A note on some properties of p^m-singular numbers of algebraic number fields

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1. INTRODUCTION

Let k be a number field of finite degree over the field $\boldsymbol{Q}$ of rational numbers. Let p be a prime number and P be the set of prime divsors $\mathfrak{p}$ of k such that $\mathfrak{p}\mid p$. Let S be a finite set of prime divisors of k such that $S\supset P$. We call an element a of the multiplicative group $k^\times$ p^m-singular with respect to S if it satisfies the following condition (i) and (ii) :

(i) a is of a locally p^m-th power at every $\mathfrak{p}\in S$.

(ii) the principal ideal (a) of k is a p^m-th power of an ideal $\mathfrak{a}$ of k.

Let $\mathcal{S}(p^m)$ be the set of all p^m-singular numbers in $k^\times$. Since $\mathcal{S}(p^m)\supset k^{\times p^m}$, we denote by $Б(p^m)$ the factor group $\mathcal{S}(p^m)/k^{\times p^m}$. We have a homomorphism $c^{(m)}$ of $Б(p^m)$ into the ideal class group of k by the condition (ii) : a map $ak^{\times p^m}\to cl(\mathfrak{a})$, where $cl(\mathfrak{a})$ denotes the ideal class containing $\mathfrak{a}$. Denote by $Б^{(0)}(p^m)$ and $Б^{(1)}(p^m)$ the image and kernel of $c^{(m)}$, respectively. We mean the canonical map $i_{m,n}: Б(p^n)\to Б(p^m)$ for $m>n$ a map $ak^{\times p^n}\to a^{p^{m-n}}k^{\times p^m}$. Let $Б(p^\infty)$ denote the inductive limit $\varinjlim Б(p^m)$. Denote by $Б^{(1)}(p^\infty)$ the subgroup $\varinjlim Б^{(1)}(p^m)$. We observe $c^{(m)}\circ i_{m,n}=c^{(n)}$, which implies $Б^{(0)}(p^n)\subset Б^{(0)}(p^m)$. Hence, we define $Б^{(0)}(p^\infty)$ to be $\cup_{m\geq 1}Б^{(0)}(p^m)$ and obtain the following exact sequence for $m=1,\cdots,\infty$:

$$(1.1)\qquad 1\to Б^{(1)}(p^m)\to Б(p^m)\to Б^{(0)}(p^m)\to 1.$$

Let H be composite of cyclic p-extensions of k which are embedded into cyclic p-extensions of k with arbitrary degree. A Galois extension of k is called S-ramified if every primes of k ramified there belong to S. Let M_S be the maximal S-ramified abelian p-extension of k. By the table in [3,I.1.c], we have $M_P\supset H$. Hence $M_S\supset H$. Set $\mathfrak{G}=\mathrm{Gal}(M_S/k)$, $\mathfrak{H}=\mathrm{Gal}(H/k)$ and $\mathfrak{T}=\mathrm{Gal}(M_S/H)$. Let μ be the p-Sylow subgroup of the torsion subgroup of $k^\times$. Denote by $k_\mathfrak{p}$ the completion of k at a prime divisor $\mathfrak{p}$. For a non-complex prime $\mathfrak{p}$, let $\mu_\mathfrak{p}$ be the p-Sylow subgroup of the torsion subgroup of $k_\mathfrak{p}^\times$. For a complex prime, set $\mu_\mathfrak{p}=\{1\}$. For a divisor $\mathfrak{m}$, denote by $k_\mathfrak{m}$ the direct product $\Pi_{\mathfrak{p}\mid\mathfrak{m}}k_\mathfrak{p}$. Let $U_\mathfrak{p}$ be the group of units of $k_\mathfrak{p}$. Denote by $U_\mathfrak{m}$ the group of units of $k_\mathfrak{m}$: $U_\mathfrak{m}=\Pi U_\mathfrak{p}$. We select one of the embeddings $k\to k_\mathfrak{m}$ and consider k a subring of $k_\mathfrak{m}$. Let E be the group of units of k. Denote by $\bar{E}_\mathfrak{m}$ the closure of E in $U_\mathfrak{m}$.

Let K be a finite Galois extension of k with Galois group G. We add sufix K or k to a symbol to specify the base field if necessary, for example, we use symbols $\mathcal{S}_K(p^m)$, $\mathcal{S}_k(p^m)$, $Б_K(p^m)$, $Б_k(p^m)$, $Б_K^{(0)}(p^m)$, $Б_k^{(0)}(p^m)$, μ_K, μ_k, *e.t.c*, if necessary. We use the symbol $\mathfrak{P}$ (resp. $\mathfrak{p}$) to denote a prime divisor of K (resp. k). $U_{\mathfrak{P}}$ (resp. $U_{\mathfrak{p}}$) denotes the group of units of $K_{\mathfrak{P}}$ (resp. $k_{\mathfrak{p}}$). A divisor of k is considered a divisor of K by extending onto K. We also write $U_{K,\mathfrak{m}}$ or $U_{k,\mathfrak{m}}$ if necessary.

We denote by the symbol $t_p^{(m)}(X)$ for an abelian group X and for a non-negative integer m the subgroup of p^m-torsions and $t_p^{(\infty)}(X)$ the p-subgroup of X, namely, $t_p^{(\infty)}(X) = \cup\, t_p^{(m)}(X)$. Dyenote by $\mu_{\mathfrak{m}}$ the p-subgroup of $U_{\mathfrak{m}}$. We abbreviate $t_p^{(m)}(\mu)$, $t_p^{(m)}(\mu_{\mathfrak{m}})$ to $\mu^{(m)}$, $\mu_{\mathfrak{m}}^{(m)}$.

The main results of the present paper are as follows.

THEOREM A. *Suppose the Leopoldt conjecture is valid for* (k, p). *Set* $\mathfrak{m} = \Pi_{\mathfrak{p}\in S}\, \mathfrak{p}$. *Then we have* $\mathfrak{T} \simeq \mu_{\mathfrak{m}}/\mu$, *and obtain the following diagram whose horizontal sequences and whose vertical sequence in the left are exact :*

$$
\begin{array}{ccccccccc}
 & & 1 & & & & & & \\
 & & \downarrow & & & & & & \\
1 & \longrightarrow & \mu_{\mathfrak{m}}^{(m)}/\mu^{(m)} & \longrightarrow & t_p^{(m)}(\mathfrak{S}) & \xrightarrow{s^{(m)}} & Б(p^m) & \longrightarrow & 1 \\
 & & \downarrow & & \| & & \downarrow{\scriptstyle c^{(m)}} & & \\
1 & \longrightarrow & t_p^{(m)}(U_{\mathfrak{m}}/\bar{E}_{\mathfrak{m}}) & \longrightarrow & t_p^{(m)}(\mathfrak{S}) & \longrightarrow & Б^{(0)}(p^m) & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle g^{(m)}} & & \downarrow{\scriptstyle s^{(m)}} & & \| & & \\
1 & \longrightarrow & Б^{(1)}(p^m) & \longrightarrow & Б(p^m) & \xrightarrow{c^{(m)}} & Б^{(0)}(p^m) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array}
$$

COROLLARY. *We have* $Б(p^\infty) \simeq t_p^{(\infty)}(\mathfrak{H})$.

