p^m-singular Numbbers and the Value of a p-adic L-Function at $s=1$ of an Abelian Field

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1. Introduction Let p be an odd prime and K be an Abelian field containing a primitive p-th root of unity. Let k be the maxiaml real subfield of K. We suppose that the conductor of K is not divisible by p^2 and that the extension degree of K is prime to p. Denote by G (resp. $\mathfrak{g}$) the Galois group of K (resp. k) over the field $\mathbf{Q}$ of rational numbers. Put $g = \sharp\mathfrak{g}$. Let $\mathbf{Q}_p$ be the field of p-adic numbers and $\mathbf{Z}_p$ be the valuation ring of $\mathbf{Q}_p$. Let $\mathbf{C}_p$ be the completion of an algebraic closure of $\mathbf{Q}_p$ with the valuation. Denote by v_p the normalized valuation of $\mathbf{C}_p$ such that $v_p(p) = 1$. Let $\hat{\mathfrak{g}}$ be the set of $\mathbf{C}_p$-irreducible characters of $\mathfrak{g}$ and $\mathcal{O}$ be the ring of values: $\mathbf{Z}_p[\chi(\sigma)\,;\,\chi\in\hat{\mathfrak{g}},\sigma\in\mathfrak{g}]$. For $\chi\in\hat{\mathfrak{g}}$, denote by $\bar{\chi}$ the character defined by $\bar{\chi}(\sigma)=\chi(\sigma^{-1})$. Let e_χ denote the idempotent associated with χ:

$$e_\chi = \frac{1}{\mathfrak{g}}\sum_{\sigma\in\mathfrak{g}}\bar{\chi}(\sigma)\sigma\in\mathcal{O}\mathfrak{g},$$

where $\mathcal{O}\mathfrak{g}$ is the group ring of $\mathfrak{g}$ with coefficients $\mathcal{O}$.

Let $B_k(p^m)$ be the subgrouop of $k^\times/k^{\times p^m}$ generated by p^m-singular number with respect to the set S of every places of k lying above p defined in [3]. Namely, it is a set of numbers which is locally p^m-th power at every place lying over p and which is contained in a p^m-th power of an ideal of k: there is an ideal $\mathfrak{a}$ such that $a\in\mathfrak{a}^{p^m}$. Denote by $B_k^{(1)}(p^m)$ a subgroup generated by units:

$$B_k^{(1)}(p^m) = E_k k^{\times p^m}/k^{\times p^m}\cap B_k(p^m)$$

where E_k is the group of units of k. Let

$$i_{m,n}: k^\times/k^{\times p^m} \longrightarrow k^\times/k^{\times p^m}$$

be a homomorphism defined to be $i_{m,n}(ak^{\times p^m}) = a^{p^{n-m}}k^{\times p^n}$ for natural numbers $m<n$. Let B_k and $B_k^{(1)}$ be the inductive limits of $B_k(p^m)$ and $B_k^{(1)}(p^m)$ with respect to $i_{m,n}$. Let A_k denote the p-Sylow subgroup of the ideal class group of k. Let $B_k^{(0)}(p^m)$ be the subgroup of A_k generated by every ideal $\mathfrak{a}$ determined for every p^m-singular number a in the above. $B_k^{(0)}$ denotes the injective limit of $B_k^{(0)}(p^m)$ with respect to a natural inclusion $B_k^{(0)}(p^m)\subset B_k^{(0)}(p^n)$ for $m<n$. The exact sequence (1.1) in [3] yields the following one:

$$1\longrightarrow B_k^{(1)}\longrightarrow B_k\longrightarrow B_k^{(0)}\longrightarrow 0.$$

By Lemma 2 in [3], we see $B_k^{(1)}$ is of finite order, and hence B_k is also. Since k is totally real, the above sequence induces an exact sequence

$$(1.1)\qquad 0 \longrightarrow B_k^{(1)} \longrightarrow B_k \longrightarrow A_k \longrightarrow 0$$

by virtue of Corollary to Theorem 6 of [3].

