Estimates for singular integrals along surfaces of revolution

ESTIMATES FOR SINGULAR INTEGRALS ALONG SURFACES OF REVOLUTION

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Abstract. We prove certain L^p estimates $(1 < p < \infty)$ for nonisotropic singular integrals along surfaces of revolution. The singular integrals are defined by rough kernels. As an application we obtain L^p boundedness of the singular integrals under a sharp size condition on their kernels. We also prove a certain estimate for a trigonometric integral, which is useful in studying nonisotropic singular integrals.

1. Introduction

Let P be an $n \times n$ real matrix whose eigenvalues have positive real parts. Let $\gamma = \text{trace } P$. Define a dilation group $\{A_t\}_{t>0}$ on $\mathbb{R}^n$ by $A_t = t^P = \exp((\log t)P)$. We assume $n \geq 2$. There is a non-negative function r on $\mathbb{R}^n$ associated with $\{A_t\}_{t>0}$. The function r is continuous on $\mathbb{R}^n$ and infinitely differentiable in $\mathbb{R}^n \setminus \{0\}$; furthermore it satisfies

(1) $r(A_t x) = tr(x)$ for all $t > 0$ and $x \in \mathbb{R}^n$;
(2) $r(x + y) \leq C(r(x) + r(y))$ for some $C > 0$;
(3) if $\Sigma = \{x \in \mathbb{R}^n : r(x) = 1\}$, then $\Sigma = \{\theta \in \mathbb{R}^n : \langle B\theta, \theta\rangle = 1\}$ for a positive symmetric matrix B, where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^n$.

Also, we have $dx = t^{\gamma-1}\, d\sigma\, dt$, that is,

$$\int_{\mathbb{R}^n} f(x)\, dx = \int_0^\infty \int_\Sigma f(A_t\theta) t^{\gamma-1}\, d\sigma(\theta)\, dt$$

for appropriate functions f, where $d\sigma$ is a C^∞ measure on Σ. See [2, 13, 17] for more details.

Let Ω be locally integrable in $\mathbb{R}^n \setminus \{0\}$ and homogeneous of degree 0 with respect to the dilation group $\{A_t\}$, that is, $\Omega(A_t x) = \Omega(x)$ for $x \neq 0$. We assume that

$$\int_\Sigma \Omega(\theta)\, d\sigma(\theta) = 0.$$

For $s \geq 1$, let Δ_s denote the collection of measurable functions h on $\mathbb{R}_+ = \{t \in \mathbb{R} : t > 0\}$ satisfying

$$\|h\|_{\Delta_s} = \sup_{j \in \mathbb{Z}} \left(\int_{2^j}^{2^{j+1}} |h(t)|^s\, dt/t \right)^{1/s} < \infty,$$

where $\mathbb{Z}$ denotes the set of integers. We define $\|h\|_{\Delta_\infty}$ as usual ($\|h\|_{\Delta_\infty} = \|h\|_{L^\infty(\mathbb{R}_+)}$).

2000 *Mathematics Subject Classification.* Primary 42B20, 42B25.
Key Words and Phrases. Nonisotropic singular integrals, extrapolation, trigonometric integrals.

Let $\Gamma : [0, \infty) \to \mathbb{R}^m$ be a continuous mapping satisfying $\Gamma(0) = 0$. We define a singular integral operator along the surface $(y, \Gamma(r(y)))$ by

$$(1.1)\qquad \begin{aligned} Tf(x,z) &= \text{p.v.} \int_{\mathbb{R}^n} f(x-y, z-\Gamma(r(y)))K(y)\,dy \\ &= \lim_{\epsilon\to 0} \int_{r(y)>\epsilon} f(x-y, z-\Gamma(r(y)))K(y)\,dy, \end{aligned}$$

where $K(y) = h(r(y))\Omega(y')r(y)^{-\gamma}$, $y' = A_{r(y)^{-1}}y$ and $h \in \Delta_1$. We assume that the principal value integral in (1.1) exists for every $(x, z) \in \mathbb{R}^n \times \mathbb{R}^m$ and $f \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^m)$ (the Schwartz class).

We denote by $L \log L(\Sigma)$ the Zygmund class of all those functions Ω on Σ which satisfy

$$\int_\Sigma |\Omega(\theta)| \log(2 + |\Omega(\theta)|)\,d\sigma(\theta) < \infty.$$

Also, we consider the $L^q(\Sigma)$ spaces and write $\|\Omega\|_q = \left(\int_\Sigma |\Omega(\theta)|^q\,d\sigma(\theta)\right)^{1/q}$ for $\Omega \in L^q(\Sigma)$ ($\|\Omega\|_\infty$ is defined as usual).

Let

$$M_\Gamma g(z) = \sup_{R>0} R^{-1} \int_0^R |g(z - \Gamma(t))|\,dt.$$

We assume that the maximal operator M_Γ is bounded on $L^p(\mathbb{R}^m)$ for all $p > 1$. See [15, 17] for examples of such functions Γ.

In this note we prove the following.

Theorem 1. *Let T be as in (1.1). Suppose that $\Omega \in L^q(\Sigma)$ for some $q \in (1, 2]$ and $h \in \Delta_s$ for some $s > 1$. Then, we have*

$$\|Tf\|_{L^p(\mathbb{R}^{n+m})} \le C_p(q-1)^{-1}\|\Omega\|_q\|h\|_{\Delta_s}\|f\|_{L^p(\mathbb{R}^{n+m})}$$

if $|1/p - 1/2| < \min(1/s', 1/2)$, where $1/s' + 1/s = 1$ and the constant C_p is independent of q and Ω.

Theorem 2. *Suppose $\Omega \in L \log L(\Sigma)$ and $h \in \Delta_s$ for some $s > 1$. Then, T is bounded on $L^p(\mathbb{R}^{n+m})$ if $|1/p - 1/2| < \min(1/s', 1/2)$.*

Theorem 2 follows from Theorem 1 by an extrapolation method. When $r(x) = |x|$ (the Euclid norm), $m = 1$ and Γ is a C^2, convex, increasing function, Theorem 2 was proved in A. Al-Salman and Y. Pan [1] (see [1, Theorem 4.1] and also [10] for a related result). In [1], it is noted that the estimates as $q \to 1$ of Theorem 1 (in their setting) can be used through extrapolation to prove the L^p boundedness of [1, Theorem 4.1], although such estimates are yet to be proved. In this note, we are able to prove Theorem 1 and apply it to prove Theorem 2.

If $\Gamma \equiv 0$ (Γ is identically 0), then T essentially reduces to the lower dimensional singular integral

$$(1.2)\qquad Sf(x) = \text{p.v.} \int_{\mathbb{R}^n} f(x-y)K(y)\,dy.$$

For this singular integral we have the following.

Theorem 3. *Let $\Omega \in L^q(\Sigma)$ and $h \in \Delta_s$ for some $q, s \in (1, 2]$. Then we have*

$$\|Sf\|_{L^p(\mathbb{R}^n)} \le C_p(q-1)^{-1}(s-1)^{-1}\|\Omega\|_q\|h\|_{\Delta_s}\|f\|_{L^p(\mathbb{R}^n)}$$

for all $p \in (1, \infty)$, where the constant C_p is independent of q, s, Ω and h.

