On a refined formula of the relative class number of a CM-field

On a Refined Formula of the Relative Class Number of a CM-Field[1]

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Abstract

We will try to generalize the result in [10] to a CM-field in this article. Let p be an odd prime number. Let F be a CM-field which is an abelian extension of a totally real subfield k. We suppose F and k are Galois extensions over $\boldsymbol{Q}$. Let ρ be the complex conjugate map. Let e be an idempotent element of the group ring of the Galois group of F/k over $\boldsymbol{Z}_p$ such that $\rho e = -e$. Our aim is to prove the following formula holds:

$$v_p(\sharp e\boldsymbol{Z}_p \otimes Cl_F) = \sum_{\chi(e)\neq 0} v_p(L(0,\chi)) + \delta_e \cdot v_p(w_F),$$

where Cl_F is the ideal class group of F and where the symbol δ_e takes value 1 if $\omega(e) \neq 0$ for the Teichimüllar character ω, otherwise it takes value 0. We shall generalize the argument in [10] directly. To achieve this aim, the most crucial obstruction is that we do not have a good analogue of the Stickelberger theorem and a p-adic analogue of Stark's conjecture in a special case, *c.f.* Theorem 3.2 in §17, [3]. This problem is not studied here.

Keywords: CM-field, relative class number formula, Iwasawa theory

Introduction

The relative class number h_F^- of a CM-field is described by the value at 0 of the L-function of a totally real number field k. Namely, if F is an abelian extension of k with degree m over the field $\boldsymbol{Q}$ of rational numbers, we have a formula

$$h_F^- = \pm Q_F w_F \prod_{\chi:\text{odd}} \frac{L(0,\chi)}{2^m},$$

where w_F is the number of roots of unity and Q_F is the unit index. Let G be the Galois group of F over k. It contains a unique complex conjugation map ρ. A character χ of G is called odd if $\chi(\rho) = -1$.

We denote by p an odd prime number in this article. We focus our thoughts on the p-part of h_F^-. Let v_p be the p-adic valuation normalized by $v_p(p) = 1$. We take an algebraic closure of the field $\boldsymbol{Q}_p$ of p-adic numbers and denote by $\boldsymbol{C}_p$ the completion of an algebraic closure of $\boldsymbol{Q}_p$ by the norm defined by the valuation. We extend v_p on $\boldsymbol{C}_p$. Let $\boldsymbol{Z}_p$ be the ring of p-adic integers. We have an idempotent element $e^- = (1-\rho)/2$ in the group ring of $\boldsymbol{Z}_pG$ of G over $\boldsymbol{Z}_p$. Then, the above formula is transformed into a p-adic formula:

$$v_p(h^-) = v_p(w_F) + \sum_{\chi(e^-)\neq 0} v_p(L(0,\chi)),$$

where the sum is taken with respect to characters χ such that $\chi(e^-) = (1-\chi(\rho))/2 \neq 0$.

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Our aim is to refine this formula relative to an arbitrary idempotent element e contained in $e^- Z_p G$. Namely, we expect the following formula holds:

$$v_p(\sharp Z_p \otimes Cl_F) = \sum_{\chi(e)\neq 0} v_p(L(0,\chi)) + \delta_\chi \cdot v_p(w_F),$$

where Cl_F is the ideal class group of F and where the symbol δ_e takes value 1 if $\omega(e) \neq 0$ for the Teichimüllar character ω, otherwise it takes value 0. In [10], we showed this formula holds if F is an imaginary abelian extension of Q and if p is unramified.

In the present paper, we try to extend and apply the method there to the CM-field. We shall make it clear that what problem is remained at this point to achieve our goal. To this purpose, it is crucial to define an analogue of Gauss sums of an imaginary abelian case. Well-behaved analogue is needed so that it represent the value of $L(0,\chi)$ and the value at 0 of the higher derivatives of p-adic L-functions, which are to be a generalizations of the Stickelberger theorem and Theorem 3.2 in §17, [3] in an imaginary abelian field, respectively. We do not study this problem.

We define lattices obtained by p-adic maps of $Z_p \otimes F^\times$ is §1. In §2, we recall the χ-parts formula of a finite module. In §3–5, we generalize the results in [10] to fit a CM-field F. This will be done in a quite natural way with compared to the imaginary abelian case. In these argument, we suppose the Gross conjecture is valid for F and p. We shall be confronted with a new problem for μ-invariants, because vanishing of it is an open problem for CM-fields thought it was proved for abelian extensions of Q.

1. p-adic maps

We shall generalize the p-adic maps which are defined in [2]. Let F be a finite Galois extension of Q with a Galois group $\mathfrak{G}$. Suppose the prime number p is decomposed into r prime factors. We choose one of them and fixed it once for all as a base point. Denote by $\mathfrak{P}$ the base point. Let $\iota_{\mathfrak{P}}$ be an embedding which defines the prime ideal $\mathfrak{P}$. Namely, the inverse image of the valuation ideal of C_p is the prime ideal. Let $\mathfrak{H}$ be the decomposition group of $\mathfrak{P}$ and let

$$\mathfrak{G} = \dot{\cup}_{i=1}^{r} \sigma_i \mathfrak{H}, \quad \sigma_1 = 1$$

be the decomposition into right cosets. The set of conjugate prime ideals of $\mathfrak{P}$ is $\{\sigma_i\mathfrak{P}\}_{i=1,\dots,r}$. An embedding

$$\iota_{\sigma_i\mathfrak{P}} = \iota_{\mathfrak{P}} \circ \sigma_i^{-1}$$

is the defining embedding of a conjugate prime $\sigma_i\mathfrak{P}$. Since F is a Galois field, the images of every $\iota_{\sigma_i\mathfrak{P}}$'s coincide with $\mathrm{Im}\,\iota_{\mathfrak{P}}$. Let Z be the decomposition field of $\mathfrak{P}$ and $\mathfrak{H}$ be the Galois group of F/Z. Denote by $\mathcal{F}$ the composition of $\mathrm{Im}\,\iota_{\mathfrak{P}}$ and Q_p in C_p. We identify the Galois group of $\mathcal{F}$ with $\mathfrak{H}$ through the embedding $\iota_{\mathfrak{P}}$. From the arithmetical point of view, we discriminate between the composite fields $\iota_{\sigma_i\mathfrak{P}}(F)Q_p$, $i = 1,\dots,r$. We call $\iota_{\sigma_i\mathfrak{P}}(F)Q_p$ the completion of F at $\sigma_i\mathfrak{P}$. We write this completion as

$$\iota_{\sigma_i\mathfrak{P}} \cdot \mathcal{F} = F_{\sigma_i\mathfrak{P}}.$$

Hence, the symbol $\iota_{\sigma_i\mathfrak{P}} \cdot \mathcal{F}$ means a right one-dimensional $\mathcal{F}$-vector space on the base $\iota_{\sigma_i\mathfrak{P}}$. We restrict the prime ideal $\mathfrak{P}$ and the corresponding embedding $\iota_{\mathfrak{P}}$ onto Z and denote by $\mathfrak{p}$ and $\iota_{\mathfrak{p}}$, respectively. The conjugate ideal of $\mathfrak{p}$ is the prime ideal $\sigma_i\mathfrak{p}$ of the subfield $\sigma_i Z$ of F, which the embedding

$$\iota_{\sigma_i\mathfrak{p}} = \iota_{\mathfrak{p}} \circ \sigma_i^{-1} = \iota_{\sigma_i\mathfrak{P}}\big|_Z$$

defines. Let $\mathcal{N}$ be the norm map with respect to the extension $\mathcal{F}/Q_p$. We have the following two commutative rectangles:

$$\begin{array}{ccc} F & \xrightarrow{\iota_{\sigma_i\mathfrak{P}}} & \iota_{\sigma_i\mathfrak{P}} \cdot \mathcal{F} \\ {\scriptstyle N_{F/\sigma Z}}\downarrow & & \downarrow{\scriptstyle \mathcal{N}} \\ \sigma_i Z & \xrightarrow[\iota_{\sigma_i\mathfrak{p}}]{} & \iota_{\sigma_i\mathfrak{p}} \cdot Q_p, \end{array} \qquad \begin{array}{ccc} F & \xrightarrow{\iota_{\sigma_i\mathfrak{P}}} & \iota_{\sigma_i\mathfrak{P}} \cdot \mathcal{F} \\ {\scriptstyle \text{inclusion}}\uparrow & & \uparrow{\scriptstyle \text{inclusion}} \\ \sigma_i Z & \xrightarrow[\iota_{\sigma_i\mathfrak{p}}]{} & \iota_{\sigma_i\mathfrak{p}} \cdot Q_p. \end{array}$$

The field F is a simple $\boldsymbol{Q}$-algebra. By extension of the ground field, the simple algebra is decomposed into a direct sum of r simple $\boldsymbol{Q}_p$-algebras:

$$\boldsymbol{Q}_p \otimes_{\boldsymbol{Q}} F \cong \sum_{i=1}^{r} \iota_{\sigma_i \mathfrak{P}} \cdot \mathcal{F} = \prod_{i=1}^{r} F_{\sigma_i \mathfrak{P}}.$$

Denote by $\mathbf{A}_F$ the direct sum of these simple $\boldsymbol{Q}_p$-algebras. The extension of the ground field inherits the Galois module structure from F. We give a $\boldsymbol{Q}_p\mathfrak{G}$-module structure on $\mathbf{A}_F$ as an induced structure from the $\boldsymbol{Q}_p\mathfrak{H}$-module $\iota_{\mathfrak{P}} \cdot \mathcal{F}$.

$$\mathcal{F} \uparrow^{\mathfrak{G}} = \boldsymbol{Q}_p\mathfrak{G} \otimes_{\boldsymbol{Q}_p\mathfrak{H}} \mathcal{F}.$$

Similarly, we have decomposition by extension of the ground field to Z:

$$\boldsymbol{Q}_p \otimes_{\boldsymbol{Q}} Z \cong \sum_{i=1}^{r} \iota_{\sigma_i \mathfrak{p}} \cdot \boldsymbol{Q}_p = \prod_{i=1}^{r} Z_{\sigma_i \mathfrak{p}} \cong \boldsymbol{Q}_p \uparrow^{\mathfrak{G}} = \boldsymbol{Q}_p\mathfrak{G} \otimes_{\boldsymbol{Q}_p\mathfrak{H}} \boldsymbol{Q}_p$$

The global field F (*resp.* Z) is embedded into $\mathbf{A}_F$ (*resp.* $\mathbf{A}_Z$) with the set of embeddings $\{\iota_{\sigma_i \mathfrak{P}}\}$ (*resp.* $\{\iota_{\sigma_i \mathfrak{p}}\}$), diagonally:

$$x \to \sum_{i=1}^{r} \iota_{\sigma_i \mathfrak{P}} \cdot \iota_{\sigma_i \mathfrak{P}}(x), \qquad \left(resp. \quad x \to \sum_{i=1}^{r} \iota_{\sigma_i \mathfrak{p}} \cdot \iota_{\sigma_i \mathfrak{p}}(x) \right).$$

The norm map $N_{F/Z}$ of the multiplicative group $\mathbf{A}_F^\times$ into $\mathbf{A}_Z^\times$ is defined to be

$$N_{F/Z}(\sum_{i=1}^{r} \iota_{\sigma_i \mathfrak{P}} \cdot x_i) = \sum_{i=1}^{r} \iota_{\sigma_i \mathfrak{p}} \cdot \mathcal{N}(x_i).$$

By Proposition 2.2 in Chap. IV, [5], we have the following commutative rectangle:

$$\begin{array}{ccc} F^\times & \longrightarrow & \mathbf{A}_F \\ {\scriptstyle N_{F/Z}}\downarrow & & \downarrow{\scriptstyle N_{F/Z}} \\ Z^\times & \longrightarrow & \mathbf{A}_Z. \end{array}$$

Let $\boldsymbol{Q}_{p,\infty}$ be the cyclotomic $\boldsymbol{Z}_p$-extension of $\boldsymbol{Q}_p$. We suppose it satisfies

Assump. 1 $\quad \boldsymbol{Q}_{p,\infty} \cap \mathcal{F} = \boldsymbol{Q}_p$.

