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1. Introduction

Let k be an imaginary finite abelian extension of the field of rational numbers $\boldsymbol{Q}$. G be the Galois group of $k/\boldsymbol{Q}$ and $\mathfrak{f}_k$ be the finite part of the conductor. We shall recall the definition of the Stickelberger elements of k, according to [4]. When a ring R and a finit group A are given, RA denote the group ring of A with coefficients in R and, for a subset X of A, $s(X)$ denotes the sum of all elements of X in RA. For finite Galois extensions M and L of $\boldsymbol{Q}$ such that $M \supset L$, an R -algebra homomorphism by restricting $\sigma \in \mathrm{Gal}(M/\boldsymbol{Q})$ onto L, we have

$$\mathrm{res}_{M/L} : R\mathrm{Gal}(M/\boldsymbol{Q}) \to R\mathrm{Gal}(L/\boldsymbol{Q}).$$

Conversely, by assigning $\sigma \in \mathrm{Gal}(L/\boldsymbol{Q})$ to $s(\mathrm{res}_{L/M}^{-1}(\sigma))$, we have another R-algebra homomorphism

$$\mathrm{cor}_{M/L} : R\mathrm{Gal}(L/\boldsymbol{Q}) \to R\mathrm{Gal}(M/\boldsymbol{Q}).$$

Let (t, K) denote the Artin Symbol for an abelian extension over $\boldsymbol{Q}$, which takes values in $\mathrm{Gal}(K/\boldsymbol{Q}) \cup \{0\}$. The symbol $\langle x \rangle$ means the value of $x-[x]$, where $[x]$ is the Gauss' symbol. For $n \in \boldsymbol{N}$, denote by ζ_n a primitive n-th root of unity and put $k_n = k \cap \boldsymbol{Q}(\zeta_n)$, $R(n) = \{t \in \boldsymbol{N} \,;\, (t, n) = 1, 1 \leqq t \leqq n-1\}$. The Stickelberger elements $\theta_{k,n}(a)$ are defined to be

$$\theta_{k,n}(a) = \sum_{t \in R(n)} \left\langle \frac{-at}{n} \right\rangle \mathrm{cor}_{k/k_n} \circ \mathrm{res}_{k_n/k_n}((t, K_n)^{-1}),\ n \in \boldsymbol{N},\ a \in \boldsymbol{Z}.$$

Especially, when $n = \mathfrak{f}_k$ and $a = -1$, we omit n and a and write θ_k for $\theta_{p,\mathfrak{f}_k}(-1)$, that is

$$\theta_k = \sum_{t \in R(\mathfrak{f}_k)} \left\langle \frac{t}{\mathfrak{f}_k} \right\rangle (t, k)^{-1}.$$

Let p be an odd prime, $\boldsymbol{Z}_p$ the ring of p-adic integers and $\boldsymbol{Q}_p$ the field of quotients of $\boldsymbol{Z}_p$. Denote by S'_k the $\boldsymbol{Z}_pG$-submodule of $\boldsymbol{Q}_pG$ generated by $\theta_{k,n}(a)$, $n \in \boldsymbol{N}$ and $a \in \boldsymbol{Z}$. The Stickelberger ideal S_k defines to be an ideal $S' \cap \boldsymbol{Z}_pG$ of $\boldsymbol{Z}_pG$ in this paper. We note the statements of Proposition 2.1, Theorem 3.1, 2.1, and 5.3 in [4] are still valid for our cases, though the Stickelberger ideals in $\boldsymbol{Z}[G]$ are argued there. Namely, letting $\varepsilon^- = (J-1)/2$, where J is the complex conjugation, and denoting by A_k the p-class group of k, we have the following statements:

(S1) *S'_k/S_k is isomorphic to the group of all p-th power roots of unity in k.*

(S1) *$\varepsilon^- S_k$ annihilates $\varepsilon^- A_k$.*

(S3) $[\varepsilon^- \boldsymbol{Z}_pG : \varepsilon^- S_k] = |\varepsilon^- A_k|$ *if the p-Sylow subgroup of G is cyclic.*

When a ring R and an R-module M are given, we call M an R-cyclic module if M is generated by one element as R-module. According to (S1)-(S3), we see $\varepsilon^- A_k \cong \varepsilon^- \boldsymbol{Z}_p\mathrm{G}/\varepsilon^- S_k$ if the p-Sylow subgroup of G is cyclic and if $\varepsilon^- A_k$ is $\boldsymbol{Z}_p G$-cyclic.

Let Δ be the p'-subgroup of G and Γ be the p-Sylow subgroup. Let φ be a $\boldsymbol{Q}_p$-irreducible character of Δ and e_φ an associated idempotent in $\boldsymbol{Q}_p G$. Since $e_\varphi \in \boldsymbol{Z}_p G$, we have

$$M = e_\varphi M \oplus (1-e_\varphi) M.$$

We call φ imaginary if $\varphi(J) = -\varphi(1)$ holds, which is equivalent to that an absolutely irreducible factor of φ is imaginary, further, $\bar{\varphi}$ denotes the $\boldsymbol{Q}_p$-irreducibe character defined by $\bar{\varphi}(\sigma) = \varphi(\sigma^{-1})$, $\sigma \in G$. We call φ faithfull if $\{\sigma \in \Delta ; \varphi(\sigma) = \varphi(1)\} = \{1\}$, or equivalently, if an absolutely irreducible factor is faithfull. Let ω denote the Teichmuller character of $\mathrm{Gal}(\boldsymbol{Q}_p(\zeta_p)/\boldsymbol{Q}_p)$. When $\zeta_p \in k$, we consider ω a character of Δ.