For $a > 0$, let

$$L_a(h) = \sup_{j\in\mathbb{Z}} \int_{2^j}^{2^{j+1}} |h(r)| \left(\log(2 + |h(r)|)\right)^a \, dr/r.$$

We define a class $\mathfrak{L}_a$ to be the space of all those measurable functions h on $\mathbb{R}_+$ which satisfy $L_a(h) < \infty$.

By Theorem 3 and an extrapolation we have the following.

Theorem 4. *Suppose $\Omega \in L\log L(\Sigma)$ and $h \in \mathfrak{L}_a$ for some $a > 2$. Then S is bounded on $L^p(\mathbb{R}^n)$ for all $p \in (1, \infty)$.*

It is noted in [5] that S is bounded on L^p, $1 < p < \infty$, if $\Omega \in L^q$ for some $q > 1$ and $h \in \Delta_2$ (see [5, Corollary 4.5]). Theorem 4 improves that result. See [13, 16] for nonisotropic singular integrals S with $h \equiv 1$ and also [3, 7, 9, 12] for related results.

In Section 2, we prove Theorems 1 and 3. The proofs are based on the method of [5]. As in [14], a key idea of the proof of Theorem 1 is to use a Littlewood–Paley decomposition depending on q for which $\Omega \in L^q$. Theorem 3 is proved in a similar fashion. Applying an extrapolation argument, we can prove Theorems 2 and 4 from Theorems 1 and 3, respectively. We give a proof of Theorem 4 in Section 3. In Section 4, we prove an estimate for a trigonometric integral, a corollary of which is used in proving Theorems 1 and 3.

Throughout this note, the letter C will be used to denote non-negative constants which may be different in different occurrences.

2. Proofs of Theorems 1 and 3

Let A^* denote the adjoint of a matrix A. Then $A_t^* = \exp((\log t)P^*)$. We write $A_t^* = B_t$. We can define a non-negative function s from $\{B_t\}$ exactly in the same way as we define r from $\{A_t\}$.

There are positive constants $c_1, c_2, c_3, c_4, \alpha_1, \alpha_2, \beta_1$ and β_2 such that

$$c_1|x|^{\alpha_1} < r(x) < c_2|x|^{\alpha_2} \quad \text{if } r(x) \geq 1,$$
$$c_3|x|^{\beta_1} < r(x) < c_4|x|^{\beta_2} \quad \text{if } 0 < r(x) \leq 1.$$

Also, we have

$$d_1|\xi|^{a_1} < s(\xi) < d_2|\xi|^{a_2} \quad \text{if } s(\xi) \geq 1,$$
$$d_3|\xi|^{b_1} < s(\xi) < d_4|\xi|^{b_2} \quad \text{if } 0 < s(\xi) \leq 1$$

for some positive numbers d_1, d_2, d_3, d_4, a_1, a_2, b_1 and b_2 (see [17]). These estimates are useful in the following.

We consider the singular integral operator T defined in (1.1). Let $E_j = \{x \in \mathbb{R}^n : \beta^j < r(x) \leq \beta^{j+1}\}$, where $\beta \geq 2$ and $j \in \mathbb{Z}$. We define a sequence of Borel measures $\{\sigma_j\}$ on $\mathbb{R}^n \times \mathbb{R}^m$ by

$$\hat{\sigma}_j(\xi, \eta) = \int_{E_j} e^{-2\pi i\langle y,\xi\rangle} e^{-2\pi i\langle \Gamma(r(y)),\eta\rangle} K(y)\, dy,$$

where $\hat{\sigma}_j$ denotes the Fourier transform of σ_j defined by

$$\hat{\sigma}_j(\xi, \eta) = \int e^{-2\pi i\langle (x,z),(\xi,\eta)\rangle}\, d\sigma_j(x, z).$$

Then $Tf(x) = \sum_{-\infty}^{\infty} \sigma_k * f(x)$.

Let $\mu_k = |\sigma_k|$, where $|\sigma_k|$ denotes the total variation of σ_k. Let $\Omega \in L^q$, $h \in \Delta_s$, $q, s \in (1, 2]$. We prove the following estimates (2.1)–(2.5):

$$\|\sigma_k\| \le C(\log \beta)\|\Omega\|_1\|h\|_{\Delta_1} \le C(\log \beta)\|\Omega\|_q\|h\|_{\Delta_s}, \tag{2.1}$$

where $\|\sigma_k\| = |\sigma_k|(\mathbb{R}^{n+m})$;

$$|\hat{\sigma}_k(\xi, \eta)| \le C\|\Omega\|_q\|h\|_{\Delta_s}(\beta^{k+d}s(\xi))^{1/b_1}, \tag{2.2}$$

where $d = b_1/\alpha_1$;

$$|\hat{\sigma}_k(\xi, \eta)| \le C(\log \beta)\|\Omega\|_q\|h\|_{\Delta_s}(\beta^k s(\xi))^{-\epsilon_0/(q's')} \tag{2.3}$$

for some $\epsilon_0 > 0$;

$$|\hat{\mu}_k(\xi, \eta)| \le C(\log \beta)\|\Omega\|_q\|h\|_{\Delta_s}(\beta^k s(\xi))^{-\epsilon_0/(q's')}, \tag{2.4}$$

where ϵ_0 is as in (2.3);

$$|\hat{\mu}_k(\xi, \eta) - \hat{\mu}_k(0, \eta)| \le C\|\Omega\|_q\|h\|_{\Delta_s}(\beta^{k+d}s(\xi))^{1/b_1}, \tag{2.5}$$

where d is as in (2.2).

First we see that

$$\|\sigma_k\|_1 = \int_{\beta^k}^{\beta^{k+1}} |h(r)|\|\Omega\|_1\, dr/r \le C(\log \beta)\|\Omega\|_1\|h\|_{\Delta_1}. \tag{2.6}$$

From this, (2.1) follows. Next, we show (2.2). Take $\nu \in \mathbb{Z}$ so that $2^\nu < \beta \le 2^{\nu+1}$. Note that

$$\hat{\sigma}_k(\xi, \eta) = \int_{\beta^k < r(x) \le \beta^{k+1}} e^{-2\pi i \langle \Gamma(r(x)), \eta\rangle}(e^{-2\pi i\langle x, \xi\rangle} - 1)h(r(x))\Omega(x')r(x)^{-\gamma}\, dx.$$

Thus

$$\begin{aligned} |\hat{\sigma}_k(\xi, \eta)| &\le C \int_{1 < r(x) \le \beta} |x||B_{\beta^k}\xi||h(\beta^k r(x))\Omega(x')|r(x)^{-\gamma}\, dx \\ &\le C\sum_{j=0}^{\nu} |B_{\beta^k}\xi|\|\Omega\|_1 2^{j/\alpha_1}\int_{2^j}^{2^{j+1}} |h(\beta^k r)|\, dr/r \\ &\le C\beta^{1/\alpha_1}|B_{\beta^k}\xi|\|\Omega\|_1\|h\|_{\Delta_1}. \end{aligned} \tag{2.7}$$

Combining (2.6) and (2.7), we have

$$|\hat{\sigma}_k(\xi, \eta)| \le C\|\Omega\|_1\|h\|_{\Delta_1}\min\left(\log \beta, \beta^{1/\alpha_1}|B_{\beta^k}\xi|\right). \tag{2.8}$$

If $s(B_{\beta^k}\xi) < 1$, then $|B_{\beta^k}\xi| \le C(\beta^k s(\xi))^{1/b_1}$. Therefore,

$$\min\left(\log \beta, \beta^{1/\alpha_1}|B_{\beta^k}\xi|\right) \le C(\beta^{k+d}s(\xi))^{1/b_1}.$$

Using this in (2.8), we have (2.2). We can prove (2.5) in the same way.

