DISSERTATION

On the hyperbolic free boundary problems related to the motion of interfaces

界面の運動に関連した双曲型自由境界問題について

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Abstract

In this study, we consider the hyperbolic type of free boundary problems. There are two types of these problems which we will present. The first one is the obstacle problems for the wave equation which are introduced by Schatzman, K.Kikuchi and Omata from differential viewpoints. The type of problems introduced by K.Kikuchi and Omata can be said the time evolutional variant of Alt–Caffarelli type free boundary problems, and whose treatment is based on the calculus of variations. During the past of couples decades, K.Kikuchi, Omata, Svadlenka, Ginder have developed many mathematical tools for the mathematical modeling of the vibration motion of the film and volume constrain problems. e.g. Discrete Morse Flow method which is based on minimizing movement method and can be an effective numerical tools. In this work, we present the new numerical scheme based on discrete Morse flow method which maybe gives improvement the numerical simulation.

The second one is the hyperbolic variant of the moving boundary problems, especially the mean curvature accelerated flow, so-called, hyperbolic mean curvature flow. When a family of smooth surfaces evolves with the acceleration that is equal to their mean curvature, we call it the mean curvature acceleration flow. These type equations correspond the mathematical modeling of the oscillation of interface e.g. the melting or crystalizing of Helium crystal or the motion of soap bubbles. There are few mathematical results e.g. the graph solutions by LeFloch-Smoczyk, etc. and numerical computation by Ginder–Svadlenka. However, the solution of this equation dose not have the energy preserving property. Here, the energy means the sum of the surface area and the kinetic energy which corresponds to the energy of the solution of the wave equation. In this thesis, we propose the new hyperbolic mean curvature flow equation whose solution conserves the energy in some sense and study their properties.

Contents

Introduction

The free boundary problems, which are one of the important subjects in the field of partial differential equations, have many connections to other mathematical fields e.g. real analysis, differential geometry, geometric measure theory. These problems also correspond the mathematical model for real phenomena, melting ice, a film with an obstacle, soap film or bubbles, fire burning, grain boundaries, etc. There are many types of the free boundary problems (see [21], [29], [30] for general references), we focus on the following three types as an introduction to this thesis.

The first type is Alt–Caffarelli type the free boundary problems which is introduced by H. W. Alt, L. A. Caffarelli and A. Freedman in [4], [5] with describing following the variational problem.

$$\begin{cases} \text{Minimize} & \displaystyle\int_\Omega (F(|\nabla u|^2) + Q^2 \chi_{u>0})\, d\mathscr{L}^N \\ \text{on} & \{u \in L^2_{\text{loc}}(\Omega) : \nabla u \in L^2(\Omega), w = u^0 \text{ on } S\} \end{cases}$$

where Ω is a connected Lipshitz domain in N-dimnesional Euclidian space $\mathbb{R}^N$. $F = F(u)$ is a given function belong to Hölder space $C^{2,1}[0,\infty)$ with $F(0) = 0$ and some technical conditions. Q is a given bounded positive measurable function, and $\chi_{u>0}$ denotes the characteristic function of $\{u > 0\} := \{x \in \Omega : u(x) > 0\}$, $d\mathscr{L}^N$ means the integration by N-dimensional Lebesgue measure. u^0 is a given non-negative function belonging to $L^2_{\text{loc}}(\Omega)$ with $\nabla u^0 \in L^2(\Omega)$ and S is a subset of $\partial\Omega$ with positive $(N-1)$-dimensional Hausdorff measure $\mathscr{H}^{N-1}$. They showed the existence and the regularity of the minimizer, and the free boundary $\partial\{u > 0\}$ is smooth in some sense by using the blow-up method, and the framework of geometric measure theory. This problem is related to the mathematical model of the jet flow. Let us explain a little details of this problem in the case of F is the identify function i.e. $F(t) = t$. Then, the first variation for the above functional leads that the minimizer u satisfies the following condition:

$$\Delta u = 0 \quad \text{in} \quad \Omega \cap \{u > 0\}, \quad |\nabla u| = Q \quad \text{on} \quad \Omega \cap \partial\{u > 0\}.$$

The second equation is called the free boundary condition which gives the information on the free boundary. As other important consequence, $\Delta u \geq 0$ holds in the distribution sense, and this leads Δu is a positive Radon measure whose support is contained the free boundary. From this fact, after the delicate arguments, the above two equations, we can rewrite to the one equation as follows (see [4] for more details):

$$\Delta u = Q\mathscr{H}^{N-1} \lfloor \partial_{\text{red}}\{u > 0\} \quad \text{in} \quad \Omega,$$

where $\partial_{\mathrm{red}}\{u>0\}$ is the reduced boundary of $\{u>0\}$ that we can consider the normal vector in appropriate sense (see [80] Section 14 for precise definition). Additionally, S. Omata [72] and S. Omata and Y. Yamaura [75] got the same results for the non-linear version of the above problem in the case of $N=2$, $F(|\nabla u|^2)$ replaced by $a^{ij}(u)D_iuD_ju$ where $a^{ij}(z)$ is a given smooth function with some elliptic condition. Moreover, Y. Yamaura [86] treated the following problem which is the above problem for the minimal surface equation with some generalization.

$$\begin{cases} \text{Minimize} & \displaystyle\int_\Omega \sqrt{1+|\nabla u|^2} + \int_\Omega Q^2 \chi_{u>0}\, d\mathscr{L}^N + \int_S |u-u^0|\, d\mathscr{H}^{N-1} \\ \text{on} & BV(\Omega), \end{cases}$$

where the first term of the above functional should be interpreted as Radon measure on Ω, and $BV(\Omega)$ denotes the space of functions of bounded variations, that is, whose distributional derivatives are Radon measure on Ω. In this case, because of the lack of the compactness of the space $W^{1,1}$, it is useful for the framework of the theory of the functions of bounded variation. See [39]. [6] for the theory of BV functions and applications.

The second type is hyperbolic variants of the above Alt–Caffarelli type free boundary problems, so called the wave type of obstacle problem which is described as follows:

Problem 0.0.1. Find $u : \Omega \times [0,T) \to \mathbb{R}$ such that

$$\begin{cases} \chi_{\overline{\{u>0\}}}\, u_{tt} - \Delta u &= -\dfrac{Q^2}{|Du|} \mathscr{H}^N \lfloor \partial\{u>0\} \quad \text{in } \Omega \times (0,T), \\ u(x,0) &= u_0(x) \quad \text{in } \Omega, \\ u_t(x,0) &= v_0(x) \quad \text{in } \Omega, \end{cases} \tag{0.0.1}$$

under suitable boundary conditions, where $\Omega \subset \mathbb{R}^N$ is a bounded Lipschitz domain, $T>0$ is the final time, u_0 denotes the initial condition, v_0 is the initial velocity, and $\{u>0\}$ is the set $\{(x,t) \in \Omega \times (0,T) : u(x,t) > 0\}$, $Du := (\nabla u, u_t)$, Q is a given constant. .

We understand the first line of (0.0.1) expresses the wave equation $u_{tt} - \Delta u = 0$ in $\{u>0\}$, the Laplace equation $\Delta u = 0$ a.e. t in $\{u<0\}$. We also have the free boundary condition, it can be formally shown that, in the energy-preserving regime, the solutions for (0.0.1) fulfill $|\nabla u|^2 - u_t^2 = Q^2$ on $\partial\{u>0\}$. Formally, when we consider the case of that $u_t \equiv 0$, then the equation (0.0.1) is reduced the Alt–Caffarelli type free boundary problem $\Delta u = Q\mathscr{H}^{N-1}\lfloor \partial\{u>0\}$ in Ω. Therefore, in this sense, we call the equation (0.0.1) the hyperbolic Alt–Caffarelli type free boundary problem.

Physically, the characteristic function $\chi_{\overline{\{u>0\}}}$ which is the coefficient of the acceleration term, means the locally coefficient of restitution is equal to zero. This problem is a natural prototype for explaining phenomena involving oscillations in the presence of an obstacle, e.g., an elastic string hitting a desk or soap bubbles moving atop the water. There are other approaches and mathematical formulations for these phenomena e.g. Amerio and Prouse [9], Schatzman [79], Citrini [24]. Recently, there is a review and some extension for these materials by Real and Figalli [76].

Similar types of problems have been treated in [53], [88], [51], [37], and that the recent paper

[74] has established a precise mathematical formulation. These papers revealed that the discrete Morse flow (also known as minimizing movements), a method based on time-discretized functionals, is an effective tool for this type of problem. More precisely, we consider the following functional:

$$I_n(u) := \int_\Omega \frac{|u - 2u_{n-1} + u_{n-2}|^2}{2h^2} \chi_{u>0}\, dx + \frac{1}{2}\int_\Omega |\nabla u|^2\, dx + \int_\Omega f(u)\, dx \quad (n = 2, 3, \cdots)$$

on an appropriate function class $\mathcal{K}$ e.g. $H^1_0(\Omega)$ if zero Dirichlet boundary condition is required. Here, $h > 0$ is the time discretized parameter, $u_1 := u_0 + hv_0$, and the function $f : \mathbb{R} \to \mathbb{R}$ corresponds the term $-Q^2|Du|^{-1}\mathscr{H}^N \lfloor \partial\{u > 0\}$ in (0.0.1). Actually, since it is not easy to consider this measure term as a functional, in the previous researches this measure term has been treated after some smoothing e.g. $f(u) := Q^2 B_\varepsilon(u)/2$ where $\varepsilon > 0$ is smoothing parameter and $B_\varepsilon(u)$ is expected that it approximates $\chi_{u>0}$. The existence of minimizer $\tilde{u}_n$, in many cases, can be proved by the direct method of calculus of variations. After cutting off this minimizer, that is, defining $u_n := \max\{\tilde{u}_n, 0\}$, we make two types of approximate solutions $\bar{u}^h$ and u^h by time-interpolating these cut off minimizers u_n. Roughly speaking, $\bar{u}^h$ is a flat type interpolating, and u^h is a zigzag type interpolating. Formally, tending $h \to 0+$, we expect the limit function of the approximate solution (u^h) is weak solution for Problem 0.0.1 in some sense. To pass this limit process, it is key point to get the energy estimate which gives the uniform boundedness of $\|u^h_t(t)\|_{L^2(\Omega)}$, $\|\nabla u^h(t)\|_{L^2(\Omega)}$ with respect to h.

On the other hand, the discrete Morse flow method can be used in the numerical simulation for the hyperbolic Alt-Caffarelli type free boundary problem. For this direction, we can refer the work by Ginder and Svadlenka [37].

We also point out that there are interesting work by Bonafini et al. [17, 18, 19]. They construct the weak solution for the fractional wave equations and its obstacle problems by using the discrete Morse flow method.

In this thesis, following [1], we provide the new type discrete Morse flow method by using the following functional for the application to Problem 0.0.1 in the case $Q = 0$:

$$I_n(u) := \int_\Omega \frac{|u - 2u_{n-1} + u_{n-2}|^2}{2h^2} \chi_{\{u>0\}\cup\{u_{n-1}>0\}\cup\{u_{n-2}>0\}}\, dx + \frac{1}{4}\int_\Omega |\nabla u + \nabla u_{n-2}|^2\, dx.$$

One of the features of this type functional is to have an energy conservation property in some sense if the absence of the free boundary, that is, the characteristic function $\chi_{\{u>0\}\cup\{u_{n-1}>0\}\cup\{u_{n-2}>0\}}$ does not appear in the above functional. To best our knowledge, we have never seen before such as the applications and the mathematical and numerical analysis of such as functional to hyperbolic free boundary problems. Moreover, this energy conservation property of this functional gives some effect for the numerical computation for not only the problem without free boundary but the problem with free boundary as we will see.

By using this new functional, we can get the following main result of this thesis same as the standard discrete Morse flow method.

THEOREM 0.0.1 (Main result 1). Let Ω be a bounded domain in $\mathbb{R}$, and $Q = 0$. Assume that $u_0, v_0 \in H_0^1(\Omega)$ and u_0 is non-negative. Then Problem 0.0.1 has a weak solution $u \in H^1((0,T); L^2(\Omega)) \cap L^\infty((0,T); H_0^1(\Omega))$ in the following sense:

$$\int_0^T \int_\Omega (-u_t\phi_t + u_x\phi_x)\, dxdt - \int_\Omega v_0\phi(x,0)dx = 0 \quad \forall \phi \in C_c^\infty(\Omega \times [0,T) \cap \{u > 0\}),$$
$$u \equiv 0 \quad \text{in } \{u \leq 0\}.$$

Here, we remark that u is continuous on $\Omega \times (0,T)$ by Sobolev imbedding in one-dimension.

The third type of free boundary problems is the hyperbolic variant of moving boundary problems. One of the simple examples of this type of problem is the mean curvature accelerated flow so-called hyperbolic mean curvature flow. Let $(\Gamma_t)_{0 \leq t < T}$ $(T \in (0,\infty])$ be a family of time dependent of $(N-1)$-dimensional hypersurfaces in $\mathbb{R}^N$. Then, the hyperbolic mean curvature flow equation is described

$$a = \kappa \qquad \text{on} \quad \Gamma_t, \tag{0.0.2}$$

where a, κ are the acceleration and the mean curvature of the surface at the point in Γ_t respectively. We sometimes call this equation(0.0.2) the HMCF equation for shortly. It is implied that the equation (0.0.2) such as the curvature dependent acceleration, is one of the mathematical models of the motion of the soap bubbles by Kang in his thesis [49] and the melting or crystallizing of helium crystal by Gurtin and Guidugli [40]. The feature of the equation (0.0.2) is including the acceleration term of the surface. Here, we have to define the acceleration of the surface. One can define it by second time derivative of the position γ_{tt} where $\gamma(\cdot,t)$ is a parametrize representation for Γ_t ([36], [49]). In this case, the equation (0.0.2) is precisely a vector type of the hyperbolic mean curvature flow equation,

$$\gamma_{tt} = \kappa \boldsymbol{n}$$

where $\boldsymbol{n}$ is the unit normal vector. Similar formulation is used in [43], [61], they define the acceleration as the time second derivative F_{tt} where F is a mapping on fixed hypersurface M times the time interval. What they have in common is to deal with the surface as mapping. However, the parametrize representation of surfaces gives restriction of the class of surfaces too much. It is not convenient to consider a more general setting. More precisely, we can not take more complicated surfaces as initial surfaces which can be considered as multi babbles in real phenomena. On the other hand, another one can define it by the normal time derivative of the normal velocity v ([40], [78]), which is the time derivative of the normal velocity along the normal path to the surfaces. This notion is considered from the general principle in the motion of the surface, that is, the normal velocity of the surface play an essential role in the motion of the surface. The notion of the normal time derivative is introduced by Hayes [44], and Thomas [83] independently (see also [22], [56] for mathematical formulation). We remark that this notion is defined only for the family of hypersurfaces. The normal time derivative of the normal velocity, denote D_tv, has following variational formula and it characterizes this notion.

THEOREM 0.0.2 (Main result 2). Let Γ_t, $t \in [0, \infty)$ are smooth hypersurfaces in $\mathbb{R}^N$, and compact for all $t \geq 0$. Assume that the normal velocity v is smooth. Then, for any $\phi \in C^1_c(\mathbb{R}^N \times (0, \infty) : \mathbb{R}_{\geq 0})$, we have

$$\frac{d}{dt} \int_{\Gamma_t} v\phi \, d\mathscr{H}^{N-1} = \int_{\Gamma_t} D_t v \, \phi + v(v\nabla\phi \cdot \boldsymbol{n} + \phi_t) - \phi v^2 \kappa \, d\mathscr{H}^{N-1}.$$

Although the essential part of its derivation is the transport identity which gives the time derivative of the integral quantities over the time depending surfaces, this variational formula suggests us the weak notion of $D_t v$. For this reason and some expectations, we adopt the normal time derivative of the normal velocity as the definition of the acceleration of the surface.

Next, we briefly review the previous researches for the hyperbolic mean curvature flow equation (0.0.2). Gurtin, Guidugli firstly treated the following equation for plane curves as the mathematical model for the melting or crystallizing of helium crystal in [40],

$$D_t v + cv = \kappa \qquad \text{on} \quad \Gamma_t \tag{0.0.3}$$

where v is the normal velocity, c is a given constant. Rotstein, Brandon, and Cohen gave the crystalline algorithm for the same equation (4.2.2) for the closed polygonal curves in [78]. As mention above, Kang treated the type of the equation (0.0.2) with various situations with numerical results by the level set method [49]. In this method, however, it is not clear how the ideas can be extended more general settings. After about twenty years of this researches, it was started to generalize these equations in the point of view of the differential geometry by He, Kong, Liu, and LeFloch, Smoczyk independently. First, He, Kong, and Liu prove the unique short time existence smooth solution of (0.0.2) in [43]. LeFloch, Smoczyk start by deriving the equations by calculating the first variation of the action containing kinetic and internal energy terms in [61]. Although LeFloch-Smoczyk's equation is different from (0.0.2) or (4.2.2), they give the weak solution in the sense of graph solutions for another type of hyperbolic mean curvature flow equations with one-dimensional setting. Another approach for the equations (0.0.2) is the numerical treatment including the multiphase settings by two of the present authors, Ginder, Svadlenka in [36]. Their method is called hyperbolic MBO-algorithm which is based on Merriman-Bence-Osher algorithm for a numerical scheme of mean curvature flow equations via level set approach developed in [65]. They provided formal justification and the error estimate in the case of a circle for the hyperbolic MBO-algorithm. At the almost same time, the another numerical simulation for the another type of relativistic hyperbolic mean curvature flow equation which is related to the motion of the relativistic string,

$$D_t v = (1 - v^2)\kappa,$$

is treated by Bonafini [16]. We can also find the study of the weak Lipschitz evolution from square in the plane by Bellettini et al. [13].

There are two purposes for this part. The first one is to establish the hyperbolic mean curvature flow with some variational property for a larger class of surfaces. By the analogy of the relationship between the mean curvature flow equation $v = \kappa$ and the heat equation $u_t = \Delta u$, it is expected that the hyperbolic mean curvature flow equation has some relations with the wave equation $u_{tt} = \Delta u$. One of the famous properties of the wave equation is energy conservation. In the wave equation, the energy is defined by

$$\frac{1}{2}\int_{\Omega} |\nabla u|^2 \, d\mathscr{L}^N + \frac{1}{2}\int_{\Omega} (u_t)^2 \, d\mathscr{L}^N$$

for functions $u : \Omega \times [0, \infty) \to \mathbb{R}$, $\Omega \subset \mathbb{R}^N$, where $\mathscr{L}^N$ denotes N-dimensional Lebesgue measure, and the solution of the wave equation conserves this energy under the zero Dirichlet boundary condition. Similarly, we define the notion of an energy of surface Γ_t, denoted by $E(t)$ as follows;

$$E(t) := \mathscr{H}^{N-1}(\Gamma_t) + \frac{1}{2}\int_{\Gamma_t} v^2 \, d\mathscr{H}^{N-1} \tag{0.0.4}$$

which is expected to be conserved for the solution of the equation(0.0.2), where $\mathscr{H}^{N-1}$ denotes $(N-1)$-dimensional Hausdorff measure (see [80] for the definition). The quantity $E(t)$ is the sum of surface area and the normal kinetic energy. Usually, kinetic energy is defined by also a mass, that is surface density, but, in this stage, we do not consider the distribution of mass on a surface. This means that we consider only the case of constant of surface density, especially, we take the unity as surface density. It is known that this is appropriate when we consider the modeling of the motion of soap bubbles and films. Recalling that the normal velocity plays an important role in the evolving surfaces, the quantity $E(t)$ is corresponding the energy for the wave equation in this sense, and we call $E(t)$ the surface evolution energy. We also give another physical motivation of the surface evolution energy by using the phenomenon of the shifting acrylic rod by soap film.

However, if we take the time derivative—strictly, the normal time derivative denoted by D_t, see section 4.1—of the normal velocity v as the acceleration a, that is, $a = D_t v$, the solution of the equation (0.0.2) dose not conserve the energy $E(t)$ in general. To derive the governing equation, we usually calculate the first variation of the action integral corresponding the energy $E(t)$. When all Γ_t is parametrized, that is $\Gamma_t = \{\gamma(\vartheta, t) \,:\, \vartheta \in \mathbb{R}^{N-1}\}$, in the same spirit of LeFloch and Smoczyk's work [61], we can calculate the first variation of the following action integral:

$$J(\gamma) := \int_0^T \int_{\gamma(\cdot,t)} \Big(\frac{|\gamma|^2}{2} - 1\Big) \, ds \, dt.$$

Moreover, under the assumption that the tangential velocity is equivalently equal to zero, we get the equation of motion for closed curve:

$$\gamma_{tt} \cdot \tilde{n} - \Big(\frac{1}{2}|\gamma_t|^2 + 1\Big)\tilde{\kappa} = 0,$$

here $\tilde{\kappa} = \tilde{\kappa}(\vartheta, t)$ is the mean curvature. As we will see in Section 6.1, the solution of this equation has the energy conservation property (Proposition 6.1.1). To extend the notion of a solution of this equation, we need to rearrange its formulation. One way to realize it, is the use of the framework of the moving hypersurface (see Section 4). Since the time derivative of the normal velocity $\tilde{v}$ in parametrize presentation can be translated the normal time derivative of the normal velocity v which is a function of the position on the hypersurface, we can rearrange the above governing equation as follows:

$$\frac{D_t v}{1+\frac{1}{2}v^2} = \kappa \tag{0.0.5}$$

By using the first variation of the surface evolution energy (Proposition 5.3.1), since the solution of this equation, if exists, conserves the surface evolution energy $E(t)$, we call this equation the energy conserving hyperbolic mean curvature flow (shortly, E-HMCF) equation. That is, as second part of this thesis, we treat the following initial problem for the equation (0.0.5).

Problem 0.0.2 (Energy conserving HMCF). Let Γ_0 be a given $(N-1)$-dimensional hypersurface in $\mathbb{R}^N$, v_0 be a given C^1 function on Γ_0, and $T \in (0,\infty]$ be a given. Find a moving hypersurface in $\mathbb{R}^N$, $\mathcal{M} := \bigcup_{0\leq t<T}(\Gamma_t \times \{t\})$ such that

$$\begin{aligned} &\frac{D_t v}{e} = \kappa \quad \text{on} \quad \Gamma_t.\ 0<t<T, \\ &v(\cdot,0) = v_0(\cdot) \quad \text{on} \quad \Gamma_0, \end{aligned}$$

where $e := 1+\frac{1}{2}v^2$.

Our second purpose is to analyze an initial problem for the energy conserving hyperbolic mean curvature flow equation (0.0.5). One of our ultimate goal is to give the weak solution of this equation (0.0.5) with no restriction to the class of surfaces as much as possible. When the surfaces are represented by the graph of function, we get the graph representation of the energy conserving hyperbolic mean curvature flow equation (0.0.5) as follows:

$$\frac{w_{tt}}{\sqrt{1+|\nabla w|^2}} + \frac{w_t \nabla w}{(1+|\nabla w|)^{3/2}} \cdot \Big(\frac{w_t \nabla^2 w \nabla w}{1+|\nabla w|^2} - 2\nabla w_t\Big) = \Big(1+\frac{w_t^2}{2(1+|\nabla w|^2)}\Big)\mathrm{div}\Big(\frac{\nabla w}{\sqrt{1+|\nabla w|^2}}\Big). \tag{0.0.6}$$

The equation (6.3.3) coincides the following *LeFloch-Smoczyk's equation* which appears in [61, Section 5] up to the coefficient of mean curvature part. LeFloch and Smoczyk showed that, in one dimensional setting, the global existence of weak solution in the sense of distribution with the entropy condition in [61]. For also the equation (6.3.3), it can be expected to get the same results by using the method in [61]. We will check about this in Section 6.4. However, it still remains the restriction of the class of surfaces, and it seem to be difficult to extend to higher dimension because their methods relied on the speciality of one dimension. Although, unfortunately, we do not reach either the definition or the existence of a weak solution for the equation (0.0.5) we believe that our formulation, and the variational formula for acceleration (Main Theorem 0.0.2) has the potential to weakly formulate the equation (0.0.5).

We conclude this introduction with the organization in this thesis. This thesis is consisted of the main two parts, named Part I, and Part II. Part I provide the hyperbolic Alt–Caffarelli type free boundary problems, which is divided into three sections. Section 1, we will provide some preliminaries for the hyperbolic Alt–Caffarelli type free boundary problems including its previous researches. It also includes the review of the regularity theory for elliptic equations by Ladyzhenskaya and the discrete Morse Flow. From section 2 to Section 3, we will deal with the new type of discrete Morse flow method by using the new functional for constructing a weak solution of the hyperbolic Alt–Caffarelli type free boundary problem, based on our paper. We will show the numerical results in sections 3.2 and 3.3. Part II provides the mean curvature accelerated flow, which is also divided into three sections. Section 4, we will prepare some notations and review previous researches of the mean curvature accelerated flow. In section 5, we will consider the acceleration of the surfaces. The variational formula of the acceleration, this is the main theorem in this thesis is also provided in this section. In section 6, we will provide some consideration and progress about the energy conserving mean curvature accelerated flow including the exact solution, graph solutions, the numerical results.

Notation

- $\mathbb{R}, \mathbb{Q}, \mathbb{N}$ means set of all real numbers, all rational numbers, all natural numbers respectively. $\mathbb{R}^N$ denotes N-dimensional Euclidian space, $\mathbb{R}_{\geq 0}$ is a set of all non-negative real number, $\mathbb{R}^{N_1 \times N_2}$ $(N_1 \times N_2)$ denotes the set of all real matrices.
- $|\alpha|$ (α : multiindex)...multiindex of order. That is, for a multiindex $\alpha = (\alpha_1, \dots, \alpha_N)$ (α_i is non-negative integer), defined by $|\alpha| := \alpha_1 + \cdots + \alpha_N$.
- $S_1 \setminus S_2$...set whose elements belonging to S_1 but not S_2.
- χ_S...characteristic function of a set S, that is, $\chi_S(x) = 1$ for $x \in S$, $\chi_S(x) = 0$ for $x \notin S$.
- id_S...idenity map on a set S.
- ∂S...boundary of a point set S.
- $\overline{S}$...closure of a point set S.
- δ_{ij} $(i, j \in \mathbb{N})$...Kronecker's delta, that is, $\delta_{ij} = 1$ if $i = j$, $= 0$ if $i \neq j$.
- $B_\rho(x)$...open ball in $\mathbb{R}^N$ with radius ρ, center x. We also use $B_\rho^N(x)$ for indicating the space dimension N. Sometime we use this notation for also closed balls if no ambiguity.
- ω_N...volunme of the unit ball in $\mathbb{R}^N$, $\frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}$. Here, Γ denotes Gamma function. We also define $\omega_s := \frac{\pi^{s/2}}{\Gamma(\frac{s}{2}+1)}$ for any positive real number s.
- $\mathscr{L}^N$ $(N \in \mathbb{N})$...N-dimensional Lebesgue measure.
- $\mathscr{H}^s$ $(s \in \mathbb{R}_{\geq 0})$...$s$-dimensional Hausdorff measure.
- $W \subset\subset U$...W, U are open sets in $\mathbb{R}^N$, $\overline{W} \subset U$ and $\overline{W}$ is compact set.
- $\mathrm{spt}\, f$...support of a function defined on some set X that is, $\overline{\{x \in X \ ; \ f(x) \neq 0\}}$
- Ω ...bounded domain in $\mathbb{R}^N$ with Lipschitz boundary.
- $|\Omega|$...Lebesgue measure of Ω.

- $\operatorname{diam}\Omega$...diameter of Ω, defined by $\sup_{x,y\in\Omega}|x-y|$.
- Q_T $(T \in (0,\infty]$ is given$)$...$\Omega \times (0,T)$.
- u_t...partial derivative of a function u with respect to time variable t, $:= \frac{\partial u}{\partial t}$.
- u_{x_i}...partial derivative of a function u with respect to space variables x_i, $:= \frac{\partial u}{\partial x_i}$.
- $D^\alpha u$ $(\alpha = (\alpha_1, \ldots, \alpha_N)$: multiindex$)$ ... defined by $\frac{\partial^{|\alpha|}u}{\partial x_1^{\alpha_1}\cdots\partial x_N^{\alpha_N}}$.
- ∇u...The gradient of a function u, that is, $\nabla u := (u_{x_1}, \ldots, u_{x_N})$.
- $\nabla^2 u$...Hessian matrix of a function u, that is, $(\nabla^2 u)_{ij} := (\frac{\partial^2 u}{\partial x_i \partial x_j})$.
- Δu...Laplacian of function u, defined by trace $\nabla^2 u$, that is, $\Delta u := \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_N^2}$.
- $C(\Omega)$...set of continuous function on Ω, also denoted by $C^0(\Omega)$.
- $C^k(\Omega)$ $(k \in \mathbb{N})$...set of k-times continuously differentiable functions on Ω.
- $C^k(\overline{\Omega})$ $(k \in \mathbb{N})$...set of functions in $C^k(\Omega)$ all of whose derivatives of order $\le k$ have continuous extension to $\overline{\Omega}$. That is, for any multiindex α with $|\alpha| \le k$, there exists $U^\alpha : \overline{\Omega} \to \mathbb{R}$ such that U^α is continuous on $\overline{\Omega}$ and $U^\alpha|_\Omega = D^\alpha u$.
- $C^\infty(\Omega)$...set of infinitely differentiable functions that is, $:= \bigcap_k C^k(\Omega)$.
- $C_c^\infty(\Omega)$...set of function belonging to $C^\infty(\Omega)$ with compact support.
- $C_c^\infty(\Omega;\mathbb{R}^N)$...set of functions $u : \Omega \to \mathbb{R}^N, u(x) = (u_1(x), \ldots, u_N(x))$ with $u_i \in C_c^\infty(\Omega)$ for each $i = 1, \ldots, N$.
- $C_c^\infty(\Omega \times [0,T))$...set of $C^\infty([0,T] \times \overline{\Omega})$ functions whose support is compact in $\Omega \times [0,T)$. Remark that the function in this space does not necesary vanish on $\Omega \times \{0\}$
- $L^p(\Omega)$ $(p \in [1,\infty))$...space of Lebesgue measurable functions which is p-th power integrable on Ω with norm $\|u\|_{L^p(\Omega)} := (\int_\Omega |u|^p\,dx)^{1/p}$.
- $L^\infty(\Omega)$...space of Lebesgue measurable functions which is essential bounded on Ω with norm $\|u\|_{L^\infty(\Omega)} :=$ ess $\sup_\Omega |u|$.
- $W^{k,p}(\Omega)$ $(p \in [1,\infty].k \in \mathbb{N})$...space of functions belonging to $L^p(\Omega)$ whose all weak derivatives order $\le k$ belong to L^p with norm $\|u\|_{W^{k,p}(\Omega)} := (\sum_{|\alpha|\le k}\int_\Omega |D^\alpha u|^p\,dx)^{1/p}$ $(p < \infty)$, $\|u\|_{W^{k,\infty}(\Omega)} := \sum_{|\alpha|\le k}$ ess $\sup_\Omega |D^\alpha u|$.
- $H^k(\Omega)$ $(k \in \mathbb{N})$...other notation for $W^{k,2}(\Omega)$.
- $H_0^1(\Omega)$ ...closure of $C_c^\infty(\Omega)$ with respect to $W^{1,2}(\Omega)$.
- $L^p(0,T;\mathscr{X})$ ...space of measurable functions $u : [0,T] \to \mathscr{X}$ $((\mathscr{X}, \|\cdot\|_{\mathscr{X}})$: Banach space$)$ with $\|u\|_{L^p(0,T;\mathscr{X})} := \|\|u\|_{\mathscr{X}}\|_{L^p((0,T))} < \infty$.

