On the Number of Generators of p-Class Groups as Galois Modules of p^nth Cyclotomic Extensions of Abelian Fields

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1. Introduction

We obtain a formula which gives the number of generators of a p-class group as a Galois module of an algebraic number field with finite degree. We will investigate this number for a certain type of a finite abelian extension of the field $\boldsymbol{Q}$ of rational numbers. We take a prime number p to be odd to avoid complicated arguments. Let M be a finite abelian extension of $\boldsymbol{Q}$ containing a primitive pth root of unity. We suppose $p|[M : \boldsymbol{Q}]$. The Galois group $G = Gal(M/\boldsymbol{Q})$ is a direct product of the p-Sylow subgroup G_p and a subgroup $\mathfrak{g}$ such that $p \nmid \sharp\mathfrak{g}$. Let C be the p-class group of M. It is a finite $\boldsymbol{Z}_p[\mathfrak{g}]$-module, where $\boldsymbol{Z}_p$ denotes the ring of p-adic integers and $\boldsymbol{Z}_p[\mathfrak{g}]$ dose the group ring of $\mathfrak{g}$ over $\boldsymbol{Z}_p$.

Let J be the Jacobson radical of the group ring $\boldsymbol{Z}_p[\mathfrak{g}]$. $\boldsymbol{Z}_p[\mathfrak{g}]/J$ is isomorphic to the group ring $\boldsymbol{F}_p[\mathfrak{g}]$, where $\boldsymbol{F}_p$ is the finite field of p-elements, *cf.* Lemma 1 of [Ya]. Hence, if a finitely generated $\boldsymbol{Z}_p[\mathfrak{g}]$-module Y is given, the quotient module Y/Y^J is an $\boldsymbol{F}_p[\mathfrak{g}]$-module. For each $\boldsymbol{F}_p[\mathfrak{g}]$-module, there is a corresponding $\boldsymbol{Q}_p[\mathfrak{g}]$-module, where $\boldsymbol{Q}_p$ is the field of fractions of $\boldsymbol{Z}_p$ and $\boldsymbol{Q}_p[\mathfrak{g}]$ is the group ring of $\mathfrak{g}$ over $\boldsymbol{Q}_p$. $\boldsymbol{Q}_p[\mathfrak{g}]$ is a semi-simple ring, and hence it decomposed into a direct sum of simple subrings R_i :

$$\boldsymbol{Q}_p[\mathfrak{g}] = \bigoplus_{i=1}^{r} R_i$$

Let e_i be identity element of R_i. We have $1 = \sum e_i$. Since $p \nmid \sharp\mathfrak{g}$, every e_i is an element of $\boldsymbol{Z}_p[\mathfrak{g}]$. By means of the canonical map $\boldsymbol{Z}_p \to \boldsymbol{F}_p$, every $\boldsymbol{F}_p[\mathfrak{g}]$-module is a $\boldsymbol{Z}_p[\mathfrak{g}]$-module canonically. We have the following decompositions of a $\boldsymbol{Z}_p[\mathfrak{g}]$-module Y and an $\boldsymbol{F}_p[\mathfrak{g}]$-module Y/Y^J:

$$Y = \bigoplus e_i Y, \qquad Y/Y^J = \bigoplus_i e_i(Y/Y^J).$$

Denote by $r_i(Y)$ the number of minimal generators of e_iY/Y^J as an $\boldsymbol{F}_p[\mathfrak{g}]$-module. Since G is abelian, e_iY is generated with $r_i(Y)$ elements as a $\boldsymbol{Z}_p[\mathfrak{g}]$-module, and dose not with $r_i(Y) - 1$ elements . The number of elements of a set of minimal generators of Y as a $\boldsymbol{Z}_p[\mathfrak{g}]$-module is equal to $\max_i r_i(Y)$.

The main object for consideration in the present paper is a quantity $r_i(C)$. A formula describing this value is proposed in [Ya]. We will determine the value concretely in case that

M is a p^nth cyclotomic extension of $K = M^{G_p}$, because the value of a_χ is closely related to the structure of the Iwasawa module of the cyclotomic $\boldsymbol{Z}_p$-extension of K. We shall show that this problem is reduced to compute the rank of p-class group of $K = M^{G_p}$ by virtue of the Frobenius reciprocity law and the genus theory. However, we do not consider the problem of computing the rank, here.

2. Characters of representations of $\mathfrak{g}$

We suppose every $\boldsymbol{Q}_p[\mathfrak{g}]$-modules (*resp.* $\boldsymbol{Z}_p[\mathfrak{g}]$-modules, *resp.* $\boldsymbol{F}_p[\mathfrak{g}]$-modules) is finitely generated. We recall correspondence between $\boldsymbol{Q}_p[\mathfrak{g}]$-modules, $\boldsymbol{Z}_p$-free $\boldsymbol{Z}_p[\mathfrak{g}]$-modules and $\boldsymbol{F}_p[\mathfrak{g}]$-modules. Let X be a finitely generated $\boldsymbol{Q}_p[\mathfrak{g}]$-module. For each X, there is a finitely generated $\boldsymbol{Z}_p$-free $\boldsymbol{Z}_p[\mathfrak{g}]$-module Y such that $X = \boldsymbol{Q}_p \otimes_{\boldsymbol{Z}_p} Y$. We obtain an $\boldsymbol{F}_p[\mathfrak{g}]$-module $Z = Y/Y^p = \boldsymbol{F}_p \otimes_{\boldsymbol{Z}_p} Y$. The correspondence $X \to Y \to Z$ is functorial and is determined up to isomorphisms in categories of $\boldsymbol{Q}_p[\mathfrak{g}]$-modules, $\boldsymbol{Z}_p[\mathfrak{g}]$-modules, and $\boldsymbol{F}_p[\mathfrak{g}]$-modules, respectively. Since $p \nmid \sharp\mathfrak{g}$, there is converse correspondence $Z \to Y \to X$. This means that we know the structure of Z or Y from that of X. $\boldsymbol{Q}_p[\mathfrak{g}]$ is a commutative ring, and hence each R_i is a field. We see a finitely generated $\boldsymbol{Q}_p[\mathfrak{g}]$-module X is a direct sum of a finite dimensional R_i-vector space e_iX. Observe $\boldsymbol{Q}_p \otimes_{\boldsymbol{Z}_p} e_iY = e_iX$. Thus, we have

$$\dim_{R_i} e_iX = r_i(Y) = r_i(Z).$$

$r_i(Z)$ is also equal to the dimension of e_iZ over the residue field of R_i. If Y is a $\boldsymbol{Z}_p[\mathfrak{g}]$-module, we define $r_i(Y)$ to be $r_i(\boldsymbol{F}_p \otimes Y)$.