The inclusion $\mu_K \subset K^\times$ induces a homomorphism $H^2(G, \mu_K) \to H^2(G, K^\times)$ of the cohomology groups. Denote by ι^2 this homomorphism.

THEOREM B. *Let K/k be a finite Galois extension with Galois group G.*

(1) *Suppose* $H^1(G, \mu_K)$ *vanishes and* ι^2 *is injective. Let* μ'_K *be a subgroup of* μ_K *such that* $\mu'_K \supset \mu_K^{p^m}$ *and* $\mu'_K/\mu_K^{p^m} \simeq (\mu_K/\mu_K^{p^m})^G$. *Then, we have an isomorphism*

$$Б_K(p^m)^G \simeq \frac{\mathcal{S}_K(p^m) \cap k^\times \mu'_K}{K^{\times p^m} \cap k^\times \mu'_K}$$

and an exact sequence

$$1 \to \mathcal{S}_K(p^m)^G \to (\mathcal{S}_K(p^m)\mu'_K)^G \to (\mu_K/\mu_K^{p^m})^G.$$

Moreover, if $\mu_K \cap \mathcal{S}_K(p^m) = \mu_K^{p^m}$, *we have the following isomorphism :*

$$Б_K(p^m)^G \simeq \frac{(\mathcal{S}_K(p^m)\mu'_K)^G}{(K^{\times p^m}\mu'_K)^G}.$$

(2) *Suppose S contains every primes which are ramified in K/k. Then, we have the following exact sequence :*

$$1 \to Б_k(p^m) \to \mathcal{S}_K(p^m)^G/k^{\times p^m} \to H^1(G, \mu_{K,\mathfrak{m}}^{(m)}).$$

Remark. 1. By Theorem I.2.36 of [2], $t_p^{(\infty)}(\mathfrak{H})$ is isomorphic to the kernel of infinitesimal and also to the Pontrjagin dual of the Hilbert kernel $\bar{H}_2(k)_p$ which is a subgroup of a twisted module of the maximal p-subgroup of $K_2(k)$.

2. Set $S=P$ in Theorem A. $t_p^{(\infty)}(\mathfrak{H})$ is isomorphic to $R_2(k)$ defined in [1]. We have k is p-rational if and only if the Leopoldt conjecture is valid for (k, p) and $Б(p^\infty)=\{1\}$, $\mu_{k,p}=\{1\}$. (*c. f.* [1])

3. Let ζ_{p^n} be a primitive p^n-th root of unity. Set $S=P$ and K be the maximal p-extnsion over k contained in $k(\zeta_{p^n})$. By Theorem B of [5], we have $Б_K^{(0)}(p)=\{1\}$ except of a finite number of n if the Greenberg conjecture is valid for (k, p). Conversely, by [5] and [6], we have the Greenberg conjectur is valid if $Б_K^{(0)}(p)=\{1\}$ and if every primes dividing p are completely ramified in $k(\zeta_{p^{n+1}})/k(\zeta_{p^n})$.

2. THE TORSION SUBGROUP OF $\mathfrak{G}$ AND THE PROOF OF THEOREM A

We observe $Б^{(1)}(p^m)$ is generated by $\mathcal{S}^{(1)}(p^m)=E\cap\mathcal{S}(p^m)$. Hence, $Б^{(1)}(p^m)\simeq\mathcal{S}^{(1)}(p^m)/E^{p^m}$. We fix an embedding $\iota_\mathfrak{p}: k\to k_\mathfrak{p}$ which preserves $\mathfrak{p}$. Denote by ι the embedding of k into $k_\mathfrak{m}$ obtained from $\iota_\mathfrak{p}: \iota=\Pi_{\mathfrak{p}|\mathfrak{m}}\iota_\mathfrak{p}$. $\bar{E}_m$ denotes the completion of $\iota(E)$. We have $\bar{E}_\mathfrak{m}=\iota(E)\bar{E}_\mathfrak{m}^n$ for an arbitrary natural number n.

LEMMA 1. *Suppose the Leopoldt conjecture is valid for* (k, p). *Then we have a non-negative integer* $t(m)$ *for* m *such that* $\iota(E)\cap U_m^{p^t}\subset\iota(E)^{p^m}$ *if and only if* $t\geq t(m)$.

PROOF: When $S=P$, the statement follows from Lemma 2 of [3]. Suppose $S\neq P$. Let $p_r: k_m\to k_p$ be the projection. We have Ker $p_r\cap\iota(E)=1$. Since $p_r(\iota(E)\cap U_\mathfrak{m}^{p^t})$ is a subgroup of finite index of $\iota_p(E)\cap U_p^{p^t}$, we have the statement for S. *q.e.d.*

By virtue of Lemma 1, we can construct a homomorphism

$$g^{(m)}: t_p^{(m)}(U_\mathfrak{m}/\bar{E}_\mathfrak{m}) \to Б^{(1)}(p^m).$$

Suppsose the Leopoldt conjecture is valid and set $\ell=\max(t(m),m)$. For $u\bar{E}_\mathfrak{m}\in t_p^{(m)}(U_\mathfrak{m}/\bar{E}_\mathfrak{m})$, We have $x\in E$ such that $\iota(x)u^{-p^m}\in\bar{E}_\mathfrak{m}^{p^\ell}$. If $u\in\bar{E}_\mathfrak{m}$, we have $u\in\iota(E)\bar{E}_\mathfrak{m}^{p^\ell}$, and hence,

$\iota(x)\in\iota(E)^{p^n}\bar{E}_{\mathrm{m}}^{p^t}$. By Lemma 1, we have $\iota(x)\in\iota(E)^{p^n}$. Moreover, we have $\iota(xy^{-1})\in\iota(E)^{p^n}$ if $\iota(x)\bar{E}_{\mathrm{m}}^{p^t}=\iota(y)\bar{E}_{\mathrm{m}}^{p^t}$. Since $x\in\mathcal{S}^{(1)}(p^m)$, we obtain $g^{(m)}$ by defining to be $g^{(m)}(u\bar{E}_{\mathrm{m}})=xk^{\times p^n}$ for $u\bar{E}_{\mathrm{m}}\in t_p^{(m)}(U_{\mathrm{m}}/\bar{E}_{\mathrm{m}})$.

LEMMA 2. *Suppose the Leopoldt conjecture is valid for (k, p). Then, $g^{(m)}$ is surjective and* Ker $g^{(m)}\simeq\mu_{\mathrm{m}}^{(m)}/\iota(\mu^{(m)})$.

