-2q}.$$

Set $n=2$. We see $C_n=\frac{m^2}{6}$ and

$$L_0=\frac{m^2(3\varepsilon^{-1}-\varepsilon^{-3})}{6}B_4(x_1)$$
$$L_4=\frac{m^2(3\varepsilon-\varepsilon^3)}{6}B_4(x_2)$$
$$L_1=\frac{4m^2}{3}B_3(x_1)B_1(x_2)$$
$$L_2=m^2Tr(\varepsilon)B_2(x_1)B_2(x_2)$$
$$L_3=\frac{4m^2}{3}B_1(x_1)B_3(x_2)$$

Therefore, the value of $\zeta((\beta c),(m),-1)$ equals

$$-\frac{|N(\beta)|c^2}{4}\cdot\frac{4m^2}{3c^2}B_3\left(\frac{c}{m}\right)+\frac{|N(\beta)|c^2}{4}\sum_{x_1,x_2}\sum_{i=0}^{4}L_i$$

We introduce a modified Carlitz sum

$$c_r(h,k;c')=\sum_{j=0}^{k-1}B_{2n-r}\left(\frac{j}{k}\right)B_r\left(\left\{\frac{hj+c'}{k}\right\}\right).$$

Put

$$\tau(s) = \left\{-\frac{c}{m} + \frac{\delta s}{mN}\right\},$$

We have

$$x_1 = \frac{j - \tau(s)}{c}.$$

Hence, $c_r(\delta, mcN; c)$ equals

$$S_r = \sum_{j=1}^{c}\sum_{s=1}^{mcN} B_{4-r}\left(\frac{s}{mcN}\right) B_r\left(\frac{j-\tau(s)}{c}\right).$$

We compute $S_4$ as follows:

$$\begin{aligned}
&\sum_{j=1}^{c}\sum_{s=1}^{mcN} B_4\left(\frac{s}{mcN}\right)\\
&\quad = c\sum_{j=0}^{c-1}\sum_{t=1}^{mN} B_4\left(\frac{t+jmN}{mcN}\right)\\
&\quad = c^{-2}\sum_{t=1}^{mN} B_4\left(\frac{t}{mN}\right)\\
&\quad = c^{-2}\left(\sum_{t=0}^{mN-1} B_4\left(\frac{t}{mN}\right) - B_4 + B_4(1)\right)\\
&\quad = -\frac{1}{30m^3N^3c^2}.
\end{aligned}$$

Since

$$\sum_{j=1}^{c} B_r\left(\frac{j-\tau(s)}{c}\right) = \sum_{j=0}^{c-1} B_r\left(\frac{j+1-\tau(s)}{c}\right) = c^{1-r}B_r(1-\tau(s)), \tag{5}$$

we also have

$$S_0 = -\frac{1}{30m^3N^3c^2}.$$

The deformation (5) is also available to $S_r$ when $1 \le r \le 3$. We obtain

$$\begin{aligned}
S_r &= c^{1-r}\sum_{s=1}^{mcN} B_{4-r}\left(\frac{s}{mcN}\right) B_r(1-\tau(s))\\
&= c^{-2}\sum_{s=1}^{mN} B_{4-r}\left(\frac{s}{mN}\right) B_r(1-\tau(s)).
\end{aligned}$$

Here, by changing $s$ to $mN - s$, the sum

$$\sum_{s=1}^{mN} B_{4-r}\left(\frac{s}{mN}\right) B_r(1-\tau(s))$$

is transformed to

$$\sum_{s=0}^{mN-1} B_{4-r}\left(\frac{mN-s}{mN}\right) B_r\left(1-\left\{-\frac{c}{m}+\frac{\delta(mN-s)}{mN}\right\}\right),$$

Hence,

$$\begin{aligned}
S_r &= (-1)^r c^{-2}\sum_{s=0}^{mN-1} B_{4-r}\left(\frac{s}{mN}\right) B_r\left(\left\{\frac{c}{m}+\frac{\delta s}{mN}\right\}\right)\\
&= (-1)^r c^{-2} c_r(\delta, mN; cN).
\end{aligned}$$

The formula in Theorem 2 is described explicitly with $c_r(\delta, mcN; c')$.

THEOREM 6. The quotient of the value of $\zeta((\beta c), (m), -1)$ by $|N(\beta)|$ is equal to the value of an expression

$$\begin{aligned}
&-\frac{m^2}{3}B_3\left(\frac{c}{m}\right) + \frac{Tr(\varepsilon^3) - 3\varepsilon}{720mN^3}\\
&-\frac{m^2}{3}\left(c_1(\delta, mN; cN) + c_3(\delta, mN; cN)\right)\\
&\qquad + \frac{m^2 Tr(\varepsilon)}{4} c_2(\delta, mN; cN).
\end{aligned}$$

We needed to obtain the value of the generalized Dedekind sum $s(h, k; c)$ in [6] to compute the value $\zeta((\alpha), (m), 0)$. Since the reciprocity formula of $s(h, k; c) + s(k, h; c)$ is known by [4] and its elementary proof was given in [5], the value of $s(h, k; c)$ can be computed rapidly by using this reciprocity law and the Euclidean algorithm. The analogous method is expected to the modified Carlitz sum. Denote by $\Delta_r(h, k; c)$ the difference of $c_r(h, k; c)$ from $c_r(h, k)$:

$$c_r(h, k; c) = c_r(h, k) + \Delta_r(h, k; c).$$

The formula of Theorem 6 is separated into two factors. We note $N = |N(\beta)|b/m$ and

$\delta$ is obtained from a solution of congruence equation (1). Hence, a part

$$\frac{Tr(\varepsilon^3-3\varepsilon)}{720mN^3}-\frac{m^2}{3}\left(c_1(\delta,mN)+c_3(\delta,mN)\right)+\frac{m^2Tr(\varepsilon)}{4}c_2(\delta,mN),$$

is dependent only on $\beta$ and $b$. This sum is computable by means of Theorem 5. The remainder sum

$$-\frac{m^2}{3}(\Delta_1(\delta,mN;cN)+\Delta_3(\delta,mN;cN))+\frac{m^2Tr(\varepsilon)}{4}\Delta_2(\delta,mN;cN)-\frac{m^2}{3}B_3\left(\frac{c}{m}\right)$$

is determined with depending on $c$.

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