Let Φ be a representation of $\mathfrak{g}$. It is a homomorphism

$$\Phi : \mathfrak{g} \longrightarrow GL_m(\boldsymbol{Q}_p)$$

of groups, where $GL_m(\boldsymbol{Q}_p)$ is a general linear group of degree m over $\boldsymbol{Q}_p$. If $m = 1$, we call a one-dimensional representation. The trace of Φ is called the character. It is a function of $\mathfrak{g}$. If Φ_1 and Φ_2 are two equivalent representations, the characters give same function of $\mathfrak{g}$ into $\boldsymbol{Q}_p$. Conversely, non-equivalent representations define different characters. Hence, we denote by Φ_χ a representation having character $\chi = Tr\,\Phi_\chi$. The left regular representation on R_i is an irreducible representation, because $\mathfrak{g}$ is abelian. Its character χ_i is called an irreducible character. Every character is written as a linear combination of irreducible characters χ_i with coefficients of non-negative integers. If a character χ is coming from a $\boldsymbol{Q}_p$-linear representation of a finitely generated $\boldsymbol{Q}_p[\mathfrak{g}]$-module X and if $\chi = \sum_k m_k \chi_{j_k}$, we have a decomposition $X = \oplus_k R_{j_k}^{m_k}$. We change suffix and write R_χ, e_χ, $r_\chi(Y)$ for R_i, e_i, $r_i(Y)$ if χ is an irreducible character χ_i.

We introduce several characters to state the formula of $r_\chi(C/C^J)$. Let ζ_{p^n} be a primitive p^nth root of unity and μ_{p^n} be a cyclic group generated by ζ_{p^n}. The p^{n-m}th power map induces a surjection $\mu_{p^n} \to \mu_{p^m}$ for $n > m$. $\{\mu_{p^n}\}_{n \geq 0}$ forms a projective system with respect to these power maps. Let $\boldsymbol{T}$ be the projective limit. It is isomorphic to $\boldsymbol{Z}_p$ as a profinite group. Let K be the intermediate field of $M/\boldsymbol{Q}$ corresponding to G_p. Since $K \ni \zeta_p$, $\mathfrak{g}$ acts on μ_p. $\mathfrak{g}$ also acts on μ_{p^n} through injection $Gal(\boldsymbol{Q}(\zeta_p)/\boldsymbol{Q}) \to Gal(\boldsymbol{Q}(\zeta_{p^n})/\boldsymbol{Q})$. $\boldsymbol{T}$ becomes a $\boldsymbol{Z}_p[\mathfrak{g}]$-module by passing through the projective limit. Let ω be the character of $\boldsymbol{Q}_p \otimes_{\boldsymbol{Z}_p} \boldsymbol{T}$. Since ω is a character of a one-dimensional representation, it is irreducible. Denote by ε the character of the trivial representation $\mathfrak{g} \to \boldsymbol{Q}_p^\times$. Namely, $\varepsilon(\sigma) = 1$ for every $\sigma \in \mathfrak{g}$. We make a convention that a symbol χ always denotes an irreducible character which is equal neither to ω nor to ε.

If a representation $\Phi : \mathfrak{g} \to GL_m(\boldsymbol{Q}_p)$ is given, we obtain another representation $\hat{\Phi}$ by $\hat{\Phi}(\sigma) = \Phi(\sigma^{-1})$. Denote by Φ^* is a representation $\hat{\Phi}\omega$: $\Phi^*(\sigma) = \Phi(\sigma)^{-1}\omega(\sigma)$. If χ is the character of Φ, we denote by $\hat{\chi}$ and χ^* the characters of $\hat{\Phi}$ and Φ^*, respectively. We see $(\chi^*)^* = \chi$. Note $\hat{\chi}$ and χ^* is irreducible if χ is irreducible, *cf.* Lemma 6 of [Ya].

Let q be a prime divisor of $\boldsymbol{Q}$ (possibly an infinite prime). Denote by $\mathfrak{q}$ a prime divisor of K lying above q. Let $\mathfrak{g}_\mathfrak{q}$ be the decomposition group of $\mathfrak{q}$. When $p = q$, we use the symbol $\mathfrak{p}$ to specialize the prime $\mathfrak{p}|p$. Since $\mathfrak{g}_\mathfrak{p}$ is a subgroup of $\mathfrak{g}$, $\boldsymbol{T}$ is a $\boldsymbol{Z}_p[\mathfrak{g}_\mathfrak{p}]$-module. To denote this module, we attach subscript $\mathfrak{p}$: $\boldsymbol{T}_\mathfrak{p}$. Let $\omega_{p,\chi}$ be a quantity defined to be

$$\omega_{p,\chi} = r_\chi(\boldsymbol{Z}_p[\mathfrak{g}] \otimes_{\boldsymbol{Z}_p[\mathfrak{g}_\mathfrak{p}]} \boldsymbol{T}_\mathfrak{p}).$$

Similarly, considering $\boldsymbol{Z}_p$ a trivial $\boldsymbol{Z}_p[\mathfrak{g}_\mathfrak{q}]$-module, we define

$$\varepsilon_{q,\chi} = r_\chi(\boldsymbol{Z}_p[\mathfrak{g}] \otimes_{\boldsymbol{Z}_p[\mathfrak{g}_\mathfrak{q}]} \boldsymbol{Z}_p).$$

Note $\boldsymbol{Z}_p[\mathfrak{g}] \otimes_{\boldsymbol{Z}_p[\mathfrak{g}_\mathfrak{q}]} \boldsymbol{Z}_p$ is isomorphic to $\boldsymbol{Z}_p[\mathfrak{g}/\mathfrak{g}_\mathfrak{q}]$.

Let T be a set of prime numbers whose primes divisors in K are ramified in M/K and which are not equal to p. Let E_T be the group of T-units of K :

$$E_T = \{x \in K^\times : (x, q) = 1 \ \textit{for every prime number} \ \ q \in T\}.$$

E_T/E_T^p is a finitely generated $\boldsymbol{F}_p[\mathfrak{g}]$-module. We abbreviate $\boldsymbol{Q}_p \otimes_{\boldsymbol{Z}_p} \boldsymbol{T}$ to $\boldsymbol{Q}_p\boldsymbol{T}$. The $\boldsymbol{Q}_p[\mathfrak{g}]$-module corresponding to E_T/E_T^p is

$$(\boldsymbol{Q}_p[\mathfrak{g}/\mathfrak{g}_{\mathfrak{p}_\infty}]/\boldsymbol{Q}_p) \oplus \boldsymbol{Q}_p\boldsymbol{T} \oplus \left(\bigoplus_{q \in T} \boldsymbol{Q}_p[\mathfrak{g}/\mathfrak{g}_\mathfrak{q}]\right) \tag{1}$$

where $\mathfrak{p}_\infty$ is a prime divisor of K such that $\mathfrak{p}_\infty|\infty$. Put $\beta_{T,\chi} = r_\chi(E_T/E_T^p)$.