PROOF: Suppose $g^{(m)}(u\bar{E}_{\mathrm{m}})=k^{\times p^n}$ for $u\bar{E}_{\mathrm{m}}\in t_p^{(m)}(U_{\mathrm{m}}/\bar{E}_{\mathrm{m}})$. This means $u^{p^n}\in\bar{E}_{\mathrm{m}}^{p^n}$. Hence we have $u\in\bar{E}_{\mathrm{m}}\mu_{\mathrm{m}}^{(m)}$, *i.e.* Ker $g^{(m)}=\mu_{\mathrm{m}}^{(m)}\bar{E}_{\mathrm{m}}/\bar{E}_{\mathrm{m}}$. By Theorem 1 of [4], we have $t_p^{(1)}(\bar{E}_{\mathrm{m}})=\iota(\mu^{(1)})$, because the Leopoldt conjecture is supposed to be valid. Suppose $\mu\neq1$. Let ξ be a generator of a finite cyclic p-group $t_p^{(m)}(\bar{E}_{\mathrm{m}})$. There is a non-negative integer n such that $t_p^{(m)}(\bar{E}_m)^{p^n}=\iota(\mu)$. We have $x\in E$ such that $\xi\in\iota(x)\bar{E}_{\mathrm{m}}^{p^t}$ for $t\geq t(n)$. If $n\neq0$, we see $\xi^{p^n}\in\iota(E)^{p^n}\bar{E}_{\mathrm{m}}^{p^{t+n}}$, and hence $\xi^{p^n}\iota(x)^{-p^n}\in\iota(E)^{p^n}$. This contradicts to that ξ^{p^n} is a generator of μ. Thus, $n=0$. This proves Ker $g^{(m)}\simeq\mu_{\mathrm{m}}^{(m)}/\iota(\mu^{(m)})$. The surjectivity of $g^{(m)}$ is obvious from the definition. *q.e.d*

Let J be the idele group of k and V_{m} be the direct product of the groups $U_{\mathfrak{p}}$ of local units on $\mathfrak{p}\notin S$: $V_{\mathrm{m}}=\Pi_{\mathfrak{p}\notin S}U_{\mathfrak{p}}$. We consider $U_{\mathfrak{p}}$ and V_{m} subgroups of J with the canonical injections. $U_{\mathrm{m}}V_{\mathrm{m}}$ is the group of units of J. Denote by U'_{m} the intersection $\cap_{n\geq1}U_{\mathrm{m}}^{p^n}$. Set $\mathcal{H}_n=k^\times V_{\mathrm{m}}U'_{\mathrm{m}}E_{\mathrm{m}}J^{p^n}$. By class field theory, we have

$$\varprojlim_n J/\mathcal{H}_n\simeq\mathfrak{G}.$$

Set $\mathcal{H}=\cap_{n\geq1}\mathcal{H}_n$. Since $\varprojlim J/\mathcal{H}_n\simeq J/\mathcal{H}$, we obtain the following isomorphism:

$$\gamma:\mathfrak{G}\to J/\mathcal{H}.$$

Let L be the maximal unramified abelian p-extension of k and denote by $\mathfrak{G}_0$ the Galois group of M_S/L. The restriction γ_0 of γ onto $\mathfrak{G}_0$ gives an isomorphism

$$\gamma_0:\mathfrak{G}_0\to U_{\mathrm{m}}/U_{\mathrm{m}}\cap\mathcal{H}.$$

(Note the second isomorphism theorem of topological groups is valid for the idele group J, because it is σ-compact.)

LEMMA 3. *Let p^e be the p-part of the class number of k and $h=p^eh_1$ be the class number. We have $\mathcal{H}_{n+e}^{h_1}\subset k^\times V_{\mathrm{m}}U_{\mathrm{m}}^{p^n}$, and further, we have $\mathcal{H}\cap U_{\mathrm{m}}=\bar{E}_{\mathrm{m}}U'_{\mathrm{m}}$.*

PROOF: Since $J^h\subset k^\times V_{\mathrm{m}}U_{\mathrm{m}}$ we have $\mathcal{H}_{n+e}^{h_1}\subset k^\times V_{\mathrm{m}}\bar{E}_{\mathrm{m}}U_{\mathrm{m}}^{p^n}$. By the inclusion $\bar{E}_{\mathrm{m}}\subset E\,V_{\mathrm{m}}U_{\mathrm{m}}^{p^n}$, we see $\mathcal{H}_{n+e}^{h_1}\subset k^\times V_{\mathrm{m}}U_{\mathrm{m}}^{p^n}$. Hence, we have $\mathcal{H}_{n+e}^{h_1}\cap U_{\mathrm{m}}\subset\bar{E}_{\mathrm{m}}U_{\mathrm{m}}^{p^n}$, and further, $\mathcal{H}^{h_1}\cap U_{\mathrm{m}}\subset\bar{E}_{\mathrm{m}}U'_{\mathrm{m}}$, Since

$(h_1, p)=1$, We have $\mathcal{H}\cap U_m\subset \bar{E}_m U'_m$. We obtain $\mathcal{H}\cap U_m=\bar{E}_m U'_m$.

PROPOSITION 4. *We have* $\mathfrak{T}\simeq\mathcal{H}\mu_m/\mathcal{H}\simeq\mu_m/t_p^{(\infty)}(\bar{E}_m)$. *If the Leopoldt conjecture is valid, we have* $t_p^{(\infty)}(\bar{E}_m)\simeq\iota(\mu)$.

PROOF: We have proved $t_p^{(\infty)}(\bar{E}_m)=\iota(\mu)$ if the Leopoldt conjecture is valid, in the proof of Lemma 2. By the table in [2,I.1.c], we observe $H\supset L$. Hence, by virtue of Lemma 3, we have $\gamma_0(\mathfrak{T})$ is a subgroup of $U_m/\bar{E}_m U'_m$. Denote by X this subgroup. Let $p_r^*: U_m/\bar{E}_m U'_m\rightarrow U_p/\bar{E}_p U'_p$ be a surjection induced from $p_r: k_m\rightarrow k_p$. By the table of [2,I.1.c], we observe $p_r^*(X)=\mu_p\bar{E}_p U'_p/\bar{E}_p U'_p$. Set $\mathfrak{n}=\Pi_{\mathfrak{p}\in S\setminus P}\mathfrak{p}$. We see Ker $p_r^*=\mu_{\mathfrak{n}}\bar{E}_m U'_m/\bar{E}_m U'_m$. Therefore, we obtain $X=\mu_m\bar{E}_m U'_m/\bar{E}_m U'_m$. Since $\mu_m\cap U'_m=\{1\}$, we have $X\simeq\mu_m/t_p^{(\infty)}(\bar{E}_m)$. *q.e.d.*