Next we prove (2.3). We use a method similar to that of [5, p. 551]. Define

$$\tau(\xi) = \int_\Sigma \Omega(\theta)e^{-2\pi i\langle \xi, \theta\rangle}\, d\sigma(\theta).$$

We need the following estimates.

Lemma 1. *Let L be the degree of the minimal polynomial of P. Then, if $0 < \epsilon_0 < a_2^{-1}\min(1/2, q'/L)$, we have*

$$\int_{\beta^k}^{\beta^{k+1}} |\tau(B_r\xi)|^2 \, dr/r \le C(\log\beta)(\beta^k s(\xi))^{-\epsilon_0/q'}\|\Omega\|_q^2,$$

where C is independent of $\Omega \in L^q$, $q \in (1,2]$ and β.

In proving Lemma 1 we use the following estimate, which follows from the corollary to Theorem 5 in Section 4 via an integration by parts argument.

Lemma 2. *Let L be as in Lemma 1. Then, for $\eta, \zeta \in \mathbb{R}^n \setminus \{0\}$ we have*

$$\left|\int_1^2 \exp\left(i\langle B_t\eta, \zeta\rangle\right) \, dt/t\right| \le C\left|\langle \eta, P\zeta\rangle\right|^{-1/L}$$

for some positive constant C independent of η and ζ.

Proof of Lemma 1. Choose $\nu \in \mathbb{Z}$ such that $2^\nu < \beta \le 2^{\nu+1}$. Then, we have

$$\int_{\beta^k}^{\beta^{k+1}} |\tau(B_r\xi)|^2 \, dr/r \le \sum_{j=0}^{\nu} \int_{\beta^k 2^j}^{\beta^k 2^{j+1}} |\tau(B_r\xi)|^2 \, dr/r$$

$$= \sum_{j=0}^{\nu} \iint_{\Sigma\times\Sigma} \left(\int_1^2 \exp\left(-2\pi i\langle B_{\beta^k 2^j r}\xi, \theta-\omega\rangle\right) \, dr/r\right) \Omega(\theta)\bar{\Omega}(\omega)\, d\sigma(\theta)\, d\sigma(\omega).$$

By Lemma 2 we have

$$\left|\int_1^2 \exp\left(-2\pi i\langle B_{\beta^k 2^j r}\xi, \theta-\omega\rangle\right) \, dr/r\right| \le C\left|\left\langle B_{\beta^k 2^j}\xi, P(\theta-\omega)\right\rangle\right|^{-\epsilon},$$

where $0 < \epsilon \le 1/L$. Using Hölder's inequality, if $0 < \epsilon < \min(1/(2q'), 1/L)$, we see that

$$\iint_{\Sigma\times\Sigma} \left|\langle B_{\beta^k 2^j}\xi, P(\theta-\omega)\rangle\right|^{-\epsilon} \left|\Omega(\theta)\bar{\Omega}(\omega)\right| \, d\sigma(\theta)\, d\sigma(\omega)$$

$$\le \left(\iint_{\Sigma\times\Sigma} \left|\langle P^* B_{\beta^k 2^j}\xi, \theta-\omega\rangle\right|^{-\epsilon q'} \, d\sigma(\theta)\, d\sigma(\omega)\right)^{1/q'} \|\Omega\|_q^2 \le C|B_{\beta^k 2^j}\xi|^{-\epsilon}\|\Omega\|_q^2,$$

where the last inequality follows from (3) of Section 1 (see [5, p. 553]). Therefore

(2.9)
$$\int_{\beta^k}^{\beta^{k+1}} |\tau(B_r\xi)|^2 \, dr/r \le C\|\Omega\|_q^2 \sum_{j=0}^{\nu} |B_{\beta^k 2^j}\xi|^{-\epsilon} \quad (0 < \epsilon < \min(1/(2q'), 1/L)).$$

If $s(B_{\beta^k}\xi) \ge 1$, $|B_{\beta^k 2^j}\xi| \ge C(\beta^k 2^j s(\xi))^{1/a_2}$ $(0 \le j \le \nu)$. Thus we see that

$$\sum_{j=0}^{\nu} |B_{\beta^k 2^j}\xi|^{-\epsilon} \le \sum_{j=0}^{\nu} C(\beta^k 2^j s(\xi))^{-\epsilon/a_2} \le C(\log\beta)(\beta^k s(\xi))^{-\epsilon/a_2}, \tag{2.10}$$

where C is independent of q. By (2.9) and (2.10) we have the estimate of Lemma 1 when $s(B_{\beta^k}\xi) \ge 1$. If $s(B_{\beta^k}\xi) < 1$, the estimate of Lemma 1 follows from the inequality $|\tau(\xi)| \le \|\Omega\|_1$. This completes the proof of Lemma 1.

Now, by Hölder's inequality we have

$$(2.11)\quad |\hat{\sigma}_k(\xi,\eta)| = \left| \int_{\beta^k}^{\beta^{k+1}} e^{-2\pi i \langle \Gamma(r),\eta\rangle} h(r)\tau(B_r\xi)\, dr/r \right|$$
$$\leq \left(\int_{\beta^k}^{\beta^{k+1}} |h(r)|^s \, dr/r \right)^{1/s} \left(\int_{\beta^k}^{\beta^{k+1}} |\tau(B_r\xi)|^{s'} \, dr/r \right)^{1/s'}$$
$$\leq C(\log\beta)^{1/s} \|h\|_{\Delta_s} \|\Omega\|_1^{(s'-2)/s'} \left(\int_{\beta^k}^{\beta^{k+1}} |\tau(B_r\xi)|^2 \, dr/r \right)^{1/s'},$$

where we have used the estimate $|\tau(\xi)| \leq \|\Omega\|_1$ to get the last inequality. By (2.11) and Lemma 1 we have (2.3). The estimate (2.4) can be proved similarly.

Let $B_{qs} = (1-\beta^{-\theta\epsilon_0/(q's')})^{-1}$, where $\beta \geq 2$, $\theta \in (0,1)$ and ϵ_0 is as in (2.3) and (2.4). To prove Theorems 1 and 3, we use the following:

Proposition 1. *Suppose that $\Omega \in L^q$, $q \in (1,2]$ and $h \in \Delta_s$, $s \in (1,2]$. Let $|1/p - 1/2| < (1-\theta)/(s'(1+\theta))$. Then, we have*

$$\|Tf\|_p \leq C(\log\beta)\|h\|_{\Delta_s}\|\Omega\|_q B_{qs} B_{q2}^{|1/p-1/p'|}\|f\|_p,$$

where C is a constant independent of Ω, h, q, s and β.