Let D_T be the group of T-divisors of K. Namely, it is a free abelian group on the set of prime divisors of K not dividing any prime contained in T. We denote by $(x)_T$ the principal divisor generated by $x \in K^\times$. Let P_T be the group of principal divisors. The quotient module D_T/P_T is called the T-divisor class group of K. Let C_T be the submodule of p-torsion elements of the T-divisor class group. It is an $\boldsymbol{F}_p[\mathfrak{g}]$-module. Put $\gamma_{T,\chi} = r_\chi(C_T)$.

Let $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_s\}$ be the set of every extension onto K of the prime divisor p of $\boldsymbol{Q}$. Let $K_{\mathfrak{p}_i}$ be the completion of K at $\mathfrak{p}_i$. For each $\mathfrak{p}_i$, there is an embedding $\iota_i : K \to K_{\mathfrak{p}_i}$. These embeddings induce a diagonal map $K \to \prod_i K_{\mathfrak{p}_i}$. Denote by ι this diagonal map. Let $U_{\mathfrak{p}_i}$ be the group of units of $K_{\mathfrak{p}_i}$. Put $U = \prod_i U_{\mathfrak{p}_i}$. Since $\iota_i(E_T) \subset U_{\mathfrak{p}_i}$, we see $\iota(E_T) \subset U$.

Take a prolongation $\mathfrak{P}_i$ of $\mathfrak{p}_i$ onto M. Let L be an unramified abelian extension of degree p over the inertia field of $M_{\mathfrak{P}_i}/K_{\mathfrak{p}_i}$. We define a submodule $V_{\mathfrak{p}_i}$ of $U_{\mathfrak{p}_i}$ to be

$$V_{\mathfrak{p}_i} = (M_{\mathfrak{P}_i}L)^{\times p} \cap U_{\mathfrak{p}_i} \tag{2}$$

It contains $U^p_{\mathfrak{p}_i}$. Define a submodule V of U to be a direct product of $V_{\mathfrak{p}_i}$ for $i = 1, \ldots, s$. U and V are $\mathfrak{g}$-modules. Thus, U/V is an $\boldsymbol{F}_p[\mathfrak{g}]$-module. ι induces an $\boldsymbol{F}_p[\mathfrak{g}]$-module homomorphism

$$\varphi_T : E_T/E_T^p \longrightarrow U/V$$

Denote by $B_{1,T}$ the kernel of φ_T. We have an exact sequence

$$1 \to B_{1,T} \to E_T/E_T^p \to U/V \to U/\iota(E_T)V \to 1 \tag{3}$$

For each $c \in C_T$, we have $\mathfrak{a} \in c$ such that $\mathfrak{a}^p = (a)_T$ and $(a,p) = 1$. Since $\iota(a)$ is an element of U, we have $\iota(aE_T)V \in U/V$. This coset $\iota(aE_T)V$ is uniquely determined by c. Hence, an $\boldsymbol{F}_p[\mathfrak{g}]$-homomorphism $\rho_T : C_T \to U/\iota(E_T)V$ is defined to be $\rho_T(c) = \iota(aE_T)V$. Denote by $B_{0,T}$ the kernel of ρ_T. We have an exact sequence

$$1 \to B_{0,T} \to C_T \to U/\iota(E_T)V. \tag{4}$$

We define quantities $\alpha_{T,\chi}$, $b_{0,T,\chi}$, $b_{1,T,\chi}$ to be

$$\begin{aligned} \alpha_{T,\chi} &= r_\chi(\operatorname{coker} \rho_T) \\ b_{0,T,\chi} &= r_\chi(B_{0,T}) \\ b_{1,T,\chi} &= r_\chi(B_{1,T}) \end{aligned}$$

respectively. The following formula obtained in [Ya] is fundamental in our arguments:

Theorem 2.1 *For each irreducible character χ such that $\chi \neq \omega, \varepsilon$, we have*

$$a_{\chi^*} = b_{0,T,\chi} + b_{1,T,\chi} = \alpha_{T,\chi} + \beta_{T,\chi} + \gamma_{T,\chi} + \omega_{p,\chi} - 1$$

Furthermore, if $\omega_{p,\chi} = 1$, we have $\alpha_{T,\chi} = 0$.

3. Auxiliary lemmas

Let M_0 be the fixed field with $\mathfrak{g}$. We have $M = M_0K$ and $Gal(M_0/\boldsymbol{Q}) = G_p$. Let $\mathfrak{g}_\chi$ be the kernel of the representation:

$$\mathfrak{g}_\chi = \{\sigma \in \mathfrak{g} : \Phi(\sigma) \ \textit{is identity matrix}\}.$$

We can reduce Φ_χ onto $\mathfrak{g}/\mathfrak{g}_\chi$ and denote by the same symbol Φ_χ this reduction, *cf.* §1 of [Ya]. We also denote by χ the character of the reduced representation. Let K_χ be the intermediate field of $K/\boldsymbol{Q}$ corresponding to $\mathfrak{g}_\chi$. In Lemma 4 of [Ya], we show the following reduction property holds:

Lemma 3.1 *Put $M_\chi = M_0 K_\chi$. Let C_χ be the p-class group of M_χ. We have $r_\chi(C/C^J) = r_\chi(C_\chi/C_\chi^J)$.*

The induced module $\boldsymbol{Q}_p[\mathfrak{g}] \otimes_{\boldsymbol{Q}_p[\mathfrak{g}_\mathfrak{q}]} \boldsymbol{Q}_p$ of the trivial $\boldsymbol{Q}_p[\mathfrak{g}_\mathfrak{q}]$-module $\boldsymbol{Q}_p$ is isomorphic to $\boldsymbol{Q}_p[\mathfrak{g}/\mathfrak{g}_\mathfrak{q}]$. We see $\varepsilon_{q,\chi} = \dim_{R_\chi} \boldsymbol{Q}_p[\mathfrak{g}/\mathfrak{g}_\mathfrak{q}]$ for a prime divisor $\mathfrak{q}$. Sine $\boldsymbol{Q}_p[\mathfrak{g}/\mathfrak{g}_\mathfrak{q}]$ is a homomorphic image of $\boldsymbol{Q}_p[\mathfrak{g}]$, we have

$$\varepsilon_{q,\chi} = \dim_{R_\chi} \boldsymbol{Q}_p[\mathfrak{g}/\mathfrak{g}_\mathfrak{q}] \leq 1.$$

Lemma 3.2 *We have $\omega_{p,\chi} \leq 1$. $\omega_{p,\chi} = 1$ if and only if p is decomposed completely in K_{χ^*}.*