- $W^{k,p}(0,T;\mathscr{X})$ ...space of $L^p(0,T\,;\,\mathscr{X})$ functions such that the weak derivative u_t belongs to $L^p(0,T\,;\,\mathscr{X})$.
- $\mathscr{X}^*$...Dual space of Banach space $(\mathscr{X},\|\cdot\|_{\mathscr{X}})$, that is, a set of bounded linear functional on $(\mathscr{X},\|\cdot\|_{\mathscr{X}})$.
- $L^p(X,\mu)$ $(p\in[1,\infty))$...A set of μ-measurable functions on a measure space function (X,Σ,μ) which are p-th power μ-integrable.
- Sometimes we omit the parentheses for denoting the functional space in the one dimensional settings e.g. $C^\infty[a,b) := C^\infty([a,b))$, $L^p(a,b) := L^p((a,b))$ etc..
- $\|\nabla u\|_{L^2(\Omega)}$ means $\||\nabla u|\|_{L^2(\Omega)}$.

Part I

Hyperbolic Alt–Caffarelli type free boundary problems

Chapter 1

Preliminaries I

1.1 Regularity theory for elliptic equations

In this section, we will review the regularity theory for elliptic equations with focus on Hölder regularity which is developed by O. Ladyzhenskaya and N. Uraltseva according to [59]. This theory will be used in the justification for the first variation of the minimizing functional with free boundary in Section 1.3. More precisely, when we tale the first variation of the functional with free boundary, to get the equation in the positive part of minimizer, i.e. $\{u > 0\} := \{x \in \Omega : u(x) > 0\}$, we should take the support of a test function to be in $\{u > 0\}$. To do this, we have to ensure the set $\{u > 0\}$ is an open set in $\mathbb{R}^N$, and the regularity of minimizer is needed.

Throughout this section, we assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary. First of all, we recall the definition of Hölder continuity.

DEFINITION 1.1.1 (Hölder continuity). Let $0 < \gamma \leq 1$. We say a function $u : \Omega \to \mathbb{R}$ is *Hölder continuous* in Ω with exponent γ if there exists a constant C such that

$$|u(x) - u(y)| \leq C|x - y|^{\gamma} \quad (x, y \in \Omega).$$

In particular, we say u is *Lipschitz continuous* in the case $\gamma = 1$. We denote the set of functions which is Hölder continuous in Ω with exponent γ by $C^{0,\gamma}(\Omega)$. We also use the notation $C^{0,\gamma}_{\mathrm{loc}}(\Omega)$ which is the set of *locally Hölder continuous* functions, that is, $u \in C^{0,\gamma}_{\mathrm{loc}}(\Omega)$ if $u \in C^{0,\gamma}(\Omega')$ for any $\Omega' \subset\subset \Omega$.

Since we now assume that Ω is bounded, it holds that $C^{0,\gamma'}(\Omega) \subset C^{0,\gamma}(\Omega)$ if $0 < \gamma < \gamma' \leq 1$. Indeed, for any $u \in C^{0,\gamma'}(\Omega)$, we have

$$|u(x) - u(y)| \leq C|x - y|^{\gamma'} = C|x - y|^{\gamma}|x - y|^{\gamma' - \gamma} \leq C(\mathrm{diam}\Omega)^{\gamma' - \gamma}|x - y|^{\gamma}$$

provided $x \neq y$. The final inequality holds also for $x = y$. Thus, $u \in C^{0,\gamma}(\Omega)$.

It has been important problem to show Hölder continuity of the weak solutions of elliptic partial differential equations. One of the famous criterion to get Hölder continuity is to show that the function belongs to the next class of functions, that is, De Giorgi class.

DEFINITION 1.1.2 (De Giorgi class $\mathfrak{B}_p(\Omega, M, \gamma, d, \frac{1}{q})$ ([59, Section 2.6])). Let M, γ, d be arbitrary positive numbers, and $p \in (1, N]$ and $q > N \geq 2$. We say a function $u \in W^{1,p}(\Omega)$ belongs to the class $\mathfrak{B}_p(\Omega, M, \gamma, \delta, \frac{1}{q})$ if u satisfies the following conditions.

(i) $\sup_\Omega |u| \leq M$.

(ii) For $w = \pm u$,

$$\int_{A_{k,r-\sigma r}} |\nabla w|^p \, dx \leq \gamma \Big[\frac{1}{(\sigma r)^{p(1-\frac{N}{q})}} \sup_{B_r} (w-k)^p + 1 \Big] |A_{k,r}|^{1-\frac{N}{q}}$$

for all $\sigma \in (0,1)$, $B_r \subset \Omega$, and k with $k \geq \max_{B_r} w - d$, where $A_{k,r} := \{x \in B_r; w(x) > k\}$, and B_r is a ball of radius r.

In this definition, we also include the case $q = \infty$. Then, we shall write simply $\mathfrak{B}_p(\Omega, M, \gamma, d)$ and the inequality in the above definition (ii) becomes

$$\int_{A_{k,r-\sigma r}} |\nabla w|^p \, dx \leq \gamma \Big[\frac{1}{(\sigma r)^p} \sup_{B_r} (w-k)^p + 1 \Big] |A_{k,r}|.$$

Especially, in the case of $p = 2$, and setting $\rho := r - \sigma r$, we can also rewrite

$$\int_{A_{k,\rho}} |\nabla w|^2 \, dx \leq \gamma \Big[\frac{1}{(r-\rho)^2} \sup_{B_r} (w-k)^2 + 1 \Big] |A_{k,r}|.$$

The important theorem by O. Ladyzhenskaya and N. Uraltseva (called by also De Girorgi's embedding theorem) is described as follows:

THEOREM 1.1.1 (De Girorgi's embedding theorem ([59, Section 2, Theorem 6.1])). For any $u \in \mathfrak{B}_p(\Omega, M, \gamma, d, \frac{1}{q})$, and for any $B_{\rho_0}(x_0) \subset \Omega$ with $\rho_0 \leq 1$, the following statement holds.

For any $B_\rho(x_0)$ with $\rho \leq \rho_0$,

$$\operatorname*{osc}_{B_\rho(x_0)} u \leq c \Big(\frac{\rho}{\rho_0} \Big)^\alpha \tag{1.1.1}$$

where c and α depend only N, M, γ, d, q.

Theorem 1.1.1 tell us that a function belonging to the class $\mathfrak{B}_p(\Omega, M, \gamma, d, \frac{1}{q})$ is locally Hölder continuous. Indeed, for fixed $x_0 \in \Omega$, we chose $\rho_0 := 3^{-1} \min\{\operatorname{dist}(x_0, \partial\Omega), 1\}$. Now we have only to show that for any $\rho \leq \rho_0$, we have u is Hölder continuous on $B_\rho(x_0)$. Fix $x, y \in B_\rho(x_0)$, we have

$$|u(x) - u(y)| \leq \operatorname*{osc}_{B_{|x-y|}(x)} u \leq c \Big(\frac{|x-y|}{2\rho} \Big)^\alpha.$$

Here, in the last inequality, we applied Theorem 1.1.1 for $B_{|x-y|}(x)$. It is valid because $B_{|x-y|}(x) \subset B_{2\rho}(x) \subset B_{3\rho}(x_0) \subset \Omega$.

REMARK 1.1.1 (Other definition for De Giorgi's class). In the original De Giorgi's work ([25]) or other material (e.g. [34]), we see a slightly different definition for De Giorgi's class. Main difference with Ladyzhenskaya's method is that the boundedness of function is not required. More precisely, we say, the function $u \in W^{1,2}(\Omega)$ belongs to the class $DG(\Omega)$ if there exists a constant c such that for any $x_0 \in \Omega$, and $0 < \rho < r <$ dist$(x_0, \partial\Omega)$, the following inequality holds

$$\int_{A_{k,\rho}} |\nabla u|^2 \, dx \leq \frac{c}{(r-\rho)^2} \int_{A_{k,r}} |u-k|^2 \, dx$$

for any $k \in \mathbb{R}$ where $A_{k,r} := \{x \in B_r; u(x) > k\}$. In this case, if both of u and $-u$ belong to the class $DG(\Omega)$, we get similar result with Theorem 1.1.1([34, Theorem 8.13] for precise claim). Roughly speaking, it appears as multiplier in the right hand side of (1.1.1) that the oscillation of u in the ball with radius ρ_0. In the end, we have Hölder continuity of $u \in DG(\Omega)$.

Finally, the following lemma will be need.

LEMMA 1.1.1 ([33, V. Lemma 3.1]). Let $f = f(t)$ be a nonnegative bounded function defined in $[r_0, r_1], r_0 \geq 0$. Suppose that for $r_0 \leq t < s \leq r_1$ we have

$$f(t) \leq \{A(s-t)^{-\alpha} + B\} + \theta f(s)$$

where A, B, α, θ are nonnegative constants with $\theta \in [0,1)$. Then, for all $\rho, R \in [r_0, r_1]$ with $\rho < R$, we have

$$f(\rho) \leq c\{A(R-\rho)^{-\alpha} + B\}$$

where c is a constant depending on α, θ.

1.2 Discrete Morse flow

In this section, we see a review for the discrete Morse flow method, sometimes write DMF. This is a spacial case of the minimizing movement method, but there are many applications for various evolution equations. The minimizing movement method, is developed by E. De Giorgi [26] for the constructing solution for the following evolution equation, so called a gradient flow equation ;

$$u'(t) = -\nabla F(u(t)) \quad (t > 0) \tag{1.2.1}$$

where $u(t)$ belongs to some class $\mathcal{K} \subset X$, X is a Banach space, $F : X \to \mathbb{R}$ is a given smooth functional, ∇ denotes the Gâteaux differential, or we also call the first variation, that is,

$$\langle \nabla F(u), \varphi \rangle := \frac{d}{d\varepsilon} F(u + \varepsilon\varphi)\Big|_{\varepsilon=0} \quad \text{for any} \quad \varphi \in X,$$

where $\langle \cdot, \cdot, \rangle$ denotes a dual pair, that is, $\nabla F(u)$ is defined as a bounded linear functional on X. For the resent development and more general setting in gradient flow and minimizing movement method, refer to [7, 27]. One of the motivation of the study for these type equation, is to find a stationary point of F on X. The method which is to find the stationary point of F by taking the limit $t \to \infty$ in the solution of (1.2.1) is called *the gradient flow method* or *Morse semi flow method.*

The prototype of the discrete Morse flow method, has been developed by K. Rektorys [77] for constructing the solution of the parabolic equations. In 1991, N. Kikuchi [54] rediscovered this method and suggested that this method, that is, the discretize version of the gradient flow method could be applied for many evolutional equations.

Let us explain the basic idea of the discrete Morse flow method. We consider the following minimization problem:

$$\text{Minimize} \quad \frac{\|u-u_{n-1}\|_X^2}{2h} + F(u) \quad \text{among all } u \in \mathcal{K}$$

where the notations $F, \mathcal{K}, X$ are same. $\|\cdot\|_X$ denotes a norm in X, u_{n-1} is given, $n \in \mathbb{N}$, and $h > 0$ is time discretization parameter. The first term of this functional corresponds the time discretization part. This time discretization part can be changed by the problem settings. The formal calculation tell us that the minimizer of this functional if exists, say u_n satisfies the following equation:

$$\frac{u_n - u_{n-1}}{h} = -\nabla F(u_n). \tag{1.2.2}$$

The equation (1.2.2) can be regarded as the time discretized version of the gradient flow equation (1.2.1), for this reason, $(u_n)_{n\in\mathbb{N}}$ is called *the discrete Morse flow*. Although the term discrete Morse semiflow has been used in other materials, we omit the term 'semi' for short. Now, we define two types of the approximate solutions $\bar{u}^h$ and u^h for (1.2.1) by time interpolating minimizers as follows.

$$\bar{u}^h(t) := \begin{cases} u_0, & t = 0 \\ u_n, & t \in ((n-1)h, nh], \quad n = 1, \cdots \end{cases} \tag{1.2.3}$$

$$u^h(t) := \begin{cases} u_0, \quad t = 0 \\ \dfrac{t-(n-1)h}{h} u_n + \dfrac{nh - t}{h} u_{n-1}, \quad t \in ((n-1)h, nh], \quad n = 1, \cdots \end{cases} \tag{1.2.4}$$

The first type is the flat type approximate solution, and the second type is the zigzag type approximate solution. By using these notations, we can rewrite the equation (1.2.2). If we get some convergence results about u_t^h and $\bar{u}^h$ with respect to $h \to 0$, we can construct the solution for the equation (1.2.1). To get the convergence results, it is a key point that the energy estimate which gives the bounds for $\|u_t^h(t)\|_X$, $\|F(\bar{u}^h(t))\|_X$ with respect to h.

The discrete Morse flow method has many application e.g. the construction the gradient flow for the harmonic maps by Bethuel et al. [12], and the analysis of Alt–Caffarelli type free boundary problems by Omata and Nagasawa [69, 70], the evolutional problem corresponding to elastic-plastic energy by Zhou [89], the evolutional problem corresponding to free discontinuity problem by Yamaura [87]. The last two materials are related on the theory of the functions of bounded variation.

The discrete Morse flow method can be also applied for the hyperbolic equations. In this case, we consider the following minimization problem:

$$\text{Minimize} \quad \frac{\|u - 2u_{n-1} + u_{n-2}\|_X^2}{2h^2} + F(u) \quad \text{among all } u \in \mathcal{K}, \quad n \geq 2$$

for constructing the solution for the following initial value problem of the second order equation,

$$\begin{cases} u''(t) &= -\nabla F(u), \quad t > 0, \\ u'(0) &= v_0, \\ u(0) &= u_0. \end{cases} \tag{1.2.5}$$

In the above setting, we set $u_1 := u_0 + hv_0$, and assume $u_0, u_1 \in \mathcal{K}$. We will explain briefly the discrete Morse flow method for this type equations under the setting convex set $\mathcal{K} \subset L^2(\Omega)$ where Ω is bounded domain in $\mathbb{R}^N$. It then turns out to consider the following functional.

$$I_n(u) := \int_\Omega \frac{|u - 2u_{n-1} + u_{n-2}|^2}{2h^2}\,dx + I(u)$$

where we changed the notation for the second part of the functional. A minimizer of I_n, denoted by u_n, satisfies the following equation

$$\frac{u_n - 2u_{n-1} + u_{n-2}}{h^2} = -\nabla I(u_n),$$

and the energy estimate

$$\int_\Omega \frac{|u_n - u_{n-1}|}{2h^2}\,dx + I(u_n) \le \frac{1}{2}\int_\Omega |v_0|^2\,dx + I(u_0). \tag{1.2.6}$$

Let us give laugh explanation for second inequality(1.2.6). By convexity of $\mathcal{K}$, the function $\theta u_m + (1-\theta)u_{m-1}$ is also in $\mathcal{K}$ where $\theta \in [0,1)$, therefore, by the minimality of u_m, we have

$$\begin{aligned} I_m(u_m) &\le I_m(\theta u_m + (1-\theta)u_{m-1}) \\ &\le \int_\Omega \frac{\{\theta(u_m - u_{m-1}) + u_{m-1} - 2u_{m-1} + u_{m-2}\}^2}{2h^2}\,dx + I(\theta u_m + (1-\theta)u_{m-1}) \\ &\le \int_\Omega \frac{\{\theta(u_m - u_{m-1}) - u_{m-1} + u_{m-2}\}^2}{2h^2}\,dx + \theta I(u_m) + (1-\theta)I(u_{m-1}). \end{aligned}$$

Here we use the convexity of the functional I in the last inequality. After arranging the last inequality, we have

$$\theta\int_\Omega \frac{|u_m - u_{m-1}|^2}{2h^2}\,dx + I(u_m) \le \int_\Omega \frac{|u_{m-1} - u_{m-2}|^2}{2h^2}\,dx + I(u_{m-1}).$$

To conclude this, after letting $\theta \uparrow 1$, we have only to sum this inequality from $m = 2$ to n.

If the functional J has appropriate properties (e.g. convexity), we can get the convergence results for the approximate solution u^h and $\bar{u}^h$ with respect to $h \to 0$, and construct the weak solution for the problem (1.2.5). We now explain the details for the wave equation as example.

EXAMPLE 1.2.1 (Discrete Morse flow method for wave equation). We consider the initial boundary value problem for the following wave equation.

$$\begin{cases} u_{tt}(x,t) & = \Delta u(x,t), \quad (x,t) \in Q_T := \Omega \times (0,T), \\ u_t(x,0) & = v_0(x), \quad x \in \Omega, \\ u(x,0) & = u_0(x), \quad x \in \Omega, \\ u(\cdot,t) & = 0, \quad \text{on} \quad \partial\Omega \times (0,T) \end{cases} \tag{1.2.7}$$

where $\Omega \subset \mathbb{R}^N$ is bounded domain with smooth boundary. We set the initial conditions $u_0, v_0 \in H_0^1(\Omega)$. We construct the weak solution of this problem by using the discrete Morse flow. Firstly, we define the weak solution for (1.2.7). We say a function $u \in H^1(Q_T)$ is the *weak solution* for the problem 1.2.7 if it satisfies that for any $\varphi \in C_c^\infty(\Omega \times [0,T))$,

$$\int_0^T \int_\Omega (-u_t\varphi_t + \nabla u \cdot \nabla\varphi)\, dx\, dt - \int_\Omega v_0\varphi(x,0)\, dx = 0. \tag{1.2.8}$$

Strictly, although we have to also require that initial and boundary conditions, we omit this since we want to focus finding function satisfying (1.2.8).

Next, to apply the discrete Morse flow, we consider the functionals

$$\begin{cases} I_n(u) & := \displaystyle\int_\Omega \frac{|u - 2u_{n-1} + u_{n-2}|^2}{2h^2}\, dx + \frac{1}{2}\int_\Omega |\nabla u|^2\, dx, \quad \Big(I(u) := \frac{1}{2}\int_\Omega |\nabla u|^2\, dx\Big) \\ \text{on } \mathcal{K} & := H_0^1(\Omega), \end{cases}$$

for any $n = 2, \cdots n_0$. Here, the natural number n_0 is decided by $T = n_0 h$. These functional, since the first part is continuous in $L^2(\Omega)$, and the second part is lower semi continuous with respect to weak convergence in $H^1(\Omega)$, there exists minimizer of I_n, we name u_n. Since u_n is a minimizer of I_n, the first variation of I_n at u_n is equal to zero. That is, for any $\varphi \in H_0^1(\Omega)$

$$0 = \frac{d}{d\varepsilon} I_n(u_n + \varepsilon\varphi)\Big|_{\varepsilon=0} = \lim_{\varepsilon\to 0} \frac{I_n(u_n+\varepsilon\varphi) - I_n(u_n)}{\varepsilon},$$

continuing the calculation of the right hand side,

$$\begin{aligned} &= \lim_{\varepsilon\to 0} \int_\Omega \frac{(2u_n + \varepsilon\varphi - 4u_{n-1} + 2u_{n-2})\varphi}{2h^2}\, dx + \lim_{\varepsilon\to 0}\frac{1}{2}\int_\Omega (2\nabla u_n\cdot\nabla\varphi + \varepsilon|\nabla\varphi|^2)\, dx \\ &= \int_\Omega \frac{u_n - 2u_{n-1} + u_{n-2}}{h^2}\varphi\, dx + \int_\Omega \nabla u_n \cdot \nabla\varphi\, dx. \end{aligned}$$

Now we define the approximate solution from these minimizers same as (1.2.3), (1.2.4). Then, we can rewrite the above first variation formula by using these notation (1.2.3), (1.2.4) as follows.

$$\int_\Omega \Big[\frac{u_t^h(t) - u_t^h(t-h)}{h}\varphi + \nabla\bar{u}^h(t)\cdot\nabla\varphi\Big]\, dx = 0 \quad \text{a.e.} t \in (0,T) \quad \forall\varphi \in H_0^1(\Omega).$$

Since this relation holds for any function belonging to $L^2((0,T):H_0^1(\Omega))$, we have

$$\int_h^T\int_\Omega\Big[\frac{u_t^h(t)-u_t^h(t-h)}{h}\varphi(t)+\nabla\bar{u}^h(t)\cdot\nabla\varphi(t)\Big]\,dx\,dt=0\quad\forall\varphi\in L^2((0,T):H_0^1(\Omega)).\tag{1.2.9}$$

Now, we want to tend $h\to 0+$ in (1.2.9) to reach (1.2.8). To do so, we need more estimates regarding to approximate solutions. Fortunately by general theory as we mentioned before, we have the energy estimate (1.2.6),

$$\int_\Omega\frac{|u_n-u_{n-1}|}{2h^2}\,dx+I(u_n)\le\frac{1}{2}\int_\Omega|v_0|^2\,dx+I(u_0),$$

rewriting this inequality by using the notions $u^h,\bar{u}^h$,

$$\|u_t^h(t)\|_{L^2(\Omega)}+\|\nabla\bar{u}^h(t)\|_{L^2(\Omega)}\le C_E\quad\text{a.e.}\quad t\in(0,T)\tag{1.2.10}$$

where the constant C_E is depend only Ω,u_0,v_0. By virtue of (1.2.10), direct calculation leads the following results.

$$\|\bar{u}^h(t)-u^h(t)\|_{L^2(\Omega)}\le C_E h\quad\text{for a.e.}\quad t\in(0,T),\tag{1.2.11}$$

$$\|u^h\|^2_{L^2(Q_T)}\le\|\bar{u}^h\|^2_{L^2(Q_T)}+\frac{h}{2}\|u_0\|^2_{L^2(\Omega)},\tag{1.2.12}$$

$$\|\nabla u^h\|^2_{L^2(Q_T)}\le\|\nabla\bar{u}^h\|^2_{L^2(Q_T)}+\frac{h}{2}\|\nabla u_0\|^2_{L^2(\Omega)},\tag{1.2.13}$$

For 1.2.11, we have only to keep in mind that $u^h(t)-\bar{u}^h(t)=\frac{nh-t}{h}(u_n-u_{n-1})$ and the energy estimate (1.2.10). For 1.2.12 and 1.2.13, we have only to calculate the difference the left hand side and the first term of the right hand side respectively. Moreover, since $u^h(t)\in H_0^1(\Omega)$ for any $t\in(0,T)$, Poincaré's inequality and its integration of the both sides over $(0,T)$ lead the next inequality :

$$\|u^h\|_{L^2(Q_T)}\le C_P\|\nabla u^h\|_{L^2(Q_T)}\tag{1.2.14}$$

where the constant C_P depends only the space dimension N and Ω. Thanks to the estimates 1.2.11, 1.2.13, and 1.2.14, up to extracting subsequences, there exists $u\in H^1(Q_T)$ such that

$$\bar{u}^h\to u\quad\text{strongly in}\quad L^2(Q_T),\tag{1.2.15}$$

$$\nabla\bar{u}^h\rightharpoonup\nabla u\quad\text{weakly in}\quad(L^2(Q_T))^N,\tag{1.2.16}$$

$$u_t^h\rightharpoonup u_t\quad\text{weakly in}\quad L^2(Q_T),\tag{1.2.17}$$

as $h\to 0+$. We are now position of the limit process $h\to 0+$ in (1.2.9),

$$\int_h^T\int_\Omega\Big[\frac{u_t^h(t)-u_t^h(t-h)}{h}\varphi+\nabla\bar{u}^h\cdot\nabla\varphi\Big]\,dx\,dt=0$$

with restriction $\varphi \in C_c^\infty(\Omega \times [0,T))$ to fit our definition of the weak solution(1.2.8). Indeed, for the gradient term, by 1.2.16, we have

$$\begin{aligned}\int_h^T \int_\Omega \nabla \bar{u}^h \cdot \nabla \varphi \, dx \, dt &= \int_0^T \int_\Omega \nabla \bar{u}^h \cdot \nabla \varphi \, dx \, dt - \int_0^h \int_\Omega \nabla \bar{u}^h \cdot \nabla \varphi \, dx \, dt \\ &\to \int_0^T \int_\Omega \nabla u \cdot \nabla \varphi \, dx \, dt,\end{aligned}$$

here we remark that the second term vanishes tending $h \to 0+$ by the boundedness of $\nabla \bar{u}^h$ with respect to h. For the time discrete part, we calculate as follows.

$$\begin{aligned}\int_h^T \int_\Omega \frac{u_t^h(t) - u_t^h(t-h)}{h} \varphi \, dx \, dt &= \int_h^T \int_\Omega \frac{u_t^h(t)}{h} \varphi(\cdot,t) \, dx \, dt + \int_h^T \int_\Omega \frac{-u_t^h(t-h)}{h} \varphi(\cdot,t) \, dx \, dt \\ &= \int_h^T \int_\Omega \frac{u_t^h(t)}{h} \varphi(\cdot,t) \, dx \, dt + \int_0^{T-h} \int_\Omega \frac{-u_t^h(t)}{h} \varphi(\cdot,t+h) \, dx \, dt \\ &= \int_0^T \int_\Omega \frac{u_t^h(t)}{h} \varphi(\cdot,t) \, dx \, dt - \int_0^h \int_\Omega \frac{u_t^h(t)}{h} \varphi(\cdot,t) \, dx \, dt \\ &\quad + \int_0^T \int_\Omega \frac{-u_t^h(t)}{h} \varphi(\cdot,t+h) \, dx \, dt + \int_{T-h}^T \int_\Omega \frac{-u_t^h(t)}{h} \varphi(\cdot,t+h) \, dx \, dt \\ &= \int_0^T \int_\Omega -u_t^h(t) \frac{\varphi(\cdot,t+h) - \varphi(\cdot,t)}{h} \, dx \, dt \\ &\quad + \int_0^h \int_\Omega \frac{u_t^h(t)}{h} \varphi(\cdot,t) \, dx \, dt + \int_{T-h}^T \int_\Omega \frac{-u_t^h(t)}{h} \varphi(\cdot,t+h) \, dx \, dt \\ &\to \int_0^T \int_\Omega -u_t \varphi_t \, dx \, dt - \int_\Omega v_0 \varphi(\cdot,0) \, dx \quad h \to 0+,\end{aligned}$$

where we use three facts in the limit process $h \to 0+$. For the first term, u_t^h converges u_t weakly in $L^2(\Omega)$ and $(\varphi(\cdot,t+h) - \varphi(\cdot,t))/h$ converges φ_t strongly in $L^2(\Omega)$. For the second term follows by $u_t^h(t) = (u_1 - u_0)/h = v_0$ for $t \in (0,h)$. For the final term, we use $\varphi(\cdot,t+h) = 0$ for $t \in (T-h,T)$. In this end, we construct the weak solution in the sense of (1.2.8) for the wave equation (1.2.7) by using the discrete Morse flow method.

The first application of the discrete Morse flow method to the hyperbolic equations was given by Tachikawa [81] to construct the weak solution for the system of the following hyperbolic equations:

$$u_{tt} - \Delta u + |u|^{p-2} u = 0 \quad (p > 1).$$

One of the advantage of the discrete Morse flow method is to be able to deal well with more complicated problem e.g. volume constrain problems and free boundary problems. Here, we would like to mention the application to the volume constrain problem of the wave equation by Svadlenka and Omata [85]. They constructed the weak solution for the following wave equation with the volume preservation:

$$u_{tt}(x,t) = \Delta u(x,t) + \lambda_u(t) \quad (x,t) \in \Omega \times (0,T) \tag{1.2.18}$$

under the suitable initial and boundary conditions. The real number $\lambda_u(t)$ corresponds the volume preservation and has the following expression:

$$\lambda_u(t) := \frac{1}{V}\int_\Omega (u_{tt}(\cdot,t)u(\cdot,t) + |\nabla u(\cdot,t)|^2)\,dx,$$

where V is a given positive number. One can check a solution u for 1.2.18 preserves the volume of the graph of u, that is,

$$\int_\Omega u(\cdot,t)\,dx = V \quad \text{for any} \quad t \in (0,T).$$

For this problem, Svadlenka and Omata applied the discrete Morse Flow method by using the following functional .

$$\begin{cases} I_n(u) & := \displaystyle\int_\Omega \frac{|u-2u_{n-1}+u_{n-2}|^2}{2h^2}\,dx + \frac{1}{2}\int_\Omega |\nabla u|^2\,dx, \\ \text{on } \mathcal{K} & := \Big\{ u \in H^1(\Omega) \,:\, u|_{\partial\Omega} = g, \displaystyle\int_\Omega u\,dx = V \Big\}, \end{cases}$$

where g is a given function belonging to $L^2(\partial\Omega)$. The functional itself is exactly same as the simple wave equation which we explain in Example 1.2.1. We do not have to add the extra term for the functional, have only to add the volume preserving condition to the admissible space. Although we need some little modifications, we can same results as Example 1.2.1 for this volume preserving problem. We also point out that there is an interesting applications the discrete Morse flow method to the contact problem of the elasticity by Y. Akagawa et. al [2], they gives the extension the discrete Morse flow method to the vector valued functions. For the applications to the hyperbolic free boundary problems, we will review in Sections 1.4.

1.3 Formulation of hyperbolic Alt–Caffarelli type free boundary problems

Here we will introduce a hyperbolic variant of Alt–Caffarelli type free boundary problem according to [1] . To derive the equation, we calculate the first variation of the following action integral:

$$J(u) := \int_0^T \int_\Omega \Big((u_t)^2 \chi_{\{u>0\}} - |\nabla u|^2 - Q^2 \chi_{\{u>0\}} \Big)\,dx\,dt$$

where Q is a constant which expresses an adhesion force. When energy is conserved, i.e., when the function u does not change its value from positive to zero as time passes, we can, under appropriate assumptions, calculate the first variation, as well as the inner variation, of the functional J. However, if energy is not conserved, we can calculate neither the first variation nor the inner variation due to the presence of the Q^2-term containing the characteristic function. To overcome this difficulty, we consider a smoothing of the characteristic function within the adhesion term by a function B_ε defined by $B_\varepsilon(u) = \int_{-1}^u \beta_\varepsilon(s)\,ds$, where $\beta_\varepsilon(u) := \frac{1}{\varepsilon}\beta(\frac{u}{\varepsilon})$, and $\beta : \mathbb{R} \to [0,1]$ is a smooth function satisfying $\beta = 0$ outside $[-1,1]$, $\int_\mathbb{R} \beta(s)\,ds = 1$, and $B(0) = \frac{1}{2}$. After smoothing, we can calculate the first variation to obtain an expression for the following problem:

$$(\mathrm{P}_\varepsilon)\begin{cases} \chi_{\overline{\{u>0\}}}\, u_{tt} &= \Delta u - \frac{1}{2}Q^2\beta_\varepsilon(u) \quad \text{in } \Omega\times(0,T),\\ u(x,0) &= u_0(x) \quad \text{in } \Omega,\\ u_t(x,0) &= v_0(x) \quad \text{in } \Omega,\\ u(x,t)|_{\partial\Omega} &= \psi(x,t) \text{ with } \psi(x,0)=u_0 \ \text{ on } \partial\Omega, \end{cases}$$

where u_0, v_0 are the same as in Problem 1.1, and ψ is a given function. Now, we set the following hypotheses:

(H1) The existence of a solution u_ε to (P_ε).