We denote by $t_p(M)$ the p-primary torsion subgroup for an Abelian group M. $\mathcal{O}M$ denotes the extension of coefficients over $\mathbf{Z}$ when the order of M is finite and dose that over $\mathbf{Z}_p$ when M is a $\mathbf{Z}_p$-module. Let U_k be the direct product of the groups of local units of k at places lying over p: $U_k = \prod_{\mathfrak{p}|p} U_\mathfrak{p}$ where $U_\mathfrak{p}$ is a subgroup of $u \in k_\mathfrak{p}^\times$ such that $v_\mathfrak{p}(u) = 0$. Let E_k be the group of units of k and C_k be a subgroup of E_K consisting of cyclotomic units of k in the sence of [2]. E_k is embedded into U_k. Denote by $\overline{E}_k$ and $\overline{C}_k$ the (topological) closure in U_k of E_k and C_k, respectively. By virtue of Lemma 2 in [3], we obtain an exact sequence

$$0 \longrightarrow t_p(\overline{E}_k/\overline{C}_k) \longrightarrow t_p(U_k/\overline{C}_k t_p(U_k)) \longrightarrow B_k^{(1)} \longrightarrow 0$$

Since the extension degree of k over $\mathbf{Q}$ is prime to p, we have the order of $t_p(\overline{E}_k/\overline{C}_k)$ equals that of A_k, (see [2]). Therefore, by comparing (1.1) with this exact sequence. we obtain

$$|B_k| = |t_p(U_k/\overline{C}_k t_p(U_k))|.$$

This eqality is a motivation of presenting this article.

Let m be the order of the residue field of a prime lying above p. Since U_k^{m-1} is a $\mathbf{Z}_p$-module, we abbreviate $\mathcal{O}U_k^{m-1}$, $\mathcal{O}\overline{E}_k^{m-1}$, $\mathcal{O}\overline{C}_k^{m-1}$ to $\mathcal{O}U_k$, $\mathcal{O}\overline{E}_k$, $\mathcal{O}\overline{C}_k$, respectively. We have the following exact sequences:

$$(1.2)\qquad \begin{array}{l} 0 \longrightarrow e_\chi \mathcal{O}B_k^{(1)} \longrightarrow e_\chi \mathcal{O}B_k \longrightarrow e_\chi \mathcal{O}A_k \longrightarrow 0, \\ 0 \longrightarrow e_\chi \mathcal{O}\overline{E}_k/\mathcal{O}\overline{C}_k \longrightarrow e_\chi \mathcal{O}U_k/\left(\mathcal{O}\overline{C}_k + \mathcal{O}t_p(U_k)\right) \longrightarrow e_\chi \mathcal{O}B_k^{(1)} \longrightarrow 0. \end{array}$$

We observe the central terms in these sequences are of the same order.

Let $L_p(s,\chi)$ denote the p-adic L-function associated with $\chi \in \hat{\mathfrak{g}}$. Then, we have the following theorem:

THEOREM. *Let $\mathfrak{h}$ be the decomposition group of the place (p) of $\mathbf{Q}$ in $k/\mathbf{Q}$. For each $\chi \in \hat{\mathfrak{g}}$, we have:*

(1) $v_p(L_p(1,\chi)) = v_p(|e_\chi \mathcal{O}B_k|)$ *if* $e_\chi \mathcal{O}\mathfrak{g}/\mathfrak{h} = 0$.

(2) $v_p(L_p(1,\chi)) = v_p(|e_\chi \mathcal{O}U_k/\mathcal{O}\overline{C}_k|)$.

(3) $|e_\chi \mathcal{O}A_k| = |e_\chi \mathcal{O}\overline{E}_k/e_\chi \overline{C}_k|$ *if* $e_\chi \mathfrak{g}/\mathfrak{h} = 0$.

Note the staement (3) follows from those of (1) and (2), directly.

2. The order of $e_\chi \mathcal{O} B_k$ —— the Proof of (1)

Let μ_n be a group of every p^nth root of unity and put $K_n = K(\mu_n)$. Denote by A_n the ideal class group of K_n. Let D_n be a subgroup of A_n generated by ideals dividing the principal ideal (p). Let $\mathbf{T}$ be the Tate module. Namely a projrctive limit of finite groups μ_m with respect to canonical maps $\mu_n \to \mu_m$ for $m < n$. $\mathbf{T}$ affords a character ω of G taking values in $\mathbf{Z}_p$. Denote by χ^* the reflection $\omega\bar{\chi}$ of χ. We abbreviate $B_{K_n}(p^m)$ to $B_n(p^m)$. In [3], a non-degenerate pairing