Proof. We refer for the Frobenius reciprocity law to Theorem 19.2.11 and Corollary 19.2.12 in [Ka]. Let $<,>_\mathfrak{g}$ be the symmetric pairing of $\boldsymbol{Q}_p$-character. The value of the pairing is defined to be

$$<\varphi,\psi>_\mathfrak{g} = \frac{1}{|\mathfrak{g}|}\sum_{\sigma\in\mathfrak{g}}\varphi(\sigma)\psi(\sigma^{-1}) = \frac{1}{|\mathfrak{g}|}\sum_{\sigma\in\mathfrak{g}}\varphi(\sigma^{-1})\psi(\sigma) = <\psi,\varphi>_\mathfrak{g}$$

for characters φ and ψ. $r_\chi(Y)$ is positive if and only if $<\chi,\varphi_Y> \neq 0$ for a character φ_Y afforded with a $\boldsymbol{Q}_p[\mathfrak{g}]$-module corresponding to Y. We have $r_\chi(Y) = 1$ if $<\chi,\varphi_Y> = 1$. Similarly, we denote by $<\varphi,\psi>_{\mathfrak{g}_\mathfrak{p}}$ the symmetric product of characters of $\mathfrak{g}_\mathfrak{p}$. Let $\omega_\mathfrak{p}$ be the character afforded with a $\boldsymbol{Q}_p[\mathfrak{g}_\mathfrak{p}]$-module $\boldsymbol{Q}_p\boldsymbol{T}_\mathfrak{p}$. The induced character of $\omega_\mathfrak{p}$ is a character afforded with a $\boldsymbol{Q}_p[\mathfrak{g}]$-module $\boldsymbol{Q}_p[\mathfrak{g}] \otimes_{\boldsymbol{Q}_p[\mathfrak{g}_\mathfrak{p}]} \boldsymbol{Q}_p\boldsymbol{T}_\mathfrak{p}$. Denote by $ind_p\omega_\mathfrak{p}$ this induced character. The Frobenius reciprocity law assures that an equality

$$<\chi, ind_p\omega_\mathfrak{p}>_\mathfrak{g} = <\chi_{|\mathfrak{g}_\mathfrak{p}}, \omega_{\mathfrak{g}_\mathfrak{p}}>_{\mathfrak{g}_\mathfrak{p}}$$

holds, where $\chi_{|\mathfrak{g}_\mathfrak{p}}$ is restriction of χ onto $\mathfrak{g}_\mathfrak{p}$. Let $\varepsilon_\mathfrak{p}$ be the trivial character of $\mathfrak{g}_\mathfrak{p}$ and $ind_p\varepsilon_\mathfrak{p}$ be the induced character. We have

$$<\chi^*, ind_p\varepsilon_\mathfrak{p}>_\mathfrak{g} = <\chi^*_{|\mathfrak{g}_\mathfrak{p}}, \omega_{\mathfrak{g}_\mathfrak{p}}>_{\mathfrak{g}_\mathfrak{p}}.$$

Since $\omega_\mathfrak{p} = \omega_{|\mathfrak{g}_\mathfrak{p}}$, we have $\chi^*_{|\mathfrak{g}_\mathfrak{p}} = \hat{\chi}_{|\mathfrak{g}_\mathfrak{p}}\omega_{\mathfrak{g}_\mathfrak{p}}$. Thus, we have

$$<\chi^*_{|\mathfrak{g}_\mathfrak{p}}, \varepsilon_\mathfrak{p}>_{\mathfrak{g}_\mathfrak{p}} = <\omega_\mathfrak{p}, \chi_{|\mathfrak{g}_\mathfrak{p}}\varepsilon_\mathfrak{p}>_{\mathfrak{g}_\mathfrak{p}} = <\omega_\mathfrak{p}, \chi_{|\mathfrak{g}_\mathfrak{p}}>_{\mathfrak{g}_\mathfrak{p}}.$$

By virtue of the Frobenius reciprocity law, we obtain

$$<\chi, ind_p\omega_\mathfrak{p}>_\mathfrak{g} = <\chi^*, ind_p\varepsilon_\mathfrak{p}>_\mathfrak{g}$$

Since $ind_p\varepsilon_{\mathfrak{p}}$ is the character afforded with $\boldsymbol{Q}_p[\mathfrak{g}/\mathfrak{g}_{\mathfrak{p}}]$, we have

$$<\chi^*, ind_p\varepsilon_{\mathfrak{p}}>_{\mathfrak{g}} \leq 1$$

and the value of the symmetric pairing is not equal to 0 if and only if $\Phi_{\chi^*}(\sigma) = \Phi_{\chi^*}(1)$ for every $\sigma \in \mathfrak{g}_{\mathfrak{p}}$. This implies p is completely decomposed in K_{χ^*}, *i.e.* $\mathfrak{g}_{\mathfrak{p}} \subset \mathfrak{g}_{\chi}$. *q.e.d.*

Lemma 3.3 *Suppose $\omega_{p,\chi} = 0$. Then, we have $r_\chi(U/U^p) = 1$. Hence, if $e_\chi(\iota(E_T)V/V) \neq 0$, we have $\alpha_{T,\chi} = 0$.*

Proof. From (3.5) of §3, [Ya], the $\boldsymbol{Q}_p[\mathfrak{g}]$-module corresponding to U/U^p is

$$\boldsymbol{Q}_p[\mathfrak{g}] \oplus \left(\boldsymbol{Q}_p[\mathfrak{g}] \otimes_{\boldsymbol{Q}_p[\mathfrak{g}_{\mathfrak{p}}]} \boldsymbol{Q}_p \boldsymbol{T}_{\mathfrak{p}}\right).$$

Thus, if $\omega_{p,\chi} = 0$, we have $r_\chi(U/U^p) = r_\chi(\boldsymbol{F}_p[\mathfrak{g}]) = 1$. Moreover, if $e_\chi(\iota(E_T)V/V) \neq 0$, we have $e_\chi(\iota(E_T)V/V) = e_\chi(U/U^p)$, because of $r_\chi(U/V) \leq r_\chi(U/U^p) = 1$. Hence, $e_\chi(U/\iota(E_T)V) = 0$. $\alpha_{T,\chi} = 0$ follows from the definition. *q.e.d.*

Let τ_∞ be a generator of $\mathfrak{g}_\infty$. We call χ is real (*resp.* imaginary) if $\Phi_\chi(\tau_\infty) = \Phi_\chi(1)$ (*resp.* $\Phi_\chi(\tau_\infty) = -\Phi_\chi(1)$). By (1), we see $\beta_\chi = 1 + \sum_{q\in T} \varepsilon_{q,\chi}$ if χ is real and $\beta_\chi = \sum_{q \in T} \varepsilon_{q,\chi}$ if χ is imaginary.