The cyclotomic $\boldsymbol{Z}_p$-extension $\boldsymbol{Q}_{p,\infty}$ is union of intermediate fields $\boldsymbol{Q}_{p,n}$ which are unique cyclic extension of degree p^n. Denote by $N_\infty \boldsymbol{Q}_p^\times$ a subgroup of the multiplicative group $\boldsymbol{Q}_p^\times$ defined to be

$$N_\infty \boldsymbol{Q}_p^\times = \bigcap_{n=1}^{\infty} N_{\boldsymbol{Q}_{p,n}/\boldsymbol{Q}_p}(\boldsymbol{Q}_{p,n}^\times).$$

This subgroup is a closed subgroup by which the quotient is isomorphic to the Galois group by the norm residue symbol of local class field theory:

$$\boldsymbol{Q}_p^\times / N_\infty \boldsymbol{Q}_p^\times \cong Gal(\boldsymbol{Q}_{p,\infty}/\boldsymbol{Q}_p).$$

Let $\log_p$ be the p-adic logarithm on $\boldsymbol{Q}_p^\times$. The subgroup $N_\infty \boldsymbol{Q}_p^\times$ coincides with the kernel. Thus, we have a sequence of isomorphisms of topological groups:

$$Gal(\boldsymbol{Q}_{p,\infty}/\boldsymbol{Q}_p) \xleftarrow[\text{norm residue maps}]{\cong} \boldsymbol{Q}_p^\times / N_\infty \boldsymbol{Q}_p^\times \xrightarrow[\frac{1}{p}\log_p]{\cong} \boldsymbol{Z}_p.$$

There is a similar sequence for the cyclotomic Z_p-extension $\mathcal{F}_\infty$. We define a closed subgroup of the multiplicative group $\mathcal{F}^\times$:

$$N_\infty \mathcal{F}^\times = \bigcap_{n=1}^{\infty} N_{\mathcal{F}_n/\mathcal{F}}(\mathcal{F}_n^\times).$$

By local class field theory, the quotient $\mathcal{F}^\times/N_\infty\mathcal{F}^\times$ is isomorphic to $Q_p^\times/N_\infty Q_p^\times$ by the norm map $\mathcal{N}$ from the assumption 1. Hence, we obtain

$$\mathcal{F}^\times/N_\infty\mathcal{F}^\times \xrightarrow[\mathcal{N}]{\cong} Q_p^\times/N_\infty Q_p^\times \xrightarrow[\frac{1}{p}\log_p]{\cong} Z_p.$$

By the induced module structure, we can convert this local correspondence to that between $\mathbf{A}_F$ and $\mathbf{A}_Z$. Let $N_\infty\mathbf{A}_F^\times$ be closed subgroup of $\mathbf{A}_F^\times$ defined to be

$$N_\infty\mathbf{A}_F^\times = \dot{\sum}_{i=1}^{r} \iota_{\sigma_i\mathfrak{P}} \cdot N_\infty\mathcal{F}^\times.$$

We also define

$$N_\infty\mathbf{A}_Z^\times = \dot{\sum}_{i=1}^{r} \iota_{\sigma_i\mathfrak{P}} \cdot N_\infty Q_p^\times.$$

We see the following commutative diagram is constructed:

$$\begin{array}{ccccc} F^\times & \longrightarrow & \mathbf{A}_F^\times/N_\infty\mathbf{A}_F^\times & & \\ {\scriptstyle N_{F/Z}}\downarrow & & {\scriptstyle N_{F/Z}}\downarrow & & \\ Z^\times & \longrightarrow & \mathbf{A}_Z^\times/N_\infty\mathbf{A}_Z^\times & \xrightarrow[\cong]{\psi'} & \dot{\sum}\iota_{\sigma_i\mathfrak{p}}\cdot Z_p = (\iota_{\mathfrak{p}}\cdot Z_p)\uparrow^{\mathfrak{G}}. \end{array}$$

Denote by $\mathcal{U}$ the induced module in the right. The isomorphism of $\mathbf{A}_Z^\times/N_\infty\mathbf{A}_Z^\times$ onto $\mathcal{U}$ is defined by

$$\psi' : x = \sum \iota_{\sigma_i\mathfrak{p}} \cdot x_i \to \sum \iota_{\sigma_i\mathfrak{p}} \cdot \frac{1}{p}\log_p(x_i).$$

The diagram yields maps $\psi_F : F^\times \to \mathcal{U}$ and $\psi : Z^\times \to \mathcal{U}$. These two maps have a relation

$$\psi_F = \psi \circ N_{F/Z}.$$

Since the image of $Z^\times$ into $\mathbf{A}_Z^\times$ is dense, we have a surjective map if we extend ψ onto $Z_p \otimes Z^\times$. We denote this extension by the same symbol ψ. The map ψ_F is also extended to a surjective Z_p-homomorphism of $Z_p \otimes F^\times$ onto $\mathcal{U}$. The actions of $\mathfrak{G}$ on the modules appeared in the above diagram are compatible with these maps ψ and ψ_F. They are $Z_p\mathfrak{G}$-homomorphisms. Let $E_{1,F}$ be the group of p-units of F. Denote by $\mathcal{W}_F$ the image of $Z_p \otimes E_{1,F}$ by ψ_F.

Let L_F be the maximal unramified p-decomposed abelian extension of F. We call an extension is p-decomposed if every prime divisors of p are completely decomposed there. Let F_∞ be the cyclotomic Z_p-extension. Denote by L_{F_∞} the maximal unramified p-decomposed abelian pro-p-extension of F_∞. It is union of the maximal unramified p-decomposed abelian extension of the nth layers of the Z_p-extension:

$$L_{F_\infty} = \bigcup_{n=0}^{\infty} L_{F_n}.$$

Proposition 1.1. *Let $L_{F_\infty}^*$ be the maximal abelian extension of F contained in L_{F_∞}. We have $Gal(L_{F_\infty^*}/L_F) \cong \mathcal{U}/\mathcal{W}_F$.*

Proof: Let $\overline{E}_{1,F}$ be the topological closure of the image of $E_{1,F}$ into $\mathbf{A}_F^{\times}$. Let N_n be a subgroup

$$N_n = \dot{\sum}_{i=1}^{r} \iota_{\sigma_i \mathfrak{P}} \cdot N_{\mathcal{F}_n/\mathcal{F}}(\mathcal{F}_n^{\times})$$

of $\mathbf{A}_F^{\times}$. Note $N_\infty \mathbf{A}_F^{\times} = \cap_{n \geq 1} N_n$. By class field theory, the quotient group $\mathbf{A}_F^{\times}/\overline{E}_{1,F}N_n$ is isomorphic to the Galois group of the maximal abelian subfield contained in L_{F_n} over F. By the assumption 1, we have $\mathbf{A}_F^{\times}/N_n \cong \mathcal{U}/\mathcal{U}^{p^n}$ holds. Hence, this quotient group is mapped onto

$$\mathcal{U}/\mathcal{W}_F + p^n\mathcal{U}$$

by $\psi' \circ N_{F/Z}$, isomorphically. Taking the inverse limit, we have $\mathcal{U}/\mathcal{W}_F \cong Gal(L^*_{F_\infty}/L_F)$. *q.e.d.*

Let U be a free Z_p-module on a basis $\{\sigma_i \mathfrak{p}\}_{i=1,\dots,r}$. This module has a $Z_p\mathfrak{G}$-module structure, canonically. We have $\mathcal{U} \cong U$ as $Z_p\mathfrak{G}$-modules. Each element of $Z^{\times}$ is mapped into U by

$$x \to \sum_{i=1}^{r} v_p(\iota_{\mathfrak{p}}(\sigma_i(x)))\sigma_i\mathfrak{p}.$$

We extend this map onto $Z_p \otimes Z^{\times}$. Denote by φ this $Z_p\mathfrak{G}$-homomorphism of $Z_p \otimes Z^{\times}$ onto U. We define a homomorphism φ_F of $Z_p \otimes F^{\times}$ by

$$\varphi_F = \varphi \circ N_{F/Z}.$$

This map is not surjective in general. Let $f_{F/Q}$ be the degree of the prime $\mathfrak{P}$ in F/Q. Since F is a Galois extension, the degree of every conjugate ideal is equal to $f_{F/Q}$. Let U_F be the image of φ. We have

$$U_F = f_{F/Q}U \cong U.$$

Denote by W_F the image of $Z_p \otimes E_{1,F}$ by φ_F. Let $\tilde{L}_F$ be the maximal unramified abelian extension of F. By class field theory, we have

$$U_F/W_F \cong Gal(\tilde{L}_F/L_F). \tag{1}$$

Since $\tilde{L}_F$ is a finite abelian extension of F, we see U_F and W_F has the same Z_p-rank. On the contrary, the Z_p-rank of $\mathcal{W}_F$ is less than that of $\mathcal{U}$, because $L^*_{F_\infty}$ contains the cyclotomic Z_p-extension of L_F. Let $\tilde{L}_{F_\infty}$ be the maximal unramified abelian pro-p-extension of F_∞. Denote by $\tilde{L}^*_{F_\infty}$ the maximal abelian extension of F contained in $\tilde{L}_{F_\infty}$. We have a similar isomorphism of the Galois group $Gal(\tilde{L}^*_{F_\infty}/\tilde{L}_F)$ as that of Proposition 1.1. Let $\mathfrak{U}_n$ be the unit group of the local field $\mathcal{F}_n$. We define a subgroup of the unit group of $\mathcal{F}$:

$$N_\infty\mathfrak{U} = \bigcap_{n=1}^{\infty} N_{\mathcal{F}_n/\mathcal{F}}(\mathfrak{U}_n).$$