(H2) The existence of a function $u : \Omega\times(0,T)\to\mathbb{R}$ such that $u_\varepsilon \to u$ in an appropriate topology as $\varepsilon \downarrow 0$ and such that the following holds:

(H2.1) $u_{tt} - \Delta u = 0$ in $\Omega\times(0,T)\cap\{u>0\}$.

(H2.2) The free boundary $\partial\{u>0\}$ is regular, $\mathscr{H}^N(\mathcal{D}\cap\partial\{u>0\})<\infty$ for any $\mathcal{D}\subset\subset\Omega\times(0,T)\subset\mathbb{R}^N\times(0,T)$, and $|Du|\neq 0$ on $\Omega\times(0,T)\cap\partial\{u>0\}$. Here, $Du = (u_{x_1},...,u_{x_N},u_t)$, and $\mathscr{H}^N$ is the N-dimensional Hausdorff measure.

(H2.3) u is a subsolution in the following sense:

$$\int_0^T\int_\Omega \Big(\chi_{\overline{\{u>0\}}}\, u_{tt}\,\zeta + \nabla u\cdot\nabla\zeta\Big)\,dx\,dt \le 0$$

for arbitrary nonnegative $\zeta\in C_c^\infty(\Omega\times(0,T))$.

Starting from (P_ε), and employing (H1), (H2.1) and (H2.2), we can show that the limit function u satisfies the following free boundary condition as in [74]:

$$|\nabla u|^2 - u_t^2 = Q^2 \quad \text{on } \Omega\times(0,T)\cap\partial\{u>0\}. \tag{1.3.1}$$

Now, for any $\mathcal{D}\subset\subset\Omega\times(0,T)$, we define a linear functional f on $C_c^\infty(\mathcal{D})$ corresponding to $\Delta u - \chi_{\overline{\{u>0\}}}\,u_{tt}$ as follows:

$$f(\zeta) := -\int_{\mathcal{D}}\Big(\chi_{\overline{\{u>0\}}}\, u_{tt}\,\zeta + \nabla u\cdot\nabla\zeta\Big)\,dx\,dt.$$

Since f is a positive linear functional on $C_c^\infty(\mathcal{D})$ by (H2.3), f can be extended to a positive linear functional on $C_0(\mathcal{D})$. Riesz's representation theorem asserts there exists a unique positive Radon measure μ_f on $\mathcal{D}$ such that

$$f(\zeta) = \int_{\mathcal{D}}\zeta\,d\mu_f. \tag{1.3.2}$$

In this sense, we can say that $\Delta u - \chi_{\overline{\{u>0\}}}\,u_{tt}$ is a positive Radon measure on $\mathcal{D}$.

On the other hand, we can calculate the value of $f(\zeta)$ from (1.3.1). By splitting the integral domain into four parts, $\mathcal{D}\cap\{u>0\}$, $\mathcal{D}\cap\partial\{u>0\}$, $\mathcal{D}\cap\partial\{u=0\}^\circ$, $\mathcal{D}\cap\{u=0\}^\circ$, noting that all terms vanish except for $\mathcal{D} \cap \partial\{u > 0\}$ by the integration by parts, and $\chi_{\overline{\{u>0\}}} = 1$ on $\partial\{u > 0\}$, we can calculate

$$\begin{aligned}\int_{\mathcal{D}} \left(\chi_{\overline{\{u>0\}}}\, u_{tt}\zeta + \nabla u \cdot \nabla\zeta\right) dx\, dt &= \int_{\mathcal{D}} \left(-(\chi_{\overline{\{u>0\}}}\, \zeta)_t u_t + \nabla u \cdot \nabla\zeta\right) dx\, dt \\ &= \int_{\mathcal{D}\cap\partial\{u>0\}} \left(-\chi_{\overline{\{u>0\}}}\, \zeta \cdot u_t \cdot \frac{-u_t}{|Du|} + \zeta\nabla u \cdot \frac{-\nabla u}{|Du|}\right) d\mathscr{H}^N \\ &= \int_{\mathcal{D}\cap\partial\{u>0\}} \frac{u_t^2 - |\nabla u|^2}{|Du|}\, \zeta\, d\mathscr{H}^N \\ &= \int_{\mathcal{D}\cap\partial\{u>0\}} \frac{-Q^2}{|Du|}\, \zeta\, d\mathscr{H}^N \quad (\text{by } (1.3.1)).\end{aligned}$$

Here, we assume that the regularity of $\chi_{\overline{\{u>0\}}}$ with respect to t. From the above and the definition of f, we observe that

$$f(\zeta) = \int_{\mathcal{D}} \frac{Q^2}{|Du|}\, \zeta\, d\mathscr{H}^N \lfloor \partial\{u > 0\}. \tag{1.3.3}$$

By (1.3.2), (1.3.3), we have

$$\mu_f = \frac{Q^2}{|Du|} \mathscr{H}^N \lfloor \partial\{u > 0\}. \tag{1.3.4}$$

In this sense, the positive Radon measure $\Delta u - \chi_{\overline{\{u>0\}}}\, u_{tt}$ has its support in the free boundary $\partial\{u > 0\}$. Formally, we can rewrite (1.3.4) as follows:

$$\chi_{\overline{\{u>0\}}}\, u_{tt} - \Delta u = -\frac{Q^2}{|Du|} \mathscr{H}^N \lfloor \partial\{u > 0\}. \tag{1.3.5}$$

Summarizing the above, starting from the smoothed problem (P_ε), under the hypotheses (H1)-(H2), we formally derive a hyperbolic degenerate equation with adhesion force. This equation (1.3.5) includes all information about the hyperbolic free boundary problem, that is, the wave equation $u_{tt} - \Delta u = 0$ in the set $\{u > 0\}$, the free boundary condition $|\nabla u|^2 - u_t^2 = Q^2$ on $\Omega \times (0, T) \cap \partial\{u > 0\}$, and the Laplace equation $\Delta u = 0$ in the set $\{u < 0\}$ a.e. $t \in (0, T)$.

1.4 Previous researches of hyperbolic Alt–Caffarelli type free boundary problems

In this section, we will review the following hyperbolic Alt–Caffarelli type free boundary problems.

Problem 1.4.1. Find $u : \Omega \times [0, T) \to \mathbb{R}$ such that

$$\begin{cases} \chi_{\overline{\{u>0\}}}\, u_{tt} - \Delta u &= -\dfrac{Q^2}{|Du|} \mathscr{H}^N \lfloor \partial\{u > 0\} \quad \text{in } \Omega \times (0, T), \\ u(x, 0) &= u_0(x) \quad \text{in } \Omega, \\ u_t(x, 0) &= v_0(x) \quad \text{in } \Omega, \end{cases} \tag{1.4.1}$$

under suitable boundary conditions, where $\Omega \subset \mathbb{R}^N$ is a bounded Lipschitz domain, $T > 0$ is the final time, u_0 denotes the initial condition, v_0 is the initial velocity, and $\{u > 0\}$ is the set

$\{(x,t)\in\Omega\times(0,T): u(x,t)>0\}$.

We now summarize the history of such type problems. Firstly, K. Kikuchi and S. Omata [53] studied this problem in the one-dimensional domain $\Omega=(0,\infty)$. The problem which they treated is described as follows.

$$\begin{cases} u_{tt}-u_{xx}=0 \quad \text{in } \Omega\times(0,\infty)\cap\{u>0\}, \\ (u_x)^2-(u_t)^2=Q^2 \quad \text{on } \Omega\times(0,\infty)\cap\partial\{u>0\}. \end{cases} \tag{1.4.2}$$

Above equations (1.4.2) comes from the calculation of the first variation and inner variation of the following functional.

$$J(u):=\int_0^T\int_\Omega\Big\{\frac{1}{2}(u_t)^2\chi_{\{u>0\}}-\frac{1}{2}(u_x)^2-Q^2\chi_{u>0}\Big\}\,dx\,dt, \quad (u\in W^{1,2}(\Omega\times(0,T))).$$

This functional is well known as the action integral of Lagrangian corresponding to the tape peeling problem. We can refer [73] for the physical background and the derivation for this functional. For this problem (1.4.2), they constructed the strong solution $u\in C^2(\Omega\times(0,\infty)\cap\{u>0\})$. Quite roughly said, their method can be explained as follows. First, considering changing variables $t=\frac{1}{2}(\xi+\eta), x=\frac{1}{2}(\xi-\eta)$, we get the equation for new variables ξ,η,

$$\begin{cases} u_{\xi\eta}=0 \quad \text{in } \{u>0\}, \\ -4u_\xi u_\eta=Q^2 \quad \text{on } \partial\{u>0\}. \end{cases}$$

Second, for this equation, putting $u=\phi(\eta)+\psi(\eta)$, construct ϕ and ψ from the initial conditions and boundary conditions. They also the regularity of its free boundary $\partial\{u>0\}$ and the well-posedness of the problem under suitable compatibility conditions.

On the other hand, H.Imai et. al. [47] studied the numerical analysis for the problem (1.4.2) by the fixed domain method. Under the assumption that the free boundary is consisted of only one point, and its position is denoted by $\ell(t)$, by using the mapping $y=2x/\ell(t)-1$, we change the domain $(0,\ell(t))\times(0,T)$ to $(-1,1)\times(0,t)$ and get the new system of equations.

$$\begin{cases} u_{tt}-\dfrac{4-((y+1)\ell'(t))^2}{\ell(t)^2}u_{yy}-2(y+1)\dfrac{\ell'(t)}{\ell(t)}u_{ty}-(y+1)\dfrac{\ell(t)\ell''(t)-2(\ell'(t))^2}{\ell(t)^2}u_y=0, \\ \ell'(t)=\sqrt{1-\Big(\dfrac{Q\ell(t)}{2u_y(1,t)}\Big)}. \end{cases}$$

By discretizing the space $[-1,1]$, we have the system of ordinary differential equations whose unknowns are u_i, $v_i:=(u_i)_t$, ℓ. Then, we adopt 4-th order Runge-Kutta method for numerical computation.

Yoshiuchi et al. [88] addressed a similar problem to Problem in the case $Q=0$ including a damping term αu_t. The word 'similar' means there is a difference in the acceleration term with the previous researches and our treating problem. More precisely, in the previous researches, the main part problem is described the following equation.

$$\chi_{\{u>0\}} u_{tt} + \alpha u_t - \Delta u = 0, \tag{1.4.3}$$

where we assume that $Q = 0$ for simplicity. This equation comes from the consideration in the point of view of the mathematical modeling for the phenomena of the film motion with obstacle since the equation (1.4.2) is not sufficient to express this motion. In 2017, S. Omata [74] suggested that the term of acceleration should be replaced by $\chi_{\overline{\{u>0\}}} u_{tt}$ to include the all information for this type problems, and modified the equation to current version (1.4). More precisely, if we adopt the equation (1.4.3), we could not get the free boundary condition even though under the formal calculation. Therefore, reader should read with replacing to $\chi_{\overline{\{u>0\}}} u_{tt}$ of the acceleration term of the main equation in the previous researches before 2017 related these type problems.

We now go back to the reviews of the work by Yoshiuchi et. al [88]. Using the discrete Morse flow method, they derived an energy estimate for approximate solutions, and provided numerical results. The functional which they used is as follows.

$$\int_\Omega \frac{|u - 2u_{n-1} + u_{n-2}|^2}{2h^2} \chi_{\{u>0\} \cup \{u_{n-1}>0\}} \, dx + \alpha \int_\Omega \frac{|u - u_{n-1}|^2}{2h} \, dx + \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx,$$

among all $u \in \mathcal{K} \subset H^1(\Omega)$. After minimizing this functional, put minimizer $\tilde{u_n}$, we cut off this minimizer, $u_n := \max\{\tilde{u}_n, 0\}$. Then, do same procedure to construct the approximate solutions. The method how to time-interpolate is same as the discrete Morse flow method for the wave equation. The important feature of this functional is appearing the previous time step function u_{n-1} in the characteristic function. By this trick, they could get the energy estimate for approximate solutions.

Moreover, the discrete Morse flow method can be applied the following hyperbolic free boundary problems with volume preservation.

$$\chi_{\overline{\{u>0\}}} u_{tt} - \Delta u = \lambda_u(t) \chi_{\{u>0\}}.$$

where $\lambda_u(t)$ is the Lagrange multiplier corresponding to the volume preserving condition $\int_\Omega u \, dx = V$, V is a given positive real number, that is,

$$\lambda_u(t) := \frac{1}{V} \int_\Omega (u_{tt} u + |\nabla u|^2) \, dx.$$

E. Ginder and K. Svadlenka [37] constructed a weak solution for this problem in the one dimensional setting, again using the discrete Morse flow method with the following functional including two penalty terms.

$$\int_\Omega \frac{|u - 2u_{n-1} + u_{n-2}|^2}{2h^2} \, dx + \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx + \Psi_1(u) + \Psi_2(u) \quad \text{on } H^1_0(\Omega)$$

where Ψ_1, Ψ_2 are functionals $H^1(\Omega) \to \{0, \infty\}$ defined

$$\Psi_1(u) := \begin{cases} 0, & \text{if } u \geq 0 \text{ a.e. in } \Omega \\ \infty, & \text{otherwise} \end{cases}$$

$$\Psi_2(u) := \begin{cases} 0, & \text{if } \int_\Omega u\,dx = V \\ \infty, & \text{otherwise.} \end{cases}$$

Moreover, the following problem, stated here without initial and boundary conditions, has been treated by K.Kikuchi in [51]:

$$\begin{cases} u_{tt} - u_{xx} \geq 0 \quad \text{in } (0,1) \times (0,\infty), \\ \text{spt}\,(u_{tt} - u_{xx}) \subset \{u = 0\}, \\ u(x,t) \geq 0 \quad L^2\text{-a.e..} \end{cases}$$

He constructed a weak solution to this problem using a minimizing method in the spirit of the discrete Morse flow. This equation is corresponding to the problem treated by M.Schatzman in [79].

Finally, we conclude the interesting application and extension of the discrete Morse flow from resent work by Akagawa et. al.[2]. They consider the following functional to numerical simulation for the rolling contact problem of elasticity.

$$\frac{1}{2}\int_\Omega \frac{|\boldsymbol{\xi} - 2\boldsymbol{\xi}_{n-1} + \boldsymbol{\xi}_{n-2}|^2}{2h^2}\,dx + \frac{1}{2}\int_\Omega \Big(\frac{1}{2}\sigma[\boldsymbol{\xi}] + \sigma[\boldsymbol{\xi_{n-2}}]\Big) : \epsilon[\boldsymbol{\xi}]\,dx,$$

among all $\boldsymbol{\xi} \in W^{1,2}(\Omega\,;\mathbb{R}^2)$ with some conditions with respect to the boundary conditions and obstacle. Here, σ, ϵ are well known the notions in the theory of elasticity, the stress tensor and strain tensor respectively, and $A : B$ means the sum of the product of each components of two same size matrix, also we omit the term corresponding outer force. They gave the extension of the discrete Morse flow method to vector valued functions. Another important feature of the above functional is the presence of the previous two time step function $\boldsymbol{\xi}_{n-2}$ in the gradient term in the functional, which can be said a kind of the Crank-Nicolson scheme.

In this thesis work, will be starting from the next section, it is main idea that the including the previous two time step functions to the characteristic function in the time discretization term which is a kind of extension of the work by Yoshiuchi et. al. [88] and the gradient term which is analogous of the work by Akagawa et. al. [2].

Chapter 2

Crank-Nicolson minimization scheme I

From this chapter to the next chapter, we consider the following hyperbolic obstacle problem in accordance with [1]:

Problem 2.0.1. Find $u : \Omega \times [0, T) \to \mathbb{R}$ such that

$$\begin{cases} \chi_{\overline{\{u>0\}}} u_{tt} - \Delta u &= 0 \quad \text{in } \Omega \times (0, T), \\ u(x, 0) &= u_0(x) \quad \text{in } \Omega, \\ u_t(x, 0) &= v_0(x) \quad \text{in } \Omega, \\ u(\cdot, t) &= u_0(\cdot) \quad \text{on } \partial\Omega \quad t \geq 0, \end{cases} \tag{2.0.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded Lipschitz domain, $T > 0$ is the final time, $u_0 \in H^1(\Omega) \cap C^0(\overline{\Omega}) \cap C^{0,\alpha}_{\text{loc}}(\Omega)$ denotes given the initial condition, $v_0 \in H^1_0(\Omega)$ is also given the initial velocity, and $\{u > 0\}$ is the set $\{(x, t) \in \Omega \times (0, T) : u(x, t) > 0\}$.

For this problem, we will construct the weak solution by a new minimization scheme. Although this is essentially same with the discrete Morse flow method, we use the following functional.

$$J_m(u) = \int_{\Omega \cap \mathcal{S}_m(u)} \frac{|u - 2u_{m-1} + u_{m-2}|^2}{2h^2}\, dx + \frac{1}{4} \int_\Omega |\nabla u + \nabla u_{m-2}|^2\, dx, \tag{2.0.2}$$

on $\mathcal{K} := \{u \in H^1(\Omega) \ : \ u = u_0 \ \text{ on } \partial\Omega\}$. Here the set $\mathcal{S}_m(u)$ is defined by $\mathcal{S}_m(u) := \{u > 0\} \cup \{u_{m-1} > 0\} \cup \{u_{m-2} > 0\}$, $\{u > 0\} := \{x \in \Omega \ : \ u(x) > 0\}$, $\{u_i > 0\} := \{x \in \Omega \ : \ u_i(x) > 0\}$ $(i = m-1, m-2)$, and $m \geq 2$ is integer. Remark that although their domain is different, we use same notations for the positive part of function as the statement of Problem 2.0.1. In this chapter, we will study the properties of this functional.

2.1 Energy conservation property

In this section, we derive the energy preserving property of the Crank-Nicolson type functional, which has not been achieved in previous research. To this end, let us consider the following modified functional:

$$I_m(u) := \int_\Omega \frac{|u - 2u_{m-1} + u_{m-2}|^2}{2h^2}\, dx + \frac{1}{4}\int_\Omega |\nabla u + \nabla u_{m-2}|^2\, dx, \tag{2.1.1}$$

on the set $\mathcal{K}$. This functional can be regarded as the no-free boundary version of J_m. We note that a unique minimizer exists for each I_m whenever $I_m(u_0) < \infty$ since the functional is convex and lower semicontinuous with respect to weak convergence in L^2.

THEOREM 2.1.1 (Energy conservation). Minimizers u_k of I_k conserve the energy

$$E_k := \left\| \frac{u_k - u_{k-1}}{h} \right\|^2_{L^2(\Omega)} + \frac{1}{2}\Big(\|\nabla u_k\|^2_{L^2(\Omega)} + \|\nabla u_{k-1}\|^2_{L^2(\Omega)} \Big), \tag{2.1.2}$$

in the sense that E_k is independent of $k \geq 1$.

Proof. For $m \geq 2$, the function $(1-\lambda)u_m + \lambda u_{m-2} = u_m + \lambda(u_{m-2} - u_m)$ is admissible for every $\lambda \in [0,1]$, which justifies

$$\frac{d}{d\lambda} I_m(u_m + \lambda(u_{m-2} - u_m))\Big|_{\lambda=0} = 0.$$

Computing this derivative, we have

$$\begin{aligned}
0 = \frac{d}{d\lambda}\int_\Omega \Big[& \frac{|u_m + \lambda(u_{m-2} - u_m) - 2u_{m-1} + u_{m-2}|^2}{2h^2} \\
& + \frac{1}{4}|\nabla(u_m + \lambda(u_{m-2} - u_m)) + \nabla u_{m-2}|^2 \Big]\Big|_{\lambda=0}\, dx \\
= \int_\Omega \Big[& \frac{(u_{m-2} - u_m)(u_m - 2u_{m-1} + u_{m-2})}{h^2} + \frac{1}{2}\nabla(u_{m-2} - u_m)\cdot\nabla(u_m + u_{m-2})\Big]\, dx \\
= \int_\Omega \Big[& \frac{(u_{m-1} - u_{m-2})^2 - (u_m - u_{m-1})^2}{h^2} + \frac{1}{2}|\nabla u_{m-2}|^2 - \frac{1}{2}|\nabla u_m|^2\Big]\, dx.
\end{aligned}$$

Summing over $m = 2, ..., k$, we arrive at

$$\int_\Omega \Big[\frac{1}{h^2}(u_1 - u_0)^2 - \frac{1}{h^2}(u_k - u_{k-1})^2 + \frac{1}{2}|\nabla u_0|^2 + \frac{1}{2}|\nabla u_1|^2 - \frac{1}{2}|\nabla u_{k-1}|^2 - \frac{1}{2}|\nabla u_k|^2\Big]\, dx = 0,$$

which means

$$\begin{aligned}
E_1 &= \int_\Omega \Big[\frac{1}{h^2}(u_1 - u_0)^2 + \frac{1}{2}|\nabla u_0|^2 + \frac{1}{2}|\nabla u_1|^2\Big]\, dx \\
&= \int_\Omega \Big[\frac{1}{h^2}(u_k - u_{k-1})^2 + \frac{1}{2}|\nabla u_{k-1}|^2 + \frac{1}{2}|\nabla u_k|^2\Big]\, dx \\
&= E_k,
\end{aligned}$$

and the proof is complete. □

2.2 Minimizing method

For any integer $m \geq 2$, we introduce the following functional:

$$J_m(u) = \int_{\Omega \cap \mathcal{S}_m(u)} \frac{|u - 2u_{m-1} + u_{m-2}|^2}{2h^2}\, dx + \frac{1}{4}\int_{\Omega} |\nabla u + \nabla u_{m-2}|^2\, dx, \tag{2.2.1}$$

where $\mathcal{S}_m(u) := \{u > 0\} \cup \{u_{m-1} > 0\} \cup \{u_{m-2} > 0\}$.

We determine a sequence of functions $\{u_m\}$ iteratively by taking $u_0 \in \mathcal{K}$ and $u_1 = u_0 + hv_0 \in \mathcal{K}$, defining $\tilde{u}_m$ as a minimizer of J_m in $\mathcal{K}$, and setting $u_m := \max\{\tilde{u}_m, 0\}$.

We now study the existence and regularity of minimizers which guarantees the possibility of applying the first variation formula to J_m.

THEOREM 2.2.1 (Existence). If $J_m(u_0) < \infty$, then there exists a minimizer $\tilde{u}_m \in \mathcal{K}$ of the functional J_m.

Proof. Given u_{m-1}, u_{m-2}, we show the existence of $\tilde{u}_m$. Since the infimum of J_m is non-negative, we can take the minimizing sequence $\{u^j\} \subset \mathcal{K}$ such that $J_m(u^j) \to \inf_{u \in \mathcal{K}} J_m(u)$ as $j \to \infty$. Now, since the sequence $\{u^j - u_0\} \subset H_0^1(\Omega)$ and $\{\chi_{\mathcal{S}_m(u^j)}\}$ are bounded in $H^1(\Omega)$ and $L^\infty(\Omega)$ respectively, there exist $\tilde{u} \in H_0^1(\Omega)$ and $\gamma \in L^\infty(\Omega)$ such that, up to extracting a subsequence,

$$\begin{aligned} u^j - u_0 &\to \tilde{u} \quad \text{strongly in } L^2(\Omega), \\ \nabla(u^j - u_0) &\rightharpoonup \nabla\tilde{u} \quad \text{weakly in } L^2(\Omega), \\ \chi_{\mathcal{S}_m(u^j)} &\overset{*}{\rightharpoonup} \gamma \quad \text{weakly * in } L^\infty(\Omega), \end{aligned} \tag{2.2.2}$$

where the existence of the limit function γ follows from Banach-Alaoglu's theorem([31, Theorem 5.12]). Moreover, by the weak * convergence and L^2-strongly convergence, we have $0 \leq \gamma \leq 1$ a.e. on Ω, and $\gamma = 1$ a.e. on $\mathcal{S}_m(u)$ where $u := \tilde{u} + u_0 \in \mathcal{K}$. Indeed, by the weak * convergence, we have

$$0 \leq \int_\Omega \chi_{\mathcal{S}_m(u^j)} \varphi\, dx \leq \int_\Omega \varphi\, dx$$

for any $\varphi \in C_c^\infty(\Omega)$ with $\varphi \geq 0$, and tending $j \to \infty$ in this inequality leads for the inequality $0 \leq \gamma \leq 1$ a.e. on Ω. For the equality $\gamma = 1$ a.e. on $S_m(u)$, remark that the use of L^2-strongly convergence of u_j to u, up to extracting subsequence, we have $\chi_{\mathcal{S}_m(u^j)}(x) \to 1$ for a.e. $x \in \mathcal{S}_m(u)$ as $j \to \infty$. This fact leads that $\chi_{\mathcal{S}_m(u^j)}(x)\varphi(x) \to \varphi(x)$ for a.e. $x \in \mathcal{S}_m(u)$ as $j \to \infty$ for any $\varphi \in C_c^\infty(\Omega)$. Moreover, combining with $|\chi_{\mathcal{S}_m(u^j)}(x)\varphi(x)| \leq |\varphi(x)|$ for any j and $x \in S_m(u)$, the Dominated convergence theorem leads

$$\lim_{j\to\infty} \int_\Omega \chi_{\mathcal{S}_m(u^j)} \chi_{\mathcal{S}_m(u)} \varphi\, dx = \lim_{j\to\infty} \int_{\mathcal{S}_m(u)} \chi_{\mathcal{S}_m(u^j)} \varphi\, dx = \int_{\mathcal{S}_m(u)} \varphi\, dx.$$

On the other hand, by the weak * convergence of $\chi_{\mathcal{S}_m(u^j)}$ to γ,

$$\lim_{j\to\infty} \int_\Omega \chi_{\mathcal{S}_m(u^j)} \chi_{\mathcal{S}_m(u)} \varphi\, dx = \int_\Omega \gamma \chi_{\mathcal{S}_m(u)} \varphi\, dx = \int_{\mathcal{S}_m(u)} \gamma\varphi\, dx \quad \text{for any } \varphi \in C_c^\infty(\Omega).$$

Thus, we get

$$\int_{\mathcal{S}_m(u)} \gamma\varphi\, dx = \int_{\mathcal{S}_m(u)} \varphi\, dx,$$

which implies the equality $\gamma = 1$ a.e. on $\mathcal{S}_m(u)$.

Therefore, we get

$$\begin{aligned} J_m(u) &= \int_\Omega \frac{|u - 2u_{m-1} + u_{m-2}|^2}{2h^2} \chi_{\mathcal{S}_m(u)}\, dx + \frac{1}{4}\int_\Omega |\nabla u + \nabla u_{m-2}|^2\, dx \\ &\le \int_\Omega \frac{|u - 2u_{m-1} + u_{m-2}|^2}{2h^2} \gamma\, dx + \frac{1}{4}\int_\Omega |\nabla u + \nabla u_{m-2}|^2\, dx \\ &\le \liminf_{j\to\infty} J_m(u^j) = \inf_{u\in\mathcal{K}} J_m(u), \end{aligned}$$

where the first inequality follows from $\gamma = 1$ a.e. on $\mathcal{S}_m(u)$, and the second inequality follows from (2.2.2), this shows the function u is a minimizer of $J_m(u)$. □

The minimizers of J_m have the following subsolution property which we will use to show the regularity of minimizers.

PROPOSITION 2.2.1 (Subsolution). Any minimizer u of J_m satisfies the following inequality for arbitrary nonnegative $\zeta \in H_0^1(\Omega)$:

$$\int_{\Omega\cap\mathcal{S}_m(u)} \frac{u - 2u_{m-1} + u_{m-2}}{h^2}\zeta\, dx + \int_\Omega \nabla\frac{u + u_{m-2}}{2}\cdot\nabla\zeta\, dx \le 0. \tag{2.2.3}$$

Proof. Fixing $\zeta \in C_c^\infty(\Omega)$ with $\zeta \ge 0$, and $\varepsilon > 0$, we have

$$\begin{aligned} 0 &\le J_m(u - \varepsilon\zeta) - J_m(u) \quad \text{(by the minimality of } u) \\ &= \int_\Omega \frac{|(u-\varepsilon\zeta) - 2u_{m-1} + u_{m-2}|^2}{2h^2}\chi_{\mathcal{S}_m(u-\varepsilon\zeta)}\, dx + \frac{1}{4}\int_\Omega |\nabla(u-\varepsilon\zeta) + \nabla u_{m-2}|^2\, dx \\ &\quad - \Big(\int_\Omega \frac{|u - 2u_{m-1} + u_{m-2}|^2}{2h^2}\chi_{\mathcal{S}_m(u)}\, dx + \frac{1}{4}\int_\Omega |\nabla u + \nabla u_{m-2}|^2\, dx\Big). \end{aligned} \tag{2.2.4}$$

Noting that

$$\begin{aligned} &\chi_{\mathcal{S}_m(u-\varepsilon\zeta)} - \chi_{\mathcal{S}_m(u)} \le 0, \\ &|(u-\varepsilon\zeta) - 2u_{m-1} + u_{m-2}|^2 - |u - 2u_{m-1} + u_{m-2}|^2 = -2\varepsilon\zeta(u - 2u_{m-1} + u_{m-2}) + \varepsilon^2\zeta^2, \\ &|\nabla(u-\varepsilon\zeta) + \nabla u_{m-2}|^2 - |\nabla u + \nabla u_{m-2}|^2 = -2\varepsilon(\nabla u + \nabla u_{m-2})\cdot\nabla\zeta + \varepsilon^2|\nabla\zeta|^2, \end{aligned}$$

we continue the estimate as

$$\begin{aligned} (2.2.4) \le &\int_\Omega \{-2\varepsilon\zeta(u - 2u_{m-1} + u_{m-2}) + \varepsilon^2\zeta^2\} \times \frac{1}{2h^2}\chi_{\mathcal{S}_m(u)}\, dx \\ &+ \frac{1}{4}\int_\Omega \{-2\varepsilon(\nabla u + \nabla u_{m-2})\cdot\nabla\zeta + \varepsilon^2|\nabla\zeta|^2\}\, dx. \end{aligned}$$

Dividing by ε, letting ε decrease to zero from above, and applying a density argument concludes the proof. □

We now derive an energy estimate satisfied by the minimizers of J_m.