Lemma 3.4 *If $\omega_{p,\chi} = 1$, or if $\omega_{p,\chi} = 0$ and $\beta_{T,\chi} \neq b_{1,T,\chi}$, we have $\alpha_{T,\chi} = 0$.*

Proof. If $\omega_{p,\chi} = 1$, we have $\alpha_{T,\chi} = 0$ from Theorem 2.1. Suppose $\omega_{p,\chi} = 0$ and $\beta_{T,\chi} \neq b_{1,T,\chi}$. By the exact sequence (3), we have

$$r_\chi(U/\iota(E_T)V) = r_\chi(U/V) - r_\chi(E_T/E_T^p) + r_\chi(B_{1,T}) \leq 1 - (\beta_{T,\chi} - b_{1,T,\chi}).$$

since $\beta_{T,\chi} > b_{1,T,\chi}$, we conclude $r_\chi(U/\iota(E_T)V) = 0$, and hence $\alpha_{T,\chi} = 0$. *q.e.d.*

The proof of Theorem 2.1 given in [Ya] is based on an equality

$$a_\chi = r_\chi(Gal(H^{ab}/K)) = r_\chi(Gal(H^{ab}/M)),$$

where H^{ab} is the maximal abelian subfield of M/K. Let $\tilde{H}$ be the Hilbert class field and $\tilde{H}^{ab}$ be the maximal abelian subfield of $\tilde{H}/K$. Since M/K is abelian, $\tilde{H}^{ab}$ is the genus field. For an arbitrary finite abelian group, we denote by A' the p-Sylow subgroup. By means of this convention, we have $Gal(\tilde{H}^{ab}/K)' = Gal(H^{ab}/K)$.

Let J_K (*resp.* J_M) be the idele group of K (*resp.* M). Let U_K (*resp.* U_M) be the unit group of the idele group. By class field theory, we have

$$Gal(\tilde{H}^{ab}/K) \cong J_K/N_{M/K}(U_M)K^\times.$$

Since $J_K/N_{M/K}(J_M)K^\times \cong G_p$ is a trivial $\mathfrak{g}$-module, we have $e_\chi(J_K/N_{M/K}(J_M)K^\times) = 1$. Thus,

$$r_\chi(Gal(H^{ab}/M)) = r_\chi((N_{M/K}(J_M)K^\times/N_{M/K}(U_M)K^\times)').$$

Denote by E_K the unit group of K: $E_K = E_\emptyset$. Put

$$\mathfrak{G}_1 = U_K/N_{M/K}(U_K)E_K, \qquad \mathfrak{G}_0 = (N_{M/K}(J_M)/N_{M/K}(U_M)K^\times)'.$$

Let p^e be the exponent of G_p : the least positive integer such that $G_p^{p^e} = 1$ holds. we have the following three exact sequences

(5) $$1 \to e_\chi \mathfrak{G}_1 \to e_\chi \mathfrak{G}_0 \to e_\chi Cl_K \to 1,$$

(6) $$1 \to e_\chi \frac{E_K}{E_K \cap N_{M/K}(U_M)} \to e_\chi \frac{U_K}{N_{M/K}(U_M)} \to e_\chi \mathfrak{G}_1 \to 1,$$

$$1 \to e_\chi \frac{E_K \cap N_{M/K}(U_M)}{E_K^{p^e}} \to e_\chi \frac{E_K}{E_K^{p^e}} \to e_\chi \frac{E_K}{E_K \cap N_{M/K}(U_M)} \to 1,$$

where Cl_K is the p-class group of K. Another formula of a_χ is obtained from these sequence. Since $\gamma_{\emptyset,\chi} = r_\chi(Cl_K)$ and $a_\chi = r_\chi(\mathfrak{G}_0)$, we have :

(7) $$a_\chi = \gamma_{\emptyset,\chi} + r_\chi(U_K/N_{M/K}(U_M)) - \beta_{\emptyset,\chi} + r_\chi(E_K \cap N_{M/K}(U_M)/E_K^{p^e}) - g_\chi,$$

where g_χ is defined to be $g_\chi = g_0 - g_1 + g_2$ for

$$\begin{aligned} g_0 &= r_\chi(\text{coker}\,(Tor(\boldsymbol{F}_p, e_\chi\mathfrak{G}_0) \to Tor(\boldsymbol{F}_p, e_\chi Cl_K))), \\ g_1 &= r_\chi(\text{coker}\,(Tor(\boldsymbol{F}_p, e_\chi U_K/N_{M/K}(U_M)) \to Tor(\boldsymbol{F}_p, e_\chi\mathfrak{G}_1))), \\ g_2 &= r_\chi(\text{coker}\,(Tor(\boldsymbol{F}_p, e_\chi E_K/E_K^{p^e}) \to Tor(\boldsymbol{F}_p, e_\chi E_K/E_K \cap N_{M/K}(U_M)))). \end{aligned}$$

Note $\beta_{\emptyset,\chi} = \varepsilon_{\infty,\chi}$. Let $\mathfrak{t}_\mathfrak{q}$ be an inertia group of a prime $\mathfrak{q}$ of K. Set T_0 be the set of every prime number whose prime divisor in K is ramified in M/K. We see $T_0 \subset T \cup \{p\}$. By local class field theory, we have

(8) $$U_K/N_{M/K}(U_M) \cong \oplus_{q\in T_0} \oplus_{\mathfrak{q}|q} \mathfrak{t}_\mathfrak{q}.$$

Put $\mathfrak{t}_q = \oplus_{\mathfrak{q}|q}\mathfrak{t}_\mathfrak{q}$. This $\boldsymbol{Z}_p[\mathfrak{g}]$-module is isomorphic to $\boldsymbol{Z}_p[\mathfrak{g}] \otimes_{\boldsymbol{Z}_p[\mathfrak{g}_\mathfrak{q}]} \mathfrak{t}_\mathfrak{q}$. Since $\mathfrak{g}$ is abelian, $\mathfrak{t}_\mathfrak{q}$ is a trivial $\mathfrak{g}_\mathfrak{q}$-module. Thus, the $\boldsymbol{Q}_p[\mathfrak{g}]$-module corresponding to $\mathfrak{t}_q/\mathfrak{t}_q^p$ is the induced module $\boldsymbol{Q}_p[\mathfrak{g}] \otimes_{\boldsymbol{Q}_p[\mathfrak{g}_\mathfrak{q}]} \boldsymbol{Q}_p$. We rewrite the above formula (7) and obtain,

Lemma 3.5 *Let T_0 be the set of every prime number whose prime divisor in K is ramified in M. We have*

$$a_\chi = \gamma_{\emptyset,\chi} + \sum_{q\in T_0} \varepsilon_{q,\chi} - \varepsilon_{\infty,\chi} + r_\chi(E_K \cap N_{M/N}(U_M)/E_K^{p^e}) - g_\chi.$$

Comparing the formulas of a^*_χ of Theorem 2.1 with that of Lemma 3.5, we observe

$$\gamma_{\emptyset,\chi^*}+\sum_{q\in T_0}\varepsilon_{q,\chi^*}-\varepsilon_{\infty,\chi^*}+r_{\chi^*}(E_K\cap N_{M/K}(U_M)/E_K^{p^e})-g_{\chi^*}=\alpha_{T,\chi}+\varepsilon_{\infty,\chi}+\sum_{q\in T}\varepsilon_{q,\chi}+\gamma_{T,\chi}+\omega_{p,\chi}-1.$$