By the assumption 1, the quotient of the unit group $\mathfrak{U}$ of $\mathcal{F}$ by this subgroup is isomorphic to a subgroup $\mathcal{F}^{\times}/N_\infty\mathcal{F}^{\times}$ of finite index. Denote by $\mathfrak{U}_{\sigma_i\mathfrak{P}}$ the unit group of $F_{\sigma_i\mathfrak{P}}$. We have an injective homomorphism

$$\prod_{i=1}^{r} \mathfrak{U}_{\sigma_i\mathfrak{P}}/N_\infty\mathfrak{U}_{\sigma_i\mathfrak{P}} \to \mathbf{A}_F^{\times}/N_\infty\mathbf{A}_F^{\times}$$

of finite cokernel. Let $E_{0,F}$ be the unit group of F. The image of $E_{0,F}$ into $\mathbf{A}_F^{\times}$ is contained in $\prod \mathfrak{U}_{\sigma_i\mathfrak{P}}$. Denote by $\overline{E}_{0,F}$ the topological closure of the image. We have injection

$$\prod_{i=1}^{r} \mathfrak{U}_{\sigma_i\mathfrak{P}}/\overline{E}_{0,F}\prod_{i=1}^{r} N_\infty\mathfrak{U}_{\sigma_i\mathfrak{P}} \to \mathbf{A}_F^{\times}/\overline{E}_{0,F}N_\infty\mathbf{A}_F^{\times}$$

of finite cokernel. By class field theory, the quotient of $\prod \mathfrak{U}_{\sigma_i\mathfrak{P}}$ by $\overline{E}_{0,F}N_\infty \prod \mathfrak{U}_{\sigma_i\mathfrak{P}}$ is isomorphic to the Galois group of $\tilde{L}^*_{F_\infty}$ over $\tilde{L}_F$. Denote by $\mathcal{E}_F$ the image of $Z_p \otimes E_{0,F}$ by φ_F. Let $\mathcal{U}'$ be the image of $\prod \mathfrak{U}_{\sigma_i\mathfrak{P}}$ into $\mathcal{U}$ by $\psi' \circ N_{F/Z}$. We have

$$\mathcal{U}'/\mathcal{E}_F \cong Gal(\tilde{L}^*_{F_\infty}/\tilde{L}_F).$$

By comparing this with the isomorphism of Proposition 1.1, we obtain

$$\mathrm{rank}_{Z_p}\mathcal{W}_F/\mathcal{E}_F = \mathrm{rank}_{Z_p}Gal(\tilde{L}^*_{F_\infty}/L^*_{F_\infty}).$$

Proposition 1.2. *If F is p-decomposed over Q, then $\mathcal{W}_F$ and $\mathcal{E}_F$ have the same Z_p-rank.*

Proof: An arbitrary Z_p-extension of the p-decomposed field F is locally cyclotomic at every prime lying above p. Hence, it is a subfield of L_{F_∞}. It follows that $\tilde{L}^*_{F_\infty}$ is a finite abelian extension of $L^*_{F_\infty}$. Therefore, $\mathcal{W}_F/\mathcal{E}_F$ is of finite order. *q.e.d.*

Proposition 1.3. *Suppose F is a CM-field. Let ρ be the complex conjugation. Put $e^- = (1-\rho)/2$. If the Gross conjecture is true for F, then we have*

$$e^-\mathcal{W}_F \cong e^-W_F \cong e^-Z_p \otimes E_{1,F}/E_{0,F}.$$

Proof: F is a CM-field if and only if ρ is an element of the center of $\mathfrak{G}$. Thus, e^- is a central idempotent element of the group ring $Z_p\mathfrak{G}$. It is obvious that e^-W_F is isomorphic to $e^-Z_p \otimes E_{1,F}/E_{0,F}$. Since F is a CM-field, $e^-Z_p \otimes E_{0,F}$ is the maximal torsion submodule of $e^-Z_p \otimes E_{1,F}$. It is contained in the kernel of ψ_F. Since it follows from the Gross conjecture that $e^-Gal(L^*_{F_\infty}/F)$ is of finite order, we have $e^-\mathcal{W}_F$ has the same rank as $e^-\mathcal{U}$. Hence, e^-W_F and $e^-\mathcal{W}_F$ has also of same rank. This implies $e^-Z_p \otimes E_{1,F}/E_{0,F}$ is mapped onto $e^-\mathcal{W}_F$ by ψ_F, isomorphically. *q.e.d.*

2. χ-parts

Let G be a finite group. Let $\mathcal{K}$ be a decomposition field of G contained in C_p which is a finite extension of Q_p. Let $\mathcal{O}$ be the valuation ring. If M is a Z_p-free module, we denote by $\mathcal{K}M$ (*resp.* $\mathcal{O}M$) the extension of coefficients to $\mathcal{K}$ (*resp.* $\mathcal{O}$): $\mathcal{K}M = \mathcal{K} \otimes_{Z_p} M$ (*resp.* $\mathcal{O}M = \mathcal{O} \otimes_{Z_p} M$). If G is abelian, the χ-part of M defined for each irreducible $\mathcal{K}$-character of G is a submodule M^χ of $\mathcal{O}M$ consisting of elements m satisfying $\sigma m = \chi(\sigma)m$ for every $\sigma \in G$. A Z_pG-module M is called a Z_pG-lattice if it is Z_p-free. If M is a Z_pG-lattice, we proposed another definition of χ-part in §1, [10]. An element

$$e_\chi = \frac{\chi(1)}{|G|}\sum_{\sigma \in G}\chi(\sigma)\sigma^{-1}$$

of the group ring $\mathcal{K}G$ is a central primitive idempotent, *c.f.* Proposition 9.17 in [1]. Let $Irr_{\mathcal{K}}(G)$ be the set of all of irreducible $\mathcal{K}$-characters of G. The sum of every idempotents e_χ is the identity element of $\mathcal{K}G$:

$$\sum_{\chi \in Irr_{\mathcal{K}}(G)} e_\chi = 1.$$

Thus, the set $\{e_\chi\}$ of idempotents decomposes a $\mathcal{K}G$-module into a direct sum of submodules:

$$\mathcal{K}M = \bigoplus_{\chi \in Irr_{\mathcal{K}}(G)} e_\chi \mathcal{K}M.$$

The direct factors are χ-eigen spaces of the $\mathcal{K}$-vector space $\mathcal{K}M$. Their $\mathcal{K}$-dimension is described by a non-degenerate symmetric scalar product on the $\mathcal{K}$-subspace generated on the basis $Irr_{\mathcal{K}}(G)$ in the space of $\mathcal{K}$-valued $\mathcal{K}$-linear maps on the group ring $\mathcal{K}G$, which is defined to be

$$\langle \zeta, \eta \rangle_G = \frac{1}{|G|}\sum_{\sigma \in G}\zeta(\sigma)\eta(\sigma^{-1})$$

for characters ζ and η. If ζ is the character afforded by $\mathcal{K}M$, we have the dimension formula

$$\dim_{\mathcal{K}} \mathcal{K}M_\chi = \langle \zeta, \chi \rangle_G.$$

Since $\mathcal{O}M$ is a submodule of $\mathcal{K}M$, we define the χ-part of the Z_pG-lattice M to be the image into the eigen space $e_\chi \mathcal{K}M$ by the projection:

$$M_\chi = e_\chi \mathcal{O}M.$$

The χ-part M_χ is an $\mathcal{O}G$-lattice whose $\mathcal{O}$-rank is equal to the dimension of $e_\chi\mathcal{K}M$. If G is abelian, we have the following relation:

$$M_\chi \supset M^\chi \supset |G| M_\chi.$$

Let $\tilde{M}$ be the direct sum of M_χ in $\mathcal{K}M$. It is an $\mathcal{O}G$-free module containing $\mathcal{O}M$. Since the rank is equal to $\dim_{\mathcal{K}} \mathcal{K}M$ and since the sum of $\dim_{\mathcal{K}} e_\chi\mathcal{K}M$ equals the dimension of $\mathcal{K}M$, we have $\tilde{M}/\mathcal{O}M$ is of finite order. Denote by pr_χ^M the projection of $\mathcal{O}M$ onto M_χ. If N be a quotient of a Z_pG-lattice M by a sublattice L, we define the χ-part N_χ^* to be

$$N_\chi^* = M_\chi / L_\chi$$

and $\tilde{N}^*$ to be the outer direct sum of N_χ^* for $\chi \in Irr_{\mathcal{K}}(G)$.

Suppose N is also a Z_pG-lattice. We have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}L & \longrightarrow & \mathcal{O}M & \xrightarrow{\pi} & \mathcal{O}N & \longrightarrow & 0 \\ & & {\scriptstyle pr_\chi^L}\downarrow & & {\scriptstyle pr_\chi^M}\downarrow & & {\scriptstyle pr_\chi^N}\downarrow & & \\ 0 & \longrightarrow & e_\chi\mathcal{K}L & \xrightarrow{\iota_\chi} & e_\chi\mathcal{K}M & \longrightarrow & e_\chi\mathcal{K}N & \longrightarrow & 0 \end{array}$$

This diagram yields a long exact sequence of $\mathcal{O}G$-modules:

$$0 \to \operatorname{Ker} pr_\chi^L \to \operatorname{Ker} pr_\chi^M \to \operatorname{Ker} pr_\chi^N \xrightarrow{\delta} \operatorname{Coker} pr_\chi^L \to \operatorname{Coker} pr_\chi^M \to \operatorname{Coker} pr_\chi^N \to 0.$$

The homomorphism δ is given by

$$\iota_\chi^{-1} \circ pr_\chi^M \circ \pi^{-1} : \operatorname{Ker} pr_\chi^N \to e_\chi\mathcal{K}L/L_\chi.$$

If $\pi(m) \in \operatorname{Ker} pr_\chi^N$ for $m \in \mathcal{O}M$, we see $\iota_\chi^{-1} \circ pr_\chi^M \circ \pi^{-1}(|G|\pi(m)) \in L_\chi$, because $e_\chi|G|m \in \mathcal{O}L$. It follows that $\operatorname{Im}\delta$ is an abelian group whose exponent divides the order of G. We have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Ker} pr_\chi^L & \longrightarrow & \operatorname{Ker} pr_\chi^M & \longrightarrow & \operatorname{Ker}\delta & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{O}L & \longrightarrow & \mathcal{O}M & \longrightarrow & \mathcal{O}N & \longrightarrow & 0. \end{array}$$

By taking cokernels of the vertical homomorphisms, the following short exact sequence arises:

$$0 \to L_\chi \to M_\chi \to \mathcal{O}N/\operatorname{Ker}\delta \to 0.$$

We have

$$N_\chi^* \cong \mathcal{O}N/\operatorname{Ker}\delta.$$

Since $\operatorname{Ker}\delta$ is a submodule of the kernel of pr_χ^N, there is a canonical surjection N_χ^* onto N_χ. Denote by $T_\chi(M;L)$ the kernel of this surjection. We have an isomorphism

$$N_\chi^* \cong N_\chi \oplus T_\chi(M;L)$$

of $\mathcal{O}$-modules, because N_χ is $\mathcal{O}$-free. Note $T_\chi(M;L)$ is isomorphic to $\operatorname{Im}\delta$.