Our main result in the present paper is as follows:

Theorem. *Let k_0 be a real abelian field whose conductor and whose extension degree are prime to p. Set $k_1 = k_0(\zeta_p)$ and $k = k_0(\zeta_{p^n})$ for $n \geqq 1$. Then we have*

(1) *Let φ be an imaginary $\boldsymbol{Q}_p$-irreducible character of Δ different from ω. We have*

$$|e_\varphi A_k| = |e_\varphi \boldsymbol{Z}_p G / e_\varphi S_k|.$$

Moreover, if $e_{\omega\bar{\varphi}} A_{k_1^+} = \{1\}$, where $A_{k_1^+}$ is the p-class group of the maximal real subfield of k_1, we have

$$e_\varphi A_k = e_\varphi \boldsymbol{Z}_p G / e_\varphi S_k.$$

(2) *Let $\mathscr{L}(p)$ be a set of imaginary subfields of k such that $(\mathfrak{f}_L, \mathfrak{f}_k/\mathfrak{f}_L) = 1$ and $p \mid \mathfrak{f}_L$. $\varepsilon^- S_k$ is generated by $\{\varepsilon\text{-}\mathrm{cor}_{k/L}(\theta_L)\}_{L \in \mathscr{L}(p)}$.*

We note that the result (1) for $n=1$ follows from Theorem II.1 in [5], however, when $n>1$, this theorem dose not work well.

2. Generators of the Stickelberger ideal

Let $B_1(x)$ donote the lst Bernoulli polynomial. $B_1(x)$ satisfies the following distribution relation:

$$\sum_{t=0}^{N-1} B_1(<y+\frac{t}{N}>) = B_1(<Ny>).$$

By means of this relation, we have:

Lemma 1. *Let q be a prime number and M a natural number and set $N = Mq^m$* for $m \geqq 1$. *Then the following relations holds:*

(1) *When $q \mid M$, for $s \in R(M)$, we have*

$$\sum_{\substack{t \in R(N) \\ t = s \bmod M}} (<y+\frac{t}{N}>) = B_1(<z^m y + \frac{s}{M}>).$$

(2) *When $q \nmid M$, for $s \in R(M)$, we have*

$$\sum_{\substack{t \in R(N) \\ t \equiv s \bmod M}} B_1(<y+\frac{t}{N}>) = B_1(<q^m y + \frac{s}{M}>) - B_1(<q^{m-1} y + \frac{q^{m-1}\alpha s}{M}>),$$

where α is an integer such that $\alpha q^m \equiv 1 \bmod M$.

(3) *Let χ be a Dirichlet character and s be a multiple of the conductor of χ. We have*

$$\sum_{t\in R(f)} B_1(<\frac{t}{f}>)\chi(t) = \prod_{\substack{q|f\\ q:\text{primes}}} (1-\chi(q))B_{1,\chi},$$

where $B_{1,\chi}$ is the generalized Bernoulli number associated to χ.

Let s_{k/k_n} be a crossed section of $G \to \mathrm{Gal}(k_n/\boldsymbol{Q})$. We define a map $\mathit{s}_n : \{x\in\boldsymbol{Z} ; (x, n)=1\} \to G$ by

$$\mathit{s}_n(x) = \mathit{s}_{k/k_n}\circ \mathrm{res}_{\boldsymbol{Q}(\zeta_n)/k_n}((x,\boldsymbol{Q}(\zeta_n)^{-1})).$$

By using this map s_n, we rewrite the definition of $\theta_{k,n}$:

$$\theta_{k,n}(a) = \sum_{t\in R(n)} <\frac{-at}{n}> \mathit{s}_n(t)s(\mathrm{Gal}(k/k_n)). \tag{2.1}$$

Note

$$\varepsilon^-\theta_{k,n}(\mathrm{a}) = \varepsilon^- \sum_{\mathrm{t}\in R(\mathrm{n})} B_1(<\frac{-at}{n}>)\mathit{s}_n(t)s(\mathrm{Gal}(k/k_n)).$$

We could apply Lemma 1 by virtue of this equality. Set $\ell = \text{g.c.d.}(n,a)$, $m=n/\ell$, $b=a/\ell$. We have

$$\begin{aligned}\theta_{k,n}(a) &= \sum_{t\in R(n)} <\frac{-bt}{m}> \mathit{s}_n(t)s(\mathrm{Gal}(k/k_n)) \\ &= [\boldsymbol{Q}(\zeta_n) : k_n(\zeta_m)]\theta_{k,m}(b) \\ &= [\boldsymbol{Q}(\zeta_n) : k_n(\zeta_m)]\mathit{s}_m(b)^{-1}\theta_{k,m}.\end{aligned} \tag{2.2}$$

This implies that S_k' is generated by $\{\theta_{k,n}\}_{n\in N}$ over $\boldsymbol{Z}_pG$.

Let C_p be the completion of an algebraic closure of $\boldsymbol{Q}_p$. Let $\mathfrak{X}$ be the set of C_p-irreducible character of G. Abusing notation, we also consider $\chi\in\mathfrak{X}$ a C_p-algebra homomorphism from C_pG onto C_p defined by $\chi(\sum_{\sigma\in G} a_\sigma\sigma) = \sum a_\sigma\chi(\sigma)$. Note that $x, y\in\boldsymbol{Q}_pG$ are equal if and only if $\chi(x) = \chi(y)$ holds for every $\chi\in\mathfrak{X}$. Let $G_\chi = \{\sigma\in G ; \chi(\sigma)=1\}$ and k_χ be the intermediate field of $k/\boldsymbol{Q}$ corresponding to G_χ. Write $\mathfrak{f}_\chi$ for $\mathfrak{f}_{k_\chi}$. We consider χ a Dirichlet character of conductor $\mathfrak{f}_\chi$. Denote by $\bar{\chi}$ the character defined by $\bar{\chi}(\sigma) = \chi(\sigma^{-1})$.

