

# On sufficient conditions for the Leopoldt conjecture(Algebraic Number Theory)

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On sufficient conditions for the Leopoldt conjecture

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Introductions.

Let  $p$  be a prime number. Let  $k$  be an algebraic number field of finite degree over the rational number field  $\mathbf{Q}$ . Let  $S$  be a non-empty finite set of prime divisors of  $k$ , and let  $U_S = \prod_{p \in S} U_p$ , where  $U_p$  is the unit group of the completion of  $k$  at  $p$ . Let  $E$  be the global unit group of  $k$  and  $i : E \rightarrow U_S$  be the canonical injection. We denote by  $E_S$  the topological closure of  $i(E)$  in  $U_S$ . Let  $P$  be the set of all prime divisors of  $k$  dividing  $p$ . Let  $r$  be the  $\mathbf{Z}$ -free rank of  $E$  and  $r_p$  be the  $\mathbf{Z}_p$ -free rank of  $E_p$ . Then it is equivalent to  $r = r_p$  that the Leopoldt conjecture holds for  $(k, p)$ . Put  $\delta_p = \delta(k, p) = r - r_p$ . Then  $\delta_p \geq 0$  and it is the defect value for the Leopoldt conjecture. We study the conditions for  $\delta_p = 0$  in this paper. Now we explain notation which uses in this paper and state main results of this paper. Let  $\xi_p$  be a primitive  $p$ -th root of unity and let  $K = k(\xi_p)$ . We denote by  $\bar{S}$  the set of all prime divi-

sors of  $K$  which divide the prime divisors of  $k$  contained in  $S$ . Let  $C$  be the divisor class group of  $K$  and  $D_S$  be its subgroup generated by all divisor classes containing the prime divisors in  $\bar{S}$ . Put  $C_S = C/D_S \cdot C^p$ . Put  $G = \text{Gal}(K/k)$ . Let  $\omega : G \rightarrow \mathbb{Z}_p^\times$  be a homomorphism to the multiplicative group of the ring of  $p$ -adic integers  $\mathbb{Z}_p$  defined by  $\xi_p^\tau = \xi_p^{\omega(\tau)}$  for any  $\tau \in G$ . Put  $\varepsilon_\omega = (\sum_{\tau \in G} \omega(\tau) \tau^{-1}) / |G|$ , which is an idempotent element of group ring  $\mathbb{Z}_p[G]$  associated with  $\omega$ . Since  $C_S$  is a  $\mathbb{Z}_p[G]$ -module, we set  $C_{S,\omega} = \varepsilon_\omega(C_S)$ . For an abelian group  $A$ , we denote by  $t_p(A)$  the maximal  $p$ -group contained in the maximal torsion subgroup of  $A$ . We prove in Theorem 1 that  $\delta_p = 0$  is equivalent to existence of a finite set  $S$  of prime divisors of  $k$  satisfying the following three conditions. (1)  $S \supset P$ . (2)  $C_{S,\omega} \cong \{1\}$ . (3)  $p\text{-rank}(t_p(E_S)) = p\text{-rank}(t_p(E))$ . Put  $e_p = \#\{p \mid p \in S \text{ and } \xi_p \in k_p\}$ . We estimate  $\delta_p$  in Theorem 2 and see that  $\delta_p \leq p\text{-rank}(C_{P,\omega}) + e_p - p\text{-rank}(t_p(E))$  holds. We prove in Proposition 1 that the condition (3) holds if  $e_p = p\text{-rank}(t_p(E))$ . If the condition (2) also holds for  $P$ , the Leopoldt conjecture is true. This was known in Gras [2], Gillard [1] and Sands [8]. Miki [6] showed that the following two conditions (4) and (5) are equivalent. (4) the Galois group of the maximal  $p$ -ramified  $p$ -abelian extension of  $k$  is torsion free and the Leopoldt conjecture holds for  $(k, p)$ . (5) the condition (2) holds for  $P$  and  $e_p = p\text{-rank}(t_p(E))$  holds. We assume in Theorem 4 that  $k$  is totally imaginary if  $p = 2$  and prove that the Galois group over  $k$  of the maximal  $p$ -extension of  $k$  which

is unramified outside  $S$  is a pro- $p$ -free group if and only if the condition (2) holds for  $S$  and  $e_S = p\text{-rank}(t_p(E))$ .

**1. the necessary and sufficient condition of**

$$\delta(k : p) = 0.$$

Let  $N$  be the set of the natural numbers. Let  $i, j$  be elements of  $N$ . Let  $A$  be an abelian group. Put  $A_i = A/A^{p^i}$  for  $i \in N$ . For  $i \geq j$ , we denote by  $\varphi_{i,j}$  a homomorphism from  $A_i$  to  $A_j$  defined by  $\varphi_{i,j}(a \cdot A^{p^i}) = a \cdot A^{p^j}$  for  $a \in A$ . Then  $(A_i, \varphi_{i,j})$  is a projective system of abelian groups. We denote by  $\bar{A}$  its projective limit, which is a  $\mathbb{Z}_p$ -module. Put  $r(A) = \dim_{\mathbb{Q}_p}(\bar{A} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$ , where we denote by  $\mathbb{Q}_p$  the field of  $p$ -adic numbers. We observe that  $r = r(E)$  and  $r_p = r(E_p)$ . We abbreviate  $r(E_S)$  to  $r_S$ .

**Theorem 1.**  $\delta(k : p) = 0$  is equivalent to existence of the finite set  $S$  of prime divisors of  $k$  satisfying the following three conditions. (1)  $S \supset P$ . (2)  $C_{S, \omega} \simeq \{1\}$ . (3)  $p\text{-rank}(t_p(E_S)) = p\text{-rank}(t_p(E))$ .

To prove this theorem, we need some lemmas.