We shall give a χ-part formula. Suppose N is of finite order. Let α (*resp.* β) be the canonical map of L (*resp.* M) into $\tilde{L}$ (*resp.* $\tilde{M}$). We have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}L & \longrightarrow & \mathcal{O}M & \longrightarrow & \mathcal{O}N & \longrightarrow & 0 \\ & & \alpha\downarrow & & \beta\downarrow & & & & \\ 0 & \longrightarrow & \tilde{L} & \longrightarrow & \tilde{M} & \longrightarrow & \tilde{N}^* & \longrightarrow & 0. \end{array}$$

By the commutativity of the left square, there is an $\mathcal{O}G$-homomorphism $\gamma : \mathcal{O}N \to \tilde{N}^*$. Thus, we have a long exact sequence

$$0 \to \operatorname{Ker}\gamma \to \operatorname{Coker}\alpha \to \operatorname{Coker}\beta \to \operatorname{Coker}\gamma \to 0.$$

Further, since N is of finite order, we have

$$\ell(\tilde{N}^*) - \ell(\mathcal{O}N) = \ell(\operatorname{Coker}\gamma) - \ell(\operatorname{Ker}\gamma).$$

Namely, the following formula is obtained:

$$\ell(\tilde{N}^*) - \ell(N) = \ell(\tilde{M}/\mathcal{O}M) - \ell(\tilde{L}/\mathcal{O}L).$$

We define $\Delta(X;Y)$ for two $Z_p\mathfrak{G}$-lattices X and Y to be

$$\Delta(X;Y) = \ell(\tilde{X}/\mathcal{O}X) - \ell(\tilde{Y}/\mathcal{O}Y).$$

We have the following χ-part formula:

$$\ell(N) = \sum_{\chi \in Irr_K(G)} \ell(N_\chi^*) - \Delta(M;L). \tag{2}$$

We notice that $\Delta(M;L) = 0$ if the order of G is prime to p or if $M \cong L$.

Let Λ be either ring of formal power series $Z_p[[T]]$ or $\mathcal{O}[[T]]$. To discriminate them, we denote it by Λ_{Z_p} or $\Lambda_{\mathcal{O}}$, respectively. Let M be a finitely generated Λ-torsion module. Let M_0 be the maximal torsion submodule of M. Denote by $\overline{M}$ the quotient M/M_0. There are distinguished polynomial $f_1, \cdots, f_a$ such that $\overline{M}$ is isomorphic to a Λ-submodule of

$$E_1 = \Lambda/f_1 \times \cdots \times \Lambda/f_a.$$

The product of f_i is uniquely determined for $\overline{M}$ and is called the *characteristic polynomial* of M in this article. On the other hand, the torsion submodule M_0 is mapped into a Λ-module

$$E_0 = \Lambda/p^{m_1} \times \cdots \times \Lambda/p^{m_b}$$

with a finite kernel and with a finite cokernel. The sum of m_i is also determined uniquely for M_0 and is called the μ-invariant of M. Denote by f_M the characteristic polynomial of M and by μ_M the μ-invariant.

Lemma 2.1. *If M is isomorphic to a submodule of $E_0 \times E_1$ and if $T \nmid f$, we have*

$$\ell(M/TM) = v_p(f(0)) + \mu_M.$$

Proof: $T \nmid f_M$ implies that the multiplication by T on E_1 is an injective endomorphism. Since it is also injective on E_0, we have the multiplication by T induces injective homomorphisms on $\overline{M}$, M_0 and M, respectively. By a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_0 & \longrightarrow & M & \longrightarrow & \overline{M} & \longrightarrow & 0 \\ & & T\downarrow & & T\downarrow & & T\downarrow & & \\ 0 & \longrightarrow & M_0 & \longrightarrow & M & \longrightarrow & \overline{M} & \longrightarrow & 0 \end{array}$$

we have an exact sequence

$$0 \to M_0/TM_0 \to M/TM \to \overline{M}/T\overline{M} \to 0.$$

Set M_0 or M_1 to M' and E_0 or E_1 to E, respectively. We have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \longrightarrow & E & \longrightarrow & E/M' & \longrightarrow & 0 \\ & & T\downarrow & & T\downarrow & & & & \\ 0 & \longrightarrow & M' & \longrightarrow & E & \longrightarrow & E/M' & \longrightarrow & 0 \end{array}$$

Let α_T be the endomorphism of E/M' yielded by commutativity of the diagram. Since E/M' is of finite order, we have the following equalities from the cokernel sequence of the vertical homomorphisms:

$$\ell(E/TE) - \ell(M'/TM') = \ell(\mathrm{Coker}\,\alpha_T) - \ell(\mathrm{Ker}\,\alpha_T) = \ell(E/M') - \ell(E/M') = 0.$$

Therefore, we have

$$\ell(M_0/TM_0) = \ell(E_0/TE_0) = \sum_{i=1}^{b} \mu_i, \qquad \ell(\overline{M}/T\overline{M}) = \ell(E_1/TE_1) = v_p(f(0)).$$

Hence, $\ell(M/TM)$ equals $v_p(f_M(0)) + \mu_M$. *q.e.d.*

Lemma 2.2. *Suppose a Z_pG-lattice M is also a finitely generated Λ-torsion module and the actions of G and T are compatible. Then, the χ-part M_χ is a finitely generated $\Lambda_{\mathcal{O}}$-torsion module and*

$$\ell(M/TM) = \sum_{\chi \in Irr_K(G)} \ell(M_\chi/TM_\chi)$$

holds if $T \nmid f_M$.

Proof: By the formula (2), we have

$$\ell(M/TM) = \sum_{\chi \in Irr_K(G)} \ell(M_\chi/TM_\chi) - \Delta(M;TM).$$

Since $T \nmid f_M$, M is isomorphic to TM. It follows $\Delta(M;TM) = 0$. *q.e.d.*

We denote by $f_{M,\chi}$ the characteristic polynomial of M_χ in this lemma. If M is not lattice, we denote by $f_{M,\chi}$ the characteristic polynomial of $\overline{M}_\chi$. If G is abelian, this polynomial is also the characteristic polynomial of M^χ. We also denote by μ_M^χ the μ-invariant of the torsion submodule of M^χ in this case.

3. Z_pG-lattices

F is a CM-field if and only if the complex conjugation ρ is contained in the center of $\mathfrak{G}$. If $p > 2$, the idempotent element $e^- = (1-\rho)/2$ of the group ring $Q_p\mathfrak{G}$ is an element of $Z_p\mathfrak{G}$. We suppose

Assump. 2 $p > 2$ and F is a CM-field satisfying the Gross conjecture for p.

Assump. 3 $Q_\infty \cap F = Q$ and every $\sigma_i\mathfrak{P}$ dose not decomposed in the first layer F_1 of F_∞.

By the assumption 3, we have every $\sigma_i\mathfrak{P}$ is not decomposed in the cyclotomic Z_p-extension F_∞ and every element $\sigma \in \mathfrak{G}$ commutes with every element of $Gal(F_\infty/F)$.

Lemma 3.1. *$e^-Gal(\tilde{L}_{F_\infty}/L_{F_\infty})$ is a Z_p-lattice which is isomorphic to e^-U_F. The Galois group of F_∞/F acts on this lattice, trivially.*

Proof: Every prime ideal $\sigma_i\mathfrak{P}$ is extended onto F_n uniquely by the assumption 3. Denote by $\mathfrak{P}_n$ the extension of $\mathfrak{P}$. We fix an extension onto F_n of an element σ_i of the Galois group and denote it by $\tilde{\sigma}_i$. Since $\tilde{\sigma}_i\mathfrak{P} = \sigma_i\mathfrak{P}$, we see $\tilde{\sigma}_i\mathfrak{P}_n$ is the extension of the prime ideal $\sigma_i\mathfrak{P}$. Let $\tilde{\sigma}$ be an arbitrary extension $\sigma \in \mathfrak{G}$. There is $h \in Gal(F_n/F)$ and j for each $\tilde{\sigma}_i$ such that $\tilde{\sigma}\tilde{\sigma}_i = \tilde{\sigma}_j h$. Thus, we have $\tilde{\sigma}(\tilde{\sigma}_i\mathfrak{P}_n) = \tilde{\sigma}_j\mathfrak{P}_n$. This makes $\{\tilde{\sigma}_i\mathfrak{P}\}$ a $\mathfrak{G}$-set. Let U_n be a free Z_p-module on this set:

$$U_n = \dot{\sum}_{i=1}^{r} Z_p(\tilde{\sigma}_i\mathfrak{P}).$$

It is a $Z_p\mathfrak{G}$-lattice where $Gal(F_\infty/F)$ acts trivially. Let m be an integer being less than n. By restricting $\mathfrak{P}_n$ onto F_m, we have a prime ideal $\mathfrak{P}_m$. This restriction yields an isomorphism $U_n \cong U_m$. Let n_0 be the maximal integer such that $\mathfrak{P}$ is unramified at F_{n_0}. The ramification index of $\mathfrak{P}_n$ over F is equal to $p^{\max(n-n_0,0)}$. Since F_n is a Galois extension of Q, the ramification index of every $\tilde{\sigma}_i\mathfrak{P}_n$ is equal to that of $\mathfrak{P}$. Hence, the extension of $\mathfrak{P}_m$ onto F_n is

$$\mathfrak{P}_n^{\max(n-n_0,0)-\max(m-n_0,0)}.$$

This extension yields an injective isomorphism $U_m \to U_n$. Denote by $e_{\mathfrak{P}}$ the ramification index of $\mathfrak{P}$ over Q. Let $\iota_{\mathfrak{P}_n}$ be an extension onto F_n of the embedding $\iota_{\mathfrak{P}}$:

$$\iota_{\mathfrak{P}_n} : F_n \to C_p.$$

The prime ideal $\tilde{\sigma}\mathfrak{P}_n$ is determined by the embedding $\iota_{\mathfrak{P}_n} \circ \tilde{\sigma}_i^{-1}$. Each element $x \in F_n^\times$ is mapped into U_n:

$$x \to e_{\mathfrak{P}} \sum p^{\max(n-n_0,0)} v_p(\iota_{\mathfrak{P}_n} \circ \tilde{\sigma}_i^{-1}(x))\tilde{\sigma}_i\mathfrak{P}_n.$$

This map is enlarged to a $Z_p\mathfrak{G}$-homomorphism of $Z_p \otimes F_n^\times$ onto U_n.