For a prime number q, let T_q denote the inertia group of q in $k/\boldsymbol{Q}$ and $\lambda_q\in G$ be a representative of the Frobenius class at prime (q).

Lemma 2. *Set $\lambda_q^* = \lambda_q^{-1}s(T_q)$ and $\alpha_n = \prod_{q|n}(1-\lambda_q^*)$ for $n\in N$. Then we have the following equality:*

$$\chi(\varepsilon^-\theta_{k,n}) = \chi(\varepsilon^-\alpha_n s(G_n))B_{1,\bar{\chi}}.$$

Proof. If χ is even or if $\mathfrak{f}_\chi \nmid n$, both side of the above equality are equal to 0. Assume χ is odd and $\mathfrak{f}_\chi | n$. By virtue of (3) of Lemma 1, we have

$$\chi(\varepsilon^-\theta_{k,n}) = -\sum_{t\in R(n)} <\frac{t}{n}> \bar{\chi}(t)\chi(s(\mathrm{Gal}(k/k_n))$$

$$= -[k : k_n] \prod_{q|n} (1-\chi(q)) B_{1,\bar{\chi}}$$
$$= \chi(\varepsilon^- s(\mathrm{Gal}(k/k_n)) \alpha_n) B_{1,\bar{\chi}}.$$

Q.E.D.

Let $\mathfrak{M}$ be the set of all imaginary abelian subfields of k, $\mathscr{L}$ be the subset consisting of $L \in \mathfrak{M}$ such that $(\mathfrak{f}_L, \mathfrak{f}_k/\mathfrak{f}_L)=1$ and $\mathscr{L}(p)=\{L \in \mathscr{L} ; p \mid \mathfrak{f}_L\}$.

Lemma 3. *Let β be an element of Z_pG satisfying the following condition:*

(C) $\varepsilon^-\beta s(T_q)=0$ *for every prime q such that $p \nmid |T_q|$.*

Then, $\varepsilon^-\beta S'_k$ is generated by $\{\varepsilon_-\beta \mathrm{cor}_{k/M}(\theta_M)\}_{M\in\mathfrak{M}}$ over Z_pG.

Proof. Note $\varepsilon^-\mathrm{cor}_{k/M}(\theta_M)=0$ if M is a real subfield of k. For $n \in N$ such that $n \mid \mathfrak{f}_k$, set $M=k_n$. By the definition (2.1), we see

(2.3) $$\theta_{k,n}=\mathrm{cor}_{k/M}(\theta_M).$$

Assume $n \nmid \mathfrak{f}_k$ and $M=k_n \in \mathfrak{M}$. Set $m=\mathfrak{f}_M$ and

$$\alpha_{n,m}=\prod_{\substack{q|n \\ q\nmid m}} (1-\lambda_q^*).$$

Since $k_n=k_m$ and $\alpha_n=\alpha_m\alpha_{n,m}$, by virtue of Lemma 2, we have

$$\chi(\varepsilon^-\theta_{k,n})=\chi(\varepsilon^-\alpha_{n,m}\theta_{k,m})$$

for every $\chi \in \mathfrak{X}$. Hence $\varepsilon^-\theta_{k,m}=\varepsilon^-\alpha_{n,m}\theta_{k,m}$. The condition (C) menas $\alpha_{n,m} \in Z_pG$ and hence the lemma follows from (2.3).

Lemma 4. *Notation being same as that of Lemma 3, we assume further that the maximal real subfield whose conductor is prime to p is a real subfield. Suppose $\beta \in Z_pG$ satisfying the following condition* (C') *instead of the above* (C) :

(C') $\varepsilon^-\beta s(T_q)=0$ *for primes q such that $q \neq p$ and $p \mid |T_q|$.*

Then we have $\varepsilon^-\beta S'_k$ is generated by $\{\varepsilon_-\beta \mathrm{cor}_{k/M}(\theta_M)\}_{M\in\mathfrak{M}}$ over Z_pG.

Proof. If the conductor of $M \in \mathfrak{M}$ is prime to p, then M is real. Hence, the factor concerning p dose not appear in $\alpha_{n,m}$. The lemma follows from the previous one.

Theorem 5. *For each $M \in \mathfrak{M}$, there is $L \in \mathscr{L}$ such that $M \subset L$ and $\varepsilon^-\mathrm{cor}_{k/M}(\theta_M) \in Z_pG\varepsilon^-\mathrm{cor}_{k/L}(\theta_L)$. Therefore, for $\beta \in Z_pG$* satisfying the condition (C) *of Lemma* 3*(resp.* (C') *of Lemma* 4*), we have $\varepsilon^-\beta S'_k$ is generated by $\{\varepsilon^-\beta\mathrm{cor}_{k/L}\}_{L\in\mathscr{L}}$(resp. $\{\varepsilon^-\beta\mathrm{cor}_{k/L}\}_{L\in\mathscr{L}(p)}$).*