Since $\varepsilon_{\infty,\chi^*}+\varepsilon_{\infty,\chi}=1$, we have

Lemma 3.6 *Define $\varepsilon'_{p,\chi^*}=\varepsilon_{p,\chi^*}$ if $p\in T_0$ and $\varepsilon'_{p,\chi^*}=0$ if $p\notin T_0$. Then, we have*

$$\gamma_{\emptyset,\chi^*}-\gamma_{T,\chi}=\alpha_{T,\chi}+\sum_{q\in T}(\varepsilon_{q,\chi}-\varepsilon_{q,\chi^*})-\varepsilon'_{p,\chi}+\omega_{p,\chi}-r_{\chi^*}(E_K\cap N_{M/K}(U_M)/E_K^p)+g_{\chi^*}.$$

Lemma 3.7 *We have $a_\chi\geq\gamma_{\emptyset,\chi}$.*

Proof. Let H_K be the p-Hilbert class field of K. We have

$$1\to e_\chi Gal(H_K\cap M/K)\to e_\chi Gal(H_K/K)\to e_\chi Gal(H_KM/M)\to 1$$

Since $e_\chi Gal(H_\cap M/K)=1$, we obtain $e_\chi Gal(H_K/K)=e_\chi Gal(H_KM/M)$. $a_\chi\geq\gamma_{\emptyset,\chi}$ follows from $H^{ab}\supset H_KM$. *q.e.d.*

4. p^nth cyclotomic extensions

We apply the general results obtained in the previous section to the special case that M is a p^nth cyclotomic extension of K. However, problems of computing the rank of p-class group of K and determining values of $\alpha_{T,\chi}$ are still remained. Set $M=K(\zeta_{p^n})$ for $n\geq 2$. We have $T=\emptyset$ from Lemma 10 in §4 of [Ya].

Lemma 4.1 *The value of a_χ dose not depend on n for $n\geq 2$. In other words, its value for every $n\geq 2$ is equal to that for $n=2$.*

Proof. $V_{\mathfrak{p}_i}$ is defined as in (2). L is an unramified abelian extension of degree p over the inertia field of $M_{\mathfrak{P}_i}/K_{\mathfrak{p}_i}$. L is cyclic over $K_{\mathfrak{p}_i}$. Put $N=LM_{\mathfrak{P}_i}$. Let L^* be a subfield such that $Gal(L^*/K_{\mathfrak{p}_i})\cong \boldsymbol{Z}/p\boldsymbol{Z}\times\boldsymbol{Z}/p\boldsymbol{Z}$. L^* is a Kummer extension of $K_\mathfrak{p}$. It contains a ramified extension $K_\mathfrak{p}(\zeta_{p^2})$ and an unramified extension over $K_{\mathfrak{p}_i}$. Hence, $L^*=K_{\mathfrak{p}_i}(\zeta_{p^2},\sqrt[p]{x})$ for $x\in U_{\mathfrak{p}_i}$ such that $K_{\mathfrak{p}_i}(\sqrt[p]{x})/K_{\mathfrak{p}_i}$ is unramified. The Kummer radical of $L^*/K_{\mathfrak{p}_i}$ is

$$(L^*)^{\times p}\cap K^\times_{\mathfrak{p}_i}/K^{\times p}_{\mathfrak{p}_i}=((L^*)^{\times p}\cap U_{\mathfrak{p}_i})K^{\times p}_{\mathfrak{p}_i}/K^{\times p}_{\mathfrak{p}_i}=V_{\mathfrak{p}_i}K^{\times p}_{\mathfrak{p}_i}/K^{\times p}_{\mathfrak{p}_i}.$$

This shows that $V_{\mathfrak{p}_i}$ for every $n\geq 2$ coincides with a unique subgroup of $U_{\mathfrak{p}_i}$. Since $\alpha_{\emptyset,\chi}=r_\chi(\operatorname{coker}\rho_T)$, it dose not depend on n. Hence, by the formula of Theorem 2.1, the value of a_χ dose not depend on n. *q.e.d.*

Lemma 4.2 *If $M=K(\zeta_{p^2})$ and if p is completely decomposed in K_χ. Then, we have*

$$\begin{aligned} e_\chi U_K/N_{M/K}(U_M) &\cong e_\chi U/U^p\\ e_\chi U_K/E_KN_{M/K}(U_M) &\cong e_\chi U/\iota(E_\emptyset)U^p\end{aligned}$$

are induced from the projection $U_K\to U$.

Proof. We write $U(K)$ for U to denote the field K for which U is defined. Note $[K : K_\chi] = N_{K/K_\chi} \circ e_\chi = e_\chi \circ N_{K/K_\chi}$ holds, because we identify χ with its restriction onto $\mathfrak{g}/\mathfrak{g}_\chi$. The norm map N_{K/K_χ} induces a homomorphism $U_K/N_{M/K}(U_M) \to U_{K_\chi}/N_{M_\chi/K_\chi}(U_{M_\chi})$. Since p dose not divide $[U_{K_\chi} : N_{K/K_\chi}(U_K)]$ and $[N_{M_\chi/K_\chi}(U_{M_\chi}) : M_{M/K_\chi}(U_M)]$, restriction of this homomorphism onto e_χ-component is an isomorphism. Thus, an isomorphism

$$e_\chi U_K/N_{M/K}(U_M) \cong e_\chi U_{K_\chi}/N_{M_\chi/K_\chi}(U_{M_\chi})$$

is defined to be $xN_{M/K}(U_M) \to N_{K/K_\chi}(x)N_{M_\chi/K_\chi}(U_{M_\chi})$. Let $\pi_\chi : U_{K_\chi} \to U(K_\chi)$ is the projection map. Since p is decomposed completely in K_χ, we have $\pi_\chi(N_{M_\chi/K_\chi}(U_{M_\chi})) = U(K_\chi)^p$. Thus, π_χ induces an isomorphism $U_{K_\chi}/N_{M_\chi/K_\chi}(U_{M_\chi}) \cong U(K_\chi)/U(K_\chi)^p$. This proves the first isomorphism. The second one is obtained by a similar argument as in the above. We can show an isomorphism

$$e_\chi U(K)/U(K)^p \cong e_\chi U(K_\chi)/U(K_\chi)^p$$

and hence, $e_\chi U_K/N_{M/K}(U_M) \cong e_\chi U(K)/U(K)^p$ follows. Since $e_\chi E_K U(K)^p/U(K)^p$ is isomorphic to $e_\chi \iota_\chi(E_{K_\chi})U(K_\chi)^p/U(K_\chi)^p$ by this isomorphism, where ι_χ is a diagonal map $E_{K_\chi} \to U(K_\chi)$, we obtain $e_\chi U_K/E_K N_{M/K}(U_M) \cong e_\chi U(K)/\iota(E_K)U(K)^p$. *q.e.d.*

When $M = K(\zeta_p)$, we observe $\left(U_K/N_{M/K}(U_M)\right)^p = 1$ and $e = 1$. This implies $g_1 = g_2 = 0$. We have $g_\chi = g_0$. There is a relation between $\alpha_{\emptyset,\chi}$ and g_χ. If $\varepsilon_{p,\chi} = 0$, we have $r_\chi(U_K/N_{M/K}(U_M)) = 0$ from (8), and hence, $r_\chi(\mathfrak{G}_1) = 0$ from (6). Thus, $r_\chi(\mathfrak{G}_0) = r_\chi(Cl_K)$ from (5). We see $g_\chi = 0$ and $a_\chi = \gamma_{\chi,\emptyset}$.