Proposition 2. *Suppose that $\Gamma \equiv 0$. Let $\Omega \in L^q$, $h \in \Delta_s$, $q, s \in (1,2]$. Then, for $p \in (1+\theta, (1+\theta)/\theta)$ we have*

$$\|Tf\|_p \leq C(\log\beta)\|\Omega\|_q\|h\|_{\Delta_s} B_{qs}^{1+|1/p-1/p'|}\|f\|_p,$$

where C is a constant independent of Ω, h, q, s and β.

To prove Propositions 1 and 2, we need the following:

Proposition 3. *Let $\mu^*(f)(x) = \sup_k |\mu_k * f(x)|$. Let $\Omega \in L^q$, $q \in (1,2]$.*

(1) *If $h \in \Delta_\infty$, for $p > 1+\theta$ we have*

$$\|\mu^*(f)\|_p \leq C(\log\beta)\|\Omega\|_q\|h\|_{\Delta_\infty} B_{q2}^{2/p}\|f\|_p,$$

where C is a constant independent of Ω, h, q and β.

(2) *Suppose that $\Gamma \equiv 0$. Let $h \in \Delta_s$, $s \in (1,2]$. Then, we have*

$$\|\mu^*(f)\|_p \leq C(\log\beta)\|\Omega\|_q\|h\|_{\Delta_s} B_{qs}^{2/p}\|f\|_p$$

for $p > 1+\theta$, where C is independent of Ω, q, h, s and β.

Proof. Since the estimate $\|\mu^*(f)\|_\infty \leq C(\log\beta)\|\Omega\|_1\|h\|_{\Delta_1}\|f\|_\infty$ follows from (2.1), by interpolation, to prove (1) and (2) of Proposition 3 we may assume $p \in (1+\theta, 2]$.

First, we give a proof of part (1). Define measures ν_k on $\mathbb{R}^n \times \mathbb{R}^m$ by

$$\hat{\nu}_k(\xi,\eta) = \hat{\mu}_k(\xi,\eta) - \hat{\Psi}_k(\xi,\eta),$$

where $\hat{\Psi}_k(\xi,\eta) = \hat{\varphi}_k(\xi)\hat{\mu}_k(0,\eta)$ with $\varphi_k(x) = \beta^{-k\gamma}\varphi(A_{\beta^{-k}}x), \varphi \in C_0^\infty$. We assume that φ is supported in $\{r(x) \leq 1\}, \hat{\varphi}(0) = 1$ and $\varphi \geq 0$. Then by (2.1), (2.4) and (2.5), for $q, s \in (1,2]$, we have

$$|\hat{\nu}_k(\xi,\eta)| \leq C(\log\beta)\|\Omega\|_q\|h\|_{\Delta_s} \min\left(1, (\beta^{k+d}s(\xi))^{1/b_1}, (\beta^k s(\xi))^{-\epsilon_0/(q's')}\right).$$

We may assume that ϵ_0 is small enough so that $\epsilon_0/4 \leq 1/b_1$. Then, we see that

$$(2.12)\qquad |\hat{\nu}_k(\xi,\eta)| \leq CA\min\left(1, (\beta^{k+d}s(\xi))^{\alpha}, (\beta^k s(\xi))^{-\alpha}\right),$$

where $A = (\log\beta)\|\Omega\|_q\|h\|_{\Delta_\infty}$ and $\alpha = \epsilon_0/(2q')$.

Let

$$g(f)(x,z) = \left(\sum_{k=-\infty}^{\infty} |\nu_k * f(x,z)|^2\right)^{1/2}.$$

Then $\mu^*(f) \le g(f) + \Psi^*(|f|)$, where $\Psi^*(f) = \sup_k ||\Psi_k| * f|$. Let

$$Mg(x) = \sup_{t>0} t^{-\gamma} \int_{r(x-y)<t} |g(y)|\, dy$$

be the Hardy–Littlewood maximal function on $\mathbb{R}^n$ with respect to the function r. By the L^p boundedness of M_Γ and M, it is easy to see that $\|\Psi^*(f)\|_p \le CA\|f\|_p$ for $p > 1$. Thus to prove Proposition 3 (1) it suffices to show

$$\|g(f)\|_p \le CAB^{2/p}\|f\|_p \quad (p \in (1+\theta, 2]), \tag{2.13}$$

where A is as above and $B = B_{q2}$. By a well-known property of Rademacher's functions, (2.13) follows from

$$\|U_\epsilon(f)\|_p \le CAB^{2/p}\|f\|_p \quad (p \in (1+\theta, 2]), \tag{2.14}$$

where $U_\epsilon(f)(x,z) = \sum \epsilon_k \nu_k * f(x,z)$ with $\epsilon = \{\epsilon_k\}$, $\epsilon_k = 1$ or -1 (the inequality is uniform in ϵ).

We define two sequences $\{r_m\}_1^\infty$ and $\{p_m\}_1^\infty$ by $p_1 = 2$ and

$$\frac{1}{r_m} - \frac{1}{2} = \frac{1}{2p_m}, \quad \frac{1}{p_{m+1}} = \frac{\theta}{2} + \frac{1-\theta}{r_m} \qquad \text{for } m \ge 1.$$

Then, we have

$$\frac{1}{p_{m+1}} = \frac{1}{2} + \frac{1-\theta}{2p_m} \qquad \text{for } m \ge 1.$$

Thus $1/p_m = (1-\eta^m)/(1+\theta)$, where $\eta = (1-\theta)/2$, so $\{p_m\}$ is decreasing and converges to $1+\theta$.

For $j \ge 1$ we prove

$$\|U_\epsilon(f)\|_{p_j} \le C_j AB^{2/p_j}\, \|f\|_{p_j}. \tag{2.15}$$

To prove (2.15) we use the Littlewood–Paley theory. Let $\{\psi_k\}_{-\infty}^\infty$ be a sequence of non-negative functions in $C^\infty((0,\infty))$ such that

$$\operatorname{supp}(\psi_k) \subset [\beta^{-k-1}, \beta^{-k+1}], \quad \sum_k \psi_k(t)^2 = 1,$$

$$|(d/dt)^j \psi_k(t)| \le c_j/t^j \quad (j = 1, 2, \dots),$$

where c_j is independent of $\beta \ge 2$. Define S_k by

$$(S_k(f))\,\hat{}\,(\xi,\eta) = \psi_k(s(\xi))\hat{f}(\xi,\eta).$$

We write $U_\epsilon(f) = \sum_{j=-\infty}^\infty U_j(f)$, where $U_j(f) = \sum_{k=-\infty}^\infty \epsilon_k S_{j+k}\left(\nu_k * S_{j+k}(f)\right)$. Then by Plancherel's theorem and (2.12) we have

$$\begin{aligned} \|U_j(f)\|_2^2 &\le \sum_k C \iint_{D(j+k)\times\mathbb{R}^m} |\hat{\nu}_k(\xi,\eta)|^2 |\hat{f}(\xi,\eta)|^2\, d\xi\, d\eta \\ &\le CA^2 \min\left(1, \beta^{-2(|j|-1-d)\alpha}\right) \sum_k \iint_{D(j+k)\times\mathbb{R}^m} |\hat{f}(\xi,\eta)|^2\, d\xi\, d\eta \\ &\le CA^2 \min\left(1, \beta^{-2(|j|-1-d)\alpha}\right) \|f\|_2^2, \end{aligned} \tag{2.16}$$

where $D(k) = \{\xi \in \mathbb{R}^n : \beta^{-k-1} < s(\xi) \le \beta^{-k+1}\}$. By (2.16) we have

$$\|U_\epsilon(f)\|_2 \le \sum_{-\infty}^{\infty} \|U_j(f)\|_2 \le C \sum_{-\infty}^{\infty} A \min\left(1, \beta^{-(|j|-1-d)\alpha}\right) \|f\|_2 \le CA(1-\beta^{-\alpha})^{-1}\|f\|_2. \tag{2.17}$$

If we denote by $A(m)$ the estimate of (2.15) for $j = m$, this proves $A(1)$.