$$e_{\chi^*} A_n / A_n^{p^n} D_n \times e_\chi B_n(p^n) \to \mu_n$$

is defined. Denote by Γ_n the Galois group of K_n/K. By Theorem 10 of [3], we have

$$B_n(p^n)^{\Gamma_n} \cong B_K(p^n).$$

By the defineition of p^m-singular numbers, we observe $e_\chi \mathcal{O} B_K(p^n) \cong e_\chi \mathcal{O} B_k(p^n)$ for $\chi \in \hat{\mathfrak{g}}$. Let γ_n be a generator of Γ_n and κ be a p-adic integer such that $\zeta^{\gamma_n} = \zeta^{\kappa_n}$ for every $\zeta \in \mu_n$. Put $\gamma_n^* = \gamma_n - \kappa_n$.

LEMMA 1. *Suppose $B_k^{p^n} = \{1\}$. Then $e_\chi \mathcal{O} B_k$ and $e_{\chi^*} \mathcal{O} A_n / (p^n \mathcal{O} A_n + \gamma_n^* \mathcal{O} A_n)$ are dual to each other relative to the pairing (2.1).*

We see every prime of K_n lying above p is totally ramified in K_n/K. Let H be a decomposition group of p in $K/\mathbf{Q}$. Let $\mathbf{Q}_n$ is the maximal p-extension over $\mathbf{Q}$ in $\mathbf{Q}(\mu_n)$. H is also consided the decomposition group of a prime of K_n lying over p by identifying G with $\mathrm{Gal}(K_n/\mathbf{Q}_n)$. Let $\{\sigma_i | i = 1, \cdots, r\}$ be a complete set of representaives of G/H such that $\sigma_1 = 1$. Let I_n is a free Abelian group over the set of prims of K_n lying over p. D_n is the subgroup of A_n generated by I_n. I_n is a $\mathbf{Z}_p G$-modules by extending $\mathbf{Z}_p$-linearly of a permutation on the primes induced by each element of G. This module structure yields a $\mathbf{Z}_p G$-isomomorphism between $\mathbf{Z}_p G/H$ and $\mathbf{Z}_p I_n$. Thus we have a surjection $I_n \to \mathbf{Z}_p D_n$. This means $e_{\chi^*} \mathcal{O} D_n = 0$ if there is $\sigma \in H$ such that $\chi^*(\sigma) \neq 1$.

We abbreviate U_{K_n} to U_n. Let $\mathfrak{p}$ be a prime of K lying above p. Denote by $\mathfrak{P}$ a prime of K_n dividing the enlargement of $\mathfrak{p}$. Let $U_\mathfrak{p}$ (resp. $U_\mathfrak{P}$) be the unit group of the completion of K (resp. K_n) at $\mathfrak{p}$ (resp. $\mathfrak{P}$). Since U_n is isomorphic to the induced module $\mathbf{Z}G \otimes_{\mathbf{Z}H} U_\mathfrak{P}$, $U_1/N_n U_n$ is also the induced module of $U_\mathfrak{p}/N_n U_\mathfrak{P}$, where N_n is the norm map of $K_{n,\mathfrak{P}}/K_\mathfrak{p}$. Note H is acts on $U_\mathfrak{p}/N_n U_\mathfrak{P}$ trivially. Thus $e_{\chi^*} \mathcal{O} U_1/N_n U_n = 0$ if $e_{\chi^*} \mathcal{O} G/H = 0$. Set $w_m = \gamma_n^{p^{m-1}} - 1$ for $m < n$. By arguing similarly as the proof of Lemma 11 in [4], we have an isomorphism:

$$e_{\chi^*} \mathcal{O} A_m \cong e_{\chi^*} \mathcal{O} A_n / A_n^{w_m} \tag{2.2}$$

if $e_\chi^* \mathcal{O} G/H = 0$. Let H_∞ be the projective limit with respect to norm maps. Let Γ be the Galois group of $\cup_n K_n$ over K. This group is a projective limit of Γ_n. Denote by γ and κ the limits of γ_n

and κ_n in Γ and $\mathbf{Z}_p$, respectively. Put $\gamma^* = \gamma - \kappa$.