Lemma 4.3 *Suppose $M = K(\zeta_{p^2})$ and $\varepsilon_{p,\chi} = 1$. Then, $g_\chi = 0$ if and only if $\alpha_{\emptyset,\chi} = 1$.*

Proof. From the proof of Lemma 3.2, we observe $\varepsilon_{p,\chi} = 1$ is equivalent to $\omega_{p,\chi^*} = 1$. In particular, we have $\omega_{p,\chi} = 0$ if $\varepsilon_{p,\chi} = 1$. Hence, by (3.7) of [Ya], we have $e_\chi U/V \cong e_\chi U/U^p$. Thus,

$$e_\chi U/\iota(E_\emptyset)V \cong e_\chi U/\iota(E_\emptyset)U^p.$$

Suppose $\alpha_{\emptyset,\chi} = 1$. We have $e_\chi B_{0,\emptyset} \cong e_\chi C_\emptyset$ $(= e_\chi Tor(F_p, Cl_K))$. There is an ideal $\mathfrak{a} \in c$ for each $c \in e_\chi C_\emptyset$ such that $(\mathfrak{a}, p) = 1$. Denote by $\tilde{\mathfrak{a}}$ an idele which represents $\mathfrak{a}$:$\mathfrak{a} = \tilde{\mathfrak{a}}U_K$ in the divisor group J_K/U_K. Let x be an element of $K^\times$ such that $\mathfrak{a}^p = (x)_\emptyset$. We have $u \in U_K$ such that $\tilde{\mathfrak{a}}^p = ux$. Let y be an element of $J_K/N_{M/K}(U_M)K^\times$ generated by $\tilde{\mathfrak{a}}$. Since $e_\chi c = c$, we can select $\mathfrak{a}$ and $\tilde{\mathfrak{a}}$ so that $y \in e_\chi \mathfrak{G}_0$. $\iota(x) \in \iota(E_\emptyset)V$ follows from $c \in e_\chi B_{0,\emptyset}$. By Lemma 4.2 and by the above isomorphism, we have

$$e_\chi U_K/E_K N_{M/K}(U_M) \cong e_\chi U/\iota(E_\emptyset)U^p \cong e_\chi U/\iota(E_\emptyset)V.$$

Hence, $e_\chi \iota(x) = 1$ in $e_\chi U/\iota(E_\emptyset)U^p$. Further, this implies $e_\chi x = 1$ in $e_\chi U_K/E_K N_{M/K}(U_M)$. Therefore, $y^p = 1$. Since $e_\chi y$ is a p-torsion element mapped onto c, we obtain $g_\chi = 0$.

Suppose $g_\chi = 0$. We have an exact sequence

$$1 \to Tor(\boldsymbol{F}_p, e_\chi \mathfrak{G}_1) \to Tor(\boldsymbol{F}_p, e_\chi \mathfrak{G}_0) \to e_\chi C_\emptyset \to 1.$$

Let c be an element of $e_\chi C_\emptyset$. Let $y = \mathfrak{a} N_{M/K}(U_M) K^\times$ ($\mathfrak{a} \in J_K$) be an inverse image into $e_\chi \mathfrak{G}_0$ of c such that $y^p = 1$. We may suppose $(\mathfrak{a}, p) = 1$. We see $\mathfrak{a}^p = ux$ for $u \in N_{M/K}(U_M)$ and $x \in K$. Since $N_{K/K_\chi}(u) = u_1^p u_2$ for $u_1 \in U(K_\chi)$ and $u_2 \in \prod_{\mathfrak{p}_\chi \nmid p} U_{\mathfrak{p}_\chi}$, we observe $\iota_\chi(N_{K/K_\chi}(x)) \in \iota_\chi(E_{K_\chi}) U(K_\chi)^p$. Hence, $c \in e_\chi B_{0,\emptyset}$. Thus, $e_\chi B_{0,\emptyset} \cong e_\chi C_\emptyset$. *q.e.d.*

Proposition 4.4 *Suppose $M = K(\zeta_{p^2})$.*

(1) *If p is decomposed completely in K_χ, we have*

$$a_\chi = \gamma_{\emptyset,\chi} + b_{1,\emptyset,\chi} + 1 - \varepsilon_{\infty,\chi} - g_\chi = \varepsilon_{\infty,\chi^*} + \gamma_{\emptyset,\chi^*}.$$

(2) *If p is not decomposed in K_χ, we have $a_\chi = \gamma_{\emptyset,\chi}$ and $g_\chi = 0$.*

Proof. We have the following exact sequence which presents $E_K/E_K \cap N_{M/K}(U_M)E_K^p$:

$$1 \to E_K \cap N_{M/K}(U_M)/E_K^p \to E_K/E_K^p \to E_K/E_K \cap N_{M/K}(U_M)E_K^p \to 1.$$

Hence, $r_\chi(E_K/E_K \cap N_{M/K}(U_M)E_K^p) = \varepsilon_{\infty,\chi} - r_\chi(E_K \cap N_{M/K}(U_M)/E_K^p)$. By Lemma 4.2, we observe $e_\chi E_K \cap N_{M/K}(U_M)/E_K^p$ is isomorphic to the kernel of $e_\chi E_K/E_K^p \to e_\chi U/U^p$. Thus, $b_{1,\emptyset,\chi}$ is equal to $r_\chi(E_K \cap N_{M/K}(U_M)/E_K^p)$. We obtain the formula of (1) from Lemma 3.5 and Theorem 2.1.