Proof. Let q be a prime such that $q^2 \mid \mathfrak{f}_k$. Let q^a be the maximal power of q dividing $\mathfrak{f}_k$. Take $b, m \in N$ so that $1 \leqq b < a$, $(m,q)=1$ and $m \mid \mathfrak{f}_k$. For $c \in \{a, b\}$, we write N_c for mq^c, and further, $K_c, k_c, \theta_c, R_c, G_c, \Delta_c$ for $Q(\zeta_{N_c})$, k_{N_c}, θ_{k,N_c}, $\mathrm{R}(N_c)$, $\mathrm{Cal}(k/k_{N_c})$, Δ_{N_c}, respectively. Set

$$\xi=s(\Delta_{k/k_a}(\mathrm{Gal}(k_a/k_b))).$$

Since $\xi s(\mathrm{Gal}(k/k_a))=s(\mathrm{Gal}(k/k_b))$, we have by virtue of Lemma 1,

$$\begin{aligned}\varepsilon^-\theta_a\xi&=\varepsilon^-\sum_{t\in R_b}\sum_{\substack{t\in R_a\\ t\equiv s \bmod N_a}}B_1(<\frac{s}{N_a}>)\,\Delta_a(t)\,s(\mathrm{Gal}(k/k_a))\\ &=\varepsilon^-\sum_{s\in R_b}\Delta_b(t)\,s(\mathrm{Gal}(k/k_b)B_1(<\frac{s}{N_b}>)\\ &=\varepsilon^-\theta_b.\end{aligned}$$

Therefore, by (2.3), we have

$$\varepsilon^-\mathrm{cor}_{k/k_a}(\theta_{k_a})\,\xi=\varepsilon^-\mathrm{cor}_{k/k_b}(\theta_b). \tag{2.4}$$

Let $\mathfrak{f}_k=\prod_{i=1}^{r}q_i^{e_i}$ be the factorization. By changing order of $\{q_i\}$ if necessary, we suppose

$$\mathfrak{f}_M=\prod_{i=1}^{s}q^{d_i},\ 1\leqq d_i<e_i,\ s\leqq r.$$

Set $m_0=\mathfrak{f}_M$, $m_i=m_{i-1}q_i^{e_i-d_i}$ and $M_i=k_{m_i}$. By virtue of (2.4), we have

$$\varepsilon^-\mathrm{cor}_{k/M_i}(\theta_{M_i})=\varepsilon^-\mathrm{cor}_{k/M_{i+1}}(\theta_{M_{i+1}})\,\xi_i,\ \xi_i\in\boldsymbol{Z}_pG.$$

This proves the first statement of the theorem, and the latter part follows from this satement and Lemma 3, 4.

3. The index of the Stickelberger ideal.

We assume the p-sylow subgroup of G is cyclic. Let φ be a $\mathfrak{g}_p$-irreducible character and χ be a $\boldsymbol{C}_p$-irreducible component of φ. Let $\mathcal{O}=\boldsymbol{Z}_p[\chi(\sigma)\ ;\sigma\in\mathfrak{g}]$, $\mathfrak{g}$ be the field of quotients of $\mathcal{O}$ and $\mathfrak{g}=\mathrm{Gal}(\mathfrak{g}/\boldsymbol{Q}_p)$. φ is the trace of χ, *i.e.*

$$\varphi=\sum_{\sigma\in\mathfrak{g}}\sigma\chi.$$

Let e_χ denote the idempotent in $\mathfrak{g}\Delta$ such that $e_\chi\mathfrak{g}\Delta$ affords χ. We see

$$e_\varphi=\sum_{\sigma\in\mathfrak{g}}e_{\sigma\chi}\in\boldsymbol{Z}_p\Delta.$$

Let M be an $\mathcal{O}$-module of order p^e. Let $\ell_{\mathcal{O}}(\mathcal{O}M)$ denote the length of $\mathcal{O}$-composition series of $\mathcal{O}M$. Since $(|\Delta|,p)=1$, we have $\ell_{\mathcal{O}}(\mathcal{O}M)=e=v_p(|M|)$, where v_p is an additive valuation such that $v_p(p)=1$. We see further $|\mathcal{O}M|=[\mathcal{O}:p\mathcal{O}]^e$, and $|\mathcal{O}M|=|\mathcal{O}L/\mathcal{O}N|$ for $\boldsymbol{Z}_p$-modules L, N such that $M\cong L/M$.

We obtain the following theorem from Theorem II.1 in [5] :

Theorem 6. *Let k be an imaginary abelian extension such that $p\nmid[k:\boldsymbol{Q}]$. Then, for imaginary $\varphi\neq\omega$, we have*

$$|e_\varphi A_k|=|e_\varphi(\varepsilon^-\boldsymbol{Z}_pG/\varepsilon^-S_k)|.$$

Proof. By Theorem II.1 in [4], we have

$$\ell_{\mathcal{O}}(e_\chi\mathcal{O}A_k)=v_p(B_{1,\bar{\chi}}).$$

On the other hand, by Theorem 5, we have $e_\chi(\varepsilon^-S_k)$ is generated by $\chi((\varepsilon^-\mathrm{cor}_{k/L}(\theta_L)),\ L\in\mathscr{L}$ over $\boldsymbol{Z}_p$. Since the value of $\chi(\varepsilon^-\mathrm{cor}_{k/L}(\theta_L))$ equals

$$-[k : k_\chi] \cdot \prod_{q \mid \mathfrak{f}_L} (1-\chi(q)) B_{1,\bar{\chi}}$$

when $\mathfrak{f}_\chi \mid \mathfrak{f}_L$ and, it equals 0 otherwise, we have $e_\chi(\varepsilon^- S_k) = \mathbf{Z}_p B_{1,\bar{\chi}}$.
Thus

$$\ell_{\mathcal{O}}(e_\chi \mathcal{O} A_k) = v_p(B_{1,\chi}) = \ell_{\mathcal{O}}(e_\chi(\varepsilon^- \mathbf{Z}_p G/\varepsilon^- S_k)).$$