We suppose throughout in the remainder part in this section that the Leopoldt conjecture is valid for (k, p). We shall extend $g^{(m)}$ onto $t_p^{(m)}(J/\mathcal{H})$. Set $\ell=\max(m,t(m))$. An element of $t_p^{(m)}(J/\mathcal{H})$ is represented by an idele $\mathfrak{a}$ such that $\mathfrak{a}^{p^n}\in\mathcal{H}_{n+e}$ for every n. By virtue of Lemma 3, we have $a\in k^\times$, $v\in V_m$, $u\in U_m$ such that

$$\mathfrak{a}^{h_1 p^n}=avu^{p^n}, \tag{2.1}$$

holds. Suppose we have another such expression:

$$\mathfrak{a}^{h_1 p^n}=a'v'u'^{p^t},\ a'\in k^\times,\ v'\in V_m,\ u'\in U_m$$

and suppose $n, t\geq\ell$. We have $aa'^{-1}\in E^{p^m}$ by virtue of Lemma 1. Hence, the coset $ak^{\times p^m}$ of $k^\times/k^{\times p^m}$ is uniquely determined for $n\geq\ell$. Moreover, we see $a\in\mathcal{S}(p^m)$. If $\mathfrak{a}\in\mathcal{H}$, we have $\mathfrak{a}^{h_1}\in k^\times V_m U_m$ from Lemma 3, and this implies $a\in k^{\times p^m}$. We obtain a map $\tilde{s}^{(m)}: t_p^{(m)}(J/\mathcal{H})\rightarrow Б(p^m)$ defined to be $\tilde{s}^{(m)}(\mathfrak{a}\mathcal{H})=ak^{\times p^m}$. We observe $\tilde{s}^{(m)}$ is a homomorphism. Let t_{h_1} be the h_1-th power automorphism of $t_p^{(m)}(J/\mathcal{H})$. Let $\dot{s}^{(m)}$ be the composition $\tilde{s}^{(m)}\circ t_{h_1}^{-1}$.

Suppose $\dot{s}^{(m)}(\mathfrak{a}\mathcal{H})=1$. We have $a\in k^{\times p^m}$ in (2.1). Hence, $v\in V_m^{p^m}$ and $\mathfrak{a}^{h_1}\in k^\times V_m U_m^{p^{n-m}}\mu_m^{(m)}$. Thus we have

$$\mathfrak{a}^{h_1}\in\cap_{n\geq\ell}k^\times V_m U_m^{p^{n-m}}\mu_m^{(m)}=\mathcal{H}\mu_m^{(m)}.$$

Hence, Ker $\dot{s}^{(m)}=\mathcal{H}\mu_m^{(m)}/\mathcal{H}$. We have $v\in V_m$, $\mathfrak{b}\in J$ for $a\in\mathcal{S}(p^m)$ such that $a=v\,\mathfrak{b}^{p^m}$. We see $\mathfrak{b}\mathcal{H}\in t_p^{(m)}(J/\mathcal{H})$ and $\dot{s}^{(m)}(\mathfrak{b}\mathcal{H})=ak^{\times p^m}$. Thus $\dot{s}^{(m)}$ is a surjection.

We have the following commutative diagram for $m>n$:

$$\begin{array}{ccc} t_p^{(n)}(J/\mathcal{H}) & \xrightarrow{\dot{s}^{(n)}} & \text{Б}(p^n) \\ \downarrow & & \downarrow i_{m,n} \\ t_p^{(m)}(J/\mathcal{H}) & \xrightarrow{\dot{s}^{(m)}} & \text{Б}(p^m) \end{array}$$

where $t_p^{(n)}(J/\mathcal{H}) \to t_p^{(m)}(J/\mathcal{H})$ is inclusion. Take the inductive limit and denote by $\dot{s}^{(\infty)}$ the morphism induced from $\dot{s}^{(m)}$, $m \geq 1$. We have $\dot{s}^{(\infty)}$ is surjective and Ker $\dot{s}^{(\infty)} = \mathcal{H}\mu_{\mathrm{m}}/\mathcal{H}$.

LEMMA 5. *Let $\{Z_m, f_{m,n}\}_{1\leq n\leq m<\infty}$ be a family of abelian groups and homomorphisms $f_{m,n}: Z_n \to Z_m$. Let X be an abelian group and Y be a subgroup for which there exists a surjective homomorphism $g_m: t_p^{(m)}(X) \to Z_m$ such that* Ker $g_m = t_p^{(m)}(Y)$ and $g_m \circ \iota_{m,n} = f_{m,n} \circ g_n$, *where $\iota_{m,n}$ is inclusion $t_p^{(n)}(X) \to t_p^{(m)}(X)$. Then, the exact sequence*

$$1 \to t_p^{(m)}(Y) \to t_p^{(m)}(X) \to Z_m \to 1$$

splits if and only if $t_p^{(n)}(Z_m) = \mathrm{Im}\, f_{m,n}$ holds for $1 \leq n \leq m$.

RROOF: Suppose $t_p^{(m)}(X) \simeq t_p^{(m)}(Y) \times Z_m$. Since $t_p^{(n)}(X) \simeq t_p^{(n)}(\mathrm{Y}) \times t_p^{(n)}(Z_m)$, Thus $g_m(t_p^{(n)}(X)) = t_p^{(n)}(Z_m)$ for $1 \leq n \leq m$. We have $t_p^{(n)}(Z_m) = \mathrm{Im}\, f_{m,n}$. Conversely, suppose $t_p^{(n)}(Z_m) = \mathrm{Im}\, f_{m,n}$ for $1 \leq n \leq m$. For $x \in t_p^{(m)}(X)$ such that $p^n x \in t_p^{(m)}(Y)$, we see $g_m(x) \in \mathrm{Im}\, f_{m,n}$. Take $z \in Z_n$ so that $g_m(x) = f_{m,n}(\mathrm{z})$, and also take $x' \in t_p^{(n)}(X)$ so that $g_n(x') = z$. We have $x - x' \in t_p^{(m)}(Y)$ and $p^n x = p^n(x-x') \in p^n t_p^{(m)}(Y)$. Thus we obtain $p^n t_p^{(m)}(X) \cap t_p^{(m)}(Y) \subset p^n t_p^{(m)}(Y)$. We have $p^n t_p^{(m)}(X) \cap t_p^{(m)}(\mathrm{Y}) = p^n t_p^{(m)}(\mathrm{Y})$. This proves $t_p^{(m)}(X) \simeq t_p^{(m)}(\mathrm{Y}) \times Z_m$. *q.e.d.*

$\dot{s}^{(m)}$ induces a surjection $t_p^{(m)}(\mathfrak{G}) \to \text{Б}(p^m)$. Denote this map by $s^{(m)}$.

THEOREM 6. *Suppose the Leopoldt conjecture is valid for (k, p). We have a surjective isomorphism $s^{(m)}: t_p^{(m)}(\mathfrak{G}) \to \text{Б}(p^m)$ with kernel being isomorphic to $\mu_{\mathrm{m}}^{(m)}/\iota(\mu^{(m)})$. Denote by $\bar{s}^{(m)}$ the composition $c^{(m)} \circ s^{(m)}$. We have* Ker $\bar{s}^{(m)} = t_p^{(m)}(\mathfrak{G}_0)$. *Hence we have the following exact sequences :*

(1) $$1 \to \mu_{\mathrm{m}}^{(m)}/\iota(\mu^{(m)}) \to t_p^{(m)}(\mathfrak{G}) \to \text{Б}(p^m) \to 1$$

(2) $$1 \to t_p^{(m)}(\mathfrak{G}_0) \to t_p^{(m)}(\mathfrak{G}) \to \text{Б}^{(0)}(p^m) \to 1$$