THEOREM 2.2.2 (Energy estimate). For any integer $k \geq 1$, we have

$$\left\| \frac{u_k - u_{k-1}}{h} \right\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla u_k\|^2_{L^2(\Omega)} \leq \|v_0\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla u_0\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla u_1\|^2_{L^2(\Omega)}. \tag{2.2.5}$$

Proof. Since the function $(1-\lambda)\tilde{u}_m + \lambda u_{m-2} = \tilde{u}_m + \lambda(u_{m-2} - \tilde{u}_m)$ belongs to $\mathcal{K}$ for any $\lambda \in [0,1]$, by the minimality property, we have $J_m(\tilde{u}_m) \leq J_m(\tilde{u}_m + \lambda(u_{m-2} - \tilde{u}_m))$, and thus,

$$\lim_{\lambda \to 0+} \frac{1}{\lambda}\Big(J_m(\tilde{u}_m + \lambda(u_{m-2} - \tilde{u}_m)) - J_m(\tilde{u}_m)\Big) \geq 0. \tag{2.2.6}$$

Let A_m denote the set

$$A_m := \Omega \cap (\{\tilde{u}_m > 0\} \cup \{u_{m-1} > 0\} \cup \{u_{m-2} > 0\}).$$

We investigate the behavior of the individual terms in (2.2.6). For the gradient term we get

$$\begin{aligned}
&\lim_{\lambda \to 0+} \frac{1}{4\lambda}\left(|\nabla(\tilde{u}_m + \lambda(u_{m-2} - \tilde{u}_m) + u_{m-2})|^2 - |\nabla(\tilde{u}_m + u_{m-2})|^2\right) \\
&\quad = \frac{1}{2}\nabla(\tilde{u}_m + u_{m-2}) \cdot \nabla(u_{m-2} - \tilde{u}_m)\, dx \\
&\quad = \frac{1}{2}|\nabla u_{m-2}|^2 - \frac{1}{2}|\nabla \tilde{u}_m|^2 \\
&\quad \leq \frac{1}{2}|\nabla u_{m-2}|^2 - \frac{1}{2}|\nabla u_m|^2.
\end{aligned} \tag{2.2.7}$$

For the time-discretized term, consider the set

$$B_m(\lambda) := \{\tilde{u}_m + \lambda(u_{m-2} - \tilde{u}_m) > 0\} \cup \{u_{m-1} > 0\} \cup \{u_{m-2} > 0\}.$$

Then $B_m(\lambda)$ is contained in the set A_m for any $\lambda \in [0,1]$. It is obvious for $B_m(0), B_m(1)$. For fixed $\lambda \in (0,1)$, $x \in B_m(\lambda)$, we have only to consider the case of that $x \in \{\tilde{u}_m + \lambda(u_{m-2} - \tilde{u}_m) > 0\} = \{(1-\lambda)\tilde{u}_m + \lambda u_{m-2} > 0\}$. If $x \in \{u_{m-2} > 0\}$, we have done. Otherwise, that is, if $x \in \{u_{m-2} = 0\}$, $x \in \{(1-\lambda)\tilde{u}_m + \lambda u_{m-2} > 0\} = \{(1-\lambda)\tilde{u}_m > 0\}$, then $x \in \{\tilde{u}_m > 0\} \subset A_m$.

Therefore, we find that

$$\begin{aligned}
&\frac{1}{2h^2}\int_\Omega \Big(|\tilde{u}_m + \lambda(u_{m-2} - \tilde{u}_m) - 2u_{m-1} + u_{m-2}|^2 \chi_{B_m(\lambda)} - |\tilde{u}_m - 2u_{m-1} + u_{m-2}|^2 \chi_{A_m}\Big) dx \\
&\leq \frac{1}{2h^2}\int_\Omega \Big(|\tilde{u}_m + \lambda(u_{m-2} - \tilde{u}_m) - 2u_{m-1} + u_{m-2}|^2 - |\tilde{u}_m - 2u_{m-1} + u_{m-2}|^2\Big)\chi_{A_m} dx.
\end{aligned}$$

Then we have

$$
\begin{aligned}
&\lim_{\lambda\to 0+} \frac{1}{2h^2\lambda}\int_\Omega \Big(|\tilde{u}_m+\lambda(u_{m-2}-\tilde{u}_m)-2u_{m-1}+u_{m-2}|^2-|\tilde{u}_m-2u_{m-1}+u_{m-2}|^2\Big)\chi_{A_m}dx\\
&\quad=\lim_{\lambda\to 0+}\frac{1}{2h^2\lambda}\int_{A_m}\lambda(u_{m-2}-\tilde{u}_m)(2\tilde{u}_m+\lambda(u_{m-2}-\tilde{u}_m)-4u_{m-1}+2u_{m-2})dx\\
&\quad=\frac{1}{h^2}\int_{A_m}(u_{m-2}-\tilde{u}_m)(\tilde{u}_m-2u_{m-1}+u_{m-2})dx\\
&\quad=\frac{1}{h^2}\int_{A_m}[(u_{m-1}-u_{m-2})^2-(u_{m-1}-\tilde{u}_m)^2]dx. \qquad (2.2.8)
\end{aligned}
$$

Now, $\int_{A_m}(u_{m-1}-u_{m-2})^2\,dx \le \int_\Omega(u_{m-1}-u_{m-2})^2\,dx$ since the integrand is non-negative. Moreover, $u_m=\max\{\tilde{u}_m,0\}$ and $u_{m-1}\ge 0$ imply $(u_{m-1}-\tilde{u}_m)^2\ge(u_{m-1}-u_m)^2$, therefore

$$
-\int_{A_m}(u_{m-1}-\tilde{u}_m)^2\,dx \le -\int_{A_m}(u_{m-1}-u_m)^2\,dx.
$$

Noting that, outside of A_m, both u_m and u_{m-1} vanish, we get

$$
-\int_{A_m}(u_{m-1}-u_m)^2\,dx = -\int_\Omega(u_{m-1}-u_m)^2\,dx.
$$

Returning to (2.2.8), we get the estimate for the time discretized term:

$$
\text{the right hand side of (2.2.8)} \le \frac{1}{h^2}\int_\Omega[(u_{m-1}-u_{m-2})^2-(u_{m-1}-u_m)^2]dx.
$$

Combining this result and the gradient term estimate (2.2.7), we obtain

$$
\int_\Omega\Big[\frac{1}{h^2}(u_{m-1}-u_{m-2})^2-\frac{1}{h^2}(u_{m-1}-u_m)^2+\frac{1}{2}|\nabla u_{m-2}|^2-\frac{1}{2}|\nabla u_m|^2\Big]\,dx\ge 0.
$$

Summing over $m=2,...,k$, we arrive at

$$
\int_\Omega\Big[\frac{1}{h^2}(u_1-u_0)^2-\frac{1}{h^2}(u_k-u_{k-1})^2+\frac{1}{2}|\nabla u_0|^2+\frac{1}{2}|\nabla u_1|^2-\frac{1}{2}|\nabla u_{k-1}|^2-\frac{1}{2}|\nabla u_k|^2\Big]\,dx\ge 0,
$$

which, after omitting the term $|\nabla u_{k-1}|^2\ge 0$, yields the desired estimate. □

The following theorem is obtained by a standard argument from elliptic regularity theory. For the sake of completeness, we shall briefly demonstrate it.

THEOREM 2.2.3 (Regularity). Assume, in addition, that $u_0,u_1\in L^\infty(\Omega)\cap C^{0,\alpha_0}_{\text{loc}}(\Omega)$ for some $\alpha_0\in(0,1)$, where $u_1:=u_0+hv_0$, and u_0 are non-negative. For every $\widetilde{\Omega}\subset\subset\Omega$, there exists a positive constant $\alpha\in(0,1)$ independent of m, such that the minimizers $\tilde{u}_m+u_{m-2}$ belong to $C^{0,\alpha}(\tilde{\Omega})$.

To prove this, we prepare two lemmas.

LEMMA 2.2.1. $\tilde{u}_m+u_{m-2}\in L^\infty(\Omega)$ for every $m\ge 2$.

Proof. We use mathematical induction for $m \geq 2$ to prove that $\tilde{u}_m \in L^\infty(\Omega)$ for every $m \geq 2$, and we get the boundedness of $\tilde{u}_m + u_{m-2} \in L^\infty(\Omega)$ as a by-product. For $m = 2$, setting $\psi_\delta(u) := u - \delta(u + u_0 - k)^+ \in \mathcal{K}$, where $u := \tilde{u}_2$, $(u + u_0 - k)^+ := \max\{u + u_0 - k, 0\}$, $\delta > 0$, $k \geq \max\{2\max_{\partial\Omega} u_0, 1\}$, we calculate the quantity $J_2(\psi_\delta(u)) - J_2(u)$, which is non-negative by the minimality of u. Noting that $S_m(\psi_\delta(u)) \subset S_m(u)$, we have

$$
\begin{aligned}
0 &\leq J_2(\psi_\delta(u)) - J_2(u)\\
&\leq \int_\Omega \Big(\frac{|\psi_\delta(u) - 2u_1 + u_0|^2}{2h^2} - \frac{|u - 2u_1 + u_0|^2}{2h^2} \Big) \chi_{\mathcal{S}_2(u)}\, dx\\
&\quad + \frac{1}{4}\int_\Omega \Big(|\nabla\psi_\delta(u) + \nabla u_0|^2 - |\nabla u + \nabla u_0|^2 \Big)\, dx.
\end{aligned}
$$

Dividing by δ, letting $\delta \to 0+$, and setting $A_k := \{u + u_0 > k\}$, we get

$$
\begin{aligned}
0 &\leq -\int_{A_k \cap \mathcal{S}_2(u)} \frac{u - 2u_1 + u_0}{h^2}(u + u_0 - k)\, dx - \frac{1}{2}\int_{A_k} |\nabla u + \nabla u_0|^2\, dx\\
&\leq \int_{A_k \cap \mathcal{S}_2(u)} \frac{2u_1}{h^2}(u + u_0 - k)\, dx - \frac{1}{2}\int_{A_k} |\nabla u + \nabla u_0|^2\, dx\\
&\leq \frac{C}{h^2}\Big(\frac{1}{2}\int_{A_k} (u + u_0 - k)^2\, dx + \frac{1}{2}|A_k| \Big) - \frac{1}{2}\int_{A_k} |\nabla u + \nabla u_0|^2\, dx,
\end{aligned}
$$

where we have used Young's inequality at the last line. Since $k \geq 1$, we get

$$
\int_{A_k} |\nabla u + \nabla u_0|^2\, dx \leq C\Big(\int_{A_k} (u + u_0 - k)^2\, dx + k^2|A_k| \Big).
$$

Therefore, by [59, Theorem 2.5.1] which ensures the boundedness of the function, we find that $u + u_0 \in L^\infty(\Omega)$ and hence $u = \tilde{u}_2 \in L^\infty(\Omega)$.

Next, we assume that $\tilde{u}_k \in L^\infty(\Omega)$ for all $k = 2, ..., m-1$. Since $u_k = \max\{\tilde{u}_k, 0\} \in L^\infty(\Omega)$ for all $k = 2, ..., m-1$, by repeating the above argument with $\tilde{u}_2, u_1, u_0$ replaced by $\tilde{u}_m, u_{m-1}, u_{m-2}$, respectively, we get $\tilde{u}_m + u_{m-2} \in L^\infty(\Omega)$ for $m \geq 2$. Therefore, $\tilde{u}_m \in L^\infty(\Omega)$. □

The lemma and our setting implies that there is a $\mu > 0$, which depends only on Ω, u_0, u_1, h but not on m, such that $\sup_\Omega |\tilde{u}_m + u_{m-2}| \leq \mu$. Indeed, for fixed h, we can determine the number $M = M(h)$, such that $T = Mh$. Recall that, we use only the information about $\tilde{u}_1, ..., \tilde{u}_M$ in our minimizing step. By above lemma, the minimizers $\tilde{u}_2, \tilde{u}_3, ..., \tilde{u}_M$ has essential supremum respectively, that is $\mu_m := \sup_\Omega |\tilde{u}_m + u_{m-2}| < \infty$. Then, setting $\mu := \max \mu_1, ..., \mu_M$, we get $\sup_\Omega |\tilde{u}_m + u_{m-2}| \leq \mu$ for every $m = 2, ..., M$.

LEMMA 2.2.2. Fix $d > 0$. There exists $\gamma = \gamma(\Omega, \mu, d, h) > 0$ such that for $U = \pm(\tilde{u}_m + u_{m-2})$,

$$
\int_{A_{k,r-\sigma r}} |\nabla U|^2\, dx \leq \gamma\Big[\frac{1}{(\sigma r)^2}\sup_{B_r}(U - k)^2 + 1 \Big]|A_{k,r}|
$$

for all $\sigma \in (0, 1)$, $B_r \subset \Omega$, and k with $k \geq \max_{B_r} U - d$, where $A_{k,r} := \{x \in B_r; U(x) > k\}$, and B_r is a ball of radius r.

Proof. For fixed $m \geq 2$, first we show the statement for $U = \tilde{u}_m + u_{m-2}$. We set $\zeta = \eta^2 \max\{u +$

$u_{m-2}-k,0\}$ in Proposition 3.2, where $u:=\tilde{u}_m$, k is a real number with $k\geq \max_{B_r}(u+u_{m-2})-d$, η is smooth function with $\operatorname{spt}\eta\subset B_r$, $0\leq\eta\leq 1$, $\eta\equiv 1$ on B_s, $|\nabla\eta|\leq 2/(r-s)$ in $B_r\setminus B_s$, and $s=r-\sigma r\in(0,r)$, $\sigma\in(0,1)$. Then, using the boundedness of u,u_{m-1},u_{m-2}, we get

$$\begin{aligned}0\leq &-\int_{A_{k,r}\cap\mathcal{S}_m(u)}\frac{u-2u_{m-1}+u_{m-2}}{h^2}\eta^2(u+u_{m-2}-k)\,dx\\ &-\frac{1}{2}\int_{A_{k,r}}(\nabla u+\nabla u_{m-2})\cdot(2\eta)\nabla\eta(u+u_{m-2}-k)\,dx-\frac{1}{2}\int_{A_{k,r}}|\nabla u+\nabla u_{m-2}|^2\eta^2\,dx\\ \leq &\, C|A_{k,r}|+\frac{1}{2}\Big(\frac{1}{2}\int_{A_{k,r}}|\nabla(u+u_{m-2})|^2\eta^2\,dx+2\int_{A_{k,r}}|\nabla\eta|^2(u+u_{m-2}-k)^2\,dx\Big)\\ &-\frac{1}{2}\int_{A_{k,r}}|\nabla u+\nabla u_{m-2}|^2\eta^2\,dx\\ \leq &\, C\Big[1+\frac{1}{(\sigma r)^2}\sup_{B_r}(u+u_{m-2}-k)^2\Big]|A_{k,r}|-\frac{1}{4}\int_{A_{k,s}}|\nabla(u+u_{m-2})|^2\,dx,\end{aligned}$$

where the constant C depends only on h,μ,d,Ω.

Next, we prove the same inequality for $U=-(\tilde{u}_m+u_{m-2})$. Note that $-\tilde{u}_m$ is a minimizer of the following functional:

$$J_m^-(w):=\int_{\Omega\cap\mathcal{S}_m^-(w)}\frac{|w+2u_{m-1}-u_{m-2}|^2}{2h^2}\,dx+\frac{1}{4}\int_\Omega|\nabla w-\nabla u_{m-2}|^2\,dx.$$
$$\text{in the set}\quad \mathcal{K}^-:=\big\{w\in H^1(\Omega);\, w=-u_0 \text{ on } \partial\Omega\big\},$$

where $\mathcal{S}_m^-(w)$ is defined to be the set $\{w<0\}\cup\{u_{m-1}>0\}\cup\{u_{m-2}>0\}$.

Now, for $w:=-\tilde{u}_m$, we set $\varphi:=w-\zeta\in\mathcal{K}^-$ where $\zeta:=\eta\max\{w-u_{m-2}-k,0\}$, k is a real number with $k\geq\max_{B_r}(w-u_{m-2})-d$, and η is a smooth function chosen in the same way as above. Then, by the minimality of w,

$$\begin{aligned}0&\leq J_m^-(\varphi)-J_m^-(w)\\ &\leq\int_{\Omega\cap\mathcal{S}_m^-(\varphi)}\Big(\frac{-2(w+2u_{m-1}-u_{m-2})}{2h^2}\zeta+\frac{|\zeta|^2}{2h^2}\Big)\,dx\\ &\quad+\int_\Omega\frac{|w+2u_{m-1}-u_{m-2}|^2}{2h^2}\Big(\chi_{\mathcal{S}_m^-(\varphi)}-\chi_{\mathcal{S}_m^-(w)}\Big)\,dx\\ &\qquad+\frac{1}{4}\int_\Omega|\nabla\varphi-\nabla u_{m-2}|^2\,dx-\frac{1}{4}\int_\Omega|\nabla w-\nabla u_{m-2}|^2\,dx.\end{aligned}\tag{2.2.9}$$

Note that the term in the third line is less than or equal to $\frac{1}{2h^2}\int_{\operatorname{spt}\zeta}|w+2u_{m-1}-u_{m-2}|^2\,dx$, since $\chi_{\mathcal{S}_m^-(\varphi)}-\chi_{\mathcal{S}_m^-(w)}$ is positive only for x satisfying $0\leq w(x)<\zeta(x)$. Therefore, noting that $\operatorname{spt}\zeta\subset A_{k,r}$, the first two terms on the right-hand side of (2.2.9) are less than or equal to $C|A_{k,r}|$, where C is a constant depending only Ω,μ,d,h. Indeed, we consider the following term:

$$\int_{\Omega\cap\mathcal{S}_m^-(\varphi)}\Big(\frac{-2(w+2u_{m-1}-u_{m-2})}{2h^2}\zeta+\frac{|\zeta|^2}{2h^2}\Big)\,dx+\frac{1}{2h^2}\int_{\operatorname{spt}\zeta}|w+2u_{m-1}-u_{m-2}|^2\,dx\tag{2.2.10}$$

The second term of (2.2.10): since spt $\zeta \subset A_{k,r}$, by the boundedness of w, u_{m-1}, u_{m-2},

$$\frac{1}{2h^2}\int_{\mathrm{spt}\,\zeta} |w + 2u_{m-1} - u_{m-2}|^2\, dx \leq C(\Omega, M, h)|\,\mathrm{spt}\ \zeta| \leq C(\Omega, M, h)|A_{k,r}|$$

hereafter $A_{k,r} := \{x \in B_r : w(x) - u_{m-2}(x) > k\}$.
The first term of (2.2.10): noting that this term is zero outside of $A_{k,r}$, we calculate: On $A_{k,r}$,

$$\begin{aligned}\frac{-2(w + 2u_{m-1} - u_{m-2})}{2h^2}\zeta + \frac{|\zeta|^2}{2h^2} &= \frac{(-2w - 4u_{m-1} + 2u_{m-2} + \zeta)\zeta}{2h^2} \\ &= \frac{\zeta}{2h^2}\{-2w - 4u_{m-1} + 2u_{m-2} + \eta(w - u_{m-2} - k)\} \\ &= \frac{\zeta}{h^2}\Big\{\Big(\frac{1}{2}\eta - 1\Big)w - 2u_{m-1} + \Big(1 - \frac{1}{2}\eta\Big)u_{m-2} - \frac{1}{2}k\eta\Big\}.\end{aligned}$$

Now, by the choice of k, and definition of $A_{k,r}$, we have

$$\max_{B_r}(w - u_{m-2}) - d \leq k < w - u_{m-2} \quad \text{on } A_{k,r}$$

Since $A_{k,r} \subset B_r$, the right hand side, is estimated:

$$\leq \max_{B_r}(w - u_{m-2}) < \max_{B_r}(w - u_{m-2}) + d$$

Thus, we have $|k - \max_{B_r}(w - u_{m-2})| < d$. Since $w - u_{m-2}$ is bounded, $|\max_{B_r}(w - u_{m-2})| \leq M$. So, we have:

$$|k| \leq d + |\max_{B_r}(w - u_{m-2})| < d + M. \tag{2.2.11}$$

Then, we get

$$\begin{aligned}\int_{\Omega} \zeta\, dx &= \int_{A_{k,r}} \zeta\, dx \\ &= \int_{A_{k,r}} \eta(w - u_{m-2} - k)\, dx \leq \int_{A_{k,r}} \eta(|w| + |u_{m-2}| + |k|)\, dx \\ &\leq C(\Omega, M, d)|A_{k,r}|. \quad \text{(by (2.2.11))}\end{aligned}$$

Therefore, we have:

$$
\begin{aligned}
\text{(The first term of (2.2.10))} &\leq \Big| \int_{\Omega\cap\mathcal{S}_m^-(\varphi)} \Big(\frac{-2(w+2u_{m-1}-u_{m-2})}{2h^2}\zeta + \frac{|\zeta|^2}{2h^2}\Big)\,dx \Big| \\
&\leq \int_{\Omega\cap\mathcal{S}_m^-(\varphi)} \Big| \frac{-2(w+2u_{m-1}-u_{m-2})}{2h^2}\zeta + \frac{|\zeta|^2}{2h^2}\Big|\,dx \\
&\leq \int_{A_{k,r}} \Big| \frac{-2(w+2u_{m-1}-u_{m-2})}{2h^2}\zeta + \frac{|\zeta|^2}{2h^2}\Big|\,dx \\
&\leq \int_{A_{k,r}} \frac{\zeta}{h^2}\Big\{ \Big|\frac{1}{2}\eta - 1\Big||w| + 2|u_{m-1}| + \Big(1-\frac{1}{2}\eta\Big)|u_{m-2}| + \frac{1}{2}|k|\eta \Big\}\,dx \\
&\leq C(\Omega,M,d,h)\int_{A_{k,r}} \zeta\,dx \quad \text{(by (1.2) and boundedness of } w \text{ etc.)} \\
&\leq C(\Omega,M,d,h)|A_{k,r}|.
\end{aligned}
$$

In the end, we have the estimate for (2.2.10) from above:

$$
(2.2.10) \leq C(\Omega,M,d,h)|A_{k,r}|.
$$

Then, we continue the estimate (2.2.9) as follows:

$$
\begin{aligned}
0 &\leq C|A_{k,r}| + \frac{1}{2}\int_{A_{k,r}} (1-\eta)^2|\nabla w - \nabla u_{m-2}|^2\,dx + \frac{1}{2}\int_{A_{k,r}} (w-u_{m-2}-k)^2|\nabla\eta|^2\,dx \\
&\quad - \frac{1}{4}\int_{A_{k,r}} |\nabla w - \nabla u_{m-2}|^2\,dx \\
&\leq C|A_{k,r}| + \frac{1}{2}\int_{A_{k,r}} |\nabla(w-u_{m-2})|^2\,dx + \frac{2}{(\sigma r)^2}\int_{A_{k,r}} (w-u_{m-2}-k)^2\,dx \\
&\quad - \frac{3}{4}\int_{A_{k,s}} |\nabla(w-u_{m-2})|^2\,dx.
\end{aligned}
$$

The last inequality, we use the following argument. Since $(1-\eta)^2 \leq 1-\eta$,

$$
\begin{aligned}
\frac{1}{2}\int_{A_{k,r}} (1-\eta)^2|\nabla w - \nabla u_{m-2}|^2\,dx &\leq \frac{1}{2}\int_{A_{k,r}} (1-\eta)|\nabla w - \nabla u_{m-2}|^2\,dx \\
&= \frac{1}{2}\int_{A_{k,r}} |\nabla w - \nabla u_{m-2}|^2\,dx - \frac{1}{2}\int_{A_{k,r}} -\eta|\nabla w - \nabla u_{m-2}|^2\,dx \\
&\leq \frac{1}{2}\int_{A_{k,r}} |\nabla w - \nabla u_{m-2}|^2\,dx - \frac{1}{2}\int_{A_{k,s}} |\nabla w - \nabla u_{m-2}|^2\,dx
\end{aligned}
$$

where we use the fact $\eta \equiv 1$ on $A_{k,s}$ and $A_{k,s} \subset A_{k,r}$ in the last inequality.

Therefore, we get

$$
\int_{A_{k,s}} |\nabla(w-u_{m-2})|^2\,dx \leq C|A_{k,r}| + \theta\int_{A_{k,r}} |\nabla(w-u_{m-2})|^2\,dx + \frac{8}{3}\frac{1}{(\sigma r)^2}\int_{A_{k,r}} (w-u_{m-2}-k)^2\,dx,
$$

where $\theta = \frac{2}{3} < 1$. By Lemma V. 3.1 in [33], we obtain

$$\int_{A_{k,s}} |\nabla(w - u_{m-2})|^2 \, dx \le C|A_{k,r}| + \frac{8}{3}\frac{1}{(\sigma r)^2}\int_{A_{k,r}} (w - u_{m-2} - k)^2 \, dx,$$

which is the desired estimate for $-(\tilde{u}_m + u_{m-2})$. □

Proof of Theorem 2.2.3 Lemmas 2.2.1 and 2.2.2 imply that $\tilde{u} := \tilde{u}_m + u_{m-2}$ $(m \ge 2)$ belongs to the De Giorgi class $\mathcal{B}_2(\Omega, \mu, \gamma, d)$. Thus, by De Giorgi's embedding theorem ([59, Section 2.6]), $\tilde{u}_m + u_{m-2} \in C^{0,\tilde{\alpha}}(\widetilde{\Omega})$ for some $\tilde{\alpha} \in (0,1)$ which is independent of m. □

The above theorem tell us that the minimizer $\tilde{u}_m$ is continuous on Ω because $\tilde{u}_m + u_{m-2}$ is locally Hölder continuous, and u_{m-2} is continuous. Thus, we can choose the support of test functions within the open set $\{\tilde{u}_m > 0\}$, which leads to the following first variation formula for J_m.

PROPOSITION 2.2.2 (First variation formula). Any minimizer u of J_m, $m = 2, 3, \ldots, M$, satisfies the following equation:

$$\int_\Omega \left(\frac{u - 2u_{m-1} + u_{m-2}}{h^2}\phi + \nabla\frac{u + u_{m-2}}{2} \cdot \nabla\phi \right) dx = 0 \tag{2.2.12}$$

for all $\phi \in C_c^\infty(\Omega \cap \{u > 0\})$.

Proof. Since $\{u > 0\}$ is an open set by Theorem 2.2.3, we can calculate the first variation of J_m using $u + \varepsilon\phi$ with $\phi \in C_c^\infty(\Omega \cap \{u > 0\})$ as a test function. Indeed,

$$\begin{aligned} J_m(u+\varepsilon\varphi) - J_m(u) &= \int_\Omega \frac{|(u+\varepsilon\varphi) - 2u_{m-1} + u_{m-2}|^2}{2h^2}\chi_{S_m(u+\varepsilon\varphi)} + \frac{1}{4}\int_\Omega |\nabla(u+\varepsilon\varphi) + \nabla u_{m-2}|^2 \\ &\quad - \Big(\int_\Omega \frac{|u - 2u_{m-1} + u_{m-2}|^2}{2h^2}\chi_{S_m(u)} + \frac{1}{4}\int_\Omega |\nabla u + \nabla u_{m-2}|^2\Big) \\ &= \int_\Omega \frac{2\varepsilon\varphi(u - 2u_{m-1} + u_{m-2})}{2h^2}\, dx + \frac{1}{4}\int_\Omega 2\varepsilon(\nabla u + \nabla u_{m-2}) \cdot \nabla\varphi \, dx + O(\varepsilon^2), \end{aligned}$$

where $O(\varepsilon^2)$ denotes (Constant)$\times\varepsilon^2$, and we use the fact there exists $\varepsilon_0 > 0$ such that $\chi_{\mathcal{S}_m(u+\varepsilon\phi)} = \chi_{\mathcal{S}_m(u)}$ for $|\varepsilon| < \varepsilon_0$. Thus,

$$\begin{aligned} 0 &= \lim_{\varepsilon\downarrow 0} \frac{J_m(u+\varepsilon\varphi) - J_m(u)}{\varepsilon} \\ &= \int_\Omega \left(\frac{u - 2u_{m-1} + u_{m-2}}{h^2}\phi + \nabla\frac{u + u_{m-2}}{2} \cdot \nabla\phi \right) dx \end{aligned}$$

□

Chapter 3

Crank-Nicolson minimization scheme II

3.1 Existence of weak solution

In this section, we will construct weak solutions to Problem 1.1 in the one dimensional setting. First, we state the definition.

DEFINITION 3.1.1 (**Weak solution).** For a given $T > 0$, a weak solution is defined as a function $u \in H^1((0,T); L^2(\Omega)) \cap L^\infty((0,T); H_0^1(\Omega))$ satisfying the following equality, for all test functions $\phi \in C_c^\infty(\Omega \times [0,T) \cap \{u > 0\})$:

$$\int_0^T \int_\Omega (-u_t \phi_t + \nabla u \cdot \nabla \phi)\, dxdt - \int_\Omega v_0 \phi(x,0) dx = 0. \tag{3.1.1}$$

Moreover, we require that $u \equiv 0$ is satisfied outside of $\{u > 0\}$, and that $u(0,x) = u_0(x)$ in Ω in the sense of traces.

REMARK 3.1.1. This weak solution contains two pieces of information, namely, the wave equation on the positive part $\{u > 0\}$, and harmonicity on the interior of the complement. If we assume the above weak solution preserves energy and has a regular free boundary, we can formally derive a free boundary condition solely from the definition of the weak solution. If we consider more general settings, such as including an adhesion term, the problem becomes more complicated and requires a different notion of a weak solution.

REMARK 3.1.2. If we consider the one dimensional case, that is, $\Omega \subset \mathbb{R}$ is bounded open interval, the function $u \in H^1((0,T); L^2(\Omega)) \cap L^\infty((0,T); H_0^1(\Omega))$ is continuous on $\Omega \times (0,T)$, thus we can take the test function ϕ with its support in $\{u > 0\}$. The continuity of u follows from the following inequality:

$$|u(x,t) - u(y,s)| \le C_1 \|u\|_{L^\infty((0,T);H_0^1(\Omega))} |x-y|^{\frac{1}{2}} + C_2 \|u\|_{L^\infty((0,T);H_0^1(\Omega))} \|u\|_{H^1((0,T);L^2(\Omega))} |t-s|^{\frac{1}{2}}$$

a.e. $(x,t), (y,s) \in \Omega \times (0,T)$, where $u(x,t) := [u(t)](x)$.

In constructing our weak solution, we carry out interpolation in time of the cut-off minimizers $\{u_m\}$ of J_m, and introduce the notion of approximate weak solutions. In particular, we define $\bar{u}^h$ and u^h as the maps $(0,T) \to H^1(\Omega)$ by

$$\begin{aligned}\bar{u}^h(t) &= u_m, \qquad m = 0,1,2,\dots,M\\ u^h(t) &= \frac{t-(m-1)h}{h}u_m + \frac{mh-t}{h}u_{m-1}, \qquad m = 1,2,3,\dots,M\end{aligned}$$

for $t \in ((m-1)h, mh]$. These functions allow us to construct the following approximate solution based on the first variation formula (Proposition 2.2.12).

DEFINITION 3.1.2 (Approximate weak solution). We call a sequence of functions $\{u_m\} \subset \mathcal{K}$ an **approximate weak solution** of Problem 1.1 if the functions $\bar{u}^h$ and u^h defined above satisfy

$$\begin{aligned}&\int_h^T \int_\Omega \left(\frac{u_t^h(t) - u_t^h(t-h)}{h}\phi + \nabla \frac{\bar{u}^h(t) + \bar{u}^h(t-2h)}{2} \cdot \nabla\phi \right) dx\,dt = 0\\ &\text{for all } \phi \in C_c^\infty(\Omega \times [0,T) \cap \{u^h > 0\}),\\ &u^h \equiv 0 \quad \text{in} \quad \Omega \times (0,T) \setminus \{u^h > 0\}. \qquad (3.1.2)\end{aligned}$$

We further require that the initial conditions $u^h(0) = u_0$ and $u^h(h) = u_0 + hv_0$ are fulfilled.