Hence

$$v_p(|e_\varphi A_k|) = \ell_{\mathcal{O}}(\mathcal{O} e_\varphi A_k) = \sum_{\sigma \in \mathfrak{g}} \ell_{\mathcal{O}}(e_{\sigma\chi} \mathcal{O} A_k) = \ell_{\mathcal{O}}(\mathcal{O} e_\varphi(\varepsilon^- \mathbf{Z}_p G/\varepsilon^- S_k)) = v_p(e_\varphi(\varepsilon^- \mathbf{Z}_p G/\varepsilon^- S_k)).$$

Q.E.D.

When the extension degree of $k/\mathbf{Q}$ is not prime to p, Solomon's theorem dose not work well. However, applying the main theorem of Iwasawa theory, we could extend Theorem 5.

Let k_0 be a real abelian extension whose conductor and whose extension degree are prime to p. Set $k_n = k_0(\zeta_{p^n})$ and $k_\infty = \bigcup_{n \geq 1} k_n$. In the following, denote by Γ the Galois group of k_∞/k and $G_n = \mathrm{Gal}(k_n/\mathbf{Q})$. We consider $\Delta = \mathrm{Gal}(k_1/\mathbf{Q})$. Write A_n for A_{k_n}. Let H_∞ be the projective limit of A_n with respect to the norm maps. Let γ be a topological generator of Γ and $\Lambda = \mathbf{Z}_p[[T]]$ be the ring of formal power series. By $T \to \gamma - 1$, H_∞ becomes a compact Λ-module. By Iwasawa theory, there are distinguished polynomials h_i such that

$$e_\varphi H_\infty \longrightarrow \prod_{i=1}^{r} \Lambda/(h_i)$$

is an injection with finite cokernel. Set $h = \prod h_i$. By extending coefficient to $\mathcal{O}$, we have

$$\mathcal{O} e_\varphi H_\infty = \bigoplus_{\sigma \in \mathfrak{g}} e_{\sigma\chi} \mathcal{O} H_\infty, \tag{2.5}$$

and obtain distinguished polynomials $h_{\sigma,i} \in \mathcal{O}\Lambda$, $i = 1, \cdots, r_\sigma$ such that

$$e_{\sigma\chi} \mathcal{O} H_\infty \longrightarrow \prod_{i=1}^{r_\sigma} \mathcal{O}\Lambda/(h_{\sigma,i})$$

have finite kernel and cokernel. Since $e_{\sigma\chi} \mathcal{O} H_\infty$ are $\mathcal{O}$-free, the kernels are trivial. Set $h_\sigma = \prod_{i=1}^{r_\sigma} h_{\sigma,i}$.
By (2.5), we have

$$\mathcal{O}\Lambda h = \mathcal{O}\Lambda(\prod_{\sigma \in \mathfrak{g}} h_\sigma).$$

Lemma 7. *Set* $E_\varphi / \prod_{i=1}^{r} \Lambda/(h_i)$ *and* $E_\sigma / = \prod_{i=1}^{r_\sigma} \mathcal{O}\Lambda/(h_{\sigma,i})$.
For $\pi \in \Lambda$ *such that* $|e_\varphi H_\infty / \pi e_\varphi H_\infty| < \infty$, *we have*
$|E_\varphi / \pi E_\varphi| = |e_\varphi H_\infty / \pi e_\varphi H_\infty|$,
$|E_\sigma / \pi E_\sigma| = |e_{\sigma\chi} \mathcal{O} H_\infty / \pi e_{\sigma\chi} \mathcal{O} H_\infty|$.

Proof. Set

$$A = \mathrm{Im}(e_\varphi H_\infty \to E_\varphi),\ B = \{y \in A\ ;\ \text{there are } z \in E_\varphi \text{ such that } \pi y = \pi z\},$$
$$C = \mathrm{coker}(e_\varphi H_\infty \to E_\varphi),\ X = \{z \in E\ ;\ \pi z \in A\}.$$

We see $\pi X = B$ and the following sequence is exact:

$$0 \to X/A \to C \to \pi C \to 0.$$

Since the order of C is finite and E_φ is a free $\boldsymbol{Z}_p$-module, we have $|C/\pi C| = |X/A| = |\pi X/\pi A| = |B/\pi A|$. The first equality follows from the exact sequence

$$0 \to B/\pi A \to A/\pi A \to E_\varphi/\pi E_\varphi \to C/\pi C \to 0.$$

By (2.5), we see $|e_{\sigma\chi}\mathcal{O}H_\infty/\pi e_{\sigma\chi}\mathcal{O}H_\infty| < \infty$ for every σ. Set $A' = \mathrm{Im}(e_{\sigma\chi}\mathcal{O}H_\infty \to E_\sigma)$, $B' = \{y \in A'$; there are $z \in E_\sigma$ such that $\pi y = \pi z\}$, $C' = \mathrm{coker}(e_{\sigma\chi}\mathcal{O}H_\infty \to E_\sigma)$, $X' = \{z \in E_\sigma ; \pi z \in A'\}$. Replacing A, B, C, X to A', B', C', X' in the above argument, the second equality is proved similary.