Now, we assume $A(m)$ and derive $A(m+1)$ from $A(m)$. Note that

$$\nu^*(f) \le \mu^*(|f|) + \Psi^*(|f|) \le g(|f|)(x) + 2\Psi^*(|f|),$$

where $\nu^*(f)(x) = \sup_k ||\nu_k| * f(x)|$. Since $\|g(f)\|_{p_m} \le CAB^{2/p_m}\|f\|_{p_m}$ by $A(m)$, we have

$$\|\nu^*(f)\|_{p_m} \le CAB^{2/p_m}\|f\|_{p_m}.$$

Also, $\|\nu_k\| \le CA$ by (2.1). Thus, by the proof of Lemma for Theorem B in [5, p. 544], we have the vector valued inequality:

$$\left\| \left(\sum |\nu_k * g_k|^2 \right)^{1/2} \right\|_{r_m} \le C(AB^{2/p_m} \sup_k \|\nu_k\|)^{1/2} \left\| \left(\sum |g_k|^2 \right)^{1/2} \right\|_{r_m} \le CAB^{1/p_m} \left\| \left(\sum |g_k|^2 \right)^{1/2} \right\|_{r_m}. \tag{2.18}$$

By (2.18) and the Littlewood–Paley inequality, we have

$$\|U_j(f)\|_{r_m} \le C \left\| \left(\sum_k |\nu_k * S_{j+k}(f)|^2 \right)^{1/2} \right\|_{r_m} \le CAB^{1/p_m}\|f\|_{r_m}. \tag{2.19}$$

Here we note that the bounds for the Littlewood-Paley inequality are independent of $\beta \ge 2$. Interpolating between (2.16) and (2.19), we have

$$\|U_j(f)\|_{p_{m+1}} \le CAB^{(1-\theta)/p_m} \min\left(1, \beta^{-\theta\alpha(|j|-1-d)}\right) \|f\|_{p_{m+1}}.$$

Thus

$$\|U_\epsilon(f)\|_{p_{m+1}} \le \sum_j \|U_j(f)\|_{p_{m+1}} \le CAB^{(1-\theta)/p_m}(1-\beta^{-\theta\alpha})^{-1}\|f\|_{p_{m+1}} \le CAB^{2/p_{m+1}}\|f\|_{p_{m+1}},$$

which proves $A(m+1)$. By induction, this completes the proof of (2.15).

Now we prove (2.14). Let $p \in (1+\theta, 2]$ and let $\{p_m\}_1^\infty$ be as in (2.15). Then we have $p_{N+1} < p \le p_N$ for some N. By interpolation between the estimates in (2.15) for $j = N$ and $j = N+1$ we have (2.14). This completes the proof of Proposition 3 (1).

Part (2) of Proposition 3 can be proved in the same way. We take $A = (\log\beta)\|\Omega\|_q\|h\|_{\Delta_s}$ and $\alpha = \epsilon_0/(q's')$ in (2.12). Then, since

$$\|\Psi^*(f)\|_p \le C(\log\beta)\|\Omega\|_1\|h\|_{\Delta_1}\|f\|_p \qquad \text{for } p > 1$$

if $\Gamma \equiv 0$, the proof of part (1) can be used to get (2.13) with $A = (\log\beta)\|\Omega\|_q\|h\|_{\Delta_s}$ as above and $B = B_{qs}$, and the conclusion of part (2) follows from (2.13). □

Proof of Proposition 1. To prove Proposition 1 we may assume $1 < s < 2$. As in [1], here we apply an idea in the proof of [6, Theorem 7.5]. We consider measures τ_k defined by

$$\hat{\tau}_k(\xi,\eta) = \int_{E_k} e^{-2\pi i\langle y,\xi\rangle} e^{-2\pi i\langle \Gamma(r(y)),\eta\rangle} |h(r(y))|^{2-s} |\Omega(y')| r(y)^{-\gamma}\, dy.$$

Then, the Schwarz inequality implies

$$|\sigma_k * f|^2 \le C(\log\beta)\|h\|_{\Delta_s}^s \|\Omega\|_1 \tau_k * |f|^2. \tag{2.20}$$

Define measures λ_k by

$$\hat{\lambda}_k(\xi,\eta) = \int_{E_k} e^{-2\pi i\langle y,\xi\rangle} e^{-2\pi i\langle \Gamma(r(y)),\eta\rangle} |\Omega(y')| r(y)^{-\gamma}\, dy.$$

Since $|h|^{2-s} \in \Delta_{s/(2-s)}$ and $\||h|^{2-s}\|_{\Delta_{s/(2-s)}} = \|h\|_{\Delta_s}^{2-s}$, if $u = s/(2-s)$ by Hölder's inequality we have

$$|\tau_k * f| \le C(\log\beta)^{1/u} \|h\|_{\Delta_s}^{2-s} \|\Omega\|_1^{1/u} (\lambda_k * |f|^{u'})^{1/u'}.$$

Therefore, if $1+\theta < r/u' = 2r(s-1)/s$, by applying (1) of Proposition 3 to $\{\lambda_k\}$ we see that

$$\|\tau^*(f)\|_r \le C(\log\beta)\|h\|_{\Delta_s}^{2-s} \|\Omega\|_q B_{q2}^{2/r} \|f\|_r, \tag{2.21}$$

where $\tau^*(f) = \sup_k |\tau_k * f|$. Thus, if $|1/v - 1/2| = 1/(2r) < 1/(s'(1+\theta))$, using (2.20), (2.21) and arguing as in the proof of Lemma for Theorem B in [5, p. 544], we see that

$$\left\| \left(\sum |\sigma_k * g_k|^2 \right)^{1/2} \right\|_v \le C(\log\beta)\|h\|_{\Delta_s} \|\Omega\|_q B_{q2}^{1/r} \left\| \left(\sum |g_k|^2 \right)^{1/2} \right\|_v . \tag{2.22}$$

We decompose $Tf = \sum_{j=-\infty}^{\infty} V_j f$, where $V_j f = \sum_{k=-\infty}^{\infty} S_{j+k}\left(\sigma_k * S_{j+k}(f)\right)$. Then, using (2.22) and the Littlewood–Paley theory, we see that

$$\|V_j f\|_v \le C(\log\beta)\|h\|_{\Delta_s} \|\Omega\|_q B_{q2}^{1/r} \|f\|_v, \tag{2.23}$$

where $|1/v - 1/2| = 1/(2r) < 1/(s'(1+\theta))$. On the other hand, by (2.1)–(2.3) we have

$$|\hat{\sigma}_k(\xi,\eta)| \le C(\log\beta)\|\Omega\|_q \|h\|_{\Delta_s} \min\left(1, (\beta^{k+d} s(\xi))^{\kappa}, (\beta^k s(\xi))^{-\kappa}\right),$$

where $\kappa = \epsilon_0/(q's')$, and hence, similarly to the proof of (2.16), we can show that

$$\|V_j f\|_2 \le C(\log\beta)\|h\|_{\Delta_s} \|\Omega\|_q \min\left(1, \beta^{-(|j|-1-d)\kappa}\right) \|f\|_2. \tag{2.24}$$