**Lemma 1.**  $r_p = r_S$  if  $P \subset S$ .

Proof. Let  $\pi : U_S \longrightarrow U_P$  be the canonical projection. Let  $S'$  be the subset of  $S$  consisting of all elements which are not contained in  $P$ . The kernel of  $\pi$  is  $U_{S'}$ . Let  $V_1$  resp.  $V_2$  be any open subgroup of  $U_P$  resp.  $U_{S'}$ . We have  $E \cdot V_1 \cdot V_2 = E_S \cdot V_1 \cdot V_2$  since  $V_1 \cdot V_2$  is an open subgroup of  $U_S$ . Hence  $\pi(E) \cdot V_1 = \pi(E_S) \cdot V_1$ . Therefore the topological closure in  $U_P$  of  $\pi(E)$  is equal to that of  $\pi(E_S)$ . Since  $E_S$  is compact,  $\pi(E_S)$  is also compact. Thus we have  $E_P = \pi(E_S)$ . Put  $V(p) = t_p(k_p^X)$  for  $p \in S$ . Put  $p^m = \max\{|V(p)| \mid p \in S'\}$ . Then we have

$$(U_{S'} \cap E_S) \cdot (E_S)^{p^n} / (E_S)^{p^n} \simeq t_p(U_{S'} \cap E_S).$$

Hence we have an exact sequence

$$1 \longrightarrow t_p(U_{S'} \cap E_S) \longrightarrow E_S / E_S^{p^n} \xrightarrow{\pi_n} E_P / E_P^{p^n} \longrightarrow 1,$$

where  $\pi_n$  is a homomorphism defined by  $\pi_n(\varepsilon \cdot E_S^{p^n}) = \pi(\varepsilon) \cdot E_P^{p^n}$  for  $\varepsilon \in E_S$ . We take the projective limit of this exact sequence. Then we see  $r_P = r_S$  by the definition. Q. E. D.

Put  $A_S^{(2)} = (E \cap (E_S)^p) / E^p$ .

Lemma 2. Suppose  $S \supset P$ . Then we have

$$\delta_p = p\text{-rank}(t_p(E_S)) - p\text{-rank}(t_p(E)) + p\text{-rank}(A_S^{(2)}).$$

Proof. Let  $X$  be a set of representatives of all left coset in  $E$  with respect to  $E^p$ . Then  $X$  is a finite set.

Put  $F = \bigcup_{\varepsilon \in X} \varepsilon \cdot E_S^p$ . Then  $F$  is compact because it is a finite union of compact sets  $\varepsilon \cdot E_S^p$ . Hence we have  $F = E_S$  because  $E_S \supset F \supset E$ . Let  $f : E/E^p \rightarrow E_S/E_S^p$  be a homomorphism defined by  $f(\varepsilon \cdot E) = \varepsilon \cdot E_S^p$  for  $\varepsilon \in X$ . Then  $f$  is surjective and  $\ker(f) = \{ \varepsilon \cdot E^p \mid \varepsilon \in E_S^p \} = (E \cap E_S^p)/E^p$ . Therefore we have  $E/(E \cap E_S^p) \cong E_S/E_S^p$ . Since  $E/E^p$  is an elementary  $p$ -abelian group, we have  $E/E^p \cong E/(E \cap E_S^p) \oplus A_S^{(2)}$ . Hence  $E/E^p \cong E_S/E_S^p \oplus A_S^{(2)}$ . Since  $r = p\text{-rank}(E/E^p) - p\text{-rank}(t_p(E))$  and  $r_S = p\text{-rank}(E_p/E_p^p) - p\text{-rank}(t_p(E_S))$ , We have

$$\delta_p = r - r_S = p\text{-rank}(t_p(E_S)) - p\text{-rank}(t_p(E)) + p\text{-rank}(A_S^{(2)}).$$

Q. E. D.

We define  $U_k^S(p)$  by

$$U_k^S(p) = \{ \alpha \in k^\times \mid \text{There exists an ideal } \mathfrak{a} \text{ of } k \text{ such}$$

$$\text{that } \mathfrak{a}^p = (\alpha), \text{ and } \alpha \in (k_{\mathfrak{p}}^\times)^p \text{ for any } \mathfrak{p} \},$$

where we denote by  $k_{\mathfrak{p}}$  the completion of  $k$  at  $\mathfrak{p}$ . We denote by  $\bar{S}$  the set  $\{ \mathfrak{P} \mid \mathfrak{P} \text{ is a prime divisor of } K \text{ which divides a prime divisor of } k \text{ contained in } S \}$ . Let  $U_K^{\bar{S}}(p)$  be a set of elements of  $K^\times$  such that there exists an ideal  $\mathfrak{A}$  of  $K$  satisfying  $\mathfrak{A}^p = (\alpha)$ , and  $\alpha \in (K_{\mathfrak{P}}^\times)^p$  for any  $\mathfrak{P} \in \bar{S}$ , where we denote by  $K_{\mathfrak{P}}$  the completion of  $K$  at  $\mathfrak{P}$ .

$$\text{Lemma 3. } U_k^S(p) \cong N_{K/k}(U_K^{\bar{S}}(p)) \cdot (K^\times)^p / (k^\times)^p,$$

where we denote by  $N_{K/k}$  the norm map from  $K$  to  $k$ .

Proof. Let  $\alpha \in U_K^{\bar{S}}(p)$  and  $\mathfrak{A}$  be an ideal of  $K$  such that  $\mathfrak{A}^p = (\alpha)$ . We denote by  $J_p$  the semi-local product  $\prod_{\mathfrak{P}|p} K_{\mathfrak{P}}^{\times}$ . Since  $N_{K/k}(\mathfrak{A})^p = N_{K/k}((\alpha))$  and  $N_{K/k}(\alpha) \in N_{K/k}(J_p^p) \subset (k^{\times})^p$ , we have  $N_{K/k}(U_K^{\bar{S}}(p)) \subset U_k^S(p)$ . Since  $(U_k^S(p))^{[K:k]} \subset N_{K/k}(U_k^S(p)) \subset U_k^S(p)$  and  $(p, [K:k]) = 1$ , we have  $U_k^S(p) \cdot (K^{\times})^p = N_{K/k}(U_K^{\bar{S}}(p)) \cdot (K^{\times})^p$ . Let  $j : k^{\times}/(k^{\times})^p \rightarrow K^{\times}/(K^{\times})^p$  be a homomorphism defined by  $j(\alpha \cdot (k^{\times})^p) = \alpha \cdot (K^{\times})^p$  for  $\alpha \in k^{\times}$ . Let  $\bar{k}$  be the algebraic closure of  $k$ , and let  $\mu_p$  be the group of all  $p$ -th roots of unity in  $K$ . Since we have the isomorphisms  $H^1(\bar{k}/k, \mu_p) \simeq k^{\times}/(k^{\times})^p$  and  $H^1(\bar{k}/K, \mu_p) \simeq K^{\times}/(K^{\times})^p$ , we see that the injection of the cohomology groups  $\text{Inj} : H^1(\bar{k}/k, \mu_p) \rightarrow H^1(\bar{k}/K, \mu_p)$  induces  $j$ . Hence  $\ker(j) \simeq H^1(K/k, \mu_p)$  by the exact sequence of the restriction and the injection of the first cohomology groups. Since  $[K:k]$  is prime to  $p$ , we have  $\ker(j) \simeq \{1\}$ . Thus  $U_k^S(p)/(k^{\times})^p$  is isomorphic to  $U_k^S(p) \cdot (K^{\times})^p / (K^{\times})^p$  by  $j$ . Therefore we have  $U_k^S(p)/(k^{\times})^p \simeq N_{K/k}(U_K^{\bar{S}}(p)) \cdot (K^{\times})^p / (K^{\times})^p$ .