Let E_{1,F_n} be the group of p-units of F_n. Denote by W_n the image of $Z_p \otimes E_{1,F_n}$ into U_n. By class field theory, the Galois group of the maximal unramified abelian p-extension of F_n over the maximal p-decomposed intermediate field is isomorphic to U_n/W_n:

$$Gal(\tilde{L}_{F_n}/L_{F_n}) \cong U_n/W_n.$$

We take the inverse limits :

$$Gal(\tilde{L}_{F_\infty}/L_{F_\infty}) \cong \varprojlim U_n/\varprojlim W_n.$$

The inverse limits in the right are taken relative to the norm maps. Since each norm map $N_{F_n/F_m} : U_n \to U_m$ is isomorphic when $m \geq n_0$, we have $\varprojlim U_n \cong N_{F_{n_0}/F}(U_{n_0}) = p^{n_0}U_0$, which is mapped onto $p^{n_0}U_F$, isomorphically. We shall compute $\varprojlim e^- W_n$. Let E_{0,F_n} be the unit group of F_n. We have an exact sequence

$$0 \to e^- Z_p \otimes E_{0,F_n} \to Z_p \otimes E_{1,F_n} \to e^- W_n \to 0.$$

Taking the cohomology long exact sequence, we obtain

$$\begin{aligned}\to H^{-1}(Gal(F_n/F), e^-W_n) \to H^0(Gal(F_n/F), e^-Z_p \otimes E_{0,F_n}) \to H^0(Gal(F_n/F), e^-Z_p \otimes E_{1,F_n})\\ \to H^0(Gal(F_n/F), e^-W_n) \to H^1(Gal(F_n/F), e^-Z_p \otimes E_{0,F_n}).\end{aligned}$$

Let μ be the group of p-power roots of unity contained in F_n. We see $e^-Z_p \otimes E_{0,F_n} \cong \mu$. By the assumption 3, we see $\mu = \{1\}$ or $\mu = <\zeta_{p^n}>$. In the latter case, we have $H^i(Gal(F_n/F), \mu) = 0$ for $i = 0, 1$. Thus, the cohomology long exact sequence is reduced to the isomorphism of 0-dimensional parts:

$$\frac{e^- Z_p \otimes E_{1,F_0}}{e^- Z_p \otimes N_{F_n/F}(E_{1,F_n})} \xrightarrow{\cong} \frac{e^- W_n}{p^n e^- W_n}.$$

Denote by $e^- W_0'$ the image of $e^- Z_p \otimes E_{0,F_0}$ into $e^- W_n$ by the injection of U_0 into U_n. We see U_0 is isomorphic to $p^{\max(n-n_0)} U_n$. It follows from the above isomorphism that $e^- W_n = e^- W_0' + p^n e^- W_n$. Hence,

$$e^- W_n = e^- W_0'.$$

We have a sequence of surjective homomorphisms:

$$e^- U_n \to e^- U_n / e^- W_n = e^- U_n / e^- W_0' \to e^- U_n / p^{\max(n-n_0)} e^- U_n.$$

By taking the inverse limits, we obtain

$$p^{n_0} e^- U_0 \to e^- Gal(\tilde{L}_{F_\infty} / L_{F_\infty}) \to p^{n_0} e^- U_0.$$

This implies $p^{n_0} e^- U_0 \cong e^- Gal(\tilde{L}_{F_\infty}/L_{F_\infty})$. Since $p^{n_0} e^- U_0 \cong p^{n_0} e^- U_F \cong e^- U_F$, $e^- Gal(\tilde{L}_{F_\infty}/L_{F_\infty})$ is a $Z_p G$-lattice which is isomorphic to $e^- U_F$. *q.e.d.*

Lemma 3.2. *$e^- Gal(\tilde{L}_{F_\infty}/F_\infty)$ and $e^- Gal(L_{F_\infty}/F_\infty)$ are finitely generated Λ_{Z_p}-torsion module and a $Z_p \mathfrak{G}$-module. The action of Λ_{Z_p} commutes with the action of $\mathfrak{G}$. Moreover, an arbitrary Λ_{Z_p}-submodule is either trivial or of infinite order.*

Proof: We fix a generator γ of $Gal(F_\infty/F)$ as a Z_p-module and take an extension $\tilde{\gamma}$ onto $\tilde{L}_{F_\infty}$. It acts on $e^- Gal(\tilde{L}_{F_\infty}/F_\infty)$ and on $e^- Gal(L_{F_\infty}/F_\infty)$ by $\tilde{\gamma} x \tilde{\gamma}^{-1}$. The action of the ring Λ is defined by $T(x) = \gamma(x) - x$, here we write additively. By this definition, these two modules become compact and finitely generated Λ-modules, *c.f.* [7]. By the assumption 3, we have $Gal(F_\infty/Q) \cong Gal(F_\infty/F) \times \mathfrak{G}$. Thus, the action of Λ commutes with the action of $\mathfrak{G}$.

The last assertion is proved for both modules in parallel. Let Cl_n be the ideal class group of F_n. Let Cl_n' be the quotient by the subgroup generated by every $\tilde{\sigma}_i \mathfrak{P}$'s. Put

$$C_n = \begin{cases} \text{the } p\text{-Sylow subgroup of } Cl_n, \\ \text{the } p\text{-Sylow subgroup of } Cl_n', \end{cases} \qquad E_n = \begin{cases} E_{0,F_n}, \\ E_{1,F_n}. \end{cases}$$

To prove the assertion for $e^- Gal(\tilde{L}_{F_n}/F_\infty)$, we use the upper objects, and to do for another, we need the lowers. In the subextension of F_∞/F, we have a natural map

$$i_{m,n} : C_m \to C_n$$

for $m < n$. The kernel of this homomorphism is isomorphic to a subgroup of the cohomology group $H^1(Gal(F_n/F_m), E_n)$. Since the cohomology group is p-primary torsion and since the complex conjugation map ρ is contained in the center of $Gal(F_n/Q)$, we see

$$e^- H^1(Gal(F_n/F_m), E_n) \cong H^1(Gal(F_n/F_m), E_n^{1-\rho}) \cong H^1(Gal(F_n/F_m), e^- Z_p \otimes E_n).$$

When $E_n = E_{0,F_n}$, we have $H^1(Gal(F_n/F_m), e^- Z_p \otimes E_{0,F_n}) = 0$. When $E_n = E_{1,F_n}$, by the exact sequence,

$$H^1(Gal(F_n/F_m), e^- Z_p \otimes E_{0,F_n}) \to H^1(Gal(F_n/F_m), e^- Z_p \otimes E_{1,F_n}) \to H^1(Gal(F_n/F_m), e^- W_n),$$

we also have $H^1(Gal(F_n/F_m), e^- Z_p \otimes E_{1,F_n})$ vanishes. Therefore, $i_{m,n}$ is injective on $e^- C_m$. Let X be the inverse limit of $e^- C_n$ by norm maps:

$$X = \varprojlim e^- C_n = \begin{cases} e^- Gal(\tilde{L}_{F_\infty}/F_\infty) \\ e^- Gal(L_{F_\infty}/F_\infty). \end{cases}$$

We abbreviate the norm map with respect to F_n/F_m to $N_{n,m}$. When $m \geq n_0$, $N_{n,m}$ is a surjective map of e^-C_n onto e^-C_m. Let X_0 be a Λ-submodule of X of finite order. There is n_1 such that $Gal(F_\infty/F_{n_1})$ acts on X_0, trivially. Let B_n be the image of X_0 by the canonical projection of $X_0 \subset \prod C_n$ onto C_n. $Gal(F_\infty/F_{n_1})$ also acts on B_n, trivially. Take an arbitrary element x of X_0:

$$x = (b_0, b_1, \cdots).$$

When $m \geq \max(n_0, n_1)$, we have $b_m = N_{m+n,m}(b_{m+n})$, and hence

$$i_{m,m+n}(b_m) = b_{n+m}^{p^n}.$$

Thus, if n satisfies $X_0^{p^n} = 1$, we have $b_m = 1$. This implies $X_0 = 1$. *q.e.d.*

We shall define three $Z_p\mathfrak{G}$-lattices which have important role in the subsequent sections. Put

$$X'_F = e^-Gal(\tilde{L}_{F_\infty}/F_\infty), \qquad Y'_F = e^-Gal(L_{F_\infty}/F_\infty).$$

They are not Z_p-free in general. Denote by $X_{F,0}$ (*resp.* $Y_{F,0}$) the maximal torsion submodule of X'_F (*resp.* Y'_F). We have the following $Z_p\mathfrak{G}$-lattices:

$$A_F = e^-Gal(\tilde{F}_{F_\infty}/L_{F_\infty}), \qquad X_F = X'_F/X_{F,0}, \qquad Y_F = X'_F/Y_{F,0}.$$

We use additive notation for these modules. By the assumption 3, the multiplication by T on Y'_F is injective. Thus, we have

$$
\begin{gathered}
0 \to A_F \to X'_F \to Y'_F \to 0, \\
X_{F,0} \cong Y_{F,0}, \\
0 \to A_F \to X_F \to Y_F \to 0.
\end{gathered}
\tag{3}
$$

4. Iwasawa theory and representations of the group $\mathfrak{G}$

Let $\{\zeta_{p^n}\}_{n\geq 1}$ be a set of primitive root of unity satisfying $\zeta_{p^n}^{p^{n-m}} = \zeta_{p^m}$ for $n > m$. Let $\boldsymbol{T}$ be the inverse limit of $\{\zeta_{p^n}\}$. It is a topological group which is isomorphic to the additive group of Z_p. We suppose

Assump. 4 F is an abelian extension over a totally real subfield k which is Galois over Q.

Let G be the Galois group of F/k. Let Z be the decomposition field of the prime ideal $\mathfrak{P}$ in F/k. We denote by H the Galois group of F over the decomposition field and put $\mathfrak{g} = G/H$.

Under the assumption 4, $F_\infty(\zeta_p)$ is an abelian extension over k and its Galois group acts on $\boldsymbol{T}$. By the assumption 3, this representation of $Gal(F_\infty(\zeta_p)/k)$ afforded by $\boldsymbol{T}$ is decomposed into a product of the representation of the finite abelian group $Gal(F(\zeta_p)/k)$ and that of $Gal(F_\infty/F)$. Let ω be the character of the representation of $Gal(F(\zeta_p)/k)$. Let $\hat{G} = \mathrm{Hom}(G, \mathcal{K}^\times)$ be the group of characters of G. We call $\chi \in \hat{G}$ an odd character if $\chi(\rho) = -1$ holds. Denote by $\hat{G}(-1)$ the set of all odd characters contained in $\hat{G}$. We have the following formula

$$v_p(L_p(0, \chi\omega)) = v_p(L(0, \chi)) + \sum_{\mathfrak{p}|p} v_p(1 - \chi(\mathfrak{p})) \tag{4}$$

for each $\chi \in \hat{G}(-1)$, where we denote by $\mathfrak{p}$ a prime ideal of k, *c.f.* (1.1) in [6]. The value $\chi(\mathfrak{p})$ is 0 if the prime ideal is ramified in the fixed field by $\mathrm{Ker}\,\chi$, which is denoted by F_χ. If $\mathfrak{p}$ is unramified, the value $\chi(\mathfrak{p})$ is equal to the value at the Frobenius automorphism of F_χ over k. Thus, $\chi(\mathfrak{p}) = 1$ is equivalent to that the prime is completely decomposed in F_χ/k. Let $\mathfrak{p}$ be restriction of the prime ideal $\mathfrak{P}$ onto k. Let s be the degree of the decomposition field Z_0 in k/Q. Let

$$Gal(k/Q) = \dot{\bigcup}_{i=1}^{s} \sigma_i Gal(k/Z_0)$$

be a disjoint union of right cosets. The set of conjugate ideals of $\mathfrak{p}$ is $\{\sigma_i\mathfrak{p}\}_{i=1,\cdots,s}$.