If $|1/p - 1/2| < (1-\theta)/(s'(1+\theta))$, then we can find numbers v and r such that $|1/v - 1/2| = 1/(2r) < 1/(s'(1+\theta))$ and $1/p = \theta/2 + (1-\theta)/v$. Thus, interpolating between (2.23) and (2.24), we have

$$\|V_j f\|_p \le C(\log\beta)\|h\|_{\Delta_s} \|\Omega\|_q B_{q2}^{(1-\theta)/r} \min\left(1, \beta^{-\theta(|j|-1-d)\kappa}\right) \|f\|_p.$$

Therefore

$$\|Tf\|_p \le \sum_j \|V_j f\|_p \le C(\log\beta)\|h\|_{\Delta_s} \|\Omega\|_q B_{q2}^{(1-\theta)/r} B_{qs} \|f\|_p. \tag{2.25}$$

This completes the proof of Proposition 1, since $(1-\theta)/r = |1/p - 1/p'|$.

Proof of Proposition 2. The L^2 estimates follow from Proposition 1, so on account of duality and interpolation we may assume that $1+\theta < p \le 4/(3-\theta)$. For $p_0 \in (1+\theta, 4/(3-\theta)]$ we can find $r \in (1+\theta, 2]$ such that $1/p_0 = 1/2 + (1-\theta)/(2r)$.

If $\Gamma \equiv 0$, by (2) of Proposition 3 and (2.1), arguing as in (2.18), we have (2.22) with B_{q2} replaced by B_{qs} for the number v satisfying $1/v - 1/2 = 1/(2r)$ (note that $1/p_0 = \theta/2 + (1-\theta)/v$). Thus, arguing as in the proof of Proposition 1, we have (2.25) with $p = p_0$ and B_{qs} in place of B_{q2}. This completes the proof of Proposition 2.

Now we can give proofs of Theorems 1 and 3. To prove Theorem 1, we may assume that $1 < s \le 2$. Let $\beta = 2^{q'}$ in Proposition 1. Then, since θ is an arbitrary number in $(0,1)$, we have Theorem 1 for $s \in (1,2]$.

Next, take $\beta = 2^{q's'}$ in Proposition 2. Then, we have

$$\|Tf\|_p \le C(q-1)^{-1}(s-1)^{-1}\|\Omega\|_q\|h\|_{\Delta_s}\|f\|_p$$

for $p \in (1,\infty)$, since $(1+\theta, (1+\theta)/\theta) \to (1,\infty)$ as $\theta \to 0$. From this the result for S in Theorem 3 follows if we take functions of the form $f(x,z) = k(x)g(z)$.

3. Extrapolation

We can prove Theorems 2 and 4 by an extrapolation method similar to the one used in [14]. We give a proof of Theorem 4 for the sake of completeness (Theorem 2 can be proved in the same way). We fix $p \in (1,\infty)$ and f with $\|f\|_p \le 1$. Let S be as in (1.2). We also write $Sf = S_{h,\Omega}(f)$. Put $U(h,\Omega) = \|S_{h,\Omega}(f)\|_p$. Then we see that

$$\begin{aligned} U(h,\Omega_1+\Omega_2) &\le U(h,\Omega_1) + U(h,\Omega_2), \\ U(h_1+h_2,\Omega) &\le U(h_1,\Omega) + U(h_2,\Omega), \end{aligned} \tag{3.1}$$

for appropriate functions $\Omega, h, \Omega_1, \Omega_2, h_1$ and h_2. Set

$$\begin{aligned} E_1 &= \{r \in \mathbb{R}_+ : |h(r)| \le 2\}, \\ E_m &= \{r \in \mathbb{R}_+ : 2^{m-1} < |h(r)| \le 2^m\} \qquad \text{for } m \ge 2. \end{aligned}$$

Then $h = \sum_{m=1}^{\infty} h\chi_{E_m}$. Put $e_m = \sigma(F_m)$ for $m \ge 1$, where

$$\begin{aligned} F_m &= \{\theta \in \Sigma : 2^{m-1} < |\Omega(\theta)| \le 2^m\} \qquad \text{for } m \ge 2, \\ F_1 &= \{\theta \in \Sigma : |\Omega(\theta)| \le 2\}. \end{aligned}$$

Let $\Omega_m = \Omega\chi_{F_m} - \sigma(\Sigma)^{-1}\int_{F_m}\Omega\,d\sigma$. Then $\Omega = \sum_{m=1}^{\infty}\Omega_m$. Note that $\int_\Sigma \Omega_m\,d\sigma = 0$. Applying Theorem 3, we see that

$$U(h\chi_{E_m}, \Omega_j) \le C(q-1)^{-1}(s-1)^{-1}\|h\chi_{E_m}\|_{\Delta_s}\|\Omega_j\|_q \tag{3.2}$$

for all $s, q \in (1,2]$.

Now we follow the extrapolation argument of A. Zygmund [18, Chap. XII, pp. 119–120]. For $k \in \mathbb{Z}$, put

$$\begin{aligned} E(k,m) &= \{r \in (2^k, 2^{k+1}] : 2^{m-1} < |h(r)| \le 2^m\} \qquad \text{for } m \ge 2, \\ E(k,1) &= \{r \in (2^k, 2^{k+1}] : 0 < |h(r)| \le 2\}. \end{aligned}$$

Then

$$\begin{aligned} \int_{E(k,m)} |h(r)|^{(m+1)/m}\,dr/r &\le Cm^{-a}\int_{E(k,m)} |h(r)|\left(\log(2+|h(r)|)\right)^a\,dr/r \\ &\le Cm^{-a}L_a(h), \end{aligned}$$

and hence

$$\|h\chi_{E_m}\|_{\Delta_{1+1/m}} \le Cm^{-am/(m+1)}L_a(h)^{m/(m+1)} \tag{3.3}$$

for $m \geq 1$. Also we have

$$\|\Omega_j\|_{1+1/j} \leq C2^j e_j^{j/(j+1)}. \tag{3.4}$$

From (3.1)–(3.4) we see that

$$\begin{aligned}
U(h,\Omega) &\leq \sum_{m\geq 1}\sum_{j\geq 1} U\left(h\chi_{E_m},\Omega_j\right) \leq C\sum_{m\geq 1}\sum_{j\geq 1} jm\|h\chi_{E_m}\|_{\Delta_{1+1/m}}\|\Omega_j\|_{1+1/j}\\
&\leq C(1+L_a(h))\sum_{m\geq 1}\sum_{j\geq 1} m^{1-am/(m+1)} j2^j e_j^{j/(j+1)}\\
&= C(1+L_a(h))\left(\sum_{m\geq 1} m^{1-am/(m+1)}\right)\left(\sum_{j\geq 1} j2^j e_j^{j/(j+1)}\right).
\end{aligned}$$

When $a > 2$, it is easy to see that $\sum_{m\geq 1} m^{1-am/(m+1)} < \infty$. Also, we have

$$\begin{aligned}
\sum_{j\geq 1} j2^j e_j^{j/(j+1)} &= \sum_{e_j<3^{-j}} + \sum_{e_j\geq 3^{-j}}\\
&\leq \sum_{j\geq 1} j2^j 3^{-j^2/(j+1)} + \sum_{j\geq 1} j2^j e_j 3^{j/(j+1)}\\
&\leq C + C\int_{\Sigma} |\Omega(\theta)|\log(2+|\Omega(\theta)|)\, d\sigma(\theta).
\end{aligned}$$

Collecting the results, we conclude the proof of Theorem 4.