LEMMA 2. *If $e_{\chi^*}\mathcal{O}G/H = 0$, we have the following isomorphism:*

$$e_\chi \mathcal{O}B_k \;\cong\; e_{\chi^*}\mathcal{O}H_\infty/\gamma^*\mathcal{O}H_\infty.$$

Proof. Observe $e_\chi \mathcal{O}B_K \cong e_\chi \mathcal{O}B_k$. Since $e_{\chi^*}\mathcal{O}D_m = 0$, we have

$$e_\chi \mathcal{O}B_k \;\cong\; e_{\chi^*}\mathcal{O}A_m/\left(p^m\mathcal{O}A_m + \gamma^*\mathcal{O}A_m\right)$$

for sufficiently large integer m. By the isomorphism (2.2), we obtain

$$e_\chi \mathcal{O}B_k \;\cong\; e_{\chi^*}\mathcal{O}A_n/\left(p^m\mathcal{O}A_n + w_m\mathcal{O}A_n + \gamma^* A_n\right)$$

for $n > m$. Since the canonical map $\mathcal{O}H_\infty \to \mathcal{O}A_n$ is surjective, this isomorphism induces the following one:

$$e_\chi \mathcal{O}B_k \;\cong\; e_{\chi^*}\mathcal{O}H_\infty/\left(p^m\mathcal{O}H_\infty + w_m\mathcal{O}H_\infty + \gamma^*\mathcal{O}H_\infty\right).$$

By letting $m \to \infty$, we obtain the isomorphism to be proved. *q.e.d.* .

Recall we suppose $p \nmid g$ and $\chi \neq 1$. We suppose $\chi \in \hat{\mathfrak{g}}$ additionally. Let q be the least common multiple of p and the conductor of χ. Select γ so that $\kappa = 1 + q$. Let $\Lambda = \mathcal{O}[[T]]$ be the ring of formal power series on indeterminant T. The action of Λ on $\mathcal{O}H_\infty$ defined by $Tx = (\gamma - 1)x$ makes this module a compact Λ-module. Since K is Abelian over $\mathbf{Q}$, we have an elementary Λ-module E_{χ^*} where $e_{\chi^*}\mathcal{O}H_\infty$ injects with finite cokernel. The characteristic polynomial h of $e_{\chi^*}\mathcal{O}H_\infty$ is a product of distinguished polynomials h_i such that E_{χ^*} is isomorphic to $\prod \Lambda/h_i$. Then, by virtue of the main theorem of Iwasawa theory (see [1]), there is $u \in \Lambda^\times$ such that

$$L_p(s,\chi) \;=\; h((1+q)^s - 1)u((1+q)^s - 1).$$

Substituting 1 for s, we have

$$h(\kappa - 1) \;=\; L_p(1,\chi)u(\kappa - 1)^{-1}. \tag{2.3}$$

Note $h(\kappa - 1) \neq 0$ follows from (2.2). Let π be an element of Λ being prime to h. By virtue of Lemma 7 and 8 in [4], we have

$$\begin{aligned} |e_{\chi^*}H_\infty/\pi e_{\chi^*}H_\infty| &= |E_{\chi^*}/\pi E_{\chi^*}| \\ &= |\Lambda/(h,\pi)|. \end{aligned}$$

Since $h(\kappa - 1) \neq 0$ implies h is prime to $T + 1 - \kappa$, we set $\pi = T + 1 - \kappa$. We have $\Lambda/(h,\pi) \cong \mathcal{O}/h(\kappa - 1)\mathcal{O}$. By Lemma 2 and (2.3), we obtain

$$\begin{aligned} v_p(|e_\chi \mathcal{O}B_k|) &= v_p(|e_{\chi^*}\mathcal{O}H_\infty/\pi e_{\chi^*}\mathcal{O}H_\infty|) \\ &= v_p(h(\kappa - 1)) \\ &= v_p(L_p(1,\chi)). \end{aligned}$$

The statemeent (1) of the theorem follows.