We make the $Z_p\mathfrak{G}$-lattice A_F a Z_pG-lattice by restriction: $A_F\downarrow_G$. If $\chi \in \hat{G}(-1)$ dose not belong to the character group $\hat{\mathfrak{g}}$, we have $e_\chi \mathcal{K} A_F = 0$. Thus, $A_{F,\chi} = 0$. Put

(5) $$a_\chi = \dim_{\mathcal{K}} e_\chi \mathcal{K} A_F$$

for $\chi \in \hat{G}$. This value is equal to the $\mathcal{O}$-rank of $A_{F,\chi}$.

Lemma 4.1. $a_\chi = \sharp\{\sigma_i : \chi(\sigma_i\mathfrak{p}) = 1\}$.

Proof: The $Z_p\mathfrak{G}$-module A_F is isomorphic to the induced module from the trivial $Z_p\mathfrak{H}$-module Z_p. Let $\varepsilon_{\mathfrak{H}}$ be the trivial character of $\mathfrak{H}$ and $\varepsilon_{\mathfrak{H}}\uparrow^{\mathfrak{G}}$ be the induced character. We have

$$a_\chi = \langle \varepsilon_{\mathfrak{H}}\uparrow^{\mathfrak{G}}\downarrow_G, \chi\rangle_G.$$

Furthermore, by Mackey's subgroup theorem, *c.f.* Theorem 19.6 in [1], we have

$$\varepsilon_{\mathfrak{H}}\uparrow^{\mathfrak{G}}\downarrow_G = \sum_{i=1}^{s} \varepsilon_{\sigma_i\mathfrak{H}\sigma_i^{-1}} \downarrow_{\sigma_i\mathfrak{H}\sigma_i^{-1}\cap G}\uparrow^G .$$

Since G is a normal subgroup of $\mathfrak{G}$, we have

$$\sigma_i\mathfrak{H}\sigma_i^{-1} \cap G = \sigma_i(\mathfrak{H}\cap G)\sigma_i^{-1} = \sigma_i H \sigma_i^{-1}.$$

Thus, we have

$$a_\chi = \sum_{i=1}^{s} \langle \varepsilon_{\sigma_i H\sigma_i^{-1}}\uparrow^G, \chi\rangle_G = \sum_{i=1}^{s} \langle \varepsilon_{\sigma_i H\sigma_i^{-1}}, \chi\downarrow_{\sigma_i H\sigma_i^{-1}}\rangle_{\sigma_i H\sigma_i^{-1}}$$

because the restriction of $\varepsilon_{\sigma_i\mathfrak{H}\sigma_i^{-1}}$ onto $\sigma_i H\sigma_i^{-1}$ is the trivial character of $\sigma_i H\sigma_i^{-1}$. $\sigma_i H\sigma_i^{-1}$ is contained in $Gal(F/F_\chi)$ if and only if

$$\langle \varepsilon_{\sigma_i H\sigma_i^{-1}}, \chi\downarrow_{\sigma_i H\sigma_i^{-1}}\rangle_{\sigma_i H\sigma_i^{-1}} = 1.$$

In the sequel, a_χ equals the number of completely decomposed $\sigma_i\mathfrak{p}$ in F_χ/F. *q.e.d.*

Lemma 4.2. *Let e be an idempotent element containd in e^-Z_pG. Let $f_{F/k}$ be the relative degree of the prime ideal $\mathfrak{P}$ in the extension F/k. We have*

$$\sum_{\substack{\chi(e)\neq 0\\ \chi(\sigma_i\mathfrak{p})\neq 1}} v_p(1-\chi(\sigma_i\mathfrak{p})) = v_p(f_{F/k})\dim_{\mathcal{K}} e\mathcal{K}\mathfrak{g}.$$

Proof: Since k is a Galois field, the relative degree of an arbitrary conjugate prime of $\mathfrak{p}$ equals $f_{F/k}$. Since the proof of the formula for $i > 1$ is similar as that of $i = 1$, we show the formula for $\mathfrak{p}$. If $\mathfrak{p}$ is ramified in F_χ, we have $\chi(\mathfrak{p}) = 0$. Such character is excluded. Hence, we suppose $\chi(\mathfrak{p}) \neq 0$. Let F_0 be the inertia field of $\mathfrak{p}$ in F/k. Since F_0 contains the decomposition field Z, it is sufficient to prove the formula under the assumption that $\mathfrak{p}$ is unramified in F. Since the abelian group G is a direct product of the p-Sylow subgroup G_p and a p'-subgroup G_0, every character χ is a product of a character $\chi_p \in \hat{G}_p$ and a character $\chi_0 \in \hat{G}_0$. Thus, $\chi \in \hat{G}(-1)$ is equivalent to that $\chi_0 \in \hat{G}_0(-1)$. Denote by $\mathfrak{g}_p$ the p-Sylow subgroup of $\mathfrak{g}$. Moreover, $\chi \notin \hat{\mathfrak{g}}$ is equivalent to that it satisfies either condition of

(i) $\chi_0 \notin \hat{\mathfrak{g}}$, (ii) $\chi_0 \in \hat{\mathfrak{g}}$ and $\chi_p \notin \hat{\mathfrak{g}}_p$.

If $\chi \in \hat{G}(-1)$ satisfies the condition (i), we see $\chi(\mathfrak{p})$ is not any p-power root of 1. Thus, $v_p(1-\chi(\mathfrak{p})) = 0$. Suppose χ satisfies the condition (ii). H is a cyclic group of order $f_{F/k}$ generated by the Frobenius automorphism. Hence, the p-Sylow subgroup H_p is also cyclic. Let σ be a generator of H_p and let $\tilde{\psi}$ be a generator of the character group. Put $d = v_p(f_{F/k})$. Note $\zeta = \tilde{\psi}(\sigma)$ is a primitive p^dth root of unity. We have an exact sequence of character groups of p-Sylow subgroups:

$$1 \to \hat{\mathfrak{g}}_p \to \hat{G}_p \to \hat{H}_p \to 1.$$

Let ψ be an inverse image of $\tilde{\psi}$ in $\hat{G}_p$. Let χ be a character satisfying the condition (ii) in the above. There are $\chi_1 \in \hat{\mathfrak{g}}(-1)$ and an integer i snch that $1 \le i < p^d$ and $\chi = \chi_1\psi^i$. Note $\chi(e) \neq 0$ if and only if $\chi_1(e) \neq 0$. Since $\chi(\sigma) = \zeta^i$, we have

$$\begin{aligned}
\sum_{\substack{\chi(\mathfrak{p})\neq 1 \\ \chi(e)\neq 0}} v_p(1-\chi(\mathfrak{p})) &= \sum_{\substack{\chi_1\in\hat{\mathfrak{g}} \\ \chi(e)\neq 0}} \sum_{i=1}^{p^d-1} v_p(1-\zeta^i) \\
&= \sharp\{\chi \in \hat{\mathfrak{g}} : \chi(e) \neq 0\} \sum_{i=1}^{d} \sum_{\substack{1\le j<p^i \\ (p,j)=1}} v_p(1-\zeta_{p^i}^j) \\
&= \sharp\{\chi \in \hat{\mathfrak{g}} : \chi(e) \neq 0\} \sum_{i=1}^{d} 1 = d \cdot \dim_{\mathcal{K}} e\mathcal{K}\mathfrak{g}
\end{aligned}$$

q.e.d.

Lemma 4.3. *Put* $\mathfrak{g}_i = G/\sigma_i H \sigma_i^{-1}$. *We have*

$$\sum_{i=1}^{s} \dim_{\mathcal{K}} e^-\mathcal{K}\mathfrak{g}_i = \dim_{\mathcal{K}} e^-\mathcal{K}\mathfrak{G}/\mathfrak{H}.$$

Proof: Let ε^- be the $\mathcal{K}$-character afforded by $\mathcal{K}G$-module $e^-\mathcal{K}G$. We have

$$\begin{aligned}
\dim_{\mathcal{K}} e^-\mathcal{K}\mathfrak{G}/\mathfrak{H} &= \langle \varepsilon_{\mathfrak{H}}\uparrow^{\mathfrak{G}}, \varepsilon^-\uparrow^{\mathfrak{G}}\rangle_{\mathfrak{G}} = \langle \varepsilon_{\mathfrak{H}}\uparrow^{\mathfrak{G}}\downarrow_G, \varepsilon^-\rangle_G \\
&= \langle \sum_{i=1}^{s} \varepsilon_{\sigma_i\mathfrak{H}\sigma_i^{-1}}\downarrow_{\sigma_i\mathfrak{H}\sigma_i^{-1}\cap G}\uparrow^G, \varepsilon^-\rangle_G = \sum_{i=1}^{s}\langle \varepsilon_{\sigma_i H\sigma_i^{-1}}\uparrow^G, \varepsilon^-\rangle_G \\
&= \sum_{i=1}^{s} \dim_{\mathcal{K}} e^-\mathcal{K}\mathfrak{g}_i
\end{aligned}$$

q.e.d.

Lemma 4.4. $a_\chi = 0$ *if and only if* $X_{F,\chi} \cong Y_{F,\chi}$.

Proof: We see $a_\chi = 0$ if and only if $A_{F,\chi} = 0$. We have an exact sequence of χ-parts:

$$0 \to A_{F,\chi} \to X_{F,\chi} \to Y^*_{F,\chi} \to 0.$$

If $A_{F,\chi} = 0$, $Y^*_{F,\chi}$ is isomorphic to $X_{F,\chi}$, and hence torsion free. Thus, $Y^*_{F,\chi} \cong Y_{F,\chi}$. Conversely, suppose $X_{F,\chi} \cong Y_{F,\chi}$. It follows that the homomorphism of $\mathcal{K}X_{F,\chi}$ onto $\mathcal{K}Y_{F,\chi}$ is isomorphic. Hence, $e_\chi\mathcal{K}X_F \cong e_\chi\mathcal{K}Y_F$. We have $e_\chi\mathcal{K}A_F = 0$. It follows $A_{F,\chi} = 0$. *q.e.d.*

Lemma 4.5. *Let K be a totally imaginary intermediate field of F/k. Let χ be an odd character which is trivial on $Gal(F/K)$. Then, we have $X_{F,\chi}$ is isomorphic to a sublattice of $X_{K,\chi}$ with finite index.*

Proof: Let $\tilde{L}_{ab}$ be the maximal abelian extension of K_∞ contained in $\tilde{L}_{F_\infty}$. Let $\tilde{L}_{cent}$ be the maximal central subextension of $\tilde{L}_{F_\infty}$ with respect to F_∞/K_∞. $\tilde{L}_{cent}$ is a finite abelian extension of $\tilde{L}_{ab}$. We shall show $\tilde{L}_{ab}/\tilde{L}_{K_\infty}F$ is also a finite extension. Let L be a finite abelian extension of $\tilde{L}_{K_\infty}F$ contained in $\tilde{L}_{ab}$. There is a finite abelian extension L' of an nth layer K_n of the $\boldsymbol{Z}_p$-extension K_∞ such that $L = L'\tilde{L}_{K_\infty}F$ and such that $L' \supset \tilde{L}_{K_n}F_n$. Let U_n be the group of unit ideles of K_n and $\tilde{U}_n$ be the group of those of F_n. By genus number formula, we have

$$(L' : \tilde{L}_{K_n}F_n)\Big|\,|U_n : N_{F_n/K_n}(\tilde{U}_n)|.$$

Since the number of prime factors in K_∞ of an arbitrary prime number is finite, there is the maximum of the indexes $|U_n : N_{F_n/K_n}(\tilde{U}_n)|$:

$$a = \max_n |U_n : N_{F_n/K_n}(\tilde{U}_{F_n})|.$$

We have

$$(L : \tilde{L}_{K_\infty}F)\big|(L' : \tilde{L}_{K_n}F_n)\big|a.$$

This implies the degree of L's over $\tilde{L}_{K_\infty}F$ is bounded. Therefore, $\tilde{L}_{ab}$ is a finite extension of $\tilde{L}_{K_\infty}F$. It follows that $\tilde{L}_{cent}$ is also finite over $\tilde{L}_{K_\infty}F$.