If one can pass to the limit as $h \to 0$, then the above approximate weak solutions are expected to converge to a weak solution defined above. In the one-dimensional setting, that is $\dim \Omega = 1$, by energy estimate (2.2.5) in Section 3, we obtain the following convergence result, as in [51].

LEMMA 3.1.1 (Limit of approximate weak solution). Let $\Omega \subset \mathbb{R}$ be a bounded open interval. Then, there exists a decreasing sequence $\{h_j\}_{j=1}^\infty$ with $h_j \to 0+$ (denoted as h again) and $u \in H^1((0,T); L^2(\Omega)) \cap L^\infty((0,T); H_0^1(\Omega))$ such that

$$u_t^h \rightharpoonup u_t \quad \text{weakly} * \text{ in } L^\infty((0,T); L^2(\Omega)), \qquad (3.1.3)$$

$$\nabla\bar{u}^h \rightharpoonup \nabla u \quad \text{weakly} * \text{ in } L^\infty((0,T); L^2(\Omega)), \qquad (3.1.4)$$

$$u_h \rightrightarrows u \quad \text{uniformly on } [0,T) \times \Omega, \qquad (3.1.5)$$

where ∇ means the spatial derivative, that is, $\nabla := \frac{\partial}{\partial x}$. Moreover, u is continuous on $\Omega \times (0,T)$, and satisfies the initial condition $u(x,0) = u_0(x)$.

Proof. Rewriting the energy estimate (2.2.5) with $\bar{u}^h$ and u^h, we have

$$\|u_t^h(t)\|_{L^2(\Omega)}^2 + \|\nabla\bar{u}^h(t)\|_{L^2(\Omega)}^2 \le C \quad \text{for a.e. } t \in (0,T), \qquad (3.1.6)$$

which together with the fact that $u^h - u_0$ has zero trace on $\partial\Omega$ immediately implies (3.1.3) and (3.1.4). Regarding (3.1.5), we first prove the equicontinuity of the family $\{u^h\}$ using (3.1.6) and the fact that, when Ω is an interval, for any $f \in H_0^1(\Omega)$, we have

$$\|f\|_{L^\infty(\Omega)} \le C\|f\|_{L^2(\Omega)}^{1/2}\|f'\|_{L^2(\Omega)}^{1/2}.$$

Indeed, for any $t, s \in [0,T)$

$$\begin{aligned}
\|u^h(t) - u^h(s)\|^2_{L^\infty(\Omega)} &\leq C\|u^h(t) - u^h(s)\|_{L^2(\Omega)}\|\nabla u^h(t) - \nabla u^h(s)\|_{L^2(\Omega)} \\
&\leq C\int_s^t \|u^h_t(\tau)\|_{L^2(\Omega)}\,d\tau \\
&\leq C|t-s|,
\end{aligned}$$

and thus, with setting $u^h(x,t) := [u^h(t)](x)$,

$$\begin{aligned}
|u^h(x,t) - u^h(y,s)| &\leq |u^h(x,t) - u^h(y,t)| + |u^h(y,t) - u^h(y,s)| \\
&= \left|\int_y^x \nabla u^h(\xi,t)\,d\xi\right| + |u^h(y,t) - u^h(y,s)| \\
&\leq \|\nabla u^h\|_{L^\infty((0,T);L^2(\Omega))}|x-y|^{1/2} + C|t-s|^{1/2}.
\end{aligned}$$

Moreover, the uniform boundedness of the family $\{u^h\}$ follows as a by-product. Therefore, invoking the Ascoli-Arzelà theorem concludes the proof of (3.1.5). □

The following lemma is needed to prove the existence of weak solutions.

LEMMA 3.1.2. Under the assumption of Lemma 3.1.1, define $\bar{w}^h(t) := 0$ if $t \in (0,h]$, and $\bar{w}^h(t) := \bar{u}^h(t-2h)$ when $t \in (h,T)$. Then,

$$\nabla\bar{w}^h \rightharpoonup \nabla u \quad \text{weakly} * \text{ in } L^\infty((0,T);L^2(\Omega)).$$

Proof. In the following argument, we omit the space variable x for simplicity. We fix $U \in L^1((0,T);L^2(\Omega))$ and extend it by zero outside of $(0,T)$. The extended function, denoted again by U, belongs to $L^1((-\infty,\infty);L^2(\Omega))$. We calculate as follows:

$$\begin{aligned}
&\Big|\int_0^T \langle\nabla\bar{w}^h(t), U(t)\rangle_{L^2(\Omega)}\,dt - \int_0^T \langle\nabla u(t), U(t)\rangle_{L^2(\Omega)}\,dt\Big| \\
&= \Big|\int_{-h}^{T-2h} \langle\nabla\bar{u}^h(t), U(t+2h)\rangle_{L^2(\Omega)}\,dt - \int_0^T \langle\nabla u(t), U(t)\rangle_{L^2(\Omega)}\,dt\Big|
\end{aligned}$$

$$\begin{aligned}
&\leq \Big|\int_0^{T-2h} \langle\nabla\bar{u}^h(t), U(t+2h) - U(t)\rangle_{L^2(\Omega)}\,dt\Big| + \Big|\int_0^T \langle\nabla\bar{u}^h(t) - \nabla u(t), U(t)\rangle_{L^2(\Omega)}\,dt\Big| \\
&\quad + \Big|\int_{T-2h}^T \langle\nabla\bar{u}^h(t) - \nabla u(t), U(t)\rangle_{L^2(\Omega)}\,dt\Big| + \Big|\int_{-h}^0 \langle\nabla\bar{u}^h(t), U(t+2h)\rangle_{L^2(\Omega)}\,dt\Big| \\
&\quad + \Big|\int_{T-2h}^T \langle\nabla u(t), U(t)\rangle_{L^2(\Omega)}\,dt\Big| \\
&\leq C\int_{-\infty}^{\infty} \|U(t+2h) - U(t)\|_{L^2(\Omega)}\,dt + \Big|\int_0^T \langle\nabla\bar{u}^h(t) - \nabla u(t), U(t)\rangle_{L^2(\Omega)}\,dt\Big| \\
&\quad + C\int_{T-2h}^T \|U(t)\|_{L^2(\Omega)}\,dt + C\int_h^{2h} \|U(t)\|_{L^2(\Omega)}\,dt,
\end{aligned} \tag{3.1.7}$$

where the constant C is independent of h. Letting $h \to 0+$, the second term converges to 0 by

(3.1.4), and the remaining terms vanish thanks to the integrability of U. □

We now arrive at the following theorem:

THEOREM 3.1.1 (Existence weak solutions to Problem 1.1). Let Ω be a bounded domain in $\mathbb{R}$. Then Problem 1.1 has a weak solution in the sense of Definition 3.1.1.

Proof. The proof is similar to that in [85] and [37]. Without loss of generality, we can consider $\Omega = (0, 1)$. By the definition of an approximate weak solution (3.1.2), we have

$$\begin{aligned} &\int_h^T \int_\Omega \left(\frac{u_t^h(t) - u_t^h(t-h)}{h}\varphi + \nabla \frac{\bar{u}^h(t) + \bar{u}^h(t-2h)}{2} \cdot \nabla\varphi \right) dxdt = 0 \\ &\qquad \forall \varphi \in \mathcal{C}(\bar{u}^h) := C_c^\infty(\Omega \times [0, T) \cap \{\bar{u}^h > 0\}), \\ &\qquad u^h \equiv 0 \quad \text{in} \quad \Omega \times (0, T) \setminus \{u^h > 0\}. \end{aligned} \tag{3.1.8}$$

We fix $\psi \in \mathcal{C}(u)$, where u is obtained in Lemma 5.1. Since u is continuous on $\Omega \times (0, T)$, there exists $\eta > 0$ such that $u \geq \eta$ on $\operatorname{spt}\psi$. By Lemma 3.1.1, the subsequence $\{u^h\}$ converges to u uniformly, and there exists $h_0 > 0$ such that

$$\max_{(x,t)\in\Omega\times(0,T)} |u^h(x, t) - u(x, t)| \leq \frac{\eta}{2} \quad \text{for all } \ h < h_0.$$

Therefore, we have $u^h \geq u - |u^h - u| \geq \eta/2$ on spt ψ for any $h \in (0, h_0)$. Note that $\bar{u}^h(x, t) = u^h(x, kh)$ for any $t \in ((k - 1)h, kh]$, and $\bar{u}^h \geq \eta/2 > 0$ on spt ψ for any $h \in (0, h_0)$. This implies that (3.1.8) holds for any test function $\varphi \in \mathcal{C}(u)$ whenever $h < h_0$. The time-discretized term can be rearranged as

$$\begin{aligned} &\int_h^T \frac{u_t^h(t) - u_t^h(t-h)}{h}\varphi(t)\, dt \\ = \ &\int_0^T u_t^h(t)\frac{\varphi(t) - \varphi(t+h)}{h}\, dt - \frac{1}{h}\int_0^h u_t^h(t)\varphi(t+h)\, dt + \frac{1}{h}\int_{T-h}^T u_t^h(t)\varphi(t+h)\, dt. \end{aligned}$$

Hence, using Lemma 3.1.1 and Lemma 3.1.2, and passing to $h \to 0+$ in (3.1.8), we obtain

$$\int_0^T \int_\Omega (-u_t\varphi_t + \nabla u \cdot \nabla\varphi)\, dxdt - \int_\Omega v_0\varphi(x, 0)dx = 0 \qquad \forall\varphi \in \mathcal{C}(u),$$

which was our goal. Moreover, by the construction, $u \equiv 0$ is satisfied outside of $\{u > 0\}$, and that $u(0, x) = u_0(x)$ in Ω. □

3.2 Numerical results for the one-dimensional problem

In this section, we present several numerical results for the equation

$$\chi_{\overline{\{u>0\}}} u_{tt} - \Delta u = 0. \tag{3.2.1}$$

The numerical computation in this section is due to Professor Yoshiho Akagawa of National Institute of Technology, Gifu collage. Main purpose in this section comparing the minimization of the Crank-Nicolson type functional

$$J_m(u) := \int_{\Omega\cap(\{u>0\}\cup\{u_{m-1}>0\}\cup\{u_{m-2}>0\})} \frac{|u-2u_{m-1}+u_{m-2}|^2}{2h^2}\,dx + \frac{1}{4}\int_\Omega |\nabla u + \nabla u_{m-2}|^2\,dx.$$

and the original discrete Morse flow method of [73], which uses the functional

$$\widetilde{J}_m(u) := \int_{\Omega\cap(\{u>0\}\cup\{u_{m-1}>0\})} \frac{|u-2u_{m-1}+u_{m-2}|^2}{2h^2}\,dx + \frac{1}{2}\int_\Omega |\nabla u|^2\,dx.$$

In the numerical calculation, we simply use the functional I_m without the restriction of the integration domain and the corresponding functional $\widetilde{I}_m$ for the original discrete Morse flow method. Subsequently, for a minimizer $\tilde{u}_m$, $m \geq 2$, of I_m or $\widetilde{I}_m$, we define

$$u_m := \max\{\tilde{u}_m, 0\}.$$

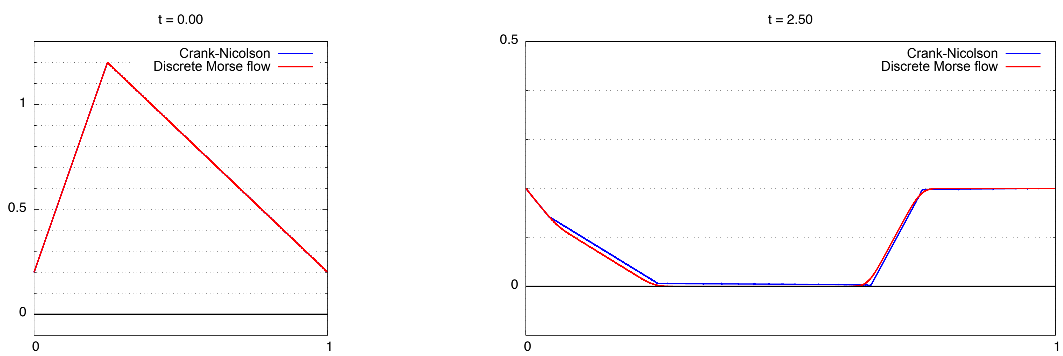
We regard u_m as a numerical solution at time level $t = mh$. The minimization problems are discretized by the finite element method, where the approximate minimizer is a continuous function over the domain and piece-wise linear over each element.

In the one-dimensional case, equation (3.2.1) has been employed in describing the dynamics of a string hitting a plane with zero reflection constant. In two dimensions, the graph of the solution may be considered as representing a soap film touching a water surface. Another important application of this model is the volume constrained problem describing the motion of scalar droplets over a flat surface (see, e.g., [37], and next section).

Having in mind the model of a string hitting an obstacle, let us first consider problem (3.2.1) in the open interval $\Omega = (0,1)$, with the initial condition

$$u_0(x) := \begin{cases} 4x + 0.2 & \text{if } 0 \leq x < 1/4, \\ -\dfrac{4}{3}(x-1) + 0.2 & \text{otherwise ,} \end{cases}$$

and $v_0 \equiv 0$. Figure 3.1 shows the behavior of the numerical solution for both methods. For the Crank-Nicolson method, the corners in the graph of the solution are kept, even as time progresses. This is not the case for the discrete Morse flow method, where corners are smoothed.



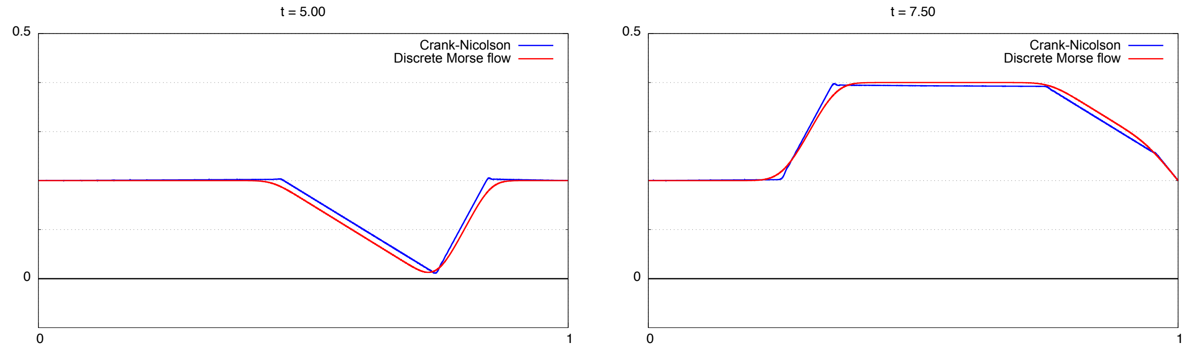


Figure 3.1: Numerical solution at four distinct times for the Crank-Nicolson method (blue) and the original discrete Morse flow method (red). The time step size is $h = 1.0 \times 10^{-4}$ and the spacial mesh size is $\Delta x = h$.

Figure 3.2 shows that the free boundary condition $u_x^2 - u_t^2 = 0 \quad \text{on } \Omega \times (0,T) \cap \partial\{u > 0\}$ is satisfied when the string peels off the obstacle.

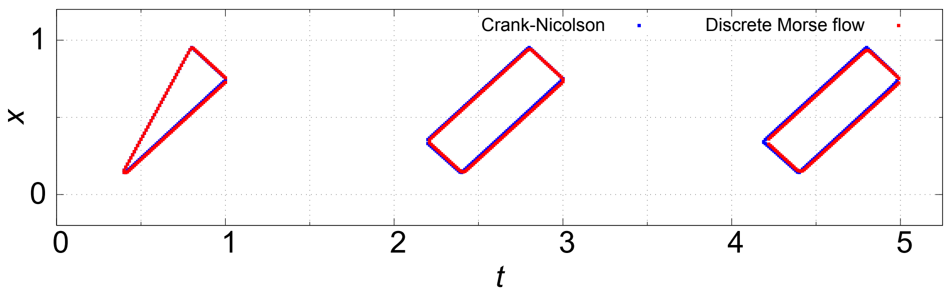


Figure 3.2: The free boundary $\{(t,x); u(x,t) = 0\}$ corresponding to the motion in Fig. 1.

Figure 3.3 shows that the energy is lost when the string touches the obstacle, while the energy is preserved before and after the contact of the string with the obstacle. For the sake of comparison, we note that the energy of the solution obtained by discrete Morse flow decays even during the non-contact stage.

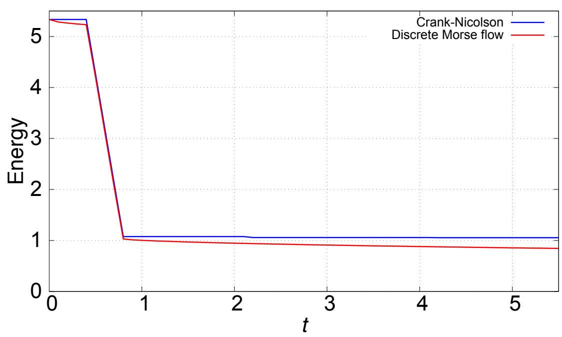


Figure 3.3: The evolution of the energy of the numerical solution for both methods.

To test the energy decay tendency of both methods, we solved the problem without free boundary with the initial condition $u_0 = \sin(2n\pi x)$, and $v_0 \equiv 0$. It was found that, for the original discrete Morse flow, energy decay becomes prominent with decreasing time resolution and increasing wave frequency. On the other hand, as can be observed in Figure 3.4, the Crank-Nicolson method preserves energy independent of the time resolution and wave frequency.

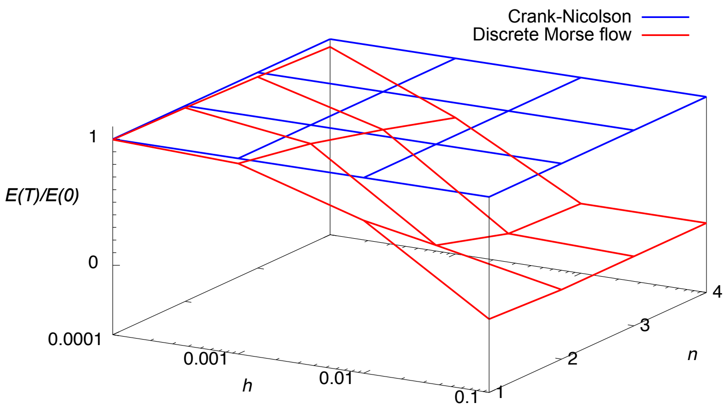


Figure 3.4: Comparison of energy decay tendency for both methods using the initial data $u_0 = \sin(2n\pi x)$ and $v_0 \equiv 0$. Here, $\Delta x = h$ is used.

Although the Crank-Nicolson method displays excellent energy-preserving properties, it appears to include an incorrect phase-shift, as is the case with the original discrete Morse flow. We summarize the features of both methods in Table 3.1.

	C-N	DMF
energy	conserved	decays
free boundary condition	holds	holds
high harmonic wave	preserved	decays
including constraints	possible	possible
phase shift	occurs	occurs

Table 3.1: Main features of the two methods compared in this section.

3.3 Numerical results for higher dimensions and more general problems

In this section, we investigate the energy preservation properties of the proposed scheme in the two dimensional setting. In particular, the functional (2.1.1) is used to approximate a solution of the wave equation with initial conditions $u_0(x, y) = \sin(\pi x)\sin(\pi y), v_0(x, y) = 0$ and Dirichlet zero boundary condition, where the domain $\Omega = (0, 1) \times (0, 1)$. The numerical computation in this section is due to Professor Elliott Ginder of Meiji University.

The functional value corresponding to a given function u is approximated using P_1 finite elements, We triangulate the domain Ω into a finite number of elements $\Omega = \cup_j^N e_j$, where each e_j is a triangular subdomain of Ω. The approximation is as follows:

$$I_m(u) \approx \sum_{j=1}^{N} \int_{e_j} \Big(\frac{|\hat{u} - 2\hat{u}_{m-1} + \hat{u}_{m-2}|^2}{2h^2} + \frac{|\nabla \hat{u} + \nabla \hat{u}_{m-2}|^2}{4} \Big) dx$$

where the notation $\hat{u}$ refers to the P_1 approximation of u restricted to an arbitrary element e. and the functional minimization is performed using a steepest descent algorithm. Here Ω has been uniformly partitioned into $N = 5684$ elements. (as shown in Figure 3.5).

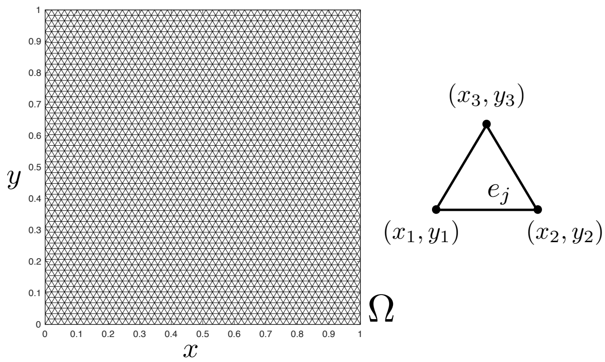


Figure 3.5: (Left) The triangulation of the domain $\Omega = (0,1) \times (0,1)$. (Right) A typical element.

Using several different values of the time step h, we compared the energy of the numerical solution obtained using the Crank-Nicolson scheme with that obtained from the standard discrete Morse flow. The total energy is computed using the finite element method on the functional:

$$\mathcal{E}_n^h(u) = \int_\Omega \Big(\frac{1}{2} \left| \frac{u - u_{n-1}}{h} \right|^2 + \frac{|\nabla u|^2}{2} \Big) \, dx. \tag{3.3.1}$$

The results are shown in Figure 3.6, where the time steps were $h = 0.0005 + 0.005 \times k$, k = 0,1,2,3,4,5. Our results confirm the energy preservation properties of the proposed scheme.

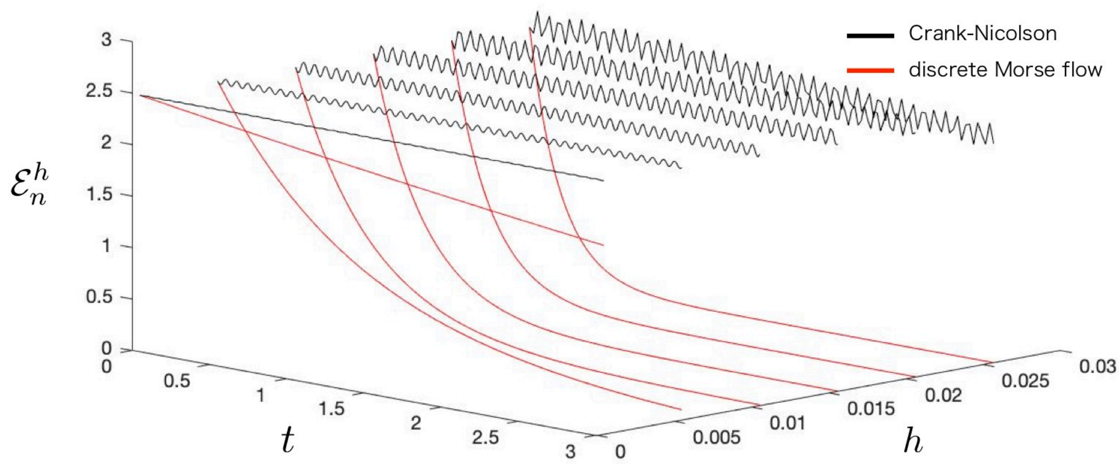


Figure 3.6: Comparison of the Crank-Nicolson scheme with the original discrete Morse flow. Time is denoted by $t = nh$.

We have also used the proposed method to investigate the numerical solution of a more complicated model equation describing droplet motions. The target equations correspond to volume constrained formulations of the original problem. In particular, volume and non-negativity constraints are added to the functionals by means of indicator functions:

$$\mathcal{J}_m^i(u) = \frac{1}{2}\int_\Omega \left(\frac{|u - 2u_{m-1}^i + u_{m-2}^i|^2}{h^2} + \frac{1}{2}|\nabla u + \nabla u_{m-2}^i|^2 + \right) dx + \Psi_1(u) + \Psi_2^{i,m}(u), \tag{3.3.2}$$

where each indicator function is defined as follows:

$$\Psi_1(u) = \begin{cases} 0, & \text{if } u(x) \geq 0 \text{ for } \mathcal{L}^N\text{-a.e. } x \in \Omega \\ \infty, & \text{otherwise} \end{cases}, \quad \Psi_2^{i,m}(u) = \begin{cases} 0, & \text{if } \int_\Omega u(x)dx = V_m^i \\ \infty, & \text{otherwise.} \end{cases}$$

Here V_m^i denotes the volume of droplet i at time step m.

By minimizing functionals $\mathcal{J}_m^i$ for each droplet, we are able to compute approximate solutions to the volume constrained problem. The results are shown in Figure 3.7. For each i, the initial condition is prescribed as a spherical cap, and we observe the droplets oscillate while coalescing into larger groups.

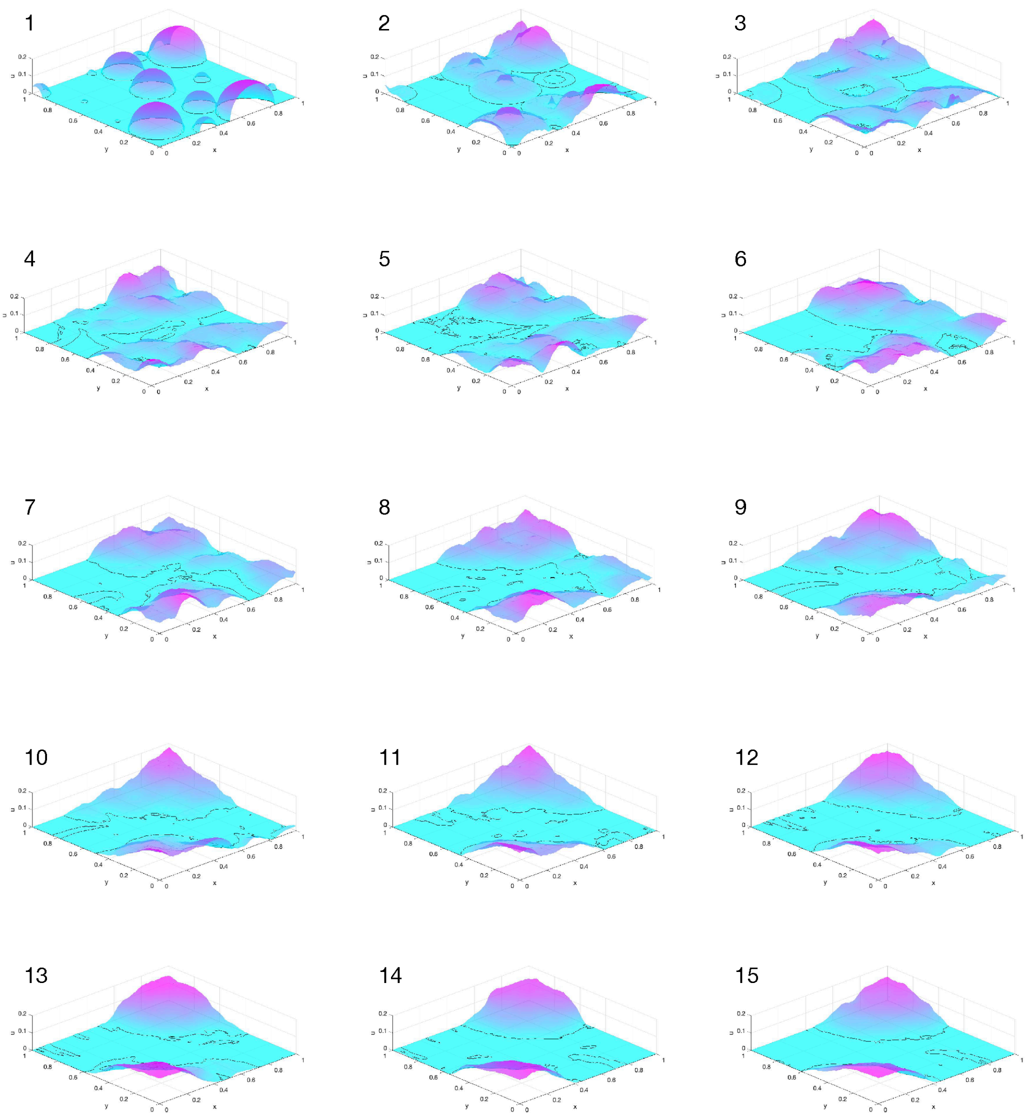


Figure 3.7: Crank-Nicolson type minimizing movement approximation of droplet motion. Time is designated by the integer values within the figure (the initial condition corresponds to number 1) and the free boundary is illustrated as the black curves.

Part II

Mean curvature accelerated flow

Chapter 4

Preliminaries II

4.1 Theory of moving hypersurface

We prepare some notations about the theory of moving hypersurfaces from [56]. First of all, we recall the definitions of basic notions for C^k-class hypersuface in Euclidian space.

DEFINITION 4.1.1 (C^k**-class hypersurface).** Let $k \in \mathbb{N}$. We say a subset $\Gamma \subset \mathbb{R}^d$ ($d = N$ or $N+1$) is a C^k-class hypersuface in $\mathbb{R}^d$ if for each $x \in \Gamma$, there exists,

$$\begin{cases} \mathcal{O} & : \text{a nonempty bounded domain in}\, \mathbb{R}^d \\ \mathcal{U} & : \text{a nonempty bounded domain in}\, \mathbb{R}^{d-1} \\ \varphi \in C^k(\mathcal{U}\,;\,\mathbb{R}^d) \end{cases} \tag{4.1.1}$$

such that $x \in \mathcal{O}$ and they satisfy the following conditions:

$$\varphi : \mathcal{U} \to \Gamma \cap \mathcal{O} \text{ is bijective, and } \operatorname{rank}(\nabla_{\boldsymbol{\xi}}^T \varphi(\boldsymbol{\xi})) = d-1 \ (\forall \boldsymbol{\xi} \in \mathcal{U}). \tag{4.1.2}$$

We also define the following set which can be said the set of all local embedding C^k-coordinate of the hypersurface Γ:

$$\mathrm{C}_k(\Gamma) := \{(\mathcal{O}, \mathcal{U}, \varphi) : (\mathcal{O}, \mathcal{U}, \varphi)\, \text{satisfies (4.1.1) and (4.1.2)}\}.$$

DEFINITION 4.1.2 (C^ℓ**-function on hypersurface).** Let $k \in \mathbb{N}$ and ℓ be an integer with $0 \le \ell \le k$. For a C^k-class hypersurface Γ in $\mathbb{R}^d$ and a function defined on Γ, we say f is C^ℓ-class function on Γ if $f \circ \varphi \in C^\ell(\mathcal{U})$ for any $(\mathcal{O}, \mathcal{U}, \varphi) \in \mathrm{C}_k(\Gamma)$. The set of all C^ℓ-class functions on Γ is denoted by $C^\ell(\Gamma)$.