Lemma 8. *Set* $\omega_n = (T+1)^{p^{n-1}} - 1$ *and take* $f_1, f_2 \in \Lambda$ *(resp.* $g_1, g_2 \in \mathcal{O}\Lambda$*). Put* $f = f_1 f_2$ *(resp.* $g = g_1 g_2$*). If* g *and* ω_n *are prime to each other, we have*

$$|\Lambda/(f_1, \omega_1)| \cdot |\Lambda/(f_2, \omega_n)| = |\Lambda/(f, \omega_n)|$$
$$|\mathcal{O}\Lambda/(g_1, \omega_n)| \cdot |\mathcal{O}\Lambda/g_2, \omega_n)| = |\mathcal{O}\Lambda/(g, \omega_n)|$$

Proof. We have a chain $\Lambda \supset (f_1, \omega_n) \supset (f, \omega_n)$. The first equality follows from $(f_1, \omega_n)/(f_1, \omega_n) \cong \Lambda/(f_2, \omega_n)$. Similarly, we have the second one.

We abuse notoation and consider γ is also a generator of $\mathrm{Gal}(k_n/k_1)$ of order p^{n-1} by restricting γ onto k_n. Let ρ_σ denote a $C_p\mathrm{Gal}(k_n/k_1)$-algebra homomorphism defined by

$$\rho_\sigma(\sum_{i=0}^{p^{n-1}-1} a_i\gamma^i) = \sum \sigma\chi(a_i)\gamma^i,\ a_i \in C_p\Delta.$$

We recall Iwasawa's construction of p-adic L-funstions. Let q_0 be the conductor of $\sigma\chi$. Note $p \mid q_0$. Denote by $\theta_{\sigma,n}$ the Stickelberger ideal θ_{K_n} for $K_n = \boldsymbol{Q}(\zeta_{q_0}, \zeta_{p^n})$ and set $\xi_{\sigma,n} = \rho_\sigma(\theta_{\sigma,n})$. To construct p-adic L-funtion, we have to fix γ to $\varprojlim_{n\in N}(1+q_0, K_n)$. By isomorphisms $\mathcal{O}\Lambda/(\omega_n) \cong \rho_\sigma(\boldsymbol{Z}_pG_n)$, we have the following identification:

$$\mathcal{O}\Lambda = \varinjlim \mathcal{O}\Lambda/(\omega_n) = \varinjlim \rho_\sigma(\boldsymbol{Z}_pG_n).$$

Then $\{\xi_{\sigma,n}\}_{n\in N}$ converges to a power series $f_\sigma \in \Lambda$ which satisfies the formula:

$$L_p(s, \omega(\sigma\chi)^{-1}) = f_\sigma((1+q_0)^s - 1).$$

(See [7] chap. 7). Furthermore, since $\xi_{\sigma,n}$ maps $\xi_{\sigma,m}$ for $n > m$ by the canonical map $\rho_\sigma(\boldsymbol{Z}_pG_n) \longrightarrow \rho_\sigma(\boldsymbol{Z}_pG_m)$, the isomorphism $\mathcal{O}\Lambda/(\omega_n) \cong \rho_\sigma(\boldsymbol{Z}_pG_n)$ induces

(2.6) $$f_\sigma \bmod \mathcal{O}\Lambda\omega_n \longrightarrow \xi_{\sigma,n}.$$

Set $m = \mathfrak{f}_{\sigma\chi}p^{-1}$ and $N = p^n m$. Let M be a multiple of m such that $(M,p) = 1$. There is unique $a_i \in R(p^n)$ such that $Ma_i \equiv (1+p)^i \bmod p^n$ for $0 \leqq i < p^{n-1}$. Set

(2.7) $$y(t, s, i ; M) = tp^n + s^{p^n}a_iM$$

Observe $\{y(t, s, i ; M) ; t \in R(m), s \in R(p), 0 \leqq i < p^{n-1}\}$ gives a complete system of representatives of $(Z/NZ)^\times$.

Lemma 9. *Let* $\psi : G_n \to C_p$ *be a character which assigns a generator* γ *of* Γ *to* ζ_{p^m} *for* $0 \leqq m < n$. *Then we have*

$$B_{1,\overline{\chi}\psi}=\sum_{0\leq i<p^{n-1}}\sum_{s\in R(p)}\sum_{t\in R(m)}B_1(<\frac{y(t,\ s,\ i\ ;M)}{N}>)\chi(tp^n+s^{p^n}M)^{-1}\psi(a_iM)^{-1}.$$

Hence, f_σ is prime to ω_n for every n.

Proof. The equality follows from Lemma 1. Set $g_0=T$ and $g_m=\omega_m/\omega_{m-1}$ for $m\geqq 1$. g_m is a irreducible polynomial and $g_m(\xi_{p^m}-1)=0$. Then, the equality of the lemma and (2.6) implies f_σ is prime to g_m. Q.E.D.

Theorem 10. *Let φ be an odd $\boldsymbol{Q}_p$-irreducible character such that $\varphi\neq\omega$ and set $\omega_n=(T+1)^{p^{n-1}}-1$. Abbreviate S_{k_n} to S_n. Then we have*

$$|e_\varphi H_\infty/\omega_n e_\varphi H_\infty|=|e_\varphi \boldsymbol{Z}_p G_n/e_\varphi S_n|.$$

Proof. We shall prove the following equality ;

$$(2.8)\qquad |\mathcal{O}\Lambda/(h_\sigma,\ \omega_n)|=|\rho_\sigma(\boldsymbol{Z}_pG_n)/\rho_\sigma(S_n)|.$$

Before proving this, we shall show that the theorem follows from this equality.