Let I be the ideal of the group ring $\boldsymbol{Z}_pG$ generated by $\{\sigma - 1 : \sigma \in Gal(F/K)\}$. We have

$$Gal(\tilde{L}_{cent}/F_\infty) \cong X'_F/I \cdot X'_F.$$

Put $M' = e^- Gal(\tilde{L}_{F_\infty}/\tilde{L}_{K_\infty}F)$ and $Z'_K = e^- Gal(\tilde{L}_{K_\infty}/\tilde{L}_{K_\infty} \cap F_\infty)$. Z'_K is a submodule of X'_K with finite index. Since $I \cdot X'_F = e^- Gal(\tilde{L}_{F_\infty}/\tilde{L}_{cent})$, there is a commutative diagram comprising exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M'/I \cdot X'_F & \longrightarrow & X'_F/I \cdot X'_F & \longrightarrow & Z'_K & \longrightarrow & 0 \\
 & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \\
0 & \longrightarrow & e^- Gal(\tilde{L}_{cent}/\tilde{L}_{K_\infty}F) & \longrightarrow & e^- Gal(\tilde{L}_{cent}/F_\infty) & \longrightarrow & e^- Gal(\tilde{L}_{K_\infty}F/F_\infty) & \longrightarrow & 0.
\end{array}$$

Denote by Z_K the quotient of Z'_K by the maximal torsion submodule. Since $e^- Gal(\tilde{L}_{cent}/\tilde{L}_{K_\infty}F)$ is of finite order, the quotient of $X'_F/I \cdot X'_F$ by its maximal torsion submodule is isomorphic to Z_K. By taking the homology long exact sequence on a short exact sequence:

$$0 \to X_{F,0} \to X'_F \to X_F \to 0,$$

we have

$$\to H_1(Gal(F/K), X_F) \to X_{F,0}/I \cdot X_{F,0} \to X'_F/I \cdot X'_F \to X_F/I \cdot X_F \to 0.$$

Hence, the quotient of $X'_F/I \cdot X'_F$ by its maximal torsion submodule is isomorphic to the quotient of $X_F/I \cdot X_F$ by its torsions. Denote by X'' a submodule of X_F such that $X''/I \cdot X_F$ is the torsion module. Since X_F is a $\boldsymbol{Z}_p$-free module of finite rank, $X''/I \cdot X_F$ is of finite order. We have a commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X'' + I \cdot X_F & \longrightarrow & X_F & \longrightarrow & Z_K & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & e_\chi I \cdot \mathcal{K}X_F & \longrightarrow & e_\chi \mathcal{K}X_F & \longrightarrow & e_\chi \mathcal{K}X_K & \longrightarrow & 0
\end{array}$$

Since $e_\chi I \cdot \mathcal{K}G = 0$, we see $e_\chi I \cdot \mathcal{K}X_F = 0$. This implies $(X'' + I \cdot X_F)_\chi = 0$. It follows $X_{F,\chi} \cong (Z_K)^*_\chi$. We have $X_{F,\chi} \cong Z_{K,\chi}$. *q.e.d.*

5. The main result

We shall approach the problem to generalize the formula in [9] to the CM-field. We shall study the factor

$$\sum_{\mathfrak{p}|p} v_p(1-\chi(\mathfrak{p}))$$

in the formula (4). We suppose every character $\chi \in \hat{G}(-1)$ satisfies

Assump. 5 If $\chi(\sigma_i \mathfrak{p}) = 1$ holds for one of σ_i's, then every σ_i's satisfies $\chi(\sigma_i \mathfrak{p}) = 1$.

This assumption holds if the decomposition field Z is a Galois extension over $\boldsymbol{Q}$.

Lemma 5.1. *the assumption 5 holds if F is a cyclic extension of k.*

Proof: In a cyclic extension, there is a unique intermediate field of a given degree. Since k is a Galois field, the degree of $\sigma_i Z$ over k is equal to that of Z. Hence, $Z = \sigma_i Z$. It follows that Z is a Galois extension over $\boldsymbol{Q}$. *q.e.d.*

Let μ_F^χ (*resp.* μ_Z^χ) be the μ-invariant of $X_F'^\chi$ (*resp.* $X_Z'^\chi$). The main theorem of Iwasawa theory, we have

$$v_p(L_p(0,\omega\chi^{-1})) = v_p(f_{F,\chi}(0)) + \mu_F^\chi \tag{6}$$

if $\chi \neq \omega$. We consider the case of $\chi \neq \omega$. We could realize that the argument is easily modified to adapt it for this excluded case from the formula

$$v_p(L(0,\varepsilon)) = v_p(f_{F,\omega}(0)) + v_p(w_F),$$

where w_F be the number of roots of unity contained in F, *c.f.* the formula (1.3) and Theorem 1.2 in [6]. Let e be an idempotent element contained in $e^- \boldsymbol{Z}_p G$ such that $\omega(e) = 0$. Put

$$\Phi_e = \{\chi \in \hat{G}(-1) : \chi(e) \neq 0\}, \qquad \Phi_e' = \Phi_e \backslash \hat{\mathfrak{g}}.$$

We set $M' = e^- Gal(L_{F_\infty}/L_{Z_\infty}F)$. We have an exact sequence of Galois groups:

$$0 \to M' \to e^- Y_{F_\infty}' \to e^- Y_{Z_\infty}' \to 0.$$

Since the characteristic polynomial of eY_F' and eY_Z' are not divisible by T by the assumption 2. We have a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & eM'/TeM' & \longrightarrow & eY_F'/TeY_F' & \longrightarrow & eY_Z'/TeY_Z' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & eGal(L_{F_\infty}^*/L_{Z_\infty}^*F) & \longrightarrow & eGal(L_{F_\infty}^*/F_\infty) & \longrightarrow & eGal(L_{Z_\infty}^*/Z_\infty) & \longrightarrow & 0. \end{array} \tag{7}$$

Let g_χ be the characteristic polynomial of $Y_{F,\chi}$. Since eA_F is a trivial Λ-module by Lemma 3.1, its characteristic polynomial is T^{a_χ} from (5). Hence, the characteristic polynomial of $X_{F,\chi}$ is $g_\chi T^{a_\chi}$. If $\chi \notin \hat{\mathfrak{g}}$, we see $a_\chi = 0$ by Lemma 4.1. It follows that g_χ is also the characteristic polynomial of $X_{F,\chi}$ from Lemma 4.4. Namely, $f_\chi = g_\chi$. Note $X_{Z,\chi} = Y_{Z,\chi} = 0$, because $e_\chi \mathcal{K} X_Z$ and $e_\chi \mathcal{K} Y_Z$ vanish. If $\chi \in \hat{\mathfrak{g}}(-1)$, the characteristic polynomial $f_{Z,\chi}$ of $X_{Z,\chi}$ is equal to that of $X_{F,\chi}$ by Lemma 4.5. Namely, $f_{Z,\chi} = g_\chi T^{a_\chi}$. Since

$$a_\chi = \dim_{\mathcal{K}} e_\chi \mathcal{K} A_F = \dim_{\mathcal{K}} e_\chi \mathcal{K} A_Z$$

by Lemma 4.1, the characteristic polynomial of $Y_{Z,\chi}$ is equal to $g_\chi = f_{Z,\chi}/T^{a_\chi}$.

Let g_F (*resp.* g_Z) be the characteristic polynomial of eY_F (*resp.* eY_Z). We have

$$g_F = \prod_{\chi\in\Phi_e} g_\chi, \qquad \left(resp. \quad g_Z = \prod_{\chi\in\Phi_e\cap\hat{\mathfrak{g}}} g_\chi \right). \tag{8}$$

Let μ_F (*resp.* μ_Z) be the μ-invariant of eY'_F (*resp.* eY'_Z). By (3), μ_F (*resp.* μ_Z) is also the μ-invariant of eX'_F (*resp.* eX'_Z).

Theorem 5.2. *We have the following formula*

$$\ell(eGal(L^*_{F_\infty}/L^*_{Z_\infty}F)) = \sum_{\chi\in\Phi'_e} v_p(L(0,\chi^{-1})) + v_p(f_{F/k})\cdot \mathrm{rank}_{Z_p} eA_F + (\mu_F - \sum_{\chi\in\Phi_e}\mu_F^\chi) - \mu_Z.$$

Proof: By the assumption 3, we have $L_{Z_\infty}\cap F_\infty = Z_\infty$. Hence,

$$\ell(eGal(L^*_{F_\infty}/L^*_{Z_\infty}F)) = \ell(eY'_F/TeY'_F) - \ell(eY'_Z/TeY'_Z).$$

By Lemma 2.1, the value in the right equals

$$v_p(g_F(0)) - v_p(g_Z(0)) + \mu_F - \mu_Z.$$

Furthermore, by Lemma 2.2 and (8), we see

$$\begin{aligned} v_p(g_F(0)) - v_p(g_Z(0)) &= \sum_{\chi\in\Phi_e} v_p(g_\chi(0)) - \sum_{\chi\in\Phi_e\cap\hat{\mathfrak{g}}} v_p(g_\chi(0)) \\ &= \sum_{\chi\in\Phi'_e} v_p(g_\chi(0)). \end{aligned}$$

Since $f_\chi = g_\chi$ for $\chi\in\Phi'_e$, we have

$$\sum_{\chi\in\Phi'_e} v_p(g_\chi(0)) + \sum_{\chi\in\Phi'_e}\mu_F^\chi = \sum_{\chi\in\Phi'_e} v_p(L_p(0,\chi^{-1}\omega))$$

from (6). By virtue of the formula (4) and Lemma 4.2, we can transform the sum of $v_p(L_p(0,\chi^{-1}\omega))$'s in the right hand side of this equality into a sum relative to values L-functions:

$$\begin{aligned} \sum_{\chi\in\Phi'_e} v_p(L_p(0,\chi^{-1}\omega)) &= \sum_{\chi\in\Phi'_e} v_p(L(0,\chi^{-1})) + \sum_{\chi\in\Phi'_e}\sum_{\mathfrak{p}|p} v_p(1-\chi^{-1}(\mathfrak{p})) \\ &= \sum_{\chi\in\Phi'_e} v_p(L(0,\chi^{-1})) + s\cdot v_p(f_{F/k})\dim_{\mathcal{K}} e\mathcal{K}\mathfrak{g}. \end{aligned}$$