The sequence (1) *splits if and only if $i_{m,n}(\text{Б}(p^n)) = t_p^{(n)}(\text{Б}(p^m))$ holds for $1 \leq n \leq m$, and similarly, That of* (2) *also splits if and only if $t_p^{(n)}(\text{Б}^{(0)}(p^m)) = \text{Б}^{(0)}(p^n)$ holds for $1 \leq n \leq m$.*

PROOF: Since $\mathcal{H} \cap U_{\mathrm{m}} = \bar{E}_{\mathrm{m}} U'_{\mathrm{m}}$, we have $\mathcal{H} \cap \mu_{\mathrm{m}}^{(m)} = t_p^{(m)}(\bar{E}_{\mathrm{m}}) = \iota(\mu^{(m)})$. We have Ker $\dot{s}^{(m)} = \mu_{\mathrm{m}}^{(m)}/\iota(\mu^{(m)})$. Hence, Ker $s^{(m)} = \mu_{\mathrm{m}}^{(m)}/\iota(\mu^{(m)})$. Let $i^{(m)}$ be the injection $t_p^{(m)}(U_{\mathrm{m}}/\bar{E}_{\mathrm{m}} U'_{\mathrm{m}}) \to t_p^{(m)}(J/\mathcal{H})$. Suppose $a \in E$ in (2.1). We observe $\mathfrak{a} \in V_{\mathrm{m}} U_{\mathrm{m}}$. Thus $\mathfrak{a}\mathcal{H} \in \mathrm{Im}\, i^{(m)}$. This proves Ker $\bar{s}^{(m)} = \mathfrak{G}_0$.

Set $i_{m,n}$ on $f_{m,n}$ and $Z_m = Б(p^m)$, $X = t_p^{(m)}(\mathfrak{G})$ and $Y = \mu_{\mathfrak{m}}^{(m)}/\iota(\mu^{(m)})$ in Lemma 5. Then, we obtain the condition concerning to spliting of the exact sequence (1). Set the canonical injection $Б^{(0)}(p^n) \to Б^{(0)}(p^m)$ on $f_{m,n}$ and $Z_m = Б^{(0)}(p^m)$, $X = t_p^{(m)}(\mathfrak{G})$ and $Y = t_p^{(m)}(\mathfrak{G}_0)$ in Lemma 5. Then, we obtain the statement concerning to the spliting of the exact sequence (2). *q.e.d.*

COROLLARY. *Let Z be compositum of all $\mathbf{Z}_p$-extensions of k. Suppose the Leopoldt conjecture is valid for (k, p). Then, we have*

$$Б^{(0)}(p^\infty) \simeq \mathrm{Gal}(L/L \cap Z).$$

PROOF: Let p^m be the exponent of $t_p^{(\infty)}(\mathfrak{G})$. We observe

$$\mathrm{Gal}(M_S/ZL) \simeq t_p^{(\infty)}(\mathfrak{G}_0) \simeq t_p^{(\infty)}(\mathfrak{T}).$$

Hence, by Theorem 6, we have $\mathrm{Gal}(ZL/Z) \simeq Б^{(0)}(p^\infty)$. *q.e.d.*

PROOF OF THEOREM A: By (1.1) and Theorem 6, we have the exactness of the horizontal sequence of the diagram in Theorem A. The exactness of the right vertical sequence follows from Lemma 2. By Proposition 4 and Theorem 6, we $t_p^{(\infty)}(\mathfrak{T}) \simeq \mu_{\mathfrak{m}}/\iota(\mu)$ and observe $t_p^{(\infty)}(\mathfrak{H}) \simeq Б(p^\infty)$ from the top row in the diagram.

3. THE PROOF OF THEOREM B

Let K be a Galois extension of k with Galois group G. S is a finite set of prime divisors of k such that $S \supset P$. Set $\mathfrak{m} = \Pi_{\mathfrak{p} \in S} \mathfrak{p}$. Let t_{p^m} denote the p^m-power endomorphism for an arbitrary abelian group.

LEMMA 7. *Let A be a G-module. If the image of the homomorphism*

$$H^1(G, t_p^{(m)}(A)) \to H^1(G, t_p^{(\infty)}(A))$$

induced from $t_p^{(m)}(A) \subset t_p^{(\infty)}(A)$ is trivial, we have $A^{p^m G} = A^{G p^m} t_p^{(\infty)}(A^{p^m G})$.

PROOF: By an exact sequence

$$1 \to t_p^{(m)}(A) \to A \xrightarrow{t_{p^m}} A^{p^m} \to 1$$

we have the following diagram whose rows are exact:

$$\begin{array}{ccccccccc} 1 & \to & t_p^{(m)}(A)^G & \to & t_p^{(\infty)}(A)^G & \to & t_p^{(\infty)}(A^{p^m})^G & \to & X & \to & 1 \\ & & \| \downarrow & & \downarrow & & \downarrow & & \downarrow i & & \end{array}$$

$$1 \to t_p^{(m)}(A)^G \to A^G \to A^{p^m G} \to H^1(G, t_p^{(m)}(A))$$

where X is the kernel of $H^1(G, t_p^{(m)}(A)) \to H^1(G, t_p^{(\infty)}(A))$. Here, set $Y = A^G/t_p^{(m)}(A)^G$. We have the following diagram

$$\begin{array}{ccccccccc} 1 & \to & t_p^{(\infty)}(Y) & \to & t_p^{(\infty)}(A^{p^m})^G & \to & X & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow i & & \\ 1 & \to & Y & \to & A^{p^m G} & \to & H^1(G, t_p^{(m)}(A)). & & \end{array}$$

By the snake lemma, we have an exact sequence

$$1 \to Y/t_p^{(\infty)}(Y) \to A^{p^m G}/t_p^{(\infty)}(A^{p^m})^G \to \text{Coker } i.$$

Hence we have $A^{p^m G} = A^{G p^m} t_p^{(\infty)}(A^{p^m})^G$ if Coker $i = \{1\}$. *q.e.d.*

ι^2 denotes the homomorphism $H^2(G, \mu_K^{(m)}) \to H^2(G, K^\times)$. μ'_K is a subgroup of μ_K such that $\mu'_K/\mu_K^{p^m} = (\mu_K/\mu_k^{p^m})^G$. Let p^e be the order of μ_K.