Remark. For a positive number a and a function h on $\mathbb{R}_+$, let

$$N_a(h) = \sum_{m\geq 1} m^a 2^m d_m(h),$$

where $d_m(h) = \sup_{k\in\mathbb{Z}} 2^{-k}|E(k,m)|$ ($E(k,m)$ is as above). We define a class $\mathcal{N}_a$ to be the space of all measurable functions h on $\mathbb{R}_+$ which satisfy $N_a(h) < \infty$. Then, it can be shown that if $h \in \mathcal{L}_a$ for some $a > 2$, then $h \in \mathcal{N}_1$. By a method similar to that used in this section, we can show the L^p boundedness of S in Theorem 4 under a less restrictive condition that $h \in \mathcal{N}_1$ and $\Omega \in L\log L$ (see [14]).

4. An estimate for a trigonometric integral

Let A be an $n \times n$ real matrix and

$$\phi_A(t) = (t-\gamma_1)^{m_1}(t-\gamma_2)^{m_2}\dots(t-\gamma_k)^{m_k}$$

be the minimal polynomial of A, where $\gamma_i \neq \gamma_j$ if $i \neq j$. Let $a_i(t) = (t-\gamma_i)^{m_i}$ for $i = 1,2,\dots,k$. Then, we can find polynomials $b_i(t)$ ($i = 1,2,\dots,k$) such that

$$\frac{1}{\phi_A(t)} = \sum_{i=1}^{k}\frac{b_i(t)}{a_i(t)}.$$

For each i, $1 \leq i \leq k$, let P_i be the polynomial defined by

$$P_i(t) = \frac{b_i(t)}{a_i(t)}\phi_A(t).$$

We consider the $n \times n$ matrices $P_i(A)$, which are defined as usual (see [8]).

Let

$$V_i = \{z \in \mathbb{C}^n : (A-\gamma_i E)^{m_i} z = 0\} \quad (i = 1,2,\dots,k),$$

where E denotes the unit matrix. Then, the vector space $\mathbb{C}^n$ can be decomposed into a direct sum as

$$\mathbb{C}^n = V_1 \oplus V_2 \oplus \cdots \oplus V_k.$$

Each of the matrices $P_i(A)$ is the projection onto V_i; indeed, we have the following (see [8]): $P_i(A)z \in V_i$ for all $z \in \mathbb{C}^n$, for $i = 1, 2, \dots, k$, and

$$P_1(A) + P_2(A) + \cdots + P_k(A) = E,$$
$$P_i^2(A) = P_i(A), \quad P_i(A)P_j(A) = 0 \quad \text{if } i \neq j \quad (1 \leq i, j \leq k).$$

For $z = (z_i)$ and $w = (w_i)$ in $\mathbb{C}^n$, we write $\langle z, w\rangle = \sum_{i=1}^n z_i w_i$. Let

$$J(A, \eta, \zeta) = \sum_{i=1}^{k} \sum_{j=0}^{m_i - 1} \left|\langle (A - \gamma_i E)^j P_i(A)\eta, A^*\zeta\rangle\right| \tag{4.1}$$

for $\eta, \zeta \in \mathbb{R}^n$. In this section, we prove the following:

Theorem 5. *Let $\eta, \zeta \in \mathbb{R}^n \setminus \{0\}$ and $0 < a < b$. Suppose that $J(A, \eta, \zeta) \neq 0$ and the numbers a, b are in a fixed compact subinterval of $(0, \infty)$. Then, we have*

$$\left|\int_a^b \exp\left(i\langle t^A \eta, \zeta\rangle\right) \, dt\right| \leq C J(A, \eta, \zeta)^{-1/N},$$

where $N = \deg \phi_A = m_1 + m_2 + \cdots + m_k$ and the constant C is independent of η, ζ, a and b.

Since $\sum_{i=1}^k P_i(A) = E$, using the triangle inequality, we see that

$$|\langle \eta, A^*\zeta\rangle| \leq \sum_{i=1}^{k} |\langle P_i(A)\eta, A^*\zeta\rangle| \leq J(A, \eta, \zeta).$$

Therefore, Theorem 5 implies the following:

Corollary. *Let η, ζ, a, b and N be as in Theorem 5. Then, we have*

$$\left|\int_a^b \exp\left(i\langle t^A \eta, \zeta\rangle\right) \, dt\right| \leq C \left|\langle A\eta, \zeta\rangle\right|^{-1/N}$$

when $\langle A\eta, \zeta\rangle \neq 0$.

This is used to prove Lemma 2 in Section 2.

We define the curve $X(t) = t^A \eta$ for a fixed $\eta \in \mathbb{R}^n \setminus \{0\}$. Then, E. M. Stein and S. Wainger [17] proved the following (see [11, 16] for related results):

Theorem A. *Suppose that the curve X does not lie in an affine hyperplane. Then*

$$\left|\int_a^b \exp\left(i\langle X(t), \zeta\rangle\right) \, dt\right| \leq C \left|\zeta\right|^{-1/n},$$

where C is independent of $\zeta \in \mathbb{R}^n \setminus \{0\}$; furthermore, if a and b are in a fixed compact subinterval of $(0, \infty)$, the constant C is also independent of a and b.

Now, we see that Theorem 5 implies Theorem A. Since $P_i(A)z \in V_i$ $(z \in \mathbb{C}^n)$, we have $(A - \gamma_i E)^m P_i(A) = 0$ if $m \geq m_i$ $(i = 1, 2, \dots, k)$. Therefore

$$\exp((\log t)A)P_i(A) = \exp((\log t)\gamma_i E)\exp((\log t)(A - \gamma_i E))P_i(A)$$
$$= t^{\gamma_i} \sum_{j=0}^{m_i - 1} \frac{(\log t)^j}{j!}(A - \gamma_i E)^j P_i(A).$$

Thus, using $\sum_{i=1}^k P_i(A) = E$, we see that

$$t^A = \sum_{i=1}^k t^{\gamma_i} \left[\sum_{j=0}^{m_i-1} \frac{(\log t)^j}{j!}(A - \gamma_i E)^j \right] P_i(A). \tag{4.2}$$

The assumption on X of Theorem A can be rephrased as follows: the function $\psi(t) = \langle t^A \eta, \zeta \rangle$ is not a constant function on $(0, \infty)$ for every $\zeta \in \mathbb{R}^n \setminus \{0\}$. If $\psi(t)$ is not a constant function, then $\psi'(t)$ is not identically 0. Thus, since $t(d/dt)\psi(t) = \langle t^A \eta, A^* \zeta \rangle$, by (4.2) we have $J(A, \eta, \zeta) > 0$, where $J(A, \eta, \zeta)$ is as in (4.1). Let $C_0 = \min_{|\zeta|=1} J(A, \eta, \zeta)$ and note that $C_0 > 0$. Then, from Theorem 5, it follows that

$$\left| \int_a^b \exp\left(i\langle X(t), \zeta \rangle\right) \, dt \right| \leq C C_0^{-1/N} |\zeta|^{-1/N}.$$

This implies Theorem A, since $N \leq n$ (in fact, it is not difficult to see that $N = n$ if X satisfies the assumption of Theorem A).