We apply the reduction property. Since p is not decomposed in K_χ, the product of inertia groups is $\{1\}$. We have $e_\chi U_{K_\chi}/N_{M_\chi/K_\chi}(U_{M_\chi}) = 1$. By (5), we see $e_\chi \mathfrak{G}_0 \cong e_\chi Cl_K$. Thus, $g_\chi = 0$ and $a_\chi = \gamma_{\emptyset,\chi}$. We obtain (2). *q.e.d.*

Let U_χ be a direct product of $U_{\mathfrak{p}_\chi}$ of the completion at prime $\mathfrak{p}_\chi$ of K_χ lying above p: $U_\chi = \prod_{\mathfrak{p}_\chi | p} U_{\mathfrak{p}_\chi}$. Suppose $\ell = [K_\chi : \boldsymbol{Q}]$ is a prime dividing $p-1$. We observe χ and χ^* are one-dimensional. We consider χ and ω Diriclet characters. Put $m = (p-1)/\ell$. There is χ_0 such that $\chi = \chi_0 \omega^{am}$ and whose conductor is prime to p. Put $b = ord\ \chi_0$. Suppose $\ell \neq p-1$. Note $\chi^* = \chi_0^{-1}\omega^{1-am}$. The order of ω^{1-am} is $(p-1)/(1-am, p-1)$. Hence, if p is not decomposed in K_{χ_0} or if b divides $(p-1)/(1-am, p-1)$, p is not decomposed in K_{χ^*}.

Theorem 4.5 *If $p = 3$ and χ is a quadratic character, the value of a_χ is determined from the rank $\gamma_{\emptyset,\chi}$ and $\gamma_{\emptyset,\chi^*}$. If $p > 3$ and $[K_\chi : \boldsymbol{Q}]$ is a prime dividing $p-1$, we have*

(1) *If p is decomposed in K_χ, we have $\omega_{p,\chi} = 0$, $\omega_{p,\chi^*} = 1$, $\alpha_{\chi^*} = 0$ and*

$$\begin{aligned} a_\chi &= \varepsilon_{\infty,\chi^*} + \gamma_{\emptyset,\chi^*} = \gamma_{\emptyset,\chi} + b_{1,\emptyset,\chi} + 1 - \varepsilon_{\infty,\chi} - g_\chi, \\ a_{\chi^*} &= \alpha_\chi + \varepsilon_{\infty,\chi} + \gamma_{\emptyset,\chi} - 1 = \gamma_{\emptyset,\chi^*}. \end{aligned}$$

(2) *If p is not decomposed in K_{χ^*} and K_χ, we have $g_\chi = g_{\chi^*} = 0$ and $\alpha_{\emptyset,\chi} + \alpha_{\emptyset,\chi^*} = 1$.*

Proof. If $p = 3$, we see χ^* is also quadratic. If p is decomposed completely in K_χ, it is not decomposed in K_{χ^*}. We have $a_\chi = \varepsilon_{\infty,\chi^*} + \gamma_{\emptyset,\chi^*}$ and $a_{\chi^*} = \gamma_{\emptyset,\chi^*}$ from Proposition 4.4. If p is not decomposed in K_χ and K_{χ^*}, we have $a_\chi = \gamma_{\emptyset,\chi}$ and $a_{\chi^*} = \gamma_{\emptyset,\chi^*}$.

Suppose $p > 3$ and the degree of K_χ is a prime dividing $p-1$. If p is decomposed in K_χ, we have (1) from Proposition 4.4. If p is not decomposed in K_χ and K_{χ^*}, we may assume χ^* is imaginary. We see $\varepsilon_{\infty,\chi^*} = 0$. By virtue of Theorem 2.1 and Proposition 4.4, we have

$$\begin{aligned} a_\chi &= \alpha_{\emptyset,\chi^*} + \gamma_{\emptyset,\chi^*} - 1 = \gamma_{\emptyset,\chi} \\ a_{\chi^*} &= \alpha_{\emptyset,\chi} + \gamma_{\emptyset,\chi} = \gamma_{\emptyset,\chi^*}. \end{aligned}$$

We observe $\alpha_{\emptyset,\chi} + \alpha_{\emptyset,\chi^*} = 1$. *q.e.d.*

Corollary 4.6 *In case (1) of the above theorem, we have*

$$\gamma_{\emptyset,\chi^*} - \gamma_{\emptyset,\chi} = b_{1,\emptyset,\chi} - g_\chi.$$

Proof. By computing $a_\chi - a_{\chi^*}$, we obtain

$$\varepsilon_{\infty,\chi^*} = b_{1,\emptyset,\chi} + 2(1-\varepsilon_{\infty,\chi}) - (g_\chi + \alpha_{\emptyset,\chi}).$$

By the formula of a_{χ^*}, we also have

$$\gamma_{\emptyset,\chi^*} - \gamma_{\emptyset,\chi} = \alpha_{\emptyset,\chi} + \varepsilon_{\infty,\chi} - 1.$$

If χ is real, we have $0 = b_{1,\emptyset,\chi} - (g_\chi + \alpha_{\emptyset,\chi})$ and $\gamma_{\emptyset,\chi^*} - \gamma_{\emptyset,\chi} = \alpha_{\emptyset,\chi}$. If χ is imaginary, we have $b_{1,\emptyset,\chi} = 0$ and $1 = g_\chi + \alpha_{\emptyset,\chi}$, $\gamma_{\emptyset,\chi^*} - \gamma_{\emptyset,\chi} = \alpha_{\emptyset,\chi} - 1$. Form these relations, we obtain $\gamma_{\emptyset,\chi^*} - \gamma_{\emptyset,\chi} = b_{1,\emptyset,\chi} - g_\chi$. *q.e.d.*

Remark 4.7 *When p is not decomposed in K_χ and is not decomposed completely in K_{χ^*}, p is decomposed in $K_{\chi_0^d}$ for $d = ord\,\omega^{1-am}$ and $K_{\chi_0^d} \neq \boldsymbol{Q}$.*

We remark the connection between a_χ and Iwasawa's λ-invariant. Denote by C_n the p-class group of $K(\zeta_{p^n})$. Let C_∞ be the projective limit of $\{C_n\}$ with respect to norm maps. C_∞ is called the Iwasawa module of K. It is a $\boldsymbol{Z}_p[\mathfrak{g}]$-module. If χ is imaginary, $e_\chi C_\infty$ is a $\boldsymbol{Z}_p$-free module of finite rank. This rank λ_χ is called the Iwasawa λ-invariant. Put $M = K(\zeta_{p^n})$. The value of $a_\chi = r_\chi(C_n/C_n^J) = r_\chi(C_2/C_2^J)$.

Lemma 4.8 *Let χ be an imaginary character. If $\lambda_\chi \neq 0$, we have $a_\chi \geq 1$. Moreover, if $\lambda_\chi \leq 1$, we have $a_\chi = \lambda_\chi$.*

Remark 4.9 *If χ is real, λ_{χ^*} is computed by virtue of the Iwasawa construction of p-adic L-functions, cf. [Wa]. When K is a composite of quadratic field $\boldsymbol{Q}(\sqrt{m})$ and $\boldsymbol{Q}(\zeta_p)$, where m is a square free integer, we compute a_χ and a_{χ^*} numerically for the quadratic character for χ of K.*

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