Lemma 8. *Suppose $H^1(G, \mu_K)$ vanishes and ι^2 is injective for $i=1,2$.*

(1) *We have an isomorphism*

$$Б_K(p^m)^G \simeq \frac{S_K(p^m) \cap k^\times \mu'_K}{K^{\times p^m} \cap k^\times \mu'_K}.$$

(2) *We also have an isomorphism*

$$\frac{(S_K(p^m)\mu'_K)^G}{\mu'^G_K} \simeq \frac{S_K(p^m) \cap k^\times \mu'_K}{S_K(p^m) \cap \mu'_K}.$$

Furthermore, we have

$$\frac{(K^{\times p^m}\mu'_K)^G}{\mu'^G_K} \simeq \frac{(K^{\times p^m} \cap k^\times \mu'_K)(S_K(p^m) \cap \mu'_K)}{S_K(p^m) \cap \mu'_K}.$$

Proof: By taking the cohomology long exact sequence from the following commutative diagrams whose horizontal rows are exact:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mu_K^{p^m} & \longrightarrow & \mu_K & \longrightarrow & \mu_K/\mu_K^{p^m} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & K^{\times p^m} & \longrightarrow & K^\times & \longrightarrow & K^\times/K^{\times p^m} & \longrightarrow & 1 \end{array}$$

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mu_K^{p^m} & \longrightarrow & K^{\times p^m} & \longrightarrow & K^{\times p^e} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & = \downarrow & & \end{array}$$

$$1 \longrightarrow \mu_K \longrightarrow K^{\times} \longrightarrow K^{\times p^e} \longrightarrow 1$$

we have the following commutative diagram whose vertical and horizontal rows are also exact:

$$\begin{array}{ccccccccc}
\mu_k & \to & (\mu_K/\mu_K^{p^m})^G & \to & H^1(G, \mu_K^{p^m}) & \to & H^1(G, \mu_K) & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
k^{\times} & \to & (K^{\times}/K^{\times p^m})^G & \to & H^1(G, K^{\times p^m}) & \to & H^1(G, K^{\times}) & & \\
 & & & & \downarrow & & & & \\
 & & H^1(G, K^{\times}) & \to & H^1(G, K^{\times p^e}) & \to & H^2(G, \mu_K) & \xrightarrow{\iota^2} & H^2(G, K^{\times})
\end{array}$$

Suppose $H^1(G, \mu_K)=\{1\}$ and Ker $\iota^2=\{1\}$. Since $H^1(G, K^{\times})=\{1\}$, we have $H^1(G, K^{\times p^e})=\{1\}$. This implies $(K^{\times}/K^{\times p^m})^G$ is generated by $k^{\times}K^{\times p^m}/K^{\times p^m}$ and $(\mu_K/\mu_K^{p^m})^G$. Namely,

$$(K^{\times}/K^{\times p^m})^G \simeq k^{\times}\mu'_K K^{\times p^m}/K^{\times p^m}.$$

Hence, we have

$$\text{Б}_K(p^m) \cap (K^{\times}/K^{\times p^m})^G \simeq \frac{\mathcal{S}_K(p^m) \cap k^{\times}\mu'_K}{K^{\times p^m} \cap k^{\times}\mu'_K}.$$

An element $x_0 \in (\mathcal{S}_K(p^m)\mu'_K)^G$ is written as a product xz of $x \in \mathcal{S}_K(p^m)$ and $z \in \mu'_K$. We see $x \in \mathcal{S}_K(p^m) \cap k^{\times}\mu'_K$. Conversely, $x \in \mathcal{S}_K(p^m) \cap k^{\times}\mu'_K$ can be written as a product $x_0 z$ of $x_0 \in (\mathcal{S}_K(p^m)\mu'_K)^G$ and $z \in \mu'_K$. Hence, by correspondence $x_0\mu_K'^G \to x(\mathcal{S}_K(p^m) \cap \mu'_K)$, we have an isomorphism

$$\frac{(\mathcal{S}_K(p^m)\mu'_K)^G}{\mu_K'^G} \simeq \frac{\mathcal{S}_K(p^m) \cap k^{\times}\mu'_K}{\mathcal{S}_K(p^m) \cap \mu'_K}$$

because $x_0=xz \in \mu_K'^G$ if and only if $x \in \mathcal{S}_K(p^m) \cap \mu'_K$. By this isomorphism, we also have

$$\frac{(K^{\times p^m}\mu'_K)^G}{\mu_K'^G} \simeq \frac{(K^{\times p^m} \cap k^{\times}\mu'_K)(\mathcal{S}_K(p^m) \cap \mu'_K)}{\mathcal{S}_K(p^m) \cap \mu'_K}.$$

The proof is completed.

LEMMA 9. *Suppose S contains every prime divisors of k which are ramified in K. Then, we have an injection*

$$\mathcal{S}_K(p^m)^G/\mathcal{S}_k(p^m) \to H^1(G, \mu_{K,\mathfrak{m}}^{(m)}).$$

Furthermore, if the image of $H^1(G, \mu_{K,\mathfrak{m}}^{(m)}) \to H^1(G, \mu_{K,\mathfrak{m}})$ *is trivial, we have an isomorphism*

$$H^1(G, \mu_{K,\mathfrak{m}}^{(m)}) \simeq \mu_{K,\mathfrak{m}}^{p^m G}/\mu_{k,\mathfrak{m}}^{p^m}.$$

PROOF: Let $J'_{k,\mathfrak{m}}$ and $J'_{K,\mathfrak{m}}$ denote the direct product $\Pi_{\mathfrak{p}\nmid\mathfrak{m}} k_{\mathfrak{p}}^{\times}$ and $\Pi_{\mathfrak{P}\nmid\mathfrak{m}} K_{\mathfrak{P}}^{\times}$, respectively. Set $I_{k,\mathfrak{m}}=$

$J'_{k,\mathfrak{m}}/V_{k,\mathfrak{m}}$ and $I_{K,\mathfrak{m}}=J'_{K,\mathfrak{m}}/V_{K,\mathfrak{m}}$. We consider $I_{k,\mathfrak{m}}$ and $I_{K,\mathfrak{m}}$ subgroups of the divisor groups of k and K, respectively, and also consider $I_{k,\mathfrak{m}}\subset I_{K,\mathfrak{m}}$. Since S contains every ramified primes, we have $I_{K,\mathfrak{m}}^{p^nG}=I_{k,\mathfrak{m}}^{p^n}$ for $n\geq 0$. The p^m-th power endomorphism of J_K induces an exact sequence

$$1\to\mu_{k,\mathfrak{m}}^{(m)}\to K_{\mathfrak{m}}^{\times}\times I_{K,\mathfrak{m}}\xrightarrow{t_{p^n}}K_{\mathfrak{m}}^{\times p^n}\times I_{K,\mathfrak{m}}^{p^n}\to 1$$

By taking the cohomology exact sequence, we have the following exact sequence:

$$1\to\mu_{K,\mathfrak{m}}^{(m)}\xrightarrow{t_{p^n}}k_{\mathfrak{m}}^{\times}\times I_{k,\mathfrak{m}}\to K_{\mathfrak{m}}^{\times p^nG}\times I_{k,\mathfrak{m}}^{p^n}\to H^1(G,\mu_{K,\mathfrak{m}}^{(m)})\to 1$$

and further, obtain an exact sequence

$$1\to k_{\mathfrak{m}}^{\times p^n}\times I_{k,\mathfrak{m}}^{p^n}\to K_{\mathfrak{m}}^{\times p^nG}\times I_{k,\mathfrak{m}}^{p^n}\to H^1(G,\mu_{K,\mathfrak{m}}^{(m)})\to 1$$

Since $k^{\times}\cap k_{\mathfrak{m}}^{\times p^n}\times I_{k,\mathfrak{m}}^{p^n}=\mathcal{S}_k(p^m)$ and $k^{\times}\cap K_{\mathfrak{m}}^{\times p^nG}\times I_{k,\mathfrak{m}}^{p^n}=\mathcal{S}_K(p^m)^G$, we have an exact sequence

$$1\to\mathcal{S}_k(p^m)\to\mathcal{S}_K(p^m)^G\to H^1(G,\mu_{K,\mathfrak{m}}^{(m)}).$$