Put

$$\tilde{\mu}_F = \mu_F - \sum_{\chi\in\Phi'_e}\mu_F^\chi.$$

As a consequence, we obtain a formula

$$\ell(eGal(L^*_{F_\infty}/L^*_{Z_\infty}F)) = \sum_{\chi\in\Phi'_e} v_p(L(0,\chi^{-1})) + s\cdot v_p(F_{F/k})\dim_{\mathcal{K}} e\mathcal{K}\mathfrak{g} + \tilde{\mu}_F - \mu_Z.$$

Here, by Lemma 3.1 and 4.3, we see $s\dim_{\mathcal{K}} e\mathcal{K}\mathfrak{g} = \dim_{\mathcal{K}} e\mathcal{K}\mathfrak{G}/\mathfrak{H} = \mathrm{rank}_{Z_p} eA_F$. *q.e.d.*

Let Z' be the decomposition field of $\mathfrak{P}$ in F/Q and $f_{F/Q}$ be the degree over Q of the prime $\mathfrak{P}$. Z' is a subfield of the decomposition field Z in F/k. We recall the p-adic maps:

$$\varphi_F = \varphi \circ N_{F/Z'}, \quad \varphi_Z = \varphi \circ N_{Z/Z'}, \quad \psi_F = \psi \circ N_{F/Z'}, \quad \psi_Z = \psi \circ N_{Z/Z'}.$$

The relative degree $f_{F/Z'}$ (*resp.* $f_{Z/Z'}$) the prime ideal $\mathfrak{P}$ (*resp.* $\mathfrak{P}_{|Z}$) in F/Z' (*resp.* Z/Z') is equal to the degree over Q. Namely, $f_{F/Z'} = f_{F/Q}$ and $f_{Z/Z'} = f_{Z/Q}$. We recall

$$U_F = f_{F/Q}U, \qquad U_Z = f_{Z/Q}U = f_{k/Q}U,$$

where $f_{k/Q}$ is the degree of the prime $\mathfrak{P}|_k$ over Q. Let M be a subgroup of $E_{1,Z}$ such that

$$|e^- Z_p \otimes E_{1,Z} : e^- Z_p \otimes M| < \infty$$

and such that it is stable by the action of G. Denote by V_F (*resp.* V_Z) the image of $e^- Z_p \otimes M$ by φ_F (*resp.* φ_Z). Similarly, we denote by $\mathcal{V}_F$ (*resp.* $\mathcal{V}_Z$) the image of $e^- Z_p \otimes M$ by ψ_F (*resp.* ψ_Z). Since M is a subgroup of the multiplicative group $Z^\times$ and since $(F : Z) = f_{F/k}$, we see

$$V_F = f_{F/k}V_Z, \qquad \mathcal{V}_F = f_{F/k}\mathcal{V}_Z.$$

If a well-behaved such subgroup M is selected, we could have the refined formula of the relative class number formula. We start the following equations:

$$\begin{aligned} \ell(eU_F/eV_F) &= \ell(eU_Z/eV_Z) - \ell(eU_Z/f_{F/k}eU_Z) + \ell(eV_Z/f_{F/k}eV_Z) \\ &= \ell(eU_Z/eV_Z) + \big(v_p(f_{F/k}) - v_p(f_{F/k})\big)\,\mathrm{rank}_{Z_p} eU_Z. \\ \ell(e\mathcal{U}/e\mathcal{V}_F) &= \ell(e\mathcal{U}/e\mathcal{V}_Z) + \ell(e\mathcal{V}_Z/f_{F/k}\mathcal{V}_Z) \\ &= \ell(e\mathcal{U}/e\mathcal{V}_Z) + v_p(f_{F/k})\mathrm{rank}_{Z_p} e\mathcal{U}. \end{aligned}$$

By Proposition 1.1, the isomorphism (1) and Proposition 1.3, we have

$$\begin{aligned} \ell(eU_F/eV_F) - \ell(e\mathcal{U}/e\mathcal{V}_F) &= \ell(eU_F/eW_F) - \ell(e\mathcal{U}/e\mathcal{W}_F) \\ &= \ell(eGal(\tilde{L}_F/L_F)) - \ell(eGal(L^*_{F_\infty}/L_F)) \\ &= \ell(eGal(\tilde{L}_F/F)) - \ell(eGal(L^*_{F_\infty}/F)). \end{aligned}$$

Since $eGal(L^*_{F_\infty}/F)$ is isomorphic to eY'_F/TeY'_F, the following formula is obtained:

(9) $$\ell(eGal(\tilde{L}_F/F)) = \ell(eY'_F/TeY'_F) + \ell(eU_Z/eV_Z) - \ell(e\mathcal{U}/e\mathcal{V}_Z) - v_p(f_{F/k})\mathrm{rank}_{Z_p} eU_Z.$$

By setting $F = Z$, we have the following result from Lemma 2.1 and 2.2:

Theorem 5.3. *We have the following formula:*

$$\ell(eGal(\tilde{L}_Z/Z)) = \ell(eU_Z/eV_Z) - \ell(e\mathcal{U}/e\mathcal{V}_Z) + \sum_{\chi \in \hat{\mathfrak{G}} \cap \Phi_e} v_p(g_\chi(0)) + \mu_Z.$$

The difference of $\ell(eY'_F/TeY'_F)$ from $\ell(eY'_Z/TeY'_Z)$ is given in Theorem 5.2. By (9), we have an interesting formula.

Corollary 5.4.

$$\ell(eGal(\tilde{L}_F/F)) = \ell(eGal(\tilde{L}_Z/Z)) + \sum_{\chi \in \Phi'_e} v_p(L(0, \chi^{-1})) + \tilde{\mu}_F - \mu_Z.$$

Proof: By (7), (9) and Theorem 5.2, 5.3, we have

$$\begin{aligned}\ell(eGal(\tilde{L}_F/F)) &= (\ell(eY'_Z/TeY'_Z)+\ell(eM'/TeM'))+\ell(eU_Z/eV_Z)-\ell(e\mathcal{U}/e\mathcal{V}_Z)-v_p(f_{F/k})\mathrm{rank}_{\mathbf{Z}_p}eA_F\\ &= \ell(eY'_Z/TeY'_Z)+\ell(eU_Z/eV_Z)-\ell(e\mathcal{U}/e\mathcal{V}_Z)+\sum_{\chi\in\Phi'_e}v_p(L(0,\chi^{-1}))+\tilde{\mu}_F-\mu_Z\\ &= \ell(eGal(\tilde{L}_Z/Z))+\sum_{\chi\in\Phi'_e}v_p(L(0,\chi^{-1}))+\tilde{\mu}_F-\mu_Z.\end{aligned}$$

q.e.d.

By this corollary, our task is divided into two parts. One of them is to prove $\tilde{\mu}_F=0$ holds for F and the second to prove the refined formula for the decomposition field Z. A little progress is obtained by the following proposition:

Proposition 5.5. *Let χ be a character belonging to $\hat{\mathfrak{g}}(-1)\backslash\{\omega\}$. Put $m_\chi=\mu_F^\chi=\mu_Z^\chi$. Then, we have*

$$m_\chi+v_p(g_\chi(0))=v_p(L_p^{(a_\chi)}(0,\chi^{-1}\omega))-a_\chi-v_p(a_\chi!).$$

Proof: We select the generator γ of $Gal(F_\infty/F)$ so that $\zeta_{p^n}^\gamma=\zeta_{p^n}^{p+1}$ holds for every n. By the main theorem of Iwasawa theory,

$$L_p(1-s,\chi^{-1}\omega)=p^{\mu_\chi}f_\chi(\varphi(T))u_\chi(\varphi(T))\big|_{T=(1+p)^s-1},$$

where u_χ is a unit power series and

$$\varphi(T)=(1+p)(1+T)^{-1}-1$$

c.f. (1.3), Theorem 1.2 and Theorem 1.4 in [6]. Put

$$D=\frac{1}{\log_p(1+p)}\frac{d}{ds}.$$

Since $f_\chi=g_\chi T^{a_\chi}$, we have

$$\begin{aligned}D^{a_\chi}L_p(1-s,\chi^{-1}\omega)\Big|_{s=1} &= \left(\frac{-1}{\log_p(1+p)}\right)^{a_\chi}L_p^{(a_\chi)}(0,\chi^{-1}\omega),\\ D^{a_\chi}f_\chi(\varphi((1+p)^s-1))u_\chi(\varphi((1+p)^s-1))\Big|_{s=1} &= a_\chi!g_\chi(0)u_\chi(0).\end{aligned}$$

Hence,

$$v_p(L_p^{(a_\chi)}(0,\chi^{-1}\omega))-a_\chi=m_\chi+v_p(a_\chi!)+v_p(g_\chi(0)).$$

q.e.d.

At the conclusion of this article, we will discuss what questions are needed to settle to take steps forward more concretely. If the formula

$$\ell(eGal(\tilde{L}_F/F))=\sum_{\chi(e)\neq 0}v_p(L(0,\chi^{-1}))$$

held, we would have

$$\sum_{\chi\in\Phi_e\cap\hat{\mathfrak{g}}}v_p(L(0,\chi^{-1}))=\ell(eGal(\tilde{L}_Z/Z))+\tilde{\mu}_F-\mu_Z$$

from Corollary 5.4. Furthermore, by Theorem 5.3 and Proposition 5.5, the value of the sum in the left would equal

$$\ell(eU_Z/eV_Z) - \ell(e\mathcal{U}/e\mathcal{V}_Z) + \sum_{\chi\in\Phi_e\cap\hat{\mathfrak{g}}} \left(v_p(L_p^{(a_\chi)}(0,\chi^{-1}\omega)) - a_\chi - v_p(a_\chi!) - m_\chi\right) + \mu_Z.$$

Hence, we would have

(10)

$$\ell(eU_Z/eV_Z)-\ell(e\mathcal{U}/e\mathcal{V}_Z) = \sum_{\chi\in\Phi_e\cap\hat{\mathfrak{g}}} v_p(L(0,\chi^{-1}))-\sum_{\chi\in\Phi_e\cap\hat{\mathfrak{g}}} \left(v_p(L_p^{(a_\chi)}(0,\chi^{-1}\omega)) - a_\chi - v_p(a_\chi!)\right)+\sum_{\chi\in\Phi'_e} m_\chi-\mu_Z.$$

At this point, we remain the following problems unsolved:

(i) to verify that the μ-invariant satisfies $\tilde{\mu}_F = 0$ and $\mu_Z = \sum_{\chi\in\Phi'_e} m_\chi$.

(ii) to construct a subgroup M which has a good arithmetical property and to prove a relation (10) holds.

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