Q. E. D.

Put  $L = K(\sqrt[p]{\alpha} \mid \alpha \in U_K^{\bar{S}}(p))$ . Then  $L$  is the maximal elementary  $p$ -abelian extension of  $K$  such that any prime divisor contained in  $\bar{S}$  is completely decomposed and  $\text{Gal}(L/K)$  is isomorphic to  $C_S$  by class field theory. Thus we identify  $C_S$  with  $\text{Gal}(L/K)$ .

Lemma 4.  $\text{Hom}(C_{S,\omega}, \mu_p) \simeq N_{K/k}(U_K^{\bar{S}}(p)) \cdot (K^\times)^p / (K^\times)^p,$

where we denote by  $\mu_p$  the group of all  $p$ -th roots of unity contained in  $K$ .

Proof. Denote by  $\bar{\alpha}$  the left coset  $\alpha \cdot (K^\times)^p$  for  $\alpha \in U_K^{\bar{S}}(p)$ . Let  $x \in C_S$ . Set  $\langle \bar{\alpha}, x \rangle = \sqrt{\alpha}^{-x-1}$ . Then it defines a non-degenerate pairing on  $(U_K^{\bar{S}}(p)/(K^\times)^p) \times C_S$ . Let  $\sigma \in G$ . Then  $\langle \sigma(\bar{\alpha}), \sigma(x) \rangle = \langle \bar{\alpha}, x \rangle^{\omega(\sigma)}$ . Hence we have  $\langle \bar{\alpha}, \varepsilon_\omega(x) \rangle^m = \langle N_G(\bar{\alpha}), x \rangle$ , where we denote by  $N_G$  the norm map of a  $G$ -module, and  $m = [K:k]$ . Let  $H = \{\alpha \in U_K^{\bar{S}}(p) \mid N_G(\alpha) \in (K^\times)^p\}$ . Then we see that  $H/(K^\times)^p$  is the annihilator of  $C_{S,\omega}$ . Therefore we have

$$\text{Hom}(C_{S,\omega}, \mu_p) \simeq N_{K/k}(U_K^{\bar{S}}(p)) \cdot (K^\times)^p / (K^\times)^p,$$

since  $\text{Hom}(C_{S,\omega}) \simeq U_K^{\bar{S}}(p)/H \cdot (K^\times)^p \simeq N_G(U_K^{\bar{S}}(p)) \cdot (K^\times)^p / (K^\times)^p.$

Q. E. D

Corollary.  $U_K^{\bar{S}}(p)/(K^\times)^p \simeq C_{S,\omega}.$

Proof of Theorem 1. By Corollary to Lemma 4, we see that  $U_K^{\bar{S}}(p)/(K^\times)^p$  is equivalent to  $C_{S,\omega} = \{1\}$ . Assume that  $S$  satisfies the condition (1), (2) and (3) of Theorem 1. Since  $U_K^{\bar{S}}(p)/(K^\times)^p = \{1\}$ , we have  $A_S^{(2)} = \{1\}$  by the proof of Corollary to Lemma 7. Hence  $\delta_p = p\text{-rank}(t_p(E_S)) - p\text{-rank}(t_p(E))$ . Therefore  $\delta_p = 0$  by the condition (3). Conversely we assume that  $\delta_p = 0$ . Let  $S$  be a finite set of prime divisors of  $k$

containing  $P$ . Since  $t_p(E_S) \supset t_p(E)$ , we have  $p\text{-rank}(t_p(E_S)) - p\text{-rank}(t_p(E)) \geq 0$ . Hence we have  $p\text{-rank}(t_p(E_S)) - p\text{-rank}(t_p(E)) = 0$  by Lemma 2. The condition (3) holds for any  $S$  containing  $P$ . We take a sufficiently large set  $S$  so that  $C_{S, \omega} \simeq \{1\}$ . Then the condition (2) holds. Q. E. D.

**2. Some sufficient conditions for the conditions of Theorem 1.**

Put  $e_p = \#\{p \mid p \in P \text{ and } \xi_p \in k_p\}$ .

**Proposition 1.**  $p\text{-rank}(t_p(E_P)) = p\text{-rank}(t_p(E))$  if  $e_p = p\text{-rank}(t_p(E))$  or if  $\xi_p \notin k_p$  for any  $p \in P$ .

**Proof.** Since  $e_p \geq p\text{-rank}(t_p(E_P)) \geq p\text{-rank}(t_p(E))$ , we have  $p\text{-rank}(t_p(E_P)) = p\text{-rank}(t_p(E))$  if  $e_p = p\text{-rank}(t_p(E))$ . We see that  $e_p = p\text{-rank}(t_p(E))$  holds if  $\xi_p \in k$  and  $|P| = 1$  or if  $\xi_p \notin k_p$  for any  $p \in P$ .

Q. E. D.

**Proposition 2.** Suppose that  $\xi_p \notin k_p$  for any  $p \in P$  and that the  $p$ -class field tower of  $K$  is finite. Then the Leopoldt conjecture holds for  $(k, p)$ .

**Proof.** Let  $K = K_0 \subsetneq K_1 \subsetneq K_2 \cdots \subsetneq K_n = K_{n+1}$  be the  $p$ -class field tower of  $K$ . Then  $K_n/k$  is a Galois extension. Since  $K_n/K$  is a  $p$ -extension and  $K/k$  is a cyclic extension

whose extension degree is prime to  $p$ , there exists  $\tau \in \text{Gal}(K/k)$  whose order is just  $[K:k]$ . Let  $M$  be the fixed field of  $\tau$  in  $K_n$ . Since  $[K_n:M] = [K:k]$  and  $K_n \supset M(\xi_p)$ , we have  $[K_n:M(\xi_p)] \mid [K:k]$ . On the other hand, since  $M(\xi_p) \supset K$ , we have  $[K_n:M(\xi_p)] \mid [K_n:K]$ . Since  $[K:k]$  is prime to  $[K_n:K]$ , we have  $[K_n:M(\xi_p)] = 1$ . Hence  $K_n = M(\xi_p)$ . Let  $P_M$  be the set of all prime divisors of  $M$  dividing  $p$ . Then we see that the condition (2) of Theorem 1 holds for  $(M, P_M)$ . Let  $\mathfrak{P} \in P_M$  and  $M_{\mathfrak{P}}$  be the completion of  $M$  at  $\mathfrak{P}$ . Let  $\mathfrak{p}$  be a prime divisor of  $k$  divided by  $\mathfrak{P}$ . Since  $\xi_p \notin k_{\mathfrak{p}}$  and  $M_{\mathfrak{P}}/k_{\mathfrak{p}}$  is a  $p$ -extension, we have  $\xi_p \notin M_{\mathfrak{P}}$ . By Proposition 1, we have that the condition (3) holds for  $(M, P)$ . Therefore the Leopoldt conjecture holds for  $M$ . Hence it also holds for  $k$ . Q. E. D.