Thus $\mathcal{S}_K(p^m)^G/\mathcal{S}_k(p^m)$ injects into $H^1(G,\mu_{K,\mathfrak{m}}^{(m)})$. Suppose $\mathrm{Im}(H^1(G,\mu_{K,\mathfrak{m}}^{(m)})\to H^1(G,\mu_{K,\mathfrak{m}}))=\{1\}$. By the exact sequence

$$1\to\mu_{k,\mathfrak{m}}^{(m)}\to\mu_{k,\mathfrak{m}}\to\mu_{K,\mathfrak{m}}^{p^nG}\to H^1(G,\mu_{K,\mathfrak{m}}^{(m)})\to H^1(G,\mu_{K,\mathfrak{m}})$$

obtained from the cohomology long exact sequence associated with

$$1\to\mu_{K,\mathfrak{m}}^{(m)}\to\mu_{K,\mathfrak{m}}\xrightarrow{t_{p^n}}\mu_{K,\mathfrak{m}}^{p^n}\to 1,$$

we have $H^1(G,\mu_{K,\mathfrak{m}}^{(m)})\simeq\mu_{K,\mathfrak{m}}^{p^nG}/\mu_{k,\mathfrak{m}}^{p^n}$. *q.e.d.*

PROOF OF THEOREM B: By Lemma 8, we have

$$Б_K(p^m)^G\simeq\frac{\mathcal{S}_K(p^m)\cap k^{\times}\mu'_K}{K^{\times p^n}\cap k^{\times}\mu'_K}.$$

If $\mathcal{S}_K(p^m)\cap\mu_K\subset\mu_K^{p^n}$, we have $\mathcal{S}_K(p^m)\cap\mu'_K\subset K^{\times p^n}$, and hence obtain

$$Б_K(p^m)^G\simeq\frac{(\mathcal{S}_K(p^m)\mu'_K)^G}{(K^{\times p^n}\mu'_K)^G}$$

from Lemma 8. Let Δ be a subgroup of $\mathcal{S}_K(p^m)\times\mu'_K$ defined to be

$$\Delta=\{(x,x^{-1})\,|\,x\in\mu'_K\cap\mathcal{S}_K(p^m)\}.$$

Let $\pi((x,y))=y$ be the projection of $\mathcal{S}_K(p^m)\times\mu'_K$ onto μ'_K. Let $f:\mathcal{S}_K(p^m)\mu'_K\to\mathcal{S}_K(p^m)\times\mu'_K/\Delta$ be a homomorphism induced from the canonical map of $\mathcal{S}_K(p^m)\to\mathcal{S}_K(p^m)\times\mu'_K$ and that of $\mu'_K\to\mathcal{S}_K(p^m)\times\mu'_K$. We observe Ker $\pi\circ f=\mathcal{S}_K(p^m)$. Let g be the restriction of $\pi\circ f$ onto $(\mathcal{S}_K(p^m)\mu'_K)^G$. We have the following exact sequence:

$$1 \to \mathcal{S}_K(p^m)^G \to (\mathcal{S}_K(p^m)\mu'_K)^G \xrightarrow{g} \mu'_K/\mu_K^{p^m}.$$

Hence, we have (1). (2) follows from Lemma 9, immediately.

4. APPLICATIONS

Let k_n be the maximal p-extension over k in $k(\zeta_{p^n})$. Set $K=k_n$. We have $H^i(G, \mu_K)=\{1\}$ for $i>0$ if p is odd or if -1 is norm form k_4. Moreover, we also have $H^i(\mathrm{Gal}(K_{\mathfrak{P}}/k_{\mathfrak{p}}), \mu_{K,\mathfrak{P}})=\{1\}$ for $i>0$, where $\mathfrak{P}$ is an extension of a prime $\mathfrak{p}$ of k. Hence, we have $H^i(G, \mu_{K,\mathfrak{m}})=\{1\}$.

THEOREM 10. *Set $K=k_n$. Suppose -1 is norm from k_4 when $p=2$. Let m be a natural number for which we have $(\mathcal{S}_K(p^m)\mu'_K)^G=(K^{\times p^m}\mu'_K)^G\mathcal{S}_K(p^m)^G$ and $\mathcal{S}_K(p^m)\cap\mu_K=\mu_K^{p^m}$. Let p^e be the exponent of $\mu_{K,\mathfrak{m}}$. We have*

$$\text{Б}_K(p^m)^G \simeq \mathcal{S}_K(p^m)^G/k^{\times p^m}\mu_K^{p^mG}.$$

Moreover, if $m\geq e$, we have

$$\text{Б}_K(p^m)^G \simeq \text{Б}_k(p^m).$$

PROOF: By $\mu_K\cap\mathcal{S}_K(p^m)=\mu_K^{p^m}$, we have

$$(K^{\times p^m}\mu'_K)^G\cap\mathcal{S}_K(p^m)^G\subset K^{\times p^mG}.$$

Hence

$$\text{Б}_K(p^m)^G \simeq \mathcal{S}_K(p^m)^G/K^{\times p^mG}.$$

from Theorem B. Since $H^1(G, \mu_K)=\{1\}$, we have $K^{\times p^mG}=k^{\times p^m}\mu_K^{p^mG}$ from Lemma 7. If $m\geq e$, we have $\mathcal{S}_K(p^m)^G=\mathcal{S}_k(p^m)$ from Theorem B, because $H^1(G, \mu_{K,\mathfrak{m}})=\{1\}$. Since $\mu_K^{p^m}=\{1\}$, we obtain $\text{Б}_K(p^m)^G\simeq\text{Б}_k(p^m)$. *q.e.d.*

Let k_0 be a finite algebraic extension of $\boldsymbol{Q}$ such that $k_0\not\ni\zeta_p$. Set $k=k_0(\zeta_p)$. Denote by Δ the Galois group of k/k_0. Let $\boldsymbol{Z}_p$ be the ring of p-adic integers and $t(\boldsymbol{Z}_p^\times)$ denote the torsion subgroup of the multiplicative group $\boldsymbol{Z}_p^\times$. The group generated by ζ_p is a $\boldsymbol{Z}_p$-module and there is $c\in t(\boldsymbol{Z}_p^\times)$ for each $\sigma\in\Delta$ such that $\zeta_p^\sigma=\zeta_p^c$. We obtain a p-adic irreducible character ω of Δ defined to be $\omega(\sigma)=c$. Let ε_i be the associated idempotent with ω^i of the group ring $\boldsymbol{Z}_p[\Delta]$ of Δ over $\boldsymbol{Z}_p$. Set $K=k(\zeta_{p^n})$. We observe $\varepsilon_i\mu_K=\{1\}$ if $i\not\equiv 1 \bmod |\Delta|$.