In the following, we give a proof of Theorem 5. Let $I = [\alpha, \beta]$ be a compact interval in $\mathbb{R}$. Consider the differential equation

$$y^{(k)} + a_1 y^{(k-1)} + a_2 y^{(k-2)} + \cdots + a_k y = 0 \quad \text{on } I, \tag{4.3}$$

where $a_1, a_2, \dots, a_k$ are complex constants. Let $\{\varphi_1, \varphi_2, \dots, \varphi_k\}$ be a basis for the space S of all solutions of (4.3). Then, we use the following to prove Theorem 5.

Proposition 4. *Let φ be a real valued function such that $\varphi' \in S$. Suppose that $\varphi' = d_1\varphi_1 + d_2\varphi_2 + \cdots + d_k\varphi_k$, where $d_1, d_2, \dots, d_k$ are complex constants, which are uniquely determined by φ'. Then, we have*

$$\left| \int_\alpha^\beta e^{i\varphi(t)} \, dt \right| \leq C \left(|d_1| + |d_2| + \cdots + |d_k| \right)^{-1/k},$$

where C is independent of φ; also the constant C is independent of α, β if they are within a fixed finite interval of $\mathbb{R}$.

To prove Proposition 4 we use the following two lemmas. Both of them are well-known.

Lemma 3. *Let φ be a solution of* (4.3). *Suppose that φ is not identically* 0. *Then, there exists a positive integer K independent of φ such that φ has at most K zeros in I.*

Lemma 4 (van der Corput). *Let $f : [c, d] \to \mathbb{R}$ and $f \in C^j([c, d])$ for some positive integer j, where $[c, d]$ is an arbitrary compact interval in $\mathbb{R}$. Suppose that $\inf_{u \in [c,d]} |(d/du)^j f(u)| \geq \lambda > 0$. When $j = 1$, we further assume that f' is monotone on $[c, d]$. Then*

$$\left| \int_c^d e^{if(u)} \, du \right| \leq C_j \lambda^{-1/j},$$

where C_j is a positive constant depending only on j. (See [17, 18]).

We now give a proof of Proposition 4. We consider linear combinations $c_1\varphi_1 + c_2\varphi_2 + \cdots + c_k\varphi_k$, where $c_1, c_2, \ldots, c_k \in \mathbb{C}$. We write $\psi = c_1\varphi_1 + c_2\varphi_2 + \cdots + c_k\varphi_k$ and define

$$N_1(\psi) = |c_1| + |c_2| + \cdots + |c_k|,$$
$$N_2(\psi) = \min_{t\in I}\left(|\psi(t)| + |\psi'(t)| + \cdots + |\psi^{(k-1)}(t)|\right).$$

Let $U = \{(c_1, c_2, \ldots, c_k) \in \mathbb{C}^k : |c_1| + |c_2| + \cdots + |c_k| = 1\}$. We consider a function F on $I \times U$ defined by

$$F(t, c_1, c_2, \ldots, c_k) = |\psi(t)| + |\psi'(t)| + \cdots + |\psi^{(k-1)}(t)|.$$

Then, the function F is continuous and positive on $I \times U$ (see [4]). Thus, if we put

$$C_0 = \min_{(t,c_1,c_2,\ldots,c_k)\in I\times U} F(t, c_1, c_2, \ldots, c_k),$$

then we see that $C_0 > 0$ and $N_2(\psi) \geq C_0 N_1(\psi)$.

Therefore, if φ is as in Proposition 4, we have

$$\min_{t\in I}\left(|\varphi'(t)| + |\varphi''(t)| + \cdots + |\varphi^{(k)}(t)|\right) \geq C_0 N_1(\varphi'). \tag{4.4}$$

By (4.4), for any $t \in I$, there exists $\ell \in \{1, 2, \ldots, k\}$ such that

$$|(d/dt)^\ell \varphi(t)| \geq C N_1(\varphi'), \quad C > 0.$$

Applying Lemma 3 suitably, we can decompose $I = \cup_{m=1}^H I_m$, where H is a positive integer independent of φ and $\{I_m\}$ is a family of non-overlapping subintervals of I such that for any interval I_m there is $\ell_m \in \{1, 2, \ldots, k\}$ satisfying $|(d/dt)^{\ell_m}\varphi(t)| \geq |(d/dt)^j\varphi(t)|$ on I_m for all $j \in \{1, 2, \ldots, k\}$, so $|(d/dt)^{\ell_m}\varphi(t)| \geq CN_1(\varphi')$ on I_m, and such that φ' is monotone on each I_m. Therefore, by Lemma 4 we have

$$\begin{aligned}\left|\int_\alpha^\beta e^{i\varphi(t)}\,dt\right| &= \left|\sum_{m=1}^H \int_{I_m} e^{i\varphi(t)}\,dt\right| \leq C\sum_{m=1}^H \min\left(|I_m|, N_1(\varphi')^{-1/\ell_m}\right)\\ &\leq CN_1(\varphi')^{-1/k}.\end{aligned}$$

Since $N_1(\varphi') = |d_1| + |d_2| + \cdots + |d_k|$, this completes the proof of Proposition 4.

Proof of Theorem 5. By the change of variables $t = e^s$ and an integration by parts argument, we can see that to prove Theorem 5 it suffices to show

$$\left|\int_\alpha^\beta \exp\left(i\langle e^{tA}\eta, \zeta\rangle\right)\,dt\right| \leq CJ(A, \eta, \zeta)^{-1/N} \tag{4.5}$$

for an appropriate constant $C > 0$, where $[\alpha, \beta]$ is an arbitrary compact interval in $\mathbb{R}$. Let $\psi(t) = \langle e^{tA}\eta, \zeta\rangle$. Then, $\psi'(t) = \langle e^{tA}\eta, A^*\zeta\rangle$, and hence, by (4.2) we have

$$\psi'(t) = \sum_{i=1}^k \sum_{j=0}^{m_i-1} c_{ij}(\eta, \zeta)t^j e^{\gamma_i t},$$

where

$$c_{ij}(\eta, \zeta) = \frac{1}{j!}\langle (A - \gamma_i E)^j P_i(A)\eta, A^*\zeta\rangle.$$

It is known that N functions $t^j e^{\gamma_i t}$ $(0 \leq j \leq m_i - 1,\ 1 \leq i \leq k)$ form a basis for the space of solutions for the ordinary differential equation of order N with characteristic polynomial ϕ_A (see [4]). Thus, the estimate (4.5) immediately follows from Proposition 4, since $\sum_{i=1}^{k} \sum_{j=0}^{m_i-1} |c_{ij}(\eta,\zeta)| \approx J(A,\eta,\zeta)$.

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