**Propositin 3.** Let  $k_0$  be an algebraic number field such that  $[k : \mathbb{Q}]$  is finite. Suppose that  $k_0$  is a cyclic extension of  $k$  of degree  $p$ . Let  $S_0$  be the finite set of prime divisors of  $k_0$  such that the condition (2) of Theorem 1 holds for  $(k_0, S_0)$ . Let  $S$  be the set of all prime divisors of  $k$  which divide prime divisors contained in  $S_0$ . Put  $K_0 = k_0(\xi_p)$ . Let  $R$  be the set of primes divisors  $\mathfrak{p}$  of  $k_0$  satisfying the following two conditions. (1)  $\mathfrak{p}$  is contained in  $S_0$  or an extension of  $\mathfrak{p}$  to  $K$  is ramified at  $K/K_0$ . (2)  $\mathfrak{p}$  is completely decomposed at  $K_0$ .

Then the condition (2) of Theorem 1 for  $(k_0, S_0)$  implies that for  $(k, S)$  if  $R = \phi$ .

Put  $X_\omega = \varepsilon_\omega(X)$  for a  $\mathbb{Z}_p[G]$ -module  $X$ . To prove this proposition, we need the following two lemmas.

Lemma 5. Let  $0 \rightarrow N \rightarrow M \rightarrow P \rightarrow 0$  be an exact sequence of  $\mathbb{Z}_p[G]$ -modules. Then we have the exact sequence

$$0 \rightarrow N_\omega \rightarrow M_\omega \rightarrow P_\omega \rightarrow 0.$$

Proof. We regard  $N$  as a submodule of  $M$ . We have an exact sequence  $0 \rightarrow N \cap M_\omega \rightarrow M_\omega \rightarrow P_\omega \rightarrow 0$ . Let  $y$  be an element of  $M$  such that  $\varepsilon_\omega(y) \in N$ . Then  $\varepsilon_\omega(y) = \varepsilon_\omega \cdot \varepsilon_\omega(y) \in N_\omega$  since  $\varepsilon_\omega \cdot \varepsilon_\omega = \varepsilon_\omega$ . Hence  $N \cap M_\omega \subset N_\omega$ . Since  $N \subset N \cap M_\omega$ , we have  $N_\omega = N \cap M_\omega$ . Q. E. D.

Let  $L$  be the class field of  $K$  whose Galois group over  $K$  is isomorphic to  $C_S$  by class field theory. We denote by  $L^*$  the maximal abelian extension of  $k$  contained in  $L$ . Let  $J_K$  be the idèle group of  $K = k(\xi_p)$  and  $U_K$  be its unit group. Put  $W_K = U_K \cdot \prod_{\mathfrak{p} \in \bar{S}} K_{\mathfrak{p}}^\times$ , where we denote by  $\bar{S}$  the set of all prime divisors of  $K$  dividing prime divisors contained in  $S_0$ . Then we have  $C_S \cong J_K/W_K \cdot (J_K)^p \cdot K^\times$ . Put  $K_0 = k_0(\xi_p)$ .

Lemma 6. Let  $\sigma$  be a generator of  $\text{Gal}(K/K_0)$ . Then we have

$$C_S / (C_S)^{\sigma^{-1}} \cong N_{K/K_0} (J_K) \cdot K_0^\times / N_{K/K_0} (W_K \cdot (J_K^p)) \cdot K_0^\times$$

Proof. We identify  $\text{Gal}(L/K)$  to  $C_S$ . Let  $M$  be the fixed field of  $C_S^{\sigma^{-1}}$  in  $L$ .  $\text{Gal}(M/K_0)$  is an abelian group since  $K/K_0$  is a cyclic extension. Hence  $L^* \supset M$ . Put  $H = \text{Gal}(L/L^*)$ . Then  $H \supset (C_S)^{\sigma^{-1}}$  because  $\sigma$  acts trivially on  $\text{Gal}(L^*/K)$ . Hence  $L^* \subset M$ . Thus we have  $L^* = M$  and  $\text{Gal}(L^*/K) \cong C_S / (C_S)^{\sigma^{-1}}$ . On the other hand, we have  $\text{Gal}(L^*/K) \cong N_{K/K_0}(J_K) \cdot K_0^\times / N_{K/K_0}(w_K \cdot (J_K^D)) \cdot K_0^\times$  by translation theorem of class field theory, because  $\text{Gal}(L/K) \cong J_K / w_K \cdot (J_K)^D \cdot K^\times$ . Q. E. D.

Proof of Proposition 3. We denote by  $\mathfrak{p}$  resp.  $\mathfrak{P}$  a prime divisor of  $K_0$  resp.  $K$ . We denote by  $K_{0\mathfrak{p}}$  resp.  $K_{\mathfrak{P}}$  the completion of  $K_0$  resp.  $K$  at  $\mathfrak{p}$  resp.  $\mathfrak{P}$ . Let  $U_{\mathfrak{p}}$  resp.  $U_{\mathfrak{P}}$  be its unit group. Let  $\mathfrak{p}_0$  be a prime divisor of  $k_0$ . We define  $V_{\mathfrak{p}_0}$  and  $W_{\mathfrak{p}_0}$  by

$$V_{\mathfrak{p}_0} = \prod_{\mathfrak{p}|\mathfrak{p}_0} K_{0\mathfrak{p}}^\times, \quad W_{\mathfrak{p}_0} = \prod_{\mathfrak{P}|\mathfrak{p}_0} K_{\mathfrak{P}}^\times \quad \text{if } \mathfrak{p}_0 \in S_0,$$

$$V_{\mathfrak{p}_0} = \prod_{\mathfrak{p}|\mathfrak{p}_0} U_{\mathfrak{p}}, \quad W_{\mathfrak{p}_0} = \prod_{\mathfrak{P}|\mathfrak{p}_0} U_{\mathfrak{P}} \quad \text{if } \mathfrak{p}_0 \notin S_0.$$