Let A_K be the ideal class group of K and D_K be a subgroup generated by prime divisors $\mathfrak{P}$ of K such that $\mathfrak{P}|p$. Set $C_K(p^n)=A_K/D_KA_K^{p^n}$. When $K\ni\zeta_{p^n}$, we have the Kummer duality between $C_K(p^n)$ and $\text{Б}_K(p^n)$. Denote by ε_i^* the idemopotent associated with ω^{1-i}. We observe $\varepsilon_i^*C_K(p^n)$ is the dual of $\varepsilon_i\text{Б}_K(p^n)$.

PROPOSITION 11. *Let p be odd. Let k_0 be a totally real number field and set $k=k_0(\zeta_p)$ and $K=k(\zeta_{p^m})$. Suppose the Leopoldt conjecture is valid for (K, p). Then, for $1\leq n\leq m$, we have the following statement :*

(1) *We have*

$$\varepsilon_i \mathrm{B}_K(p^n) \simeq \varepsilon_i \mathrm{B}_K^{(0)}(p^n)$$

for odd i such that $i\not\equiv 1$ mod. $|\Delta|$. When $\varepsilon_i=\varepsilon_1$, we have $\varepsilon_i \mathrm{B}_K(p^n)\simeq \varepsilon_i \mathrm{B}_K^{(0)}(p^n)$ if $\mu_K \cap \mathcal{S}_K(p^n)=\mu_K^{p^n}$.

(2) *We obtain the following exact sequence* :

$$1 \to \varepsilon_i(\mu_{K,\mathfrak{m}}^{(n)}/\iota(\mu_K^{(n)})) \to \varepsilon_i t_p^{(n)}(\mathfrak{G}_K) \to \varepsilon_i \mathrm{B}_K^{(0)}(p^n) \to 1$$

PROOF: Since $\varepsilon_i E_K/E_K^{p^n} \simeq \varepsilon_i \mu_K/\mu_K^{p^n}$ for odd i, we have $\varepsilon_i \mathrm{B}_K^{(1)}(p^n)=\{1\}$. By (1.1), we have $\varepsilon_1 \mathrm{B}_K(p^n) \simeq \varepsilon_1 \mathrm{B}_K^{(0)}(p^n)$ if $\varepsilon_i \neq \varepsilon_1$. Suppose $\mu_K \cap \mathcal{S}_K(p^n)=\mu_K^{p^n}$. We have $\varepsilon_1 \mathrm{B}_K^{(1)}=\{1\}$. Hence $\varepsilon_1 \mathrm{B}_K(p^n)\simeq\varepsilon_1 \mathrm{B}_K^{(0)}(p^n)$. This proves (1). The exact sequence of (2) follows from Theorem A. *q.e.d.*

THEOREM 12. *Let p be odd. Let k_0 be a totally real number field and set $k=k_0(\zeta_p)$, $K=k(\zeta_{p^m})$. Suppose the Leopoldt conjecture is valid for (K, p) and $\mu_K\cap \mathcal{S}_K(p^n)$ equals $\mu_K^{p^n}$ for an arbitrary natural number n. Let i be odd. Then the following conditions* (i), (ii) and (iii) are equivalent:

(i) $\varepsilon_i^* C_K=\{1\}$, (ii) $\varepsilon_i \mathrm{B}_K^{(0)}(p)=\{1\}$, (iii) $\varepsilon_i t_p^{(\infty)}(\mathfrak{H})=\{1\}$,

Moreover, if $\varepsilon_i(\mu_{K,\mathfrak{m}}/\iota(\mu_K))=\{1\}$, (iii) *is equivalent to* $\varepsilon_i t_p^{(\infty)}(\mathfrak{G})=\{1\}$.

PROOF: By Proposition 11, we have $\varepsilon_i \mathrm{B}_K(p^n)\simeq\varepsilon_i \mathrm{B}_K^{(0)}(p^n)$ for every n. We can prove that the exact sequence

$$1 \to \varepsilon_i t_p^{(n)}(\mathfrak{G}_0) \to \varepsilon_i t_p^{(n)}(\mathfrak{G}) \to \varepsilon_i \mathrm{B}_K^{(0)}(p^n) \to 1$$

splits if and only if $t_p^{(l)}(\varepsilon_i \mathrm{B}_K^{(0)}(p^n))=\varepsilon_i \mathrm{B}_K^{(0)}(p^l)$ holds for $1\leq l\leq n$, by the same argument as in the proof of Theorem A. By Proposition 11, we have $\varepsilon_i \mathrm{B}_K^{(1)}(p^n)=\{1\}$, and hence the above sequence splits. Thus $\varepsilon_i \mathrm{B}_K^{(0)}(p)=\{1\}$ is equivalent to $\mathrm{B}_K^{(0)}(p^n)=\{1\}$. Moreover, since $\varepsilon_i \mathrm{B}_K^{(0)}(p)\simeq\varepsilon_i \mathrm{B}_K(p)$, we see $t_p^{(1)}(\varepsilon_i \mathrm{B}_K^{(\infty)}(p))\simeq\varepsilon_i \mathrm{B}_K(p)$, this proves the equivalence of (ii) with (iii) by virtue of Theorem A. By the Kummer duality, we observe (i) and (ii) are equivalent. By Proposition 11 and Theorem A, we have $t_p^{(\infty)}(\mathfrak{G})=t_p^{(\infty)}(\mathfrak{H})$ if $\varepsilon_i(\mu_{K,\mathfrak{m}}/\iota(\mu_K))=\{1\}$.

Remark. 1. Set $K=k(\zeta_{p^m})$. When $i\not\equiv 1$mod. $|\Delta|$, we have $\varepsilon_i\mu_k=\{1\}$. Hence, by Lemma 8, we obtain the following isomorphisms:

$$\varepsilon_i \text{Б}_K(p^m)^G \simeq \varepsilon_i \left(\frac{(\mathcal{S}_K(p^m)\mu'_K)^G}{(K^{\times p^m}\mu'_K)^G}\right) \simeq \varepsilon_i \left(\frac{\mathcal{S}_K(p^m)^G}{k^{\times p^m}}\right).$$

Suppose $S=P$ and $H^1(G, \varepsilon_i\mu_{K,m}^{(m)})=\{1\}$. We have $\varepsilon_i(\mathcal{S}_K(p^m)^G/\mathcal{S}_k(p^m))=\{1\}$ from Theorem B. Hence we obtain

$$\varepsilon_i \text{Б}_K(p^m)^G \simeq \varepsilon_i \text{Б}_k(p^m).$$

Especially, we have

$$\varepsilon_i \text{Б}_k(p^\infty) \simeq \varepsilon_i \text{Б}_K(p^\infty)^G.$$

This implies $\varepsilon_i t_p^{(\infty)}(\mathfrak{H}_k) \simeq \varepsilon_i t_p^{(\infty)}(\mathfrak{H}_K^G)$.

2. Let k be totally real. Suppose the Leopoldt conjecture is valid for (k, p). We have an isomorphism

$$\varepsilon_i \text{Б}_k^{(0)}(p^\infty) \simeq \varepsilon_i t_p(A_k)$$

for even i from Corollary to Theorem 6.

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