By the main theorem of Iwasawa theory, we have $\mathcal{O}\Lambda f_\sigma=\mathcal{O}\Lambda h_\sigma$. Hence, Lemma 9 imlies that h_σ and ω_n are prime to each other, and further that h_σ and ω_n are also prime to each other. Thus, by virtue of Lemma 8 and (2.5), we have

$$\begin{aligned}|\mathcal{O}E_\varphi/\omega_n\mathcal{O}E_\varphi|&=|\mathcal{O}e_\varphi H_\infty/\omega_n\mathcal{O}e_\varphi H_\infty|\\&=\prod_{\sigma\in\mathfrak{g}}|e_{\sigma\chi}\mathcal{O}H_\infty/\omega_n e_{\sigma\chi}\mathcal{O}H_\infty|\\&=\prod_{\sigma\in\mathfrak{g}}|\mathcal{O}\Lambda/(h_\sigma,\ \omega_n)|.\end{aligned}$$

On the other hand, we have an isomorphism

$$\begin{aligned}e_\varphi\mathcal{O}G_n/e_\varphi\mathcal{O}S_n&\cong\bigoplus_{\sigma\in\mathfrak{g}}e_{\sigma\chi}\mathcal{O}G_n/e_{\sigma\chi}\mathcal{O}S_n.\\&\cong\bigoplus_{\sigma\in\mathfrak{g}}\rho_\sigma(\boldsymbol{Z}_pG_n)/\rho_\sigma(S_n).\end{aligned}$$

Therefore, the theorem follows from (2.8).

Now, we shall prove (2.8). Note $e_\varphi S'_n=e_\varphi S_n$, because of $\varphi\neq\omega$ and (S1). Denote by $\mathfrak{f}$ the conductor of $\sigma\chi$ and set $k=\boldsymbol{Q}(\xi_{\mathfrak{f}},\ \xi_{p^n})$. We have

$$\rho_\sigma(\mathrm{cor}_{k_n/K}(\theta_{\sigma,n}))=[k_n:K]\xi_{\sigma,n}.$$

Since $([k_n:K],\ p)=1$, we have $\xi_{\sigma,n}\in\rho_\sigma(S_n)$, and hence, by (2.6),

$$(2.9)\qquad \mathcal{O}\Lambda/(f_\sigma,\ \omega_n)\longrightarrow\rho_\sigma(\boldsymbol{Z}_pG_n)/\rho_\sigma(S_n)$$

is surjective.

Since the condition (C') is valid, we have $e_\varphi S_n$ is generated by $\{e_\varphi\mathrm{cor}_{k/L}(\theta_L)\}_{L\in\mathscr{L}(p)}$ by Theorem 5. For $L\in\mathscr{L}(p)$ such that $\mathfrak{f}\nmid\mathfrak{f}_L$, we have $\rho_\sigma(\varepsilon^-\mathrm{cor}_{k/L}\theta_L)=0$. When $\mathfrak{f}\mid\mathfrak{f}_L$, let $\mathfrak{f}_L=p^n m$ be a decomposition such that $(p,m)=1$. Set $N=p^nm$. Let $\sigma\chi=\chi_0\omega^k$ be the decomposition into the product of characters such that $p\nmid\mathfrak{f}_{\chi_0}$ and $0\leqq k<p-1$. Using the symbol $y(t,\ s,\ i\ ;m)$ of

(2.7), we have

$$\rho_\sigma(\varepsilon^- \mathrm{cor}_{k_n/L}(\theta_L)) = [k_n : L] \sum_{0\leq i<p^n} \gamma^{-i} \sum_{s\in R(p)} \omega^{-k}(s) \sum_{t\in R(m)} B_1 < \frac{y(t,s,i;m)}{N} >) \chi_0(tp^n)^{-1}.$$

By virtue of Lemma 1, the right hand side is modified as follows:

$$[k_n : L] \sum_{i=0}^{p^{n-1}-1} \prod_{q\mid m} (1-\bar{\chi}(q)) \sum_{t\in R(\mathfrak{f})} B_1(<\frac{t}{\mathfrak{f}}>) \sigma\chi^{-1}(t)\gamma^{-i}.$$

This means $\rho_\sigma(S_n) \subset \mathcal{O}\xi_{\sigma,n}$. Thus (2.9) is injective and the equality (2.8) follows.

Q.E.D.

3. $Z_p[G]$-cyclicity of A_k.

Let k be the imaginary abelian field such that the p-Sylow subgroup of G is cyclic. Let k_1 be the intermediate field corresponding to the p-Sylow subgroup of G. Let U_{k_1} (resp. U_k) be the unit group of the idele group of J_{k_1} (resp. J_k).