For each  $\mathfrak{p}$ , we choose a prime divisor  $\mathfrak{P}$  of  $K$  dividing  $\mathfrak{p}$  and denote by  $Z_{\mathfrak{p}}$  the decomposition group of  $\mathfrak{P}$  in  $\text{Gal}(K/K_0)$ . If  $\mathfrak{P}$  is ramified at  $K/K_0$ , then  $Z_{\mathfrak{p}}$  is also the inertia group of  $\mathfrak{P}$ . Hence we have by class field theory

$$(2.1) \quad v_{p_0} / N_{K/K_0} (w_{p_0}) \simeq \prod_{p|p_0} Z_p.$$

for any prime divisor  $p_0$  of  $k_0$ . We consider the  $Z_p[G]$ -module structure of this group. We see  $Z_p \simeq \{1\}$  if  $p$  is decomposed at  $K/K_0$  or if  $p_0$  is not contained in  $S_0$  and  $p$  is not ramified at  $K/K_0$ . Now we consider the group of (2.1) for  $p_0$  whose extension  $\mathbb{B}$  to  $K$  is not decomposed if  $p_0 \in S_0$  or which is ramified at  $K/k$  if  $p_0 \notin S_0$ . Let  $p$  be a fixed prime divisor of  $K_0$  dividing  $p_0$ . We denote by  $G_{p_0}$  be the decomposition group of  $p$  in  $\text{Gal}(K_0/k_0)$ . Let  $G = \cup_{i=1}^t \sigma_i \cdot G_{p_0}$  be the decomposition of  $G$  to left cosets. We assume  $\sigma_1 \in G_{p_0}$ . Let  $\tau_1$  be a generator of  $Z_p$ . Then  $\tau_i = \sigma_i \cdot \tau_1 \cdot \sigma_i^{-1}$  is a generator of  $Z_{p\sigma_i}$ . We use the additive notation for  $Z_p$  in the followings. Then

$$\prod_{p|p_0} Z_p \simeq \{ (n_1 \cdot \tau_1, n_2 \cdot \tau_2, \dots, n_t \cdot \tau_t) \mid n_i \in \mathbb{Z}/p\mathbb{Z}$$

for  $i = 1, \dots, t\}$ .

Let  $\varphi : \prod_{p|p_0} Z_p \longrightarrow (\mathbb{Z}/p\mathbb{Z})[G/G_p]$  be a  $Z_p[G]$ -isomorphism defined by  $\varphi((n_1 \cdot \tau_1, \dots, n_t \cdot \tau_t)) = \sum_{i=1}^t n_i \cdot \sigma_i \cdot G_p$ . Then we have

$$(2.2) \quad \left( \prod_{p|p_0} Z_p \right)_{\omega} \simeq ((\mathbb{Z}/p\mathbb{Z})[G/G_p])_{\omega}.$$

This module is  $\{0\}$  if  $G_p \neq \{1\}$ , and is isomorphic to  $\mathbb{Z}/p\mathbb{Z}$  if  $G_p \simeq \{1\}$ . By (2.1) and (2.2), we have  $(v_{p_0} / N_{K/K_0} (w_{p_0}))_{\omega} \simeq \{1\}$  if and only if  $p_0$  is not completely decomposed at

$K_0/k_0$ . Hence we have  $(v_{p_0}/N_{K/K_0}(w_{p_0}))_\omega \simeq \{1\}$  for any  $p_0$  because  $R = \emptyset$ . Put  $v_{K_0} = \prod_{p_0} v_{p_0}$  and  $w_K = \prod_{p_0} w_{p_0}$ , where  $p_0$  runs through the set of all prime divisors of  $k_0$ . Then we have

$$(2.3) \quad (v_{K_0}/N_{K/K_0}(w_K))_\omega \simeq \{1\}.$$

We define  $\mathbf{Z}_p[G]$ -modules  $N, M, P, Z, Y$  by

$$N = v_{K_0} \cdot N_{K/K_0}(J_K^D) \cdot K_0^\times / N_{K/K_0}(w_K \cdot (J_K^D)) \cdot K_0^\times,$$

$$M = v_{K_0} \cdot (J_{K_0}^D) \cdot K_0^\times / N_{K/K_0}(w_K \cdot (J_K^D)) \cdot K_0^\times,$$

$$P = v_{K_0} \cdot (J_{K_0}^D) \cdot K_0^\times / v_{K_0} \cdot N_{K/K_0}(J_K^D) \cdot K_0^\times,$$

$$Y = N_{K/K_0}(J_K) \cdot K_0^\times / N_{K/K_0}(w_K \cdot (J_K^D)) \cdot K_0^\times,$$

$$Z = J_{K_0} / N_{K/K_0}(w_K \cdot (J_K^D)) \cdot K_0^\times.$$

Then we have exact sequences of  $\mathbf{Z}_p[G]$ -modules

$$(2.4) \quad 1 \longrightarrow N \longrightarrow M \longrightarrow P \longrightarrow 1,$$

$$(2.5) \quad 1 \longrightarrow M \longrightarrow Z \longrightarrow Z/M \longrightarrow 1,$$

$$(2.6) \quad 1 \longrightarrow Y \longrightarrow Z \longrightarrow J_{K_0} / N_{K/K_0}(J_K) \cdot K_0^\times \longrightarrow 1.$$

Since  $N$  is a homomorphic image of  $v_{K/K_0}/N_{K/K_0}(J_K) \cdot K_0^\times$  as

$\mathbf{Z}_p[G]$ -module, we see  $N_\omega \simeq \{1\}$  by Lemma 5 and (2.3). Let  $f :$

$J_{K_0} / N_{K/K_0}(J_K) \cdot K_0^\times \longrightarrow P$  be a  $\mathbf{Z}_p[G]$ -homomorphism defined by