Let us consider a C^k ($k \ge 2$) -class hypersurface Γ. Then, we define the mean curvature at $x \in \Gamma$ in the direction of the unit normal vector $\boldsymbol{n}(x)$, denoted by $\kappa(x)$, and it is known that these quantities do not depend on the local coordinate. We remark that the mean curvature is defined by the sum of the principal curvatures, not the average of them. If we have local graph representation for Γ, that is, Γ is locally represented by $\{(\xi, w(\xi)) : \xi \in U : \text{some domain in } \mathbb{R}^{N-1}\}$ for some smooth function $w : U \to \mathbb{R}$, the unit normal vector and the mean curvature can be represented

$$\boldsymbol{n}(x) = \frac{(-\nabla_\xi w(\xi), 1)}{\sqrt{1+|\nabla_\xi w(\xi)|^2}}, \quad \kappa(x) = \mathrm{div}_\xi \Big(\frac{\nabla_\xi w(\xi)}{\sqrt{1+|\nabla_\xi w(\xi)|^2}} \Big), \quad x = (\xi, w(\xi)),$$

respectively. We also use the sign convention for the mean curvature such that $\kappa = -\frac{1}{R}$ if Γ is a circle with radius R.

For a given $T \in (0,\infty]$, we set the time interval $\mathcal{I} := [0, T)$. Let $(\Gamma_t)_{t\in\mathcal{I}}$ be the family of time dependent of nonempty oriented $(N-1)$-dimensional hypersurface in $\mathbb{R}^N$. We define *moving hypersurfaces* in $\mathbb{R}^N$, denoted $\mathcal{M}$ by $\bigcup_{t\in\mathcal{I}}(\Gamma_t \times \{t\})$ that is, $(x_0, t_0) \in M$ means that $x_0 \in \Gamma_{t_0}$, and $\mathcal{M}$ is a subset in $\mathbb{R}^N \times \mathcal{I}$.

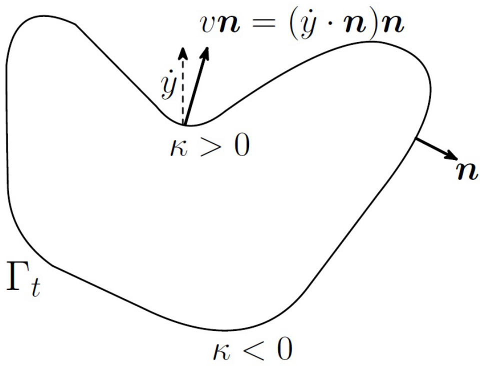


Figure 4.1: Mean curvature, and Normal velocity

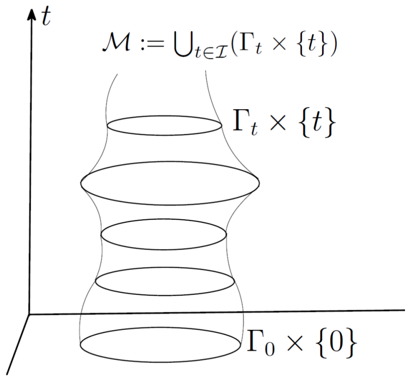


Figure 4.2: Moving hypersurfaces

For each $(x,t) \in \mathcal{M}$, we can consider the unit normal vector $\boldsymbol{n}(x,t)$, $\kappa(x,t)$ as the unit normal vector and the mean curvature at $x \in \Gamma_t$ respectively. Hereafter, we assume that $\mathcal{M}$ is C^1-class N-dimensional hypersurface in $\mathbb{R}^{N+1}$ and $\boldsymbol{n} \in C^1(\mathcal{M}, \mathbb{R}^N)$. We say that $\boldsymbol{y}$ is C^1 *trajectory* on $\mathcal{M}$ if $\boldsymbol{y} \in C^1(\mathcal{I}_0, \mathbb{R}^N)$ with $\boldsymbol{y}(t) \in \Gamma_t$ for $t \in \mathcal{I}_0$, where $\mathcal{I}_0$ is subinterval of $\mathcal{I}$. Then, for each $(x,t) \in \mathcal{M}$, we define *the normal velocity* $v(x,t)$ by $\boldsymbol{y}'(t) \cdot \boldsymbol{n}(x,t)$. Remark that v is well defined, that is, v does not depend on trajectory $\boldsymbol{y}$ (see [56], Theorem 5.5). Also, we point out that v does not depend on the local coordinate. By using the standard theory of ordinary differential equations, it is known that for any $(x_0,t_0) \in \mathcal{M}$, there uniquely exists C^1 trajectory $\boldsymbol{y}$ such that $\boldsymbol{y}'(t) = v(y(t),t)\boldsymbol{n}(y(t),t)$ (see [56], Proposition 5.8). Such $\boldsymbol{y}$ is called *the normal trajectory* on $\mathcal{M}$ through (x_0,t_0).

Next, we introduce the notion of *the normal time derivative.* This gives the answer for the question that what does mean the time derivative of the functions $f(x,t)$ on $\mathcal{M}$.

DEFINITION 4.1.3 (Normal time derivative). Let $f \in C^1(\mathcal{M}, \mathbb{R}^k)$, $k = 1, \ldots, N+1$. For $(x_0,t_0) \in \mathcal{M}$, and the normal trajectory $\boldsymbol{y}$ on $\mathcal{M}$ through (x_0,t_0),

$$D_t f(x_0,t_0) := \frac{d}{dt} f(y(t),t)\Big|_{t=t_0},$$

$D_t f$ is called *the normal time derivative* of f on $\mathcal{M}$.

The normal time derivative is introduced by W. D. Hays [44] and T. Y. Thomas [83] independently. The mathematical definition is established by E. Guritn, A.Struthers, O. Williams [42]. Remark that $D_t f$ does not depend on trajectory because the normal trajectory is unique, and even if a function does not depend on time variable, the normal time derivative may be not zero. For example, for identity map on $\mathcal{M}$, $\gamma(x,t) := x \in \mathbb{R}^{N+1}$, we have $D_t\gamma = v\boldsymbol{n}$ on $\mathcal{M}$. Intuitively, this example says that the time derivative along the normal line of the position is the normal velocity. Also, by definition of normal time derivative and the classical chain rule, we have $D_t(f^2) = 2fD_tf$ for any $f \in C^1(\mathcal{M}, \mathbb{R})$. The notion of the normal time derivative can be considered the analogous of the material derivative in the fluid dynamics as mentioned in [22]. It can be understood through the following formula. For $f \in C^1(\mathcal{Q})$ where $\mathcal{Q}$ is an open neighbourhood of $\mathcal{M}$ in $\mathbb{R}^{N+1}$, the normal time derivative can be expressed as follows:

$$D_t f(x,t) = f_t(x,t) + v(x,t)\nabla f(x,t) \cdot \boldsymbol{n}(x,t) \quad \text{for} \quad (x,t) \in \mathcal{M}. \tag{4.1.3}$$

We conclude the preliminaries with following useful formula, so called *the transport identity* :

THEOREM 4.1.1 (Transport identity). Let $f \in C^1(\mathcal{M})$ and Γ_t be compact for all $t \geq 0$, then

$$\frac{d}{dt}\int_{\Gamma_t} f(x,t)\, d\mathscr{H}^{N-1} = \int_{\Gamma_t} (D_t f - f\kappa v)\, d\mathscr{H}^{N-1}. \tag{4.1.4}$$

In particular, if $f \equiv 1$ we have the well-known first variation formula of the surface area,

$$\frac{d}{dt}\mathscr{H}^{N-1}(\Gamma_t) = -\int_{\Gamma_t} \kappa v\, d\mathscr{H}^{N-1}. \tag{4.1.5}$$

4.2 Previous research of mean curvature accelerated flow

In this section, we will review the hyperbolic mean curvature flow equation. M.E.Gurtin and P. Podio-Guidugli [40] firstly treated the following equation for plane curves as the mathematical model for the melting or crystallizing of helium crystal,

$$\rho(\theta)D_t v + \beta(\theta)v = [\psi(\theta) + \psi''(\theta)]\kappa - f \quad \text{on} \quad \Gamma_t \tag{4.2.1}$$

where ρ, β, ψ is physical quantities describing the effective density, the kinetic coefficient, the interfacial energy respectively, and f is a driving force for crystallization, and Γ_t is smooth, simple closed curves in $\mathbb{R}^2$. In their situation, the crystal is modeled by an enclosed area by Γ_t. Here, $\theta = \theta(x)$ $(x \in \Gamma_t)$ expresses the angle with $\boldsymbol{n}(x,t) = (\cos\theta(x), \sin\theta(x))$, $\boldsymbol{\tau}(x,t) = (\sin\theta(x), -\cos\theta(x))$, where $\boldsymbol{n}, \boldsymbol{\tau}$ are the unit normal vector, unit tangent vector respectively. Quite roughly said, θ measures how much tilted does $\boldsymbol{n}$ to x_1-axis. M.E.Gurtin and P. Podio-Guidugli derived the equation from some balance laws, and study some simple solutions e.g. the radial solutions for the isotropic crystal.

When we assume that ρ, β, ψ are constant, that is, the system is isotropic, and the absence of driving force $(f = 0)$, with appropriate rescaling with respect to t, the equation (4.2.1) can be reduced

$$D_t v + cv = \kappa \qquad \text{on} \quad \Gamma_t \tag{4.2.2}$$

where v is the normal velocity, c is a constant. H.G.Rotstein, S.Brandon, and A.Novick-Cohen gave the crystalline algorithm for the equation (4.2.2) for the closed polygonal curves in [78].

On the other hand, M.Kang treated the following equation as the mathematical model for the motion of bubbles with various situations with numerical results by the level set method [49].

$$\mu \frac{d\boldsymbol{u}}{dt} = -p\boldsymbol{n} - \sigma\kappa\boldsymbol{n} + f - \mu\boldsymbol{u}(\nabla \cdot \boldsymbol{u} - \boldsymbol{n} \cdot D\boldsymbol{u} \cdot \boldsymbol{n}) - \boldsymbol{u}\frac{d\mu}{dt} \tag{4.2.3}$$

where $\boldsymbol{u}$ is velocity vector, $\frac{d}{dt}$ denotes the material derivative, that is, $\frac{d}{dt} := \frac{\partial}{\partial t} + \boldsymbol{u} \cdot \nabla$. The notions μ, p, σ, f denote mass density, a factor related on the pressure, surface tension, additional sources of momentum. We can get the vector version of the hyperbolic mean curvature flow equation $\frac{d\boldsymbol{u}}{dt} = \kappa\boldsymbol{n}$ as the special case of (4.2.3). In his method, however, it is not clear how the ideas can be extended more general settings.

Over the last two decades, it was started to generalize these equations in the point of view of the differential geometry by C.L.He, D.X.Kong, K.Liu, and P.G.LeFloch, K.Smoczyk independently. Let us explain in the one-dimensional, that is, planner curves settings for simplicity. First, He, Kong, and Liu prove the unique short time existence smooth solution of the following hyperbolic mean curvature flow equation in [43].

$$\gamma_{tt} = \kappa\boldsymbol{n} \tag{4.2.4}$$

here, $\gamma : (0, \ell) \times [0, T) \to \mathbb{R}^2$ is a family of smooth curves. The equation (4.2) can be interpreted a vector version of the equation $D_t v = \kappa$ which is the equaiton (4.2.2) with $c = 0$. LeFloch and Smoczyk [61] stated from deriving the following equation by calculating the first variation of the action containing kinetic and internal energy terms.

$$\gamma_{tt} = e\kappa\boldsymbol{n} - \nabla e \tag{4.2.5}$$

where $e := \frac{1}{2}(|\gamma_t|^2 + 1)$ is the local energy density, and ∇e is defined by

$$\nabla e := \Big(\frac{\partial^2 \gamma}{\partial s \partial t} \cdot \frac{\partial \gamma}{\partial t}\Big)\boldsymbol{\tau}.$$

LeFloch and Smoczyk gave the weak solution in the sense of graph solutions for another type of hyperbolic mean curvature flow equations with one dimensional setting. Moreover, the following type of equation, so-called hyperbolic Monge-Ampère equation is investigated in [57], [58],

$$\gamma_{tt} = \kappa\boldsymbol{n} - \nabla e.$$

On the other hand, the numerical treatment for the equation including the multiphase settings is developed by E.Ginder, K.Svadlenka in [36]. Their method is called hyperbolic MBO-algorithm which is based on Merriman-Bence-Osher algorithm for a numerical scheme of mean curvature flow equations via level set approach developed in [65]. Quite roughly said, they solve the wave equation $u_{tt} = \Delta u$ under the suitable initial and boundary conditions up to small-time τ, define the next time step curve γ_1 as zero level set of $u(\cdot, \tau)$. Repeat this procedure up to the final time T. More precisely, the algorithm of hyperbolic MBO is shown as follows.

Hyperbolic MBO algorithm for closed curves

Given : initial curve γ_0, its normal velocity v_0, a final time $T > 0$, and time discretize size $\tau = T/M$ for the numerical E-HMCF equation (6.4.6) , N and k_0 are positive integers. We will solve the equation (6.4.6) up to $t = k_o\tau$ (see the following Step1, 3). Firstly we extend v_0 to a neighborhood of γ_0 e.g. using the orthogonal projection on γ_0.

Step 1. For $t \in [0, k_0\tau]$ solve the initial value problem:

$$u_{tt}(x,t) = \Delta u(x,t), \quad u(x,0) = d_0(x), \quad u_t(x,0) = -v_0(x)$$

where d_0 is the signed distance function for γ_0 which is defined by

$$d_0(x) := \begin{cases} \text{dist}\,(x,\gamma_0) & x \in \Omega_0^+ \\ 0 & x \in \gamma_0 \\ -\,\text{dist}\,(x,\gamma_0) & x \in \Omega_0^-. \end{cases}$$

here Ω_0^- is enclosed area by γ_0, and $\Omega_0^+ := \mathbb{R}^2 \setminus (\Omega_0^- \cup \gamma_0)$.

Step 2. Define γ_1 as the zero level set $u(\cdot, k_0\tau)$.

Step 3. For $n = 1, ..., N-1$ repeat

Step 3.1 For $t \in [0, k_0\tau]$ solve the initial value problem:

$$u_{tt}(x,t) = 2\Delta u(x,t), \quad u(x,0) = 2d_n(x) - d_{n-1}(x), \quad u_t(x,0) = 0$$

where d_n is the signed distance function for γ_n.

Step 3.2 Define γ_{n+1} as the zero level set $u(\cdot, k_0\tau)$.

They provided formal justification and the error estimate in the case of the circle for the hyperbolic MBO-algorithm. They also treat the multiphase version of the hyperbolic MBO-algorithm. By replacing the solving wave equation in the above Step 1, Step 3 by the minimizing movement method e.g. discrete Morse flow method, it can treat that the volume constrains problems. See [36], [38]. Another numerical simulation for the following hyperbolic mean curvature flow equation which is related to the motion of the relativistic string,

$$D_t v = (1 - v^2)\kappa,$$

is treated by Bonafini [16]. Moreover, another numerical approach to access the multiphase motion and volume-preserving problems is developed by S.Ishida et. al. [48].

Chapter 5

Acceleration and surface evolutional energy

5.1 Properties of acceleration

We define the acceleration of the surface as the normal time derivative of the normal velocity. That is,

DEFINITION 5.1.1 (**Acceleration of surfaces).** Let $\mathcal{M}$ be moving hypersurface in $\mathbb{R}^N$, $v \in C^1(\mathcal{M})$, Then, for $(x,t) \in \mathcal{M}$, we define

$$a(x,t) := D_t v(x,t).$$

We remark that this acceleration capture only to motion in a normal direction. For example, consider the rolling circle with constant speed. This circle remains the same shape, and thus the normal velocity is equal to zero everywhere, any time, but the tangential velocity is not equal to zero, so the normal acceleration is. Then, $a \equiv 0$ since $v \equiv 0$. Thus, in general, the acceleration $a = D_t v$ does not coincide with the normal acceleration. However, in the case of that the tangential velocity is equal to zero everywhere, any time, we can regard $D_t v$ as the normal acceleration in the sense of the normal time derivative of the normal velocity. In the following subsections, we investigate the properties of this acceleration and see that our observation is correct in the next section.

Now, we derive the relation between acceleration $a = D_t v$ and the classical acceleration which is defined by the second time derivative of position, $\ddot{x}$ in the case of the closed curves in $\mathbb{R}^2$. Let Γ_t be a closed curve, that is $\Gamma_t = \{\gamma(\vartheta, t) \in \mathbb{R}^2 \,:\, \vartheta \in [0, \ell]\}$. We also set that $\tilde{n} : [0,\ell] \times [0,T) \to \mathbb{R}^2$, $\tilde{\tau} : [0,\ell] \times [0,T) \to \mathbb{R}^2$, $\tilde{\kappa} : [0,\ell] \times [0,T) \to \mathbb{R}$ be unit normal vector, unit tangent vector, mean curvature respectively, and

$$\tilde{v} : [0,\ell] \times [0,\infty) \to \mathbb{R},\ \tilde{v} := \gamma_t \cdot \tilde{n},$$
$$\tilde{e} : [0,\ell] \times [0,\infty) \to \mathbb{R},\ \tilde{e} := 1 + \frac{1}{2}|\gamma_t|^2,$$
$$\tilde{w} : [0,\ell] \times [0,\infty) \to \mathbb{R},\ \tilde{w} := \gamma_t \cdot \tilde{\tau}$$

be normal velocity, energy density, tangential velocity respectively. We remark that the velocity vector γ_t has the decomposition $\gamma_t = \tilde{v}\tilde{n} + \tilde{w}\tilde{\tau}$. By directly calculation, it is easy to see that the normal acceleration $\gamma_{tt} \cdot \tilde{n}$ satisfies the following formula,

$$\gamma_{tt} \cdot \tilde{n} = \tilde{v}_t + \frac{1}{|\gamma'|}\tilde{v}'\tilde{w} + \tilde{\kappa}\tilde{w}^2. \tag{5.1.1}$$

Although we distinguish them and the notions for moving hypersurface i.e. v, n, κ, we have the following relations: Fixed $(x_0, t_0) \in \mathcal{M} := \bigcup_{t\in\mathcal{I}}(\Gamma_t \times \{t\})$, if $x_0 = \gamma(\vartheta, t_0)$ for some $\vartheta \in [0,\ell]$, then

$$\tilde{n}(\vartheta, t_0) = \boldsymbol{n}(x_0, t_0),$$
$$\tilde{\kappa}(\vartheta, t_0) = \kappa(x_0, t_0),$$
$$\tilde{v}(\vartheta, t_0) = v(x_0, t_0).$$

For the third equality follows from that we can take $\gamma(\vartheta, \cdot)$ as C^1 trajectory through (x_0, t_0) and the first equality. We now assume that the tangential velocity $\tilde{w}$ is equivalently equal to zero. Then, since $\gamma_t = \tilde{v}\tilde{n}$, we can take $\gamma(\vartheta, \cdot)$ as C^1 normal trajectory through (x_0, t_0). Thus, by definition of the normal time derivative, we get the following the relation.

$$D_t v(x_0, t_0) = \frac{d}{dt} v(\gamma(\vartheta, t), t)\bigg|_{t=t_0} = \frac{d}{dt}\tilde{v}(\vartheta, t)\bigg|_{t=t_0} = \tilde{v}_t(\vartheta, t_0).$$

Since $\gamma_{tt} \cdot \tilde{n} = \tilde{v}_t$ by (5.1.1) and $\tilde{w} \equiv 0$, we have

$$D_t v(x_0, t_0) = D_t v(\gamma(\vartheta, t_0), t_0) = \gamma_{tt}(\vartheta, t_0) \cdot \tilde{n}(\vartheta, t_0).$$

We can generalize this result for the parametrized surfaces. To check this, we have only to do minor change for the above argument. Let Γ_t be a parametrized $(N-1)$- dimensional surface, that is $\Gamma_t = \{\gamma(\vartheta, t) \in \mathbb{R}^N \ :\ \vartheta \in U \subset \mathbb{R}^{N-1}\}$. Since Γ_t has $(N-1)$ tangent vectors $\tau_1, \tau_2, ..., \tau_{N-1}$, we define tangential velocities $\tilde{w}_i : U \times [0,\infty) \to \mathbb{R}$ by $\tilde{w}_i := \gamma_t \cdot \tilde{\tau}_i$ for each $i = 1, ..., N-1$. As a same above, the velocity vector γ_t has the decomposition $\gamma_t = \tilde{v}\tilde{n} + \sum_{i=1}^{N-1} \tilde{w}_i\tilde{\tau}_i$. Direct calculation shows the normal acceleration $\gamma_{tt} \cdot \tilde{n}$ satisfies the following formula,

$$\gamma_{tt} \cdot \tilde{n} = \tilde{v}_t + \sum_{i=1}^{N-1}\Big(\frac{1}{|\gamma'|}\tilde{v}'\tilde{w}_i + \tilde{\kappa}\tilde{w}_i^2\Big).$$

Remaining calculation and argument are same as above.

Graph representation

Here, we give the representation of the acceleration of surface $D_t v$ when the surface Γ_t is the graph or the zero level set of some function.

We assume that

$$\Gamma_t = \{(\xi, w(\xi,t)) \ : \ \xi \in U \subset \mathbb{R}^{N-1}\}$$

for some sufficiently smooth function $w : U \times (0,T) \to \mathbb{R}$. Then, the normal derivative v, the unit normal vector are given by

$$v(x,t) = \frac{w_t(\xi,t)}{\sqrt{1+|\nabla_\xi w(\xi,t)|^2}}, \quad \boldsymbol{n}(x,t) = \frac{(-\nabla_\xi w(\xi,t),1)}{\sqrt{1+|\nabla_\xi w(\xi,t)|^2}}, \tag{5.1.2}$$

for $x = (\xi, w(\xi,t)) \in \Gamma_t$. We calculate the $D_t v(x_0,t_0)$ for $x_0 \in \Gamma_{t_0}$. By the definition of the time normal derivative,

$$D_t v(x_0,t_0) = \frac{d}{dt} v(y(t),t)\Big|_{t=t_0} \tag{5.1.3}$$

where y is the normal trajectory through (x_0,t_0) i.e. $y'(t) = v(y(t),t)\boldsymbol{n}(y(t),t)$. Remark that when we put $y(t) = (\eta(t), w(\eta(t),t))$, from the normality of y, we get

$$\eta'(t) = \frac{-1}{1+|\nabla_\xi w(\eta(t),t)|^2}(w_t(\eta(t),t)\nabla_\xi w(\eta(t),t)). \tag{5.1.4}$$

Then we can calculate

$$\begin{aligned}
\frac{d}{dt} v(y(t),t) &= \frac{d}{dt} \frac{w_t(\eta(t),t)}{\sqrt{1+|\nabla_\xi w(\eta(t),t)|^2}} \\
&= \frac{1}{1+|\nabla_\xi w(\eta(t),t)|^2}\Big(\frac{d}{dt}(w_t(\eta(t),t))\sqrt{1+|\nabla_\xi w(\eta(t),t)|^2} \\
&\quad - w_t(\eta(t),t)\frac{d}{dt}\sqrt{1+|\nabla_\xi w(\eta(t),t)|^2}\Big) \\
&= \frac{1}{\sqrt{1+|\nabla_\xi w(\eta(t),t)|^2}}\Big(\frac{d}{dt}(w_t(\eta(t),t))\Big) \\
&\quad - \frac{1}{1+|\nabla_\xi w(\eta(t),t)|^2}\Big(w_t(\eta(t),t)\frac{d}{dt}\sqrt{1+|\nabla_\xi w(\eta(t),t)|^2}\Big)
\end{aligned} \tag{5.1.5}$$

Before continue to a calculation after (5.1.5), we will calculate the following:

$$\begin{aligned}
\frac{d}{dt}(w_t(\eta(t),t)) &= \nabla_\xi w_t(\eta(t),t)\cdot\eta'(t) + w_{tt}(\eta(t),t) \\
&= \frac{-w_t(\eta(t),t)\nabla_\xi w(\eta(t),t)\cdot\nabla_\xi w_t(\eta(t),t)}{1+|\nabla_\xi w(\eta(t),t)|^2} + w_{tt}(\eta(t),t),
\end{aligned} \tag{5.1.6}$$

$$
\begin{aligned}
\frac{d}{dt}\sqrt{1+|\nabla_\xi w(\eta(t),t)|^2} &= \frac{1}{\sqrt{1+|\nabla_\xi w(\eta(t),t)|^2}}\Big\{\nabla_\xi w(\eta(t),t)\cdot\frac{d}{dt}(\nabla_\xi w(\eta(t),t))\Big\}\\
&= \frac{1}{\sqrt{1+|\nabla_\xi w(\eta(t),t)|^2}}\times\\
&\quad \Big\{\nabla w(\eta(t),t)\cdot(\nabla_\xi^2 w(\eta(t),t)\eta'(t)+\nabla_\xi w_t(\eta(t),t))\Big\}\\
&= \frac{\nabla_\xi w(\eta(t),t)\cdot(\nabla_\xi^2 w(\eta(t),t)\eta'(t))}{\sqrt{1+|\nabla_\xi w(\eta(t),t)|^2}}+\frac{\nabla_\xi w(\eta(t),t)\cdot\nabla_\xi w_t(\eta(t),t)}{\sqrt{1+|\nabla_\xi w(\eta(t),t)|^2}}\\
&= \frac{-w_t(\eta(t),t)\nabla_\xi w(\eta(t),t)\cdot(\nabla_\xi^2 w(\eta(t),t)\nabla_\xi w(\eta(t),t))}{(1+|\nabla_\xi w(\eta(t),t)|^2)\sqrt{1+|\nabla_\xi w(\eta(t),t)|^2}}\\
&\quad +\frac{\nabla_\xi w(\eta(t),t)\cdot\nabla_\xi w_t(\eta(t),t)}{\sqrt{1+|\nabla_\xi w(\eta(t),t)|^2}} \qquad (5.1.7)
\end{aligned}
$$

where $\nabla_\xi^2 w$ denotes the Hessian matrix of w and we used (5.1.4). Since

$$x_0 = y(t_0) = (\eta(t_0), w(\eta(t_0),t_0)) = (\xi_0, w(\xi_0,t_0)),$$

by combining (5.1.3), (5.1.6), (5.1.7), we get

$$D_t v = \frac{w_{tt}}{\sqrt{1+|\nabla w|^2}} + \frac{w_t\nabla w}{(1+|\nabla w|)^{3/2}}\cdot\Big(\frac{w_t\nabla^2 w\nabla w}{1+|\nabla w|^2} - 2\nabla w_t\Big) \qquad (5.1.8)$$

where we omitted the variables (x_0,t_0) in the left hand side, and (ξ_0,t_0) in the right hand side.

Level set representation

Next, let Γ_t be a zero level set of some sufficiently smooth function $u : \mathbb{R}^N\times[0,T)\to\mathbb{R}$, that is

$$\Gamma_t = \{x\in\mathbb{R}^N : u(x,t)=0\}.$$

At first, concerning the normal velocity v and the normal vector by the standard theory of the level set method, we have

$$v(x,t) = \frac{u_t(x,t)}{|\nabla u(x,t)|},\quad \boldsymbol{n}(x,t) = -\frac{\nabla u(x,t)}{|\nabla u(x,t)|}, \qquad (5.1.9)$$

for $x\in\Gamma_t$. We can calculate the $D_t v(x_0,t_0)$ for $x_0\in\Gamma_{t_0}$ in the same spirit of the previous section. By the definition of the time normal derivative and the first identity of (6.4.1),

$$D_t v(x_0,t_0) = \frac{d}{dt}v(y(t),t)\Big|_{t=t_0} \qquad (5.1.10)$$

where y is the normal trajectory through (x_0, t_0) i.e. $y'(t) = v(y(t), t)\boldsymbol{n}(y(t), t)$. We then calculate

$$\begin{aligned}
\frac{d}{dt}v(y(t),t) &= \frac{d}{dt}\frac{u_t(y(t),t)}{|\nabla u(y(t),t)|} \\
&= \frac{1}{|\nabla u(y(t),t)|^2}\Big(\frac{d}{dt}(u_t(y(t),t))|\nabla u(y(t),t)| - u_t(y(t),t)\frac{d}{dt}|\nabla u(y(t),t)|\Big) \\
&= \frac{1}{|\nabla u(y(t),t)|}\Big(\frac{d}{dt}(u_t(y(t),t))\Big) - \frac{1}{|\nabla u(y(t),t)|^2}\Big(u_t(y(t),t)\frac{d}{dt}|\nabla u(y(t),t)|\Big) \qquad (5.1.11)
\end{aligned}$$

Before continue to a calculation after (5.1.11), we will calculate the following:

$$\frac{d}{dt}(u_t(y(t),t)) = \nabla u_t(y(t),t)\cdot y'(t) + u_{tt}(y(t),t), \tag{5.1.12}$$

$$\begin{aligned}
\frac{d}{dt}|\nabla u(y(t),t)| &= \frac{1}{|\nabla u(y(t),t)|}\Big\{\nabla u(y(t),t)\cdot\frac{d}{dt}(\nabla u(y(t),t))\Big\} \\
&= \frac{1}{|\nabla u(y(t),t)|}\Big\{\nabla u(y(t),t)\cdot(\nabla^2 u(y(t),t)y'(t) + \nabla u_t(y(t),t))\Big\} \\
&= \frac{\nabla u(y(t),t)\cdot(\nabla^2 u(y(t),t)y'(t))}{|\nabla u(y(t),t)|} + \frac{\nabla u(y(t),t)\cdot\nabla u_t(y(t),t)}{|\nabla u(y(t),t)|}, \qquad (5.1.13)
\end{aligned}$$

where $\nabla^2 u$ denotes the Hessian matrix of u. By combining (5.1.11), (5.1.12) and (5.1.13), we get

$$\begin{aligned}
\frac{d}{dt}v(y(t),t) &= \frac{1}{|\nabla u(y(t),t)|}\Big(\nabla u_t(y(t),t)\cdot y'(t) + u_{tt}(y(t),t)\Big) \\
&- \frac{1}{|\nabla u(y(t),t)|^2}\Big\{u_t(y(t),t)\Big(\frac{\nabla u(y(t),t)\cdot(\nabla^2 u(y(t),t)y'(t))}{|\nabla u(y(t),t)|} + \frac{\nabla u(y(t),t)\cdot\nabla u_t(y(t),t)}{|\nabla u(y(t),t)|}\Big)\Big\} \\
&= \frac{u_{tt}(y(t),t)}{|\nabla u(y(t),t)|} - \frac{u_t(y(t),t)\nabla u(y(t),t)\cdot(\nabla^2 u(y(t),t)y'(t))}{|\nabla u(y(t),t)|^3} - \frac{2u_t(y(t),t)\nabla u(y(t),t)\cdot\nabla u_t(y(t),t)}{|\nabla u(y(t),t)|^3} \\
&= \frac{u_{tt}(y(t),t)}{|\nabla u(y(t),t)|} + \frac{u_t(y(t),t)^2\nabla u(y(t),t)\cdot(\nabla^2 u(y(t),t)\nabla u(y(t),t))}{|\nabla u(y(t),t)|^5} \\
&\quad - \frac{2u_t(y(t),t))\nabla u(y(t),t)\cdot\nabla u_t(y(t),t)}{|\nabla u(y(t),t)|^3} \qquad (5.1.14)
\end{aligned}$$

The second, and last equality follows from the normality of y, that is $y'(t) = v(y(t), t)\boldsymbol{n}(y(t), t)$ and (6.4.1). By (5.1.10), (5.1.14), we have:

$$\begin{aligned}
D_t v(x_0,t_0) &= \frac{u_{tt}(x_0,t_0)}{|\nabla u(x_0,t_0)|} + \frac{u_t(x_0,t_0)^2\nabla u(x_0,t_0)\cdot(\nabla^2 u(x_0,t_0)\nabla u(x_0,t_0))}{|\nabla u(x_0,t_0)|^5} \\
&\quad - \frac{2u_t(x_0,t_0))\nabla u(x_0,t_0)\cdot\nabla u_t(x_0,t_0)}{|\nabla u(x_0,t_0)|^3}
\end{aligned}$$

Omitting the variables (x_0, t_0),

$$D_t v = \frac{u_{tt}}{|\nabla u|} + \frac{u_t^2\nabla u\cdot(\nabla^2 u\nabla u)}{|\nabla u|^5} - \frac{2u_t\nabla u\cdot\nabla u_t}{|\nabla u|^3} \tag{5.1.15}$$

5.2 Variational formula of acceleration

We consider the weak formulation of the acceleration $D_t v$. For the purpose, firstly we recall that the normal velocity is characterized via the variation of weighted surface area, that is,

$$\frac{d}{dt}\int_{\Gamma_t} \phi \, d\mathscr{H}^{N-1} = \int_{\Gamma_t} v(\nabla\phi \cdot \boldsymbol{n}) + \phi_t - \phi v \kappa \, d\mathscr{H}^{N-1},$$

for $\phi \in C_c^1(\mathbb{R}^N \times (0,\infty))$ under the assumption that Γ_t is compact for all $t \geq 0$. This is one characterization of the normal velocity given by Brakke [20]. We can find other characterization based on the distribution for the sets of finite perimeter by Luckhaus–Sturzenhecker [63] and Mugnai–Seis–Spadaro [66]. Here, following Brakke's idea, we will try to characterize the acceleration via the time derivative of the integral quantity on the surface.