3. Computation of the order of $\mathcal{O}U_k/\mathcal{O}\overline{C}_k$ — the proof of (2)

Let e be the ramification index of p in $k/\mathbf{Q}$ and s be the number of primes of k lying above p. We have a decomposition of ideals:

$$(p) = (\mathfrak{p}_1 \cdots \mathfrak{p}_s)^e.$$

Note $e \,|\, p-1$ and $e < p-1$. Let $\iota_i : k \to \mathbf{C}_p$ be an embedding which determines $\mathfrak{p}_i$. Let $\tilde{k}^{(i)}$ be the completion of k at $\mathfrak{p}_i$. $\tilde{k}^{(i)}$ is the composite of $\mathrm{Im}\,\iota_i$ and $\mathbf{Q}_p$. Let $\tilde{\mathcal{O}}^{(i)}$ and $\tilde{\mathfrak{p}}^{(i)}$ be the valuation ring of $\tilde{k}^{(i)}$ and its maximal ideal. We abbreviate $\tilde{k}^{(1)}$, $\tilde{\mathcal{O}}^{(1)}$ and $\tilde{\mathfrak{p}}^{(1)}$ to $\tilde{k}$, $\tilde{\mathcal{O}}$ and $\tilde{\mathfrak{p}}$, respectively. Since $\tilde{k} = \tilde{k}^{(i)}$, we see $\tilde{\mathcal{O}} = \tilde{\mathcal{O}}^{(i)}$ and $\tilde{\mathfrak{p}} = \tilde{\mathfrak{p}}^{(i)}$. We embed k into the sth-ply product $\tilde{k}^s = \tilde{k} \times \cdots \times \tilde{k}$ by $\iota = \iota_1 \times \cdots \times \iota_s$. This embedding gives an isomorphism $U_k \cong \tilde{\mathcal{O}}^\times \times \cdots \times \tilde{\mathcal{O}}^\times$. We select each ι_i once for all and consider this embedding and isomorphism canonical. We abuse notation and denote by U_k the $\mathbf{Z}_p$-module $(1+\tilde{\mathfrak{p}})^s$. Put $V_k = \tilde{\mathcal{O}}^s = \tilde{\mathcal{O}} \times \cdots \times \mathcal{O}$.

Let $\log_p x$ be the p-adic log function defined on $\mathbf{C}_p$. $x \to \log_p x$ induces a surjection

$$1+\tilde{\mathfrak{p}} \cong \tilde{\mathfrak{p}}$$

whose kernel is $t_p(1+\tilde{\mathfrak{p}})$. Hence, $(x_1, \cdots, x_s) \to (\log_p x_1, \cdots, \log_p x_s)$ defines a $\mathbf{Z}_p$-homomorphism into V_k. Denote by the symbol Log_p this homomorphism, we have an exact sequence

$$0 \longrightarrow t_p(U_k) \longrightarrow U_k \xrightarrow{\mathrm{Log}_p} V_k.$$

Let α be an integer of k which generates a principal ideal

$$(\alpha) = \mathfrak{p}_1 \cdots \mathfrak{p}_s \mathfrak{a}, \quad (\mathfrak{a}, p) = 1.$$

αV_k is the image of Log_p: $\alpha V_k = \mathrm{Log}_p(U_k)$.

Let $\mathcal{O}_k$ be the ring of integers of k and $\mathcal{O}_{k,p}$ be the localization with respect to a multiplicatively closed subset $\{x \in \mathcal{O}^\times \,|\, (p,x) = 1\}$. An isomorphism $\mathbf{Q}_p \otimes_{\mathbf{Q}} k \cong \prod \tilde{k}^{(i)}$ induces

$$\mathbf{Z}_p \otimes_{\mathbf{Z}_{(p)}} \mathcal{O}_k \cong \prod \tilde{\mathcal{O}}^{(i)} = V_k,$$

where $\mathbf{Z}_{(p)}$ is the localization of $\mathbf{Z}$ at p. Since $\mathbf{Z}_p \otimes_{\mathbf{Z}} \mathcal{O}_k \to \mathbf{Z}_p \otimes_{\mathbf{Z}_{(p)}} \mathcal{O}_{k,p}$ is surjective and since the $\mathbf{Z}_p$-ranks of both modules are equal, this surjection is isomorphic. We identify these two modules. Let $\mathfrak{h}$ be the decomposition group of $\mathfrak{p}_1$ in $k/\mathbf{Q}$. Since V_k is an induced module of $\mathbf{Z}_p\mathfrak{h}$-module $\tilde{\mathcal{O}}$, we have

$$\mathcal{O} \otimes_{\mathbf{Z}_p} \mathcal{O}_k \xrightarrow{1\otimes\iota} \mathcal{O}V_k = \mathrm{Ind}\,\mathcal{O} \otimes_{\mathbf{Z}_p} \tilde{\mathcal{O}}$$
$$= \mathcal{O} \otimes_{\mathbf{Z}_p} \mathrm{Ind}\tilde{\mathcal{O}}.$$