$f(\alpha \cdot N_{K/K_0}(J_K) \cdot K_0^\times) = \alpha^D \cdot v_{K_0} \cdot N_{K/K_0}(J_K^D) \cdot K_0^\times$ . Then  $f$  is a sur-

jection. Since  $J_{K_0}/N_{K/K_0}(J_K) \cdot K_0$  is a trivial  $G$ -module, we have  $(J_{K_0}/N_{K/K_0}(J_K) \cdot K_0^\times)_\omega \simeq \{1\}$ . Hence  $\text{image}(f)_\omega = P_\omega \simeq \{1\}$  by Lemma 5. Then we have  $M_\omega \simeq \{1\}$  by Lemma 5 and (2.4). Hence we have  $Z_\omega \simeq (Z/M)_\omega$  by Lemma 5 and (2.5). Since it is equivalent to  $(J_{K_0}/V_{K_0} \cdot (J_{K_0}^p) \cdot K_0^\times)_\omega \simeq \{1\}$  that the condition (2) of Theorem 1 holds for  $(k_0, S_0)$ , we have  $(Z/M)_\omega \simeq \{1\}$  by the assumption of Proposition. Hence we see  $Z_\omega \simeq \{1\}$ . Therefore  $Y_\omega \simeq \{1\}$  by (2.6). Since  $Y \simeq C_S/(C_S)^{\sigma^{-1}}$  by Lemma 6, we have  $(C_S/C_S^{\sigma^{-1}})_\omega \simeq \{1\}$ . Since  $\text{Gal}(K/k_0)$  is an abelian group, we have  $\sigma \cdot \tau = \tau \cdot \sigma$  for any  $\tau \in G$ . Hence  $\sigma \cdot \varepsilon_\omega = \varepsilon_\omega \cdot \sigma$ . Thus  $(C_S^{\sigma^{-1}})_\omega = (C_{S,\omega})^{\sigma^{-1}}$ . Therefore we have  $C_{S,\omega}/(C_{S,\omega})^{\sigma^{-1}} \simeq \{1\}$  because  $(C_S/C_S^{\sigma^{-1}})_\omega \simeq C_{S,\omega}/(C_{S,\omega})^{\sigma^{-1}}$ . This implies  $C_{S,\omega} \simeq \{1\}$ . Q. E. D.

### 3. Some theorems concerned with $\delta_p(k : p)$ .

We define groups  $V_S, W_S$  and  $A_S^{(1)}$  by

$$V_S = \{ u \in U_S \mid u^p \in E_S \},$$

$$W_S = \{ u \in U_S \mid u^p = 1 \},$$

$$A_S^{(1)} = E \cap (U_S^p)/E \cap (E_S^p).$$

Lemma 8.  $1 \longrightarrow W_S/W_S \cap t_p(E_S) \longrightarrow V_S/W_S \longrightarrow A_S^{(1)} \longrightarrow 1.$

Proof. Let  $u \in V_S$ . Then there exists  $\delta \in E_S$  such that

$u^p = \delta$ . Since  $E \cdot (E_S^p) = E_S$ , we have  $\varepsilon \in E$  and  $\delta_1 \in E_S$  such that  $\delta = \varepsilon \cdot \delta_1^p$ . Then we see  $\varepsilon \in E \cap (U_S^p)$ . Let  $f : V_S \rightarrow A_S^{(1)}$  be a homomorphism defined by  $f(u) = \varepsilon \cdot (E \cap (E_S^p))$ . Then  $\ker(f) = W_S \cdot E_S$ . Let  $\varepsilon \in E \cap (U_S^p)$ . Then there exists  $u \in U_S$  such that  $u^p = \varepsilon$ . We have  $f(u) = \varepsilon \cdot (E \cap (E_S^p))$ . Hence  $f$  is surjective. Since  $W_S \cdot E_S / E_S \cong W_S / W_S \cap t_p(E_S)$ , we have an exact sequence

$$1 \longrightarrow W_S / W_S \cap t_p(E_S) \longrightarrow V_S / E_S \longrightarrow A_S^{(1)} \longrightarrow 0.$$

Q. E. D.

Let  $T$  be a finite set of prime divisors of  $k$ . We permit in the case  $T = \emptyset$ . Let  $A$  be the maximal subgroup of the ideal class group of  $k$  whose exponent is divided by  $p$ . We define  $U_k^T(p)$  for  $T = \emptyset$  by  $U_k^\emptyset(p) = \{ \alpha \in k \mid \text{There exists an ideal } \mathfrak{a} \text{ of } k \text{ such that } \mathfrak{a}^p = (\alpha) \}$ . We define a subgroup  $A_T^{(0)}$  of  $A$  by

$$A_T^{(0)} = \{ c \in A \mid c \text{ contains an ideal } \mathfrak{a} \text{ of } k \text{ such that } \mathfrak{a}^p = (\alpha) \text{ for some } \alpha \in U_k^T(p) \}.$$

Lemma 9.  $U_k^T(p) / (E \cap (U_k^T(p))) \cdot (k^\times)^p \cong A_T^{(0)}$ .

Proof. Let  $\alpha \in U_k^T(p)$ . Then there exists an ideal  $\mathfrak{a}$  of  $k$  such that  $\mathfrak{a}^p = (\alpha)$ . Let  $c$  be the divisor class containing  $\mathfrak{a}$ . This divisor class is contained in  $A_S^{(0)}$ . We define a homomorphism  $f : U_k^T(p) \rightarrow A_S^{(0)}$  by  $f(\alpha) = c$ . Then  $f$  is

surjective by the definition of  $A_S^{(0)}$ . Since  $\ker(f) = \{ \alpha \in U_k^T(p) \mid \text{There exists } \beta \in k \text{ such that } (\alpha) = (\beta^p) \}$ , we have  $\ker(f) = (E \cdot (k^\times)^p) \cap U_k^T(p) = (E \cap U_k^T(p)) \cdot (k^\times)^p$ . Therefore we see  $\text{image}(f) \cong U_k^T(p) / (E \cap U_k^T(p)) \cdot (k^\times)^p$ .

Q. E. D.

*Corollary.* Suppose  $S$  is a non-empty finite set of prime divisors of  $k$ . Then we have

$$U_k^S(p) / (k^\times)^p \cong A_S^{(0)} \oplus A_S^{(1)} \oplus A_S^{(2)}.$$

*Proof.* Since  $E \cap U_k^S(p) = E \cap U_S^p$ , we have a chain of abelian groups  $U_k^S(p) / (k^\times)^p \supset (E \cap U_k^S(p)) \cdot (k^\times)^p / (k^\times)^p \supset (E \cap E_S^p) \cdot (k^\times) / (k^\times)^p$ . The sequence of the quotient groups of this chain is isomorphic to  $A_S^{(0)}, A_S^{(1)}, A_S^{(2)}$ . Since the exponent of  $U_k^S(p) / (k^\times)^p$  divides  $p$ , it is isomorphic to  $A_S^{(0)} \oplus A_S^{(1)} \oplus A_S^{(2)}$ .

Q. E. D.

Put  $e_S = \{ p \mid p \in S \text{ and } \xi_p \in k_p \}$ . We see that  $e_S$  is equal to  $p\text{-rank}(W_S)$ .