We now consider the time derivative of

$$\int_{\Gamma_t} v \, \phi \, d\mathscr{H}^{N-1} \quad \text{for} \;\; \phi \in C_c^1(\mathbb{R}^N \times (0,\infty) : \mathbb{R}_{\geq 0}).$$

This quantity can be regarded as the weighted normal momentum of Γ_t. To un-change the sign of this quantity, we restrict the codomain of the test function ϕ to the set of the non-negative number $\mathbb{R}_{\geq 0}$. Then, we can get the following variational identity.

PROPOSITION 5.2.1 (Variational identity for acceleration). Let $\mathcal{M} = \bigcup_{t\in\mathcal{I}} \Gamma_t \times \{t\}$ be a moving hypersurface in $\mathbb{R}^N$. Assume that Γ_t is compact for all $t \geq 0$, $v \in C^1(\mathcal{M})$, put $a := D_t v$. Then, for $\phi \in C_c^1(\mathbb{R}^N \times (0,T) : \mathbb{R}_{\geq 0})$,

$$\frac{d}{dt}\int_{\Gamma_t} v\phi \, d\mathscr{H}^{N-1} = \int_{\Gamma_t} a\,\phi + v(v\nabla\phi \cdot \boldsymbol{n} + \phi_t) - \phi v^2 \kappa \, d\mathscr{H}^{N-1},$$

or equivalently,

$$\int_{\Gamma_t} v\phi \, d\mathscr{H}^{N-1}\bigg|_{t=t_1}^{t_2} = \int_{t_1}^{t_2}\int_{\Gamma_t} a\,\phi + v(v\nabla\phi \cdot \boldsymbol{n} + \phi_t) - \phi v^2 \kappa \, d\mathscr{H}^{N-1}\, dt, \tag{5.2.1}$$

for $t_1, t_2 \in [0,T)$ with $t_1 \leq t_2$.

Proof By extending ϕ with $\phi(\cdot,t) := 0$ for $t \in (-\infty,0] \cup [T,\infty)$, we have $\phi \in C^1(\mathbb{R}^{N+1})$, especially $\phi \in C^1(\mathcal{M})$. Thus, we can apply the transport identity (4.1.4) for $v\phi \in C^1(\mathcal{M})$ and the derivative formula (4.1.3) to conclude the proof. □

Observing the equality (5.2.1) in Proposition 5.2.1, for this equality (5.2.1) to be meaningful, it is sufficient that the normal velocity and the mean curvature are locally 3-th power integrable with respect to $\mathscr{H}^{N-1}\lfloor_{\Gamma_t}\times dt$. Here, we say that the function $f = f(x,t)$ $(x,t) \in \mathcal{M}$, is locally p-th power integrable with respect to $\mathscr{H}^{N-1}\lfloor_{\Gamma_t}\times dt$ if for any compact set $K \subset \mathbb{R}^{N+1}$,

$$\int_{K\cap\mathcal{M}} |f|^p \, d\mathscr{H}^{N-1}\, dt < \infty.$$

In fact, if so, by applying Hölder inequality with $p = 3/2$. $q = 3$, to get

$$\begin{aligned}\int_{t_1}^{t_2}\int_{\Gamma_t} \phi v^2\kappa \, d\mathscr{H}^{N-1}\, dt &\leq L\int_{\text{spt }\phi\cap\mathcal{M}} |v^2\kappa| \, d\mathscr{H}^{N-1}\, dt \\ &\leq L\Big(\int_{\text{spt }\phi\cap\mathcal{M}} |v^2|^p \, d\mathscr{H}^{N-1}\, dt\Big)^{\frac{1}{p}}\Big(\int_{\text{spt }\phi\cap\mathcal{M}} |\kappa|^q \, d\mathscr{H}^{N-1}\, dt\Big)^{\frac{1}{q}} \\ &= L\Big(\int_{\text{spt }\phi\cap\mathcal{M}} |v|^3 \, d\mathscr{H}^{N-1}\, dt\Big)^{\frac{2}{3}}\Big(\int_{\text{spt }\phi\cap\mathcal{M}} |\kappa|^3 \, d\mathscr{H}^{N-1}\, dt\Big)^{\frac{1}{3}},\end{aligned}$$

where $L := \sup_{\mathbb{R}^N\times[0,T)}|\phi| < \infty$. Also, other quantities

$$\int_{t_1}^{t_2}\int_{\Gamma_t} v\, a\, \phi \, d\mathscr{H}^{N-1}\, dt, \quad \int_{t_1}^{t_2}\int_{\Gamma_t} v(v\nabla\phi\cdot\boldsymbol{n} + \phi_t)\, d\mathscr{H}^{N-1}\, dt$$

are finite under these assumptions and additional assumption, $a = D_t v$ is locally integrable.

By using this formula (5.2.1), we can characterize the acceleration $D_t v$ in the following sense. Suppose that a function $\tilde{a} : \mathcal{M} \to \mathbb{R}$ satisfies the following identity,

$$\int_{\Gamma_t} v\phi \, d\mathscr{H}^{N-1}\Big|_{t=t_1}^{t_2} = \int_{t_1}^{t_2}\int_{\Gamma_t} \tilde{a}\,\phi + v(v\nabla\phi\cdot\boldsymbol{n} + \phi_t) - \phi v^2\kappa \, d\mathscr{H}^{N-1}\, dt,$$

by combining with (5.2.1), we have

$$\int_{t_1}^{t_2}\int_{\Gamma_t} (\tilde{a} - D_t v)\phi \, d\mathscr{H}^{N-1}\, dt = 0$$

for all test functions $\phi \in C_c^1(\mathbb{R}^N \times (0,T) : \mathbb{R}_{\geq 0})$. Then, we can get

$$\tilde{a} = D_t v \quad \mathscr{H}^{N-1}\times\mathscr{L}^1\text{-a.e. on} \quad \mathcal{M}.$$

5.3 Surface evolution energy and its first variation

In this section, we introduce the surface evolution energy and derive its first variation formula. It directly follows from the calculation of the time derivative of the surface evolution energy. Before deriving this, we define the surface evolution energy for time-dependent surfaces Γ_t.

DEFINITION 5.3.1 (Surface evolution energy). Let $\mathcal{M} = \bigcup_{t\in\mathcal{I}}(\Gamma_t \times \{t\})$ be a moving hypersurface in $\mathbb{R}^N$. We define the surface evolution energy $E(t)$ by a sum of the surface area and the normal kinetic energy, that is,

$$E(t) := \mathscr{H}^{N-1}(\Gamma_t) + \frac{1}{2}\int_{\Gamma_t} v(\cdot,t)^2 \, d\mathscr{H}^{N-1}. \tag{5.3.1}$$

In this stage, we simply consider the surface density as a positive function which means a mass per unit surface area, and consider the case of the constant surface density ρ, especially $\rho \equiv 1$ for simplicity. This quantity (5.3.1) is a corresponding the energy for the wave equation with zero Dirichlet boundary condition, that is,

$$\frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{1}{2}\int_\Omega |u_t|^2\,dx,$$

for function $u : \Omega \times \mathcal{I} \to \mathbb{R}$, $\Omega \subset \mathbb{R}^N$. In the following remark, we see the another physical interpretation of the surface evolution energy.

REMARK 5.3.1 (Physical interpretation of surface evolution energy). We would like to give another physical meaning of the surface evolution energy. Consider the phenomena of shifting acrylic rod by surface tension of a soap film (see Figure 5.1). We set physical parameters, $\sigma > 0$ is a constant surface tension, m is the line density of the acrylic rod, that is, mass per unit length of the acrylic rod. For simplicity, we consider reducing to one dimension setting (Figure 5.2). Set $\ell(t)$ as the position of the acrylic rod with the initial conditions $\ell(0) = \ell_0$, $\ell'(0) = 0$. By Newton's second law, we have

$$m\ell''(t) = -\sigma. \tag{5.3.2}$$

Now, we consider the following energy:

$$E_D(t) := \sigma\mathscr{H}^1(\Gamma_t) + \frac{1}{2}\int_{\Gamma_t} \rho v^2\,d\mathscr{H}^1 + \frac{1}{2}m\ell'(t)^2.$$

If $\sigma = 1$, this quantity $E_D(t)$ is exact the surface evolution energy $E(t)$ except the last term, that is boundary term $\partial\Gamma_t$. We can regard $E_D(t)$ as the surface evolution energy for Dirichlet boundary condition.

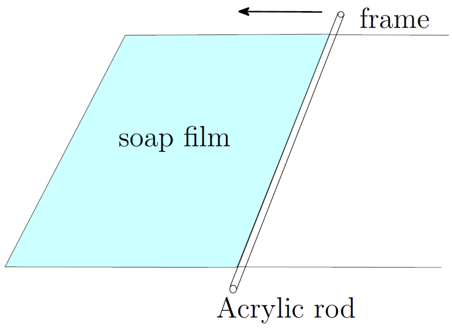


Figure 5.1: Shifting acrylic rod

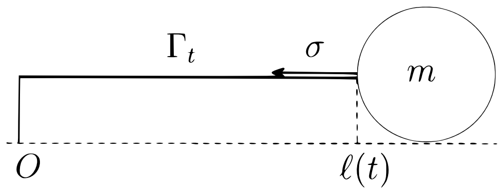


Figure 5.2: One dimension version

Since $\ell'(t) = -(\sigma/m)t < 0$ for $t > 0$ by (5.3.2), the area energy $\sigma\mathscr{H}^1(\Gamma_t) = \sigma\ell(t)$ is decreasing. On the other hand, noting that the normal velocity $v \equiv 0$, we then calculate the time derivative of E_D as follows:

$$\begin{aligned}\frac{d}{dt}E_D(t) &= \frac{d}{dt}\Big(\sigma\ell(t) + \frac{1}{2}m\ell'(t)^2\Big) = \sigma\ell'(t) + m\ell'(t)\ell''(t) = \sigma\ell'(t) - \sigma\ell'(t) \text{ (by (5.3.2))}\\ &= 0\end{aligned}$$

Therefore, the energy $E_D(t)$ is conserved. This real phenomenon shows that the decreasing of area energy is changed to the increasing of kinetic energy on the boundary.

Next, we calculate the first variation formula for the surface evolution energy in the case of compact surfaces.

PROPOSITION 5.3.1 (First variation formula for the surface evolution energy). Let $\mathcal{M} = \bigcup_{t\in\mathcal{I}} \Gamma_t \times \{t\}$ be a moving hypersurface in $\mathbb{R}^N$. Assume that Γ_t is compact for all $t \in \mathcal{I}$, $v \in C^1(\mathcal{M})$, put $a := D_t v$. Then, we have:

$$\frac{d}{dt}\Big(\mathscr{H}^{N-1}(\Gamma_t) + \frac{1}{2}\int_{\Gamma_t} v^2\, d\mathscr{H}^{N-1}\Big) = \int_{\Gamma_t} v(a - \kappa e)\, d\mathscr{H}^{N-1} \tag{5.3.3}$$

where

$$e(x,t) := 1 + \frac{1}{2}v(x,t)^2.$$

Proof. Apply the transport identity (4.1.4) for v^2 to get

$$\frac{d}{dt}\int_{\Gamma_t} v^2\, d\mathscr{H}^{N-1} = \int_{\Gamma_t} (2vD_t v - \rho\kappa v^3)\, d\mathscr{H}^{N-1} \tag{5.3.4}$$

where we used $D_t(v^2) = 2vD_t v$. Combining (4.1.5), (5.3.4), we have

$$\begin{aligned}\frac{d}{dt}E(t) &= \frac{d}{dt}\Big(\mathscr{H}^{N-1}(\Gamma_t) + \frac{1}{2}\int_{\Gamma_t} v^2\, d\mathscr{H}^{N-1}\Big)\\ &= -\int_{\Gamma_t} \kappa v\, d\mathscr{H}^{N-1} + \frac{1}{2}\int_{\Gamma_t}(2vD_t v - \kappa v^3)\, d\mathscr{H}^{N-1}\\ &= \int_{\Gamma_t} v\Big(-\kappa + D_t v - \frac{1}{2}\kappa v^2\Big)\, d\mathscr{H}^{N-1} = \int_{\Gamma_t} v\Big\{D_t v - \Big(1 + \frac{1}{2}v^2\Big)\kappa\Big\}\, d\mathscr{H}^{N-1},\end{aligned}$$

by recalling the definition of a and e, this concludes the proof. □

Moreover, we can generalize this identity by considering the weighted energy as follows.

PROPOSITION 5.3.2 (First variation formula for weighted energy). Under the same assumptions as in Proposition 2.2, for any $f \in C^1(\mathcal{M})$, we have:

$$\frac{d}{dt}\Big(\int_{\Gamma_t} f\, d\mathscr{H}^{N-1} + \frac{1}{2}\int_{\Gamma_t} fv^2\, d\mathscr{H}^{N-1}\Big) = \int_{\Gamma_t} eD_t f + fv(a - \kappa e)\, d\mathscr{H}^{N-1}. \tag{5.3.5}$$

Proof Applying the transport identity (4.1.4) for $fv^2 \in C^1(\mathcal{M})$ to get

$$\frac{d}{dt}\int_{\Gamma_t} fv^2\, d\mathscr{H}^{N-1} = \int_{\Gamma_t} D_t(fv^2) - (fv^2)\kappa v\, d\mathscr{H}^{N-1} = \int_{\Gamma_t} v^2 D_t f + 2fvD_t v - f\kappa v^3\, d\mathscr{H}^{N-1}.$$

After direct calculation, we attain the identity (5.3.5). □

Chapter 6

Energy conserving mean curvature accelerated flow

From this chapter, we consider the following the initial value problem for the energy conserving hyperbolic mean curvature flow equation.

Problem 6.0.1 (Energy conserving mean curvature accelerated flow equation). Let Γ_0 be a given $(N-1)$-dimensional hypersurface in $\mathbb{R}^N$, and v_0 be a given C^1 function on Γ_0. Find a moving surface in $\mathbb{R}^N$, $\mathcal{M} := \bigcup_{t\in\mathcal{I}}(\Gamma_t \times \{t\})$ such that

$$\frac{D_t v}{1+\frac{1}{2}v^2} = \kappa \quad \text{on} \quad \Gamma_t, \tag{6.0.1}$$

$$v(\cdot,0) = v_0(\cdot) \quad \text{on} \quad \Gamma_0. \tag{6.0.2}$$

Firstly, we consider the motivation for this equation from the motion of planner curves.

6.1 Motivation

The motion for planer curves

For the time-dependent planer curves, we can derive the governing equation by calculating the first variation of the action integral up to finite time $T > 0$. We use the notion of section5.1. Define the action integral for $\gamma : (0,\ell) \times (0,T)$ by

$$J(\gamma) := J_K(\gamma) - J_I(\gamma)$$

where

$$J_K(\gamma) := \int_0^T \int_{\gamma(\cdot,t)} \frac{|\gamma_t|^2}{2}\, ds\, dt, \quad J_I(\gamma) := \int_0^T \int_{\gamma(\cdot,t)} ds\, dt.$$

For any the test function $\varphi \in C_c^\infty((0,\ell)\times(0,T) : \mathbb{R}^2)$, we can calculate the first variations of J_I and J_K as follows,

$$\frac{d}{d\varepsilon} J_I(\gamma+\varepsilon\varphi)\bigg|_{\varepsilon=0} = \int_0^T \int_{\gamma(\cdot,t)} \tilde{\kappa}(\varphi\cdot\tilde{n})dsdt \tag{6.1.1}$$

$$\frac{d}{d\varepsilon} J_K(\gamma + \varepsilon\varphi)\Big|_{\varepsilon=0}$$
$$= -\int_0^T \int_{\gamma(\cdot,t)} \Big\{ \gamma_{tt} + \frac{\tilde{w}'}{|\gamma'|}\gamma_t - \tilde{\kappa}\tilde{v}\gamma_t + \frac{\tilde{e}'}{|\gamma'|}\tilde{\tau} + (\tilde{e}-1)\tilde{\kappa}\tilde{n} \Big\} \cdot \varphi ds dt \qquad (6.1.2)$$

where $\tilde{\tau}$ is the unit tangent vector, the notion of prime $'$ means the spatial derivative. For (6.1.1), it follows from the direct and simple calculation. Let us explain the derivation for (6.1.2) since this calculation is also direct, but complicated a little. Now, we calculate as follows.

$$\begin{aligned}
\frac{d}{d\varepsilon} J_K(\gamma + \varepsilon\varphi)\Big|_{\varepsilon=0} &= \frac{d}{d\varepsilon} \int_0^T \int_{\gamma(\cdot,t)} \frac{1}{2}|\gamma_t + \varepsilon\varphi_t|^2 \, ds \, dt \Big|_{\varepsilon=0} \\
&= \frac{d}{d\varepsilon} \int_0^T \int_0^\ell \frac{1}{2}|\gamma_t + \varepsilon\varphi_t|^2 |\gamma' + \varepsilon\varphi'| \, d\vartheta \, dt \Big|_{\varepsilon=0} \\
&= \int_0^T \int_0^\ell \gamma_t \cdot \varphi_t |\gamma'| + \frac{1}{2}|\gamma_t|^2 \frac{\gamma' \cdot \varphi'}{|\gamma'|} \, d\vartheta \, dt \\
&= -\int_0^T \int_0^\ell (|\gamma'|\gamma_t)_t \cdot \varphi + \Big\{ \frac{1}{2}|\gamma_t|^2 \frac{\gamma'}{|\gamma'|} \Big\}' \cdot \varphi \, d\vartheta \, dt
\end{aligned}$$

where the last term follows from the integration by parts. By noting that $\frac{1}{2}|\gamma_t|^2 \frac{\gamma'}{|\gamma'|} = (\tilde{e}-1)\tilde{\tau}$, we continue the calculation.

$$\begin{aligned}
&= -\int_0^T \int_0^\ell \Big\{ |\gamma'|\gamma_{tt} + \frac{\gamma' \cdot \gamma_t'}{|\gamma'|}\gamma_t + \{(\tilde{e}-1)\tilde{\tau}\}' \Big\} \cdot \varphi \, d\vartheta \, dt \\
&= -\int_0^T \int_0^\ell \Big\{ \gamma_{tt} + \frac{\tilde{w}' - \tilde{\kappa}|\gamma'|\tilde{v}}{|\gamma'|}\gamma_t + \frac{1}{|\gamma'|}\Big(\tilde{e}'\tau + (\tilde{e}-1)\tilde{\kappa}|\gamma'|\tilde{n} \Big) \Big\} \cdot \varphi |\gamma'| \, d\vartheta \, dt \\
&= -\int_0^T \int_{\gamma(\cdot,t)} \Big(\gamma_{tt} + \frac{\tilde{w}'}{|\gamma'|}\gamma_t - \tilde{\kappa}\tilde{v}\gamma_t + \frac{\tilde{e}'}{|\gamma'|}\tilde{\tau} + (\tilde{e}-1)\tilde{\kappa}\tilde{n} \Big) \cdot \varphi \, ds \, dt.
\end{aligned}$$

Therefore, by (6.1.1), (6.1.2),

$$\frac{d}{d\varepsilon} J(\gamma + \varepsilon\varphi)\Big|_{\varepsilon=0}$$
$$= -\int_0^T \int_{\gamma(\cdot,t)} \Big\{ \gamma_{tt} + \Big(\frac{\tilde{w}'}{|\gamma'|} - \tilde{\kappa}\tilde{v} \Big)\gamma_t + \frac{\tilde{e}'}{|\gamma'|}\tilde{\tau} + (\tilde{e}-2)\tilde{\kappa}\tilde{n} \Big\} \cdot \varphi \, ds \, dt. \qquad (6.1.3)$$

Thus, since the desired curve γ holds by (6.1.3)

$$\frac{d}{d\varepsilon} J(\gamma + \varepsilon\varphi)\Big|_{\varepsilon=0} = 0,$$

for any $\varphi \in C_c^\infty((0,\ell) \times (0,T) : \mathbb{R}^2)$, we have

$$\gamma_{tt} + \Big(\frac{\tilde{w}'}{|\gamma'|} - \tilde{\kappa}\tilde{v} \Big)\gamma_t + \frac{\tilde{e}'}{|\gamma'|}\tilde{\tau} + (\tilde{e}-2)\tilde{\kappa}\tilde{n} = 0.$$

Recalling $\gamma_t = \tilde{v}\tilde{n} + \tilde{w}\tilde{\tau}, \tilde{e} = \frac{1}{2}|\gamma|^2 + 1$, and if we consider the normal flow, that is, $\tilde{w} \equiv 0$, we have more simple equation

$$\gamma_{tt} - \tilde{e}\tilde{\kappa}\tilde{n} + \frac{\tilde{e}'}{|\gamma'|}\tilde{\tau} = 0. \tag{6.1.4}$$

When we consider this equation for closed curves, we have the following the proposition.

PROPOSITION 6.1.1 (HMCF equation for the closed curves). Let $\gamma : (0, \ell) \times [0, T) \to \mathbb{R}^2$ be a closed curves with zero tangential velocity and a solution of the equation

$$\gamma_{tt} - \tilde{e}\tilde{\kappa}\tilde{n} + \frac{\tilde{e}'}{|\gamma'|}\tilde{\tau} = 0. \tag{6.1.5}$$

Then, we have the following.

(i) A solution of the equation(6.1.5) conserves the following quantity,

$$\int_{\gamma(\cdot,t)} ds + \frac{1}{2}\int_{\gamma(\cdot,t)} |\gamma_t|^2 \, ds.$$

This quantity is exactly the surface evolution energy $E(t)$, which is defined in Section 5.3

(ii) Let $\mathcal{M} = \bigcup_{t\in[0,T)} \gamma(\cdot, t) \times \{t\}$, then $\mathcal{M}$ is a solution of the equation of the following equation:

$$\frac{D_t v}{e} = \kappa.$$

Proof. By multiplying the unit normal vector $\tilde{n}$ to both side of the equation (6.1.5), the solution of the equation (6.1.5) satisfies

$$\gamma_{tt} \cdot \tilde{n} - \tilde{e}\tilde{\kappa} = 0. \tag{6.1.6}$$

(i) Since

$$\frac{d}{dt}\int_{\gamma(\cdot,t)} \tilde{e}\, ds = \int_{\gamma(\cdot,t)} (\tilde{e}ds)_t,$$

we have only to show $(\tilde{e}ds)_t = 0$. By the assumption $\tilde{w} \equiv 0$ and (6.1.6), we have

$$\tilde{e}_t = \frac{d}{dts}\left(\frac{1}{2}|\gamma|^2\right) = \gamma_t \cdot \gamma_{tt} = (\tilde{v}\tilde{n}) \cdot \gamma_{tt} = \tilde{v}(\gamma_{tt} \cdot \tilde{n}) = \tilde{v}\tilde{\kappa}\tilde{e}.$$

Noting that $(ds)_t = |\gamma'|^{-1}(\tilde{w}' - \tilde{\kappa}\tilde{v}|\gamma'|)ds$, to gat

$$(\tilde{e}ds)_t = \tilde{e}_t ds + \tilde{e}(ds)_t = (\tilde{v}\tilde{\kappa}\tilde{e})ds + \tilde{e}(-\tilde{\kappa}\tilde{v})ds = 0.$$

(ii) Since $\tilde{w} \equiv 0$, we saw that $D_t v(\gamma(\vartheta, t), t) = \gamma_{tt}(\vartheta, t)$ as in Section 5.1. Also, by $e(\gamma(\vartheta, t), t) = \tilde{e}(\vartheta, t)$, and $\kappa(\gamma(\vartheta, t), t) = \tilde{\kappa}(\vartheta, t)$, (6.1.6), the conclusion follows. □

Representation of the equation for the moving hypersurfaces

In the previous section, we derive the governing equation for the motion of closed curves in $\mathbb{R}^2$ by the calculate the first variation of the acton integral corresponding the energy $E(t)$. As a result, if the tangential velocity is vanished, the solution of the equation (6.1.6) conserves the surface evolution energy $E(t)$. We point out that these argument is almost directly extended to parametrized hypersurface $\Gamma_t = \{\gamma(\vartheta, t) : \vartheta \in \mathbb{R}^{N-1}\}$.

However, the parametrize expression of the surface gives restriction of the class of surfaces to much, and it is difficult to directly extend to more general surfaces. Thus, let us rewrite the equation (6.1.6) by using the notion of the moving hypersurface. By Proposition 3.1 (ii), we can rearrange as follows:

$$\frac{D_t v}{e} = \kappa \quad \text{on } \mathcal{M} \tag{6.1.7}$$

here, $e := 1 + \frac{1}{2}v^2$, can be regard as the energy density for $E(t)$ and $\mathcal{M}$ is the moving hypersurace consisted by the closed curves, that is, $\mathcal{M} = \bigcup_{t\in\mathcal{I}} \gamma(\cdot, t) \times \{t\}$.

Now, we consider this equation (6.1.7) for general moving hypersurface $\mathcal{M}$ in $\mathbb{R}^N$. Let $\mathcal{M} = \bigcup_{t\in\mathcal{I}} \Gamma_t \times \{t\}$ be a solution of this equation (6.1.7) and we assume that all Γ_t are compact. We can check that the solution $\mathcal{M}$ conserves the surface evolution energy $E(t)$. In fact, by Proposition 2.2., we know that the following first variation formula for the surface evolution energy $E(t)$:

$$\frac{d}{dt}E(t) = \frac{d}{dt}\Big(\mathscr{H}^{N-1}(\Gamma_t) + \frac{1}{2}\int_{\Gamma_t} v^2 \, d\mathscr{H}^{N-1}\Big) = \int_{\Gamma_t} v\Big\{D_t v - \Big(1 + \frac{1}{2}v^2\Big)\kappa\Big\} \, d\mathscr{H}^{N-1} \tag{6.1.8}$$

Observing (6.1.8), it is easy to see that the solution of the equation (6.1.7) conserves the surface evolution energy $E(t)$. On the other hand, the notions of $v, \kappa, D_t v$ can be defined for also not compact surface. Motivated these facts, we call this equation (6.1.7) *the energy conserving hyperbolic mean curvature flow equation, the energy conserving mean curvature accelerated flow equation, or E-HMCF* for shortly.

6.2 Exact solutions

Let us consider some exact solution to Problem 6.0.1. The first non-trivial example is the case of that $\mathcal{M}$ consisted from the $(N-1)$-dimensional sphere $(N \geq 2)$. That is,

$$\mathcal{M} = \bigcup_{t\in\mathcal{I}} (\Gamma_t \times \{t\}), \quad \Gamma_t = \{x \in \mathbb{R}^N \ : \ |x| = R(t)\},$$

where $R \in C^2(\mathcal{I} : \mathbb{R}_{\geq 0})$ with initial conditions $R(0) = r_0 (> 0)$, $\dot{R}(0) = v_0$. Then since

$$D_t v = \ddot{R}, \quad \kappa = -\frac{N-1}{R(t)},$$

we can deduce the equation (6.3.1) to the following second order nonlinear ordinary differential equation:

$$\Big(1 + \frac{1}{2}\dot{R}(t)^2\Big)^{-1} \ddot{R}(t) = -\frac{N-1}{R(t)} \tag{6.2.1}$$

For simplicity, we only consider the cace of $N = 2$ in the following. Then the equation (6.2.1) becomes

$$\ddot{R}(t) = -\Big(1 + \frac{1}{2}\dot{R}(t)^2\Big)\frac{1}{R(t)}. \tag{6.2.2}$$

This equation(6.2.2) can be reduced to the Bernoulli type ordinary differential equation by considering the change of variables $\nu(R) := \dot{R}$. After solving this the Bernoulli type ordinary differential equation, we get the following first order ordinary differential equation.

$$\dot{R}(t) = \pm\sqrt{\frac{2\{r_0(1 + \frac{1}{2}v_0^2) - R(t)\}}{R(t)}} \tag{6.2.3}$$

Since now we consider $\dot{R}$ is a real value, the range of $R(t)$ should be $0 \leq R(t) \leq r_0(1+\frac{1}{2}v_0^2)$. We also point out that the equation (6.2.3) can be solved by separation of variables. More precisely, after rewriting

$$\frac{RdR}{\sqrt{R\{r_0(1 + \frac{1}{2}v_0^2) - R\}}} = \pm\sqrt{2}dt, \tag{6.2.4}$$

we get

$$F(R(t)) = \begin{cases} \sqrt{2}t + C_1 & \text{if } \dot{R}(t) \geq 0 \\ -\sqrt{2}t + C_2 & \text{if } \dot{R}(t) < 0 \end{cases}$$

where F is the antiderivative of the left hand side of (6.2.4), and C_1, C_2 is a constant, that is

$$F(R) = \int \frac{R}{\sqrt{R\{c - R\}}}\, dR = c\tan^{-1}\Big(\frac{R}{\sqrt{R(c - R)}}\Big) - \sqrt{R(c - R)}.$$

where $c := r_0(1 + \frac{1}{2}v_0^2)$. Thus, we have the following information about the function F.

(I1) F is a differentiable function defined on $[0, r_0(1 + \frac{1}{2}v_0^2))$, $F(0) = 0$.

(I2) $F(R) \to r_0(1 + \frac{1}{2}v_0^2)\frac{\pi}{2}$ as $R \uparrow r_0(1 + \frac{1}{2}v_0^2)$. We define $F(r_0(1 + \frac{1}{2}v_0^2)) := r_0(1 + \frac{1}{2}v_0^2)\frac{\pi}{2}$.

(I3) F is monotone increasing.

In especially, F has the inverse function F^{-1} from (I1), (I3). Now, we can investigate each case. Remark that $\dot{R}(t)$ is decreasing function because $\ddot{R}(t)$ is always negative from the original equation(6.2.2).