Let ξ be a root of unity such that $\mathcal{O} = \mathbf{Z}_p[\xi]$. Let $g \in \mathbf{Q}_p[X]$ be the minimal polynomial of ξ. Suppose g is decomposed into a product of irreducible polynomials g_i, $i = 1, \cdots, r$ in $\tilde{k}[X]$. Let ξ_i be a root of an algebraic equation $g_i = 0$ and suppose $\xi_1 = \xi$. We have

$$\mathcal{O}V_k = \mathcal{O} \otimes_{\mathbf{Z}_p} \tilde{\mathcal{O}} \cong \tilde{\mathcal{O}}[X]/g \cong \prod_{i=1}^{r} \tilde{\mathcal{O}}[\xi_i].$$

Denote by pr the composite of these isomorphisms and the projection of the direct product onto the first factor $\tilde{\mathcal{O}}[\xi]$.

Let f be the conductor of χ and ζ be a primitive f-th root of unity. Let k_χ be the intermediate field of $k/\mathbf{Q}$ corresponding to $\mathrm{Ker}\,\chi$ and $\mathcal{O}_\chi$ be the ring of integers of k_χ. Let Tr_χ be the trace map of k/k_χ. Since p is prime to $[k : k_\chi]$, we have

$$e_\chi \mathcal{O} \otimes \mathcal{O}_\chi = e_\chi \mathcal{O} \otimes \mathrm{Tr}_\chi \mathcal{O}_k = e_\chi \mathcal{O} \otimes \mathcal{O}_k.$$

Hence we suppose $k = k_\chi$. Denote by $\mathcal{O}_f$ the ring of integers of $\mathbf{Q}(\zeta)$ and Tr be the trace map with respect to $\mathbf{Q}(\zeta)/k_\chi$.

LEMMA 3. *Suppose $k = k_\chi$. Let $\tau(\bar{\chi})$ be the Gauss sum of $\bar{\chi}$. We have*

$$e_\chi \mathcal{O} \otimes \mathcal{O}_k \cong e_\chi \mathcal{O} \otimes \mathrm{Tr}\mathcal{O}_f,$$
$$e_\chi \mathcal{O} \otimes \mathcal{O}_k \cong \tilde{\mathcal{O}}\tau(\bar{\chi}).$$

Proof. Let $\mathfrak{P}_1$ be an extension of $\mathfrak{p}_1$ onto $\mathbf{Q}(\zeta)$. The completion of $\mathbf{Q}(\zeta)$ there is $\mathbf{Q}_p(\zeta)$. Let $\tilde{\mathrm{Tr}}$ be the trace map from $\mathbf{Q}_p(\zeta)$ into $\tilde{k}$. Let $\tilde{\mathcal{O}}_f$ be the valuation ring of $\mathbf{Q}_p(\zeta)$. We have

$$\mathbf{Z}_p \otimes \mathrm{Tr}\mathcal{O}_f \cong \left(\tilde{\mathrm{Tr}}\tilde{\mathcal{O}}_f\right)^{e_\chi} = \tilde{\mathrm{Tr}}\tilde{\mathcal{O}}_f \times \cdots \times \tilde{\mathrm{Tr}}\tilde{\mathcal{O}}_f.$$

Let $\tilde{k}_I$ be the inertia field in $\mathbf{Q}_p(\zeta)/\tilde{k}$ and $\tilde{\mathcal{O}}_I$ be the valuation ring. Denote by Tr_1 (resp. Tr_2) the trace of $\mathbf{Q}_p(\zeta)/\tilde{k}_I$ (resp. $\tilde{k}_I/\tilde{k}$). Note $\tilde{\mathrm{Tr}} = \mathrm{Tr}_2 \circ \mathrm{Tr}_1$. We see $\mathrm{Tr}_1(\tilde{\mathcal{O}}_I) = \tilde{\mathcal{O}}$. The assumption $p^2 \nmid f$ implies p dose not divide the ramification index, we have $\mathrm{Tr}_1\tilde{\mathcal{O}}_f \supset \tilde{\mathcal{O}}_I$. However, since every element of $\mathrm{Tr}_1\tilde{\mathcal{O}}_f$ is integral over $\mathbf{Z}_p$, we have the converse inclusion. Therefore, we obtain

$$\tilde{\mathrm{Tr}}(\tilde{\mathcal{O}}_f) = \tilde{\mathcal{O}}.$$

This proves $\mathbf{Z}_p \otimes \mathrm{Tr}\mathcal{O}_f \cong V_k$. Consequently,

$$e_\chi \mathcal{O} \otimes V_k \cong e_\chi \mathcal{O} \otimes \mathrm{Tr}\mathcal{O}_f.$$

Since $\mathcal{O} \otimes \mathcal{O}_k \cong \mathcal{O} \otimes V_k$, we have the first isomorphism.