*Theorem 2.* We have for  $S \supset P$ ,

$$\begin{aligned} \delta_p &= e_S + p\text{-rank}(C_{S, \omega}) - p\text{-rank}(t_p(E)) - p\text{-rank}(V_S/E_S) \\ &\quad - p\text{-rank}(A_S^{(0)}). \end{aligned}$$

Proof. By Lemma 2, we have

$$(3.1) \quad \delta_p = p\text{-rank}(t_p(E_S)) - p\text{-rank}(t_p(E)) + p\text{-rank}(A_S^{(2)}).$$

By Lemma 8, we have

$$(3.2) \quad p\text{-rank}(t_p(E_S)) = e_S - p\text{-rank}(W_S/E_S) + p\text{-rank}(A_S^{(1)})$$

since  $p\text{-rank}(t_p(E_S)) = p\text{-rank}(W_p \cap t_p(E_S))$ .

We substitute  $p\text{-rank}(t_p(E_S))$  in (3.1) by the right hand side of (3.2). Then

$$(3.3) \quad \delta_p = e_S - p\text{-rank}(V_S/E_S) + p\text{-rank}(t_p(E)) + \\ (p\text{-rank}(A_S^{(1)}) + p\text{-rank}(A_S^{(2)})).$$

By Corollary to Lemma 9 and Corollary to Lemma 4,

$$(3.4) \quad p\text{-rank}(A_S^{(1)}) + p\text{-rank}(A_S^{(2)}) = p\text{-rank}(C_{S,\omega}) - p\text{-rank}(A_S^{(0)})$$

We substitute  $(p\text{-rank}(A_S^{(1)}) + p\text{-rank}(A_S^{(2)}))$  in (3.3) by the right hand side of (3.4). Then

$$\delta_p = e_S - p\text{-rank}(V_S/E_S) - p\text{-rank}(t_p(E)) + p\text{-rank}(C_{S,\omega}) \\ - p\text{-rank}(A_S^{(0)}).$$

Q. E. D.

Corollary. We have  $\delta_p \leq p\text{-rank}(C_{S,\omega}) + e_S - p\text{-rank}(t_p(E))$  for  $P \subset S$ .

Let  $k_S$  be the maximal  $p$ -extension of  $k$  unramified outside  $S$ . Put  $G_S = \text{Gal}(k_S/k)$  and  $G_S^* = G_S/[G_S, G_S]$ , where

$[G_S, G_S]$  is the comutator subgroup of  $G_S$ . Put  $\hat{G}_S = \text{Hom}(G_S^*, \mathbf{Q}/\mathbf{Z})$ . Let  $f_p$  be an endomorphism of  $\mathbf{Q}/\mathbf{Z}$  defined by  $f_p(x) = p \cdot x$  for  $x \in \mathbf{Q}/\mathbf{Z}$ . We have an exact sequence

$$0 \longrightarrow \mathbf{Z}/p\mathbf{Z} \longrightarrow \mathbf{Q}/\mathbf{Z} \xrightarrow{f_p} \mathbf{Q}/\mathbf{Z} \longrightarrow 0.$$

Then we have the following cohomology long exact sequence

$$\begin{aligned} H^1(G_S, \mathbf{Q}/\mathbf{Z}) &\xrightarrow{f_p^{(1)*}} H^1(G_S, \mathbf{Q}/\mathbf{Z}) \longrightarrow H^2(G_S, \mathbf{Z}/p\mathbf{Z}) \longrightarrow \\ H^2(G_S, \mathbf{Q}/\mathbf{Z}) &\xrightarrow{f_p^{(2)*}} H^2(G_S, \mathbf{Q}/\mathbf{Z}), \end{aligned}$$

where we denote by  $f_p^{(1)*}$  and  $f_p^{(2)*}$  the induced homomorphism of the cohomology groups by  $f_p$ . Since  $G_S$  acts trivially on  $\mathbf{Q}/\mathbf{Z}$ , we have  $H^1(G_S, \mathbf{Q}/\mathbf{Z}) \simeq \hat{G}_S$ . Then  $\text{coker}(f_p^{(1)*}) = \hat{G}_S/p \cdot \hat{G}_S$  and  $\ker(f_p^{(2)*}) = \{c \in H^2(G_S, \mathbf{Q}/\mathbf{Z}) \mid p \cdot c = 0\}$ . Put  $H^2(G_S, \mathbf{Q}/\mathbf{Z})_p = \ker(f_p^{(2)*})$  and  $G_{S,p}^* = \{x \in G_S^* \mid x^p = 1\}$ . Then  $G_{S,p}^*$  is equal to  $\{x \in G_S^* \mid x^p = 1\}$ . Put  $\hat{G}_{S,p} = \text{Hom}(G_{S,p}^*, \mathbf{Q}/\mathbf{Z})$ . Then  $\text{coker}(f_p^{(1)*}) \simeq \hat{G}_S/p \cdot \hat{G}_S$  is equal to  $\hat{G}_{S,p}$ . Hence we have a short exact sequence

$$(3.5) \quad 0 \longrightarrow \hat{G}_{S,p} \longrightarrow H^2(G_S, \mathbf{Z}/p\mathbf{Z}) \longrightarrow H^2(G_S, \mathbf{Q}/\mathbf{Z})_p \longrightarrow 0.$$

Put  $g^S = p\text{-rank}(H^1(G_S, \mathbf{Z}/p\mathbf{Z}))$ ,  $r^S = p\text{-rank}(H^2(G_S, \mathbf{Q}/\mathbf{Z}))$  and  $t^S = p\text{-rank}(t_p(G_S^*))$ .

**Theorem 3.** Suppose that  $k$  is totally imaginary if  $p = 2$ . Suppose  $P \subset S$ . Let  $r_2$  be the number of the complex places of  $k$ . Then we have

$$(1) \quad p\text{-rank}(H^2(G_S, \mathbf{Z}/p\mathbf{Z})) = g^S - (r_2 + 1) = \delta_p + t^S,$$

$$(2) \quad p\text{-rank}(H^2(G_S, \mathbf{Q}/\mathbf{Z})_p) = \delta_p.$$

Proof. Let  $S_1$  be the union of  $S$  and the set of all infinite prime divisors of  $k$ . We see  $k_S = k_{S_1}$  in case of  $p = 2$  because  $k$  is totally imaginary. If there exists an intermediate field  $k'$  of  $k_{S_1}/k$  of finite degree over  $k$  such that the infinite prime divisors are ramified, then we see  $[\tilde{k}' : k] \equiv 0 \pmod{2}$ , where we denote by  $\tilde{k}'$  the Galois closure of  $k'$ . Since  $\tilde{k}' \subset k_{S_1}$ , such  $k'$  does not exist in case of  $p \neq 2$ . Thus  $k_S = k_{S_1}$ . Therefore we have, by Corollary 1 to the main theorem of Neumann [7],