Case I ($v_0 > 0$). Since $v_0 = \dot{R}(0) > 0$ and $\dot{R}$ is decreasing, there exists $t_1 > 0$ such that $\dot{R}(t) > 0$ for $t < t_1$, $\dot{R}(t_1) = 0$, and $\dot{R}(t) < 0$ for $t > t_1$. The function $R(t)$ attains the maximum $r_0(1 + \frac{1}{2}v_0^2)$ at $t = t_1$. Thus, the solution $R(t)$ has the following the implicit expression.

$$F(R(t)) = \begin{cases} \sqrt{2}t + C_1 & \text{if } t \in [0, t_1] \\ -\sqrt{2}t + C_2 & \text{if } t \geq t_1 \end{cases}$$

Since $F(r_0) = F(R(0)) = C_1$ and $t_1 = (\sqrt{2})^{-1}(F(R(t_1)) - F(r_0))$, $C_2 = \sqrt{2}t_1 + F(R(t_1)) =$

$2F(R(t_1)) - F(r_0) \geq 0$. We also remark that R is vanishes at $t = t_0$ where

$$t_0 := \frac{2F(R(t_1)) - F(r_0)}{\sqrt{2}} = \frac{2F(r_0(1 + \frac{1}{2}v_0^2)) - F(r_0)}{\sqrt{2}}.$$

Now, recalling that F^{-1} exists, the exact solution is as follows.

$$R(t) = \begin{cases} F^{-1}(\sqrt{2}t + F(r_0)) & \text{if } t \in [0, t_1] \\ F^{-1}(-\sqrt{2}t + 2F(R(t_1)) - F(r_0)) & \text{if } t \in [t_1, t_0] \end{cases}$$

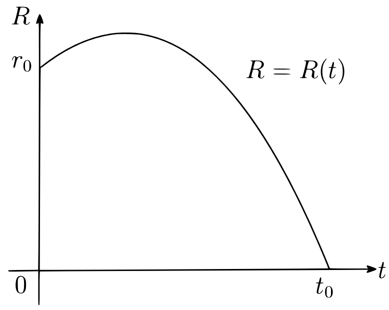


Figure 6.1: $R(t)$ in the case of $v_0 > 0$

Case II. ($v_0 \leq 0$) Since $v_0 = \dot{R}(0) \leq 0$, $\dot{R}$ is decreasing, $\dot{R}(t) < 0$ for $t > 0$. Thus, the solution $R(t)$ has the following the implicit expression.

$$F(R(t)) = -\sqrt{2}t + C_2$$

Since $C_2 = F(R(0)) = F(r_0)$, $R(t)$ vanishes at $t = t_0 := (\sqrt{2})^{-1}F(r_0)$, the exact solution is as follows.

$$R(t) = F^{-1}(-\sqrt{2}t + F(r_0)) \quad t \in [0, t_0]$$

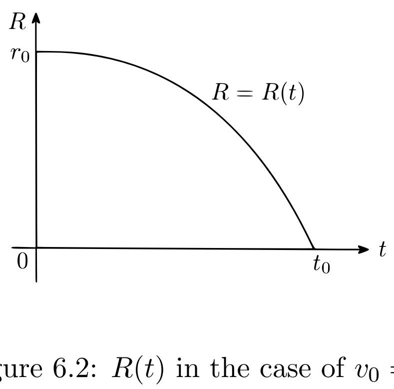


Figure 6.2: $R(t)$ in the case of $v_0 = 0$

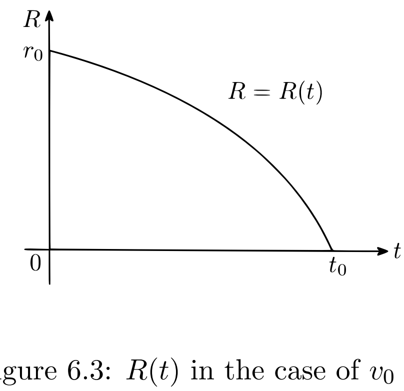


Figure 6.3: $R(t)$ in the case of $v_0 < 0$

In each of the cases, the circle Γ_t finally shrink to a point at the finite time t_0, which is depending only on the initial data r_0 and v_0, even if the surface evolution energy $E(t)$ conserves.

As the second example, we can also consider an another exact solution in the case of the axially symmetric solution for Problem 6.0.1. Let Γ_t be a infinitely long cylinder (see Figure 6.4), that is,

$$\Gamma_t = \{(R(t)\cos\alpha, R(t)\sin\alpha, \rho) \in \mathbb{R}^3 \ : \ \alpha \in [0, 2\pi], \rho \in \mathbb{R}\}.$$

Since the mean curvature of this Γ_t is equal to $-\frac{1}{R(t)}$, the equation (6.3.1) can be reduced the same equation with (6.2.2). Therefore, The cylinder Γ_t shrink to a line as Figure 6.5.

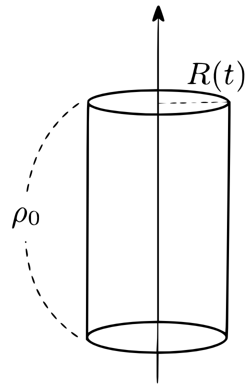


Figure 6.4: The cylinder in $\mathbb{R}^3$ with arbitrary length $\rho_0 > 0$

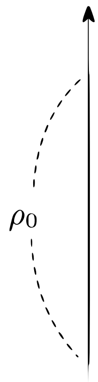


Figure 6.5: Shrink to line with length ρ_0

We also remark that these solutions are only time local solution for Problem 6.0.1, that is, we have to restrict the final time $T > 0$ to finite, that is, $T = t_0$. In the next subsection, we will compare with the E-HMCF equation and the original HMCF equation $D_t v = \kappa$ in the case of a circle.

Comparing with the original HMCF equation

In this subsection, we compare with the E-HMCF equation $(1 + \frac{1}{2}v^2)^{-1} D_t v = \kappa$ and the original HMCF equation $D_t v = \kappa$ in the case of circle from some viewpoints. Throughout this subsection, we consider the zero initial velocity i.e. $v_0 = 0$. At first, we compare the extinction time for two equations. In the case of circle, that is $\Gamma_t = \{x \in \mathbb{R}^2 \ : \ |x| = R(t)\}$, the initial value problem for the original HMCF equation $D_t v = \kappa$ is rewritten by

$$\ddot{R}(t) = -\frac{1}{R(t)}, \quad R(0) = r_0, \quad R'(0) = v_0 = 0 \tag{6.2.5}$$

The exact solution for (6.2.5) is calculated in [36] as follows:

$$R(t) = r_0 \exp\Big[-\Big\{\mathrm{erf}^{-1}\Big(\sqrt{\frac{\pi}{2}}\frac{t}{r_0}\Big)\Big\}^2\Big]$$

where erf is the error function. Since $\mathrm{erf}^{-1}(x) \to \infty$ as $x \to 1-$, we can calculate the extinction time t_{e_1} of the solution for the equation (6.2.5), that is, $t_{e_1} := r_0\sqrt{\pi/2}$. On the other hand, in the case of E-HMCF equation (6.2.2), as calculated in Section 4.1, the extinction time t_{e_2} ($= t_0$ in

Section 4.1) is as follows:

$$t_{e_2} = r_0 \frac{\pi}{2\sqrt{2}}.$$

By the inequality $2\sqrt{x} > x$ for $x \in (0, 4)$, we have

$$t_{e_2} = r_0 \frac{\pi}{2\sqrt{2}} < r_0 \sqrt{\frac{\pi}{2}} = t_{e_1}.$$

Thus, in the case o the circle, the solution of E-HMCF vanishes faster than of HMCF. We can also observe this fact by the graph of two solutions(Figure 6.6).

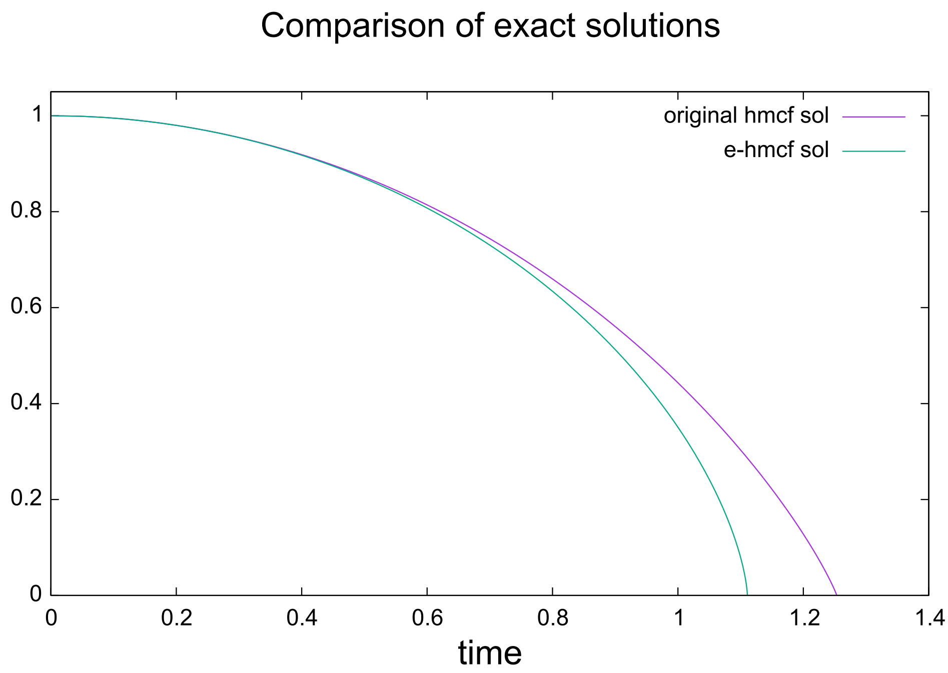


Figure 6.6: Comparison the exact solutions of E-HMCF equation and HMCF equation in the case of circle with $r_0 = 1$.

Next, we see the behavior of the surface evolution energy. Here, the surface evolution energy for the circles is,

$$E(t) := \mathscr{H}^1(\Gamma_t) + \frac{1}{2} \int_{\Gamma_t} v^2 \, d\mathscr{H}^1 = 2\pi R(t) + \pi R(t)\dot{R}(t).$$

By the structure of the equations, the both of two cases, we have $\ddot{R} < 0$, thus $\dot{R}$ is decreasing function with $R(t) < 0$ for $t > 0$ since $\dot{R}(0) = 0$. In the case of the original HMCF equation,

$$\begin{aligned} \frac{d}{dt}E(t) &= 2\pi\dot{R}(t) + \pi(\dot{R}(t)^3 + 2R(t)\dot{R}(t)\ddot{R}(t)) = \pi\dot{R}(t)^3 \quad (\text{by } (6.2.5)) \\ &< 0 \quad (\text{by } \dot{R}(t) < 0) \end{aligned}$$

So, the surface evolution energy decreases in the original HMCF. On the other hand, as mention in Section 4.1, in the E-HMCF, the surface evolution energy $E(t)$ is preserved.

6.3 Graph expression

We consider the graph solution of the following energy conserving hyperbolic mean curvature flow equation:

$$\frac{D_t v}{1+\frac{1}{2}v^2} = \kappa \quad \text{on} \quad \Gamma_t \tag{6.3.1}$$

where Γ_t is the time depending hypersurface in $\mathbb{R}^N$, v is the normal velocity, $D_t v$ is the normal time derivative of v, κ is the mean curvature. Assume that Γ_t is represented

$$\Gamma_t = \{(\xi, w(\xi,t)) \ : \ \xi \in U \subset \mathbb{R}^{N-1}\}$$

for some function $w : U \times (0,T) \to \mathbb{R}$. Then, the normal derivative v, the unit normal vector and the mean curvature are given by

$$v(x,t) = \frac{w_t(\xi,t)}{\sqrt{1+|\nabla_\xi w(\xi,t)|^2}}, \quad \boldsymbol{n}(x,t) = \frac{(-\nabla_\xi w(\xi,t),1)}{\sqrt{1+|\nabla_\xi w(\xi,t)|^2}}, \quad \kappa(x,t) = \mathrm{div}_\xi \Big(\frac{\nabla_\xi w(\xi,t)}{\sqrt{1+|\nabla_\xi w(\xi,t)|^2}} \Big), \tag{6.3.2}$$

for $x = (\xi, u(\xi,t)) \in \Gamma_t$. On the other hand, we know the graph representation for $D_t v(x_0,t_0)$ for $x_0 \in \Gamma_{t_0}$ by (5.1.8) in Section 5.1.

$$D_t v = \frac{w_{tt}}{\sqrt{1+|\nabla w|^2}} + \frac{w_t \nabla w}{(1+|\nabla w|)^{3/2}} \cdot \Big(\frac{w_t \nabla^2 w \nabla w}{1+|\nabla w|^2} - 2\nabla w_t \Big)$$

where we omitted the variables (ξ, t_0). Therefore, the graph representation of E-HMCF equation (6.3.1) becomes

$$\frac{w_{tt}}{\sqrt{1+|\nabla w|^2}} + \frac{w_t \nabla w}{(1+|\nabla w|)^{3/2}} \cdot \Big(\frac{w_t \nabla^2 w \nabla w}{1+|\nabla w|^2} - 2\nabla w_t \Big) = \Big(1 + \frac{w_t^2}{2(1+|\nabla w|^2)}\Big) \mathrm{div} \Big(\frac{\nabla w}{\sqrt{1+|\nabla w|^2}} \Big). \tag{6.3.3}$$

We remark that the equation (6.3.3) coincides with the following LeFloch, and Smoczyk's equation which appears in [61, Section 5], up to the coefficient of mean curvature part.

$$\frac{w_{tt}}{\sqrt{1+|\nabla w|^2}} + \frac{w_t \nabla w}{(1+|\nabla w|)^{3/2}} \cdot \Big(\frac{w_t \nabla^2 w \nabla w}{1+|\nabla w|^2} - 2\nabla w_t \Big) = \Big(\frac{N-1}{2} + \frac{w_t^2}{2(1+|\nabla w|^2)} \Big) \mathrm{div} \Big(\frac{\nabla w}{\sqrt{1+|\nabla w|^2}} \Big).$$

The equation (6.3.3) can be also derived by calculating the first variation of the following action integral:

$$\int_0^T \int_U \Big(\frac{w_t^2}{2\sqrt{1+|\nabla w|^2}} - \sqrt{1+|\nabla w|^2} \Big) \, dx \, dt.$$

In particular, for one-dimension i.e. $U \subset \mathbb{R}$, the equation (6.3.3) becomes

$$w_{tt} = \frac{1}{2}\Big(2 + \frac{w_t^2}{1+w_x^2}\Big) \frac{w_{xx}}{1+w_x^2} + \frac{2w_x w_t w_{xt}}{1+w_x^2} - \frac{w_x^2 w_t^2 w_{xx}}{(1+w_x^2)^2}. \tag{6.3.4}$$

Let us check that the equation (6.3.4) is hyperbolic. Because, rearranging the equation (6.3.4),

$$w_{tt} = \frac{2(1+w_x^2) + w_t^2 - 2w_x^2 w_t^2}{2(1+w_x^2)} w_{xx} + 2\frac{w_x w_t}{1+w_x^2} w_{xt}, \tag{6.3.5}$$

from this form, we have

$$\det \begin{pmatrix} -1 & \dfrac{w_x w_t}{1+w_x^2} \\ \dfrac{w_x w_t}{1+w_x^2} & \dfrac{2(1+w_x^2) + w_t^2 - 2w_x^2 w_t^2}{2(1+w_x^2)} \end{pmatrix} = \frac{-2(1+w_x^2) - w_t^2}{2(1+w_x^2)^2} = \frac{-\Big(1 + \dfrac{w_t^2}{2(1+w_x^2)}\Big)}{1+w_x^2} < 0.$$

Now, we define the weak solution of the equation (6.3.4) in the sense of distribution.

DEFINITION 6.3.1 (weak solution for 1-dimension graph). A Lipschitz function $w : U \times [0,T) \to \mathbb{R}$ is called a weak solution of (6.3.4), if and only if for any $\varphi \in C_c^\infty(U \times [0,T))$,

$$\int_0^T \int_U \Big(w_t \varphi_t - \frac{1}{2}\Big(\frac{w_t^2}{1+w_x^2} + 2\Big) w_x \varphi_x \Big) \frac{1}{\sqrt{1+w_x^2}}\, dx\, dt = 0$$

For whole domain, that is, $U = \mathbb{R}$, we expect the following the existence result of weak solution by using the general theory of the systems of conservation laws (see [46]) as in [61]: There exists a constant $\delta_0 > 0$ such that given any initial data $w_0, w_1 : \mathbb{R} \to \mathbb{R}$ such that

$$TV(w_{0,x}) + TV(w_1) < \delta_0,$$

where TV is the total variation, then the initial value problem for (6.3.4) admits a weak solution $w = w(x,t)$ in the sense of Definition 2.1.

To apply the theory of [46], we transform the equation (6.3.4) to the system of conservation laws. For the purpose, we introduce two variables a and b as follows:

$$a := \frac{w_t}{\sqrt{1+w_x^2}}, \quad b := w_x.$$

Then, by using the equation (6.3.5), the direct calculation allow us to find that two functions a, b are the conservative quantities, that is, these are solution for the system of the conservation laws:

$$\begin{aligned} a_t &= \Big(\frac{(2+a^2)b}{2\sqrt{1+b^2}}\Big)_x \\ b_t &= (a\sqrt{1+b^2})_x \end{aligned}$$

6.4 Numerical approach

In this subsection, we consider the numerical approach for Problem 6.0.1 by using the hyperbolic Meriiman-Bence-Osher (we call the hyperbolic MBO shortly) algorithm. In general, the MBO algorithm is based on the level set method, that is, we express the hypersurface as the level set of some function, and we solve the partial differential equation, so-called the level set equation, which corresponds to the original surface evolution equation. So, firstly we need the level set equation for the energy conserving hyperbolic mean curvature flow equation (6.3.1).

Level set equation for E-HMCF

Let us rewrite the energy conserving hyperbolic mean curvature flow equation (6.3.1) by using the level set function. Let Γ_t be a zero level set of some sufficiently smooth function $u : \mathbb{R}^N \times [0,\infty) \to \mathbb{R}$, that is

$$\Gamma_t = \{x \in \mathbb{R}^N \,:\, u(x,t) = 0\}.$$

We recall that

$$v(x,t) = \frac{u_t(x,t)}{|\nabla u(x,t)|}, \quad \boldsymbol{n}(x,t) = -\frac{\nabla u(x,t)}{|\nabla u(x,t)|}, \quad \kappa(x,t) = \mathrm{div}\left(\frac{\nabla u(x,t)}{|\nabla u(x,t)|}\right), \tag{6.4.1}$$

for $x \in \Gamma_t$. By the representation of $D_t v$ as (5.1.15) in section 2.3, we have the level set equation of the hyperbolic mean curvature flow equation (6.3.1) as follows:

$$\frac{u_{tt}}{|\nabla u|} + \frac{u_t^2 \nabla u \cdot (\nabla^2 u \nabla u)}{|\nabla u|^5} - \frac{2u_t \nabla u \cdot \nabla u_t}{|\nabla u|^3} = \left(1 + \frac{1}{2}\frac{u_t^2}{|\nabla u|^2}\right) \mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right) \tag{6.4.2}$$

Since the divergence term can be calculated:

$$\mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right) = \frac{\mathrm{Tr}\,(\nabla^2 u)|\nabla u|^2 - (\nabla^2 u \nabla u)\cdot \nabla u}{|\nabla u|^3},$$

where Tr denotes a trace of a matrix, we can combine two terms of u_t^2, then we have:

$$u_{tt} + \frac{1}{2}\left(\frac{3u_t(\nabla^2 u \nabla u)\cdot \nabla u}{|\nabla u|^2} - u_t\,\mathrm{Tr}\,(\nabla^2 u) - 4\nabla u \cdot \nabla u_t\right)\frac{u_t}{|\nabla u|^2} = |\nabla u|\;\mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right) \tag{6.4.3}$$

Unfortunately, it is hopeless to treat numerically this equation (6.4.3) because of the complicated structure of the equation. Therefore, we would like to change our strategy for this equation to treat numerically.

Numerical treatment for E-HMCF

We consider numerical treatment for the energy conserving hyperbolic mean curvature equation (6.3.1) by the level set method. Our purpose in this subsection is to apply the hyperbolic MBO algorithm which is introduced by Ginder and Svadlenka [36] to the energy conserving mean curvature flow equation (6.3.1). In the MBO algorithm, the surfaces are expressed by the zero-level set of the signed distance function. At first, let us recall a signed distance function for a closed surface.

DEFINITION 6.4.1 (Signed distance function). Let Γ be $(N-1)$-dimensional closed surface in $\mathbb{R}^N$. We assume that there exists two open set $\Omega^+, \Omega^- \subset \mathbb{R}^N$ such that $\mathbb{R}^N = \Omega^+ \cup \Omega^- \cup \Gamma$, and

$\Omega^+ \cap \Omega^- = \emptyset$. Then, we define a function $d : \mathbb{R}^N \to \mathbb{R}$ by

$$d(x) := \begin{cases} \mathrm{dist}\,(x, \Gamma) & x \in \Omega^+ \\ 0 & x \in \Gamma \\ -\,\mathrm{dist}\,(x, \Gamma) & x \in \Omega^- \end{cases}$$

where $\mathrm{dist}\,(x, \Gamma) := \inf_{y \in \Gamma} |x - y|$, the function d is called signed distance function.

The one of the important properties of the singed distance function d, is differentiable almost everywhere, and $|\nabla d| = 1$ where d is differentiable. For time dependent surface Γ_t, we can define $d(x, t)$ in same way. In the BMO algorithm, since we expect $u = d$ we shall consider that

$$|\nabla u(x, t)| = 1 \text{ for } x \in \mathbb{R}^N, t \geq 0. \tag{6.4.4}$$

The point is, this x does not depend on time. Now, differentiating this relation (6.4.4) with respect to time variable t, we get

$$\nabla u(x, t) \cdot \nabla u_t(x, t) = \frac{\nabla u(x, t) \cdot \nabla u_t(x, t)}{|\nabla u(x, t)|} = 0 \tag{6.4.5}$$

Then, by using (6.4.4) and (6.4.5), recalling the normality of trajectory y, we can calculate the time normal derivative of the normal velocity $D_t v$ as follows:

$$\begin{aligned} D_t v &= \frac{d}{dt} v(y(t), t) = \frac{d}{dt}\Big(\frac{u_t(y(t), t)}{|\nabla u(y(t), t)|}\Big) \\ &= \frac{d}{dt}\Big(u_t(y(t), t)\Big) = u_{tt}(y(t), t) + \nabla u_t(y(t), t) \cdot y'(t) \\ &= u_{tt}(y(t), t) - \nabla u_t(y(t), t) \cdot \frac{u_t(y(t), t) \nabla u(y(t), t)}{|\nabla u(y(t), t)|^2} = u_{tt}(y(t), t) \end{aligned}$$

Recalling that (6.4.1), (6.4.5) again, we get the numerical E-HMCF equation (6.3.1) as follows:

$$u_{tt} = \Delta u + \frac{u_t^2}{2} \Delta u = \Big(1 + \frac{u_t^2}{2}\Big) \Delta u. \tag{6.4.6}$$

Now, the hyperbolic MBO algorithm for approximate solution $\{\Gamma_n\}_{n=0}^M$ of the E-HMCF equation is as follows.

HMBO algorithm for E-HMCF

Given : initial hypersurface Γ_0, its normal velocity v_0, a final time $T > 0$, and time discretize size $\tau = T/M$ for the numerical E-HMCF equation (6.4.6) , N and k_0 are positive integers. We will solve the equation (6.4.6) up to $t = k_o \tau$ (see the following Step1, 3). Firstly we extend v_0 to a neighborhood of Γ_0 e.g. using the orthogonal projection on Γ_0.

Step 1. For $t \in [0, k_0 \tau]$ solve the initial value problem:

$$u_{tt}(x, t) = \Big(1 + \frac{1}{2} u_t^2(x, t)\Big) \Delta u(x, t), \quad u(x, 0) = d_0(x), \quad u_t(x, 0) = -v_0(x)$$

where d_0 is the signed distance function for Γ_0.

Step 2. Define Γ_1 as the zero level set $u(\cdot, k_0\tau)$.

Step 3. For $n = 1, ..., N-1$ repeat

Step 3.1 For $t \in [0, k_0\tau]$ solve the initial value problem:

$$u_{tt}(x,t) = \Big(1 + \frac{1}{2}u_t^2(x,t)\Big)\Delta u(x,t), \quad u(x,0) = d_n(x), \quad u_t(x,0) = \frac{d_n(x) - d_{n-1}(x)}{k_0\tau}$$

where d_n is the signed distance function for Γ_n.

Step 3.2 Define Γ_{n+1} as the zero level set $u(\cdot, k_0\tau)$.

In the next subsection, we show some numerical results by using the hyperbolic BMO algorithm for the energy conserving mean curvature flow equation.

Numerical results

We show some numerical examples for Problem 0.0.2 and check that the hyperbolic BMO algorithm works for also the energy conserving mean curvature flow equation(6.3.1). When we discretize the equation (6.4.6), we use central finite difference scheme for $u_{tt}, \Delta u$ as usual, and backward finite difference scheme for u_t. We set the space discretize size $h = 1/100$, and the time discretize size $\tau = 0.5h$ for the wave equation, and we choose $k_0 = 4$. In the Case I and Case II, time is from outside to inside.

Case I. (A family of circles) $r_0 = 0.5$, $v_0 = 0$. Figure 6.7 shows circles is shrinking. This result seem to coincide the exact solution which we calculated in Section 4.1.

Case II. (A family of smoothed squares) $v_0 = 0$. Initial shape is given as the zero level set of the function $u(x,y) = x^{10} + y^{10} - 0.5$. Figure 6.8 shows that curve part becomes flat, and flat part becomes the curve part, finally smoothed squares shrink.

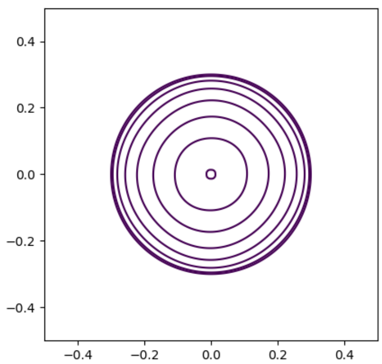


Figure 6.7: Shrinking circles

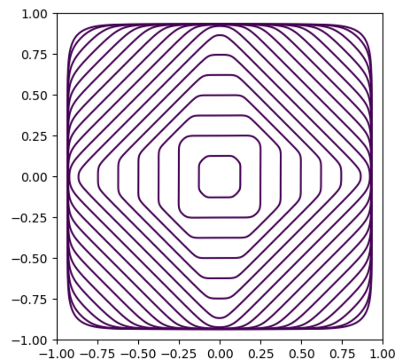


Figure 6.8: Shrinking smoothed squares

Above two examples, we can not observe the oscillation of the interface. Now, we consider non-convex closed curve, so-called, gourd-shaped as the initial shape.

Case III. (Gourd-shape) $v_0 = 0$. Initial shape is given as the zero level set of the function $u(x, y) = 13(x^2 + y^2)^2 - 8x^2 - 0.05$. Comparing with the mean curvature flow motion $v = \kappa$, we can observe the oscillation of the interface.

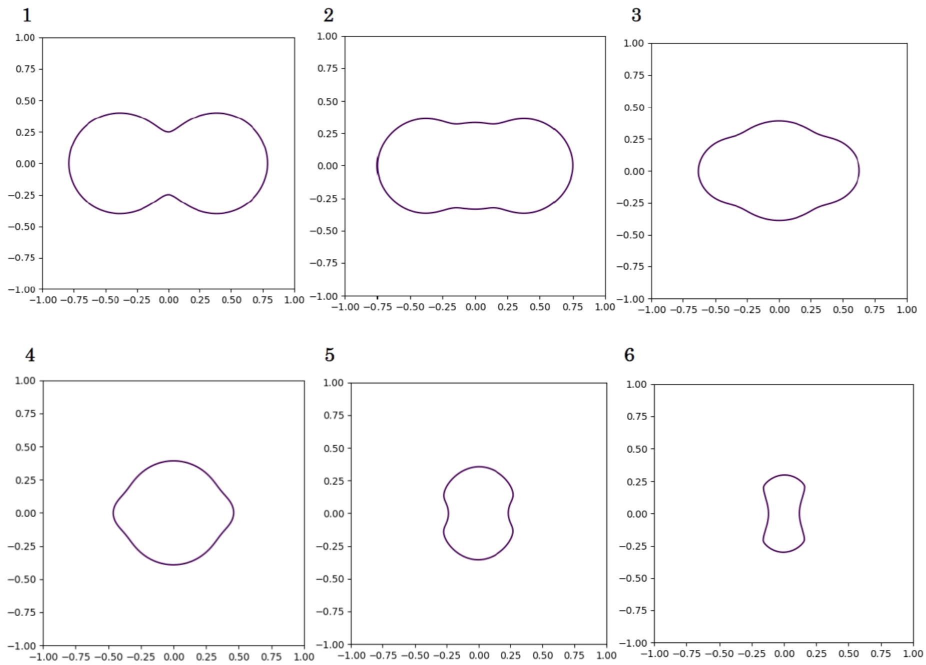


Figure 6.9: Evolution of gourd-shape by E-HMCF. Time is designated by the integer values within the figure (the initial condition corresponds to number 1).

In so far as these numerical results go, unfortunately, it can not be observed the difference between the E-HMCF and the original HMCF even though the circle case. Our numerical method is based on the hyperbolic MBO algorithm developed by Ginder–Svadlenka. In the hyperbolic MBO algorithm for the original HMCF equation, they decided the coefficient and the initial value of the wave equation in each threshold steps very carefully by using the asymptotic analysis. To improve our numerical method and results, it might be helpful to study the equation $u_{tt} = (1 + \frac{1}{2}u_t^2)\Delta u$.

Conclusion

The present thesis is devoted to the following two types of hyperbolic free boundary problems.

[1] Hyperbolic Alt-Caffarelli type free boundary problems.

The existence of weak solutions is proved by minimizing a Crank-Nicolson type functional in the one-dimensional setting. This new functional was shown to preserve the energy correctly both on continuous and discrete levels, which is of significance in numerical simulations. Future tasks include extending our result to higher dimensions and to developing computational methods for investigating the numerical properties of the free boundary problem.

[2] Mean curvature accelerated flow

The acceleration for the moving surface is established by using the notion of the normal time derivative. The variational formula for this acceleration, which is expected the generalization of the notion of acceleration, is also proved by the transport identity. Next, by analogous of the wave equation, the surface evolution energy is introduced as the sum of the surface area and the normal kinetic energy. By using the notion of this acceleration, the energy conserving hyperbolic mean curvature flow equation is derived and its solution preserves the surface evolution energy. Also, some exact solutions, comparison with the original hyperbolic mean curvature flow, and the graph solution of the energy conserving hyperbolic mean curvature flow equation is considered. The hyperbolic MBO algorithm, that is the level set approach, is effective for also the energy conserving hyperbolic mean curvature flow equation in numerical computation for some cases. The future works include the more mathematical and numerical analysis of the energy conserving HMCF equation, and with the volume-preserving condition.

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