We define $\chi(t)$ for $t \in \mathbf{Z}$ such that $(t, f) = 1$ to be $\chi(t) = \chi(\sigma_t)$, where σ_t is restriction of an automorphism of $\mathbf{Q}(\zeta)$ sending ζ onto ζ^t. $e_\chi \mathcal{O} \otimes \mathrm{Tr}\mathcal{O}_f$ is generated by $\{e_\chi 1 \otimes \mathrm{Tr}(\zeta^t) | t = 0, \cdots, f-1\}$, where

$$e_\chi 1 \otimes \mathrm{Tr}\zeta^t = \begin{cases} 0 & \text{if } (t,f) \neq 1, \\ \chi(t)\frac{1}{g}\sum_{\sigma\in\mathfrak{g}} \bar{\chi}(\sigma) \otimes \mathrm{Tr}\zeta^\sigma & \text{otherwise.} \end{cases}$$

Since $e_\chi \mathcal{O} \otimes \mathcal{O}_k$ is a free $\mathcal{O}$-module of rank 1, it must be generated by $\sum \bar{\chi}(\sigma) \otimes \mathrm{Tr}\zeta^\sigma$. Simultaneously, the image of $e_\chi \mathcal{O} \otimes \mathcal{O}_k$ with pr is a free $\mathcal{O}$-module whose rank is 1 or 0. Let N_f be the norm map from $\mathbf{Q}(\zeta)$ to k. We have

$$pr\left(\sum_{\sigma\in\mathfrak{g}} \bar{\chi}(\sigma) \otimes N_f \zeta^\sigma\right) = \tau(\bar{\chi}) \neq 0.$$

Hence $e_\chi \mathcal{O} \otimes \mathcal{O}_k \cong \mathcal{O}\tau(\bar{\chi})$. *q.e.d.*

LEMMA 4. *Suppose $k = k_\chi$. We have*

$$e_\chi \mathcal{O}\alpha V_k = \begin{cases} p e_\chi \mathcal{O} V_k & \text{if } \chi(p) \neq 0 \\ e_\chi \mathcal{O} V_k & \text{othereise.} \end{cases}$$

Proof. If $\chi(p) \neq 0$, the prime ideal (p) is not ramified in k. Hence we are able to choose $\alpha = p$. Assume $\chi(p) = 0$. Let F be the residue field of $\tilde{\mathcal{O}}$. We have the following exact sequence:

$$0 \longrightarrow e_\chi \mathcal{O} \otimes \alpha V_k \longrightarrow e_\chi \mathcal{O} \otimes V_k \longrightarrow e_\chi \mathcal{O} \otimes F^s \longrightarrow 0.$$

Recall $\mathfrak{h}$ is the inertia group of p in $k/\mathbf{Q}$. This group acts on F trivially. F^s in the right term of the above sequence is an induced module of F:

$$F^s \cong \mathbf{Z}_p \mathfrak{g} \otimes_{\mathbf{Z}_p \mathfrak{h}} F.$$

$\chi(p) = 0$ menas χ is not trivial on the inertia group. Hence

$$e_\chi \mathcal{O} \otimes \mathbf{Z}_p \mathfrak{g} \otimes_{\mathbf{Z}_p \mathfrak{h}} F = 0.$$

We have $e_\chi \mathcal{O}\alpha V_k = e_\chi \mathcal{O} V_k$ from the above exact sequence. *q.e.d.* .