$$\sum_{i=0}^2 (-1)^i p\text{-rank}(H^i(G_S, \mathbf{Z}/p\mathbf{Z})) = -r_2.$$

Hence  $p\text{-rank}(H^2(G_S, \mathbf{Z}/p\mathbf{Z})) = p\text{-rank}(H^1(G_S, \mathbf{Z}/p\mathbf{Z})) - (r_2 + 1)$ . Since the  $\mathbf{Z}_p$ -free rank of  $G_S^*$  equals  $r_2 + 1 + \delta_p$  by the theory of  $\mathbf{Z}_p$ -extensions, we see

$$p\text{-rank}(G_S^*/(G_S^*)^p) = (r_2 + 1 + \delta_p) + t^S.$$

Since  $g^S = p\text{-rank}(H^1(G_S, \mathbf{Z}/p\mathbf{Z})) = p\text{-rank}(G_S^*/(G_S^*)^p)$ , we have

$$p\text{-rank}(H^2(G_S, \mathbf{Z}/p\mathbf{Z})) = g^S - (r_2 + 1) = \delta_p + t^S.$$

Hence we have  $p\text{-rank}(H^2(G_S, \mathbf{Q}/\mathbf{Z})_p) = \delta_p$  by (3.5), because  $p\text{-rank}(\hat{G}_{S,p}) = p\text{-rank}(G_{S,p}^*) = p\text{-rank}(t_p(G_S^*))$ .

Q. E. D.

**Corollary.** Suppose  $k$  is totally imaginary if  $p = 2$ . Suppose  $P \subset S$ . Then  $G_S$  is a pro- $p$ -free group if and only if  $\delta_p = 0$  and  $t^S = 0$ .

**Proof.** By Satz 4.12 of Koch [5], we have that  $G_S$  is the pro- $p$ -free group if and only if  $H^2(G_S, \mathbf{Z}/p\mathbf{Z}) = \{0\}$ . By (1) of Theorem 3, we have that  $H^2(G_S, \mathbf{Z}/p\mathbf{Z}) = \{0\}$  is equivalent to  $\delta_p = t^S = 0$ . Q. E. D.

**Lemma 10.** Suppose  $P \subset S$ . Let  $r_2$  be the number of complex places of  $k$ . Then we have

$$g^S = e_S - p\text{-rank}(t_p(E)) + p\text{-rank}(C_{S, \omega}) + r_2 + 1.$$

**Proof.** Let  $J$  be the idèle group of  $k$ , and  $U$  be its unit group. Put  $U(S) = \prod_{p \in S} U_p$ , which is contained in  $U$ . By §3 in Miki [6], we have an exact sequence

$$(3.6) \quad 1 \longrightarrow U_k^S(p)/(k^\times)^p \longrightarrow U_k^\phi(p)/(k^\times)^p \longrightarrow U/U(S) \cdot U^p \\ \longrightarrow J/U(S) \cdot J^p \cdot k^\times \longrightarrow J/U \cdot J^p \cdot k^\times \longrightarrow 1.$$

Since  $G_S^*/(G_S^*)^p \simeq J/U(S) \cdot J^p \cdot k^\times$  by class field theory, we have  $p\text{-rank}(J/U \cdot J^p \cdot k^\times) = g_S$ . We compute  $g^S$  by (3.6). We have  $U_k^\phi(p)/E \cdot (k^\times)^p \simeq A_\phi^{(0)}$  by Lemma 9. Let  $h_p$  be the  $p$ -rank of the  $p$ -Sylow subgroup of the ideal class group of  $k$ . Then  $h_p$

=  $p$ -rank( $J/U \cdot J^p \cdot k^X$ ) by the definition of  $h_p$ . Since  $h_p$  is also equal to  $p$ -rank( $A_\phi^{(0)}$ ), we have  $p$ -rank( $J/U \cdot J^p \cdot k^X$ ) =  $p$ -rank( $A_\phi^{(0)}$ ). Hence

$$\begin{aligned}
 (3.7) \quad & p\text{-rank}(J/U \cdot J^p \cdot k^X) - p\text{-rank}(U_k^\phi(p)/(k^X)^p) = \\
 & p\text{-rank}(J/U \cdot J^p \cdot k^X) - (p\text{-rank}(U_k^\phi(p)/E \cdot (k^X)^p) + \\
 & p\text{-rank}(E \cdot (k^X)^p/(k^X)^p)) \\
 & = p\text{-rank}(E \cdot (k^X)^p/(k^X)^p) = -p\text{-rank}(E/E^p).
 \end{aligned}$$

We have  $p$ -rank( $U/U(S) \cdot U^p$ ) =  $e_S + [k:\mathbf{Q}]$  because  $U/U(S) \cdot U^p$  is isomorphic to  $U_S/U_S^p$ . Therefore by (3.6), (3.7) and corollary to Lemma 4, we have

$$\begin{aligned}
 g^S &= p\text{-rank}(U_k^S(p)/(k^X)^p) + (e_S + [k:\mathbf{Q}]) - p\text{-rank}(E/E^p) \\
 &= p\text{-rank}(C_{S,\omega}) + e_S + r_2 + 1 - p\text{-rank}(t_p(E)).
 \end{aligned}$$

Q. E. D.

**Theorem 4.** Put  $r^S = p$ -rank( $H^2(G_S, \mathbf{Z}/p\mathbf{Z})$ ). Suppose that  $k$  is totally imaginary if  $p = 2$ . Suppose that  $P \subset S$ . Then  $r^S = 0$  if and only if  $p$ -rank( $C_{S,\omega}$ ) = 0 and  $e_S = p$ -rank( $t_p(E)$ ).

**Proof.** By (1) of Theorem 3 and Lemma 10, we have  $r^S = 0$  holds if and only if  $e_S - p$ -rank( $t_p(E)$ ) +  $p$ -rank( $C_{S,\omega}$ ) = 0. Since  $e_S - p$ -rank( $t_p(E)$ )  $\geq 0$ , we have  $r^S = 0$  if and only if  $e_S - p$ -rank( $t_p(E)$ ) = 0 and  $p$ -rank( $C_{S,\omega}$ ) = 0. Q. E. D.

Remark. We see  $e_S \geq p\text{-rank}(t_p(E_S)) \geq p\text{-rank}(t_p(E))$ .  
 Hence the condition (3) of Theorem 1 holds if  $e_S = p\text{-rank}(t_p(E))$ .

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