LEMMA 5. *Suppose $k = k_\chi$ and set $\eta(\bar{\chi}) = \sum_{a=1}^{f-1} \bar{\chi}(a) \log_p(1-\zeta^a)$. Then*

$$pr \circ \mathrm{Log}_p(e_\chi \mathcal{O}\overline{C}_k) = \mathcal{O}\eta(\bar{\chi}).$$

Proof. Let D_k be a subgroup of $k^\times$ generated by $d_n \doteq N_n(1-\zeta_n)$ for $n \geq 3$, where ζ_n is a primitive n-th root of unity and N_n is the norm map from $\mathbf{Q}(\zeta_n)$ to k. Recall $C_k = D_k \cap E_k$. Let D be a subgroup of $\mathcal{O} \otimes D_k$ generated by d_f as a $\mathcal{O}\mathfrak{g}$-module. Suppose $(n,f) < f$. Since there is

$\sigma \in \mathfrak{g}$ such that $\sigma d_n = d_n$, we have $e_\chi 1 \otimes d_n = 0$. When $f|n$ and $n > f$, we have $e_\chi 1 \otimes d_n \in e_\chi D$. Hence $e_\chi D = e_\chi \mathcal{O} \otimes D_k$ is generated by $e_\chi 1 \otimes d_f$.

When f is not a power of a prime, we have $e_\chi \mathcal{O} \otimes C_k = e_\chi \mathcal{O} \otimes D_k$. Suppose f is a power of prime: $f = q^e$. We have the following exact sequence:

$$0 \longrightarrow e_\chi \mathcal{O} \otimes C_k \longrightarrow e_\chi \mathcal{O} \otimes D_k \longrightarrow e_\chi \mathcal{O} \otimes I_q$$

where I_q is the subgroup of the ideal group of k generated by ideals dividing q. Since q is totally ramified in $k/\mathbf{Q}$, $\mathfrak{g}$ acts on I_q trivially, and hence $e_\chi \mathcal{O} \otimes I_q = 0$. We have $e_\chi \mathcal{O} \otimes C_k = \mathcal{O} \otimes D_k$.

Since $\mathcal{O} \otimes C_k \cong \mathcal{O} \otimes \overline{C}_k$, we have $e_\chi \mathcal{O}\overline{C}_k$ is generated by $e_\chi 1 \otimes d_f$. We comput $pr \circ \mathrm{Log}_p(e_\chi 1 \otimes d_f)$:

$$\begin{aligned} pr \circ \mathrm{Log}_p(e_\chi 1 \otimes d_f) &= pr\left(\left(\frac{1}{g}\sum_\sigma \bar{\chi}(\sigma) \otimes \log_p(\iota_i(N_f(1-\zeta)))\right)_{1 \geq i \geq s}\right) \\ &= \frac{1}{g}\sum_\sigma \bar{\chi}(\sigma) \sum_{\substack{1 \leq b < f \\ \chi(b)=1}} \log_p(1-\zeta^b) \\ &= \frac{1}{g}\eta(\bar{\chi}). \end{aligned}$$

Since $\eta(\bar{\chi}) \neq 0$ and $e_\chi \mathcal{O}\overline{C}_k$ is a free $\mathcal{O}$-module of rank 1, we have $e_\chi \mathcal{O}\overline{C}_k \cong \mathcal{O}\eta(\bar{\chi})$. *q.e.d.*

Now, we shall prove the statement (2) of the theorem. Set $\beta = p$ when $\chi(p) \neq 0$ and $\beta = \alpha$ in otherwise. By Lemma 3, 4 and 5, we have isomorphisms

$$\frac{e_\chi \mathcal{O} U_k}{e_\chi \mathcal{O}\overline{C}_k} \cong \frac{e_\chi \mathrm{Log}_p(\mathcal{O}U_k)}{e_\chi \mathrm{Log}_p(\mathcal{O}\overline{C}_k)} \cong \frac{\beta\mathcal{O}\tau(\bar{\chi})}{\mathcal{O}\eta(\bar{\chi})}.$$

Since $\tau(\chi)\tau(\bar{\chi}) = \chi(-1)f$, we have

$$[\beta\mathcal{O}\tau(\bar{\chi}) : \mathcal{O}\eta(\bar{\chi})] = [\mathcal{O} : \mathcal{O}\eta(\bar{\chi})\tau(\chi)f^{-1}\beta^{-1}].$$

Since

$$L_p(1,\chi) = \left(1 - \frac{\chi(p)}{p}\right)\frac{\tau(\chi)}{f}\sum_{a=1}^{f-1}\log_p(1-\zeta^a),$$

the above equality proves (2).

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