Lemma 11. Let φ be a $\boldsymbol{Q}_p$-irreducible character of Δ and γ be a generator of Γ. Set $V_{k_1} = U_{k_1} \cap J_{k_1}^p k_1^\times$. Then we have the following exact sequence:

$$1 \to e_\varphi(U_{k_1}/V_{k_1} N_{k/k_1} U_k) \to e_\varphi(A_k/A_k^{\gamma-1} A_k^p) \to e_\varphi(A_{k_1}/A_{k_1}^p) \to 1$$

Proof. Set $X = N_{k/k_1}(J_k) k_1^\times$, $Y = N_{k/k_1}(U_k) k_1^\times$, $Z = U_{k_1} k_1^\times$. Denote by $t_p(M)$ the p-Sylow subgroup for a finite abelian group M. We have exact sequences:

$$1 \to e_\varphi t_p(X/Y) \to e_\varphi t_p(J_{k_1}/Y) \to e_\varphi \mathrm{Gal}(k/k_1) \to 1,$$

$$1 \to e_\varphi Z/Y \to e_\varphi t_p(J_{k_1}/Y) \to e_\varphi A_{k_1} \to 1.$$

Since $e_\varphi(\mathrm{Gal}(k/k_1)) = \{1\}$ and since X/Y is the genus group with respect to k/k_1, we have

$$e_\varphi t_p(J_{k_1}/Y) \cong e_\varphi(A_k/A_k^{\gamma-1}).$$

The lemma follows from $Z\, J_{k_1}^P / Y\, J_{k_1}^p \cong U_{k_1}/V_{k_1} N_{k/k_1} U_k$.

For a finite abelian group M of exponent dividing p, denote by dim M the dimension of M as a $\boldsymbol{F}_p$-space, where $\boldsymbol{F}_p$ is the finite field of p-elements. We abbreviate $A_k/A_k^{r-1}A_k^p$ to $\mathcal{A}_k$. By Lemma 11, we have By Lemma 11, we have

(4.1) $$\dim e_\varphi \mathcal{A}_k = \dim e_\varphi U_{k_1}/V_{k_1} N_{k/k_1} U_k + \dim e_\varphi A_{k_1}/A_{k_1}^p.$$

Note $e_\varphi A_k$ is $Z_p G$-cyclic if and only if dim $\mathcal{A}_k \leqq 1$.

We suppose $k \ni \zeta_p$ and $k = k_1(\zeta_{p^n})$, $n \geqq 2$. Let Δ_p be the decomposition group of (p) in k/k_1.

Lemma 12. *Let χ be a $\boldsymbol{C}_p$-irreducible factor of φ and $\chi = \chi_0 \omega^t$ be the decomposition of characters such that $p \nmid \mathfrak{f}_{\chi_0}$ and $0 \leqq t < p-1$. Then, if $t \neq 1$, we have $e_\varphi \boldsymbol{Z}_p \Delta = \{0\}$. Hence, $e_\varphi U_{k_1}/V_{k_1} N_{k/k_1} U_k = \{1\}$.*

Proof. Note $e_{\sigma\chi}\mathcal{O}\Delta=\{0\}$ for every $\sigma\in\mathfrak{g}$ because $\sigma\chi$ are not trivial on Δ_p. Since $e_\varphi \boldsymbol{Z}_p\Delta=\{0\}$ if and only if $e_{\sigma\chi}\mathcal{O}\Delta=\{0\}$ for every $\sigma\in\mathfrak{g}$, we have $e_\varphi \boldsymbol{Z}_p\Delta=\{0\}$. We have $U_{k_1}/U_{k_1}^p N_{k/k_1}U_k\cong \boldsymbol{F}_p\Delta/\Delta_p$, and hence $e_\varphi U_{k_1}/V_{k_1}N_{k/k_1}U_k=\{1\}$. Q.E.D.

Put $\mathcal{S}=k_1^\times\cap J_{k_1}^p\prod_{\mathfrak{p}\nmid p}U_\mathfrak{p}$, where $U_\mathfrak{p}$ denote the unit groups of the completions of k_1 at places $\mathfrak{p}$. Let E be the unit group of k_1. Set Б $=\mathcal{S}/k_1^{\times p}$, $Б^{(1)}=E\cap\mathcal{S}/E^p$. For $a\in\mathcal{S}$, there is $\mathfrak{a}\in J_{k_1}$ such that $\mathfrak{a}U_{k_1}\ni a$. By $ak_1^{\times p}\to\mathfrak{a}U_{k_1}k_1^\times$, we have a homomorphism $Б\to A_{k_1}$. Denote by $Б^{(0)}$ the image of this homomorphism. Hence, the following sequence is exact:

$$1\to Б^{(1)}\to Б\to Б^{(0)}\to 1. \tag{4.2}$$

Let D be a subgroup of the ideal class group of k_1 generated by ideals dividng p. By [6], there is a perfect duality $Б\times A_{k_1}D/A_{k_1}^pD\to\mu_p$, where μ_p is the subgroup of $k_1^\times$ generated by ζ_p. Moreover, we have $e_{\omega\bar\varphi}Б$ and $e_\varphi A_{k_1}D/A_{k_1}^pD$ are dual each other. Note $e_\varphi D/D^p=\{1\}$ if $e_\varphi Z_p\Delta/\Delta_p=\{0\}$.

Lemma 13. *Let φ be an imaginary $\boldsymbol{Q}_p$-irreducible character of Δ. Then we have*

$$\dim\, e_{\omega\bar\varphi}Б^{(1)}\leqq 1.$$

Further, if $e_\varphi \boldsymbol{Z}_p\Delta/\Delta_p=\{0\}$ and $\varphi\neq\omega$, we have

$$\dim\, e_\varphi A_k=\dim\, e_{\omega\bar\varphi}Б^{(1)}.$$

Proof. Putting $F=\boldsymbol{Z}_p\otimes_{\boldsymbol{Z}}E$, we have

$$e_{\omega\bar\varphi}(F/F^p\mu_p)\cong e_{\omega\bar\varphi}(E/E^p\mu_p)$$

here we identify $1\otimes\mu_p$ and μ_p. By assigning xE^p to $x^{p^t}E^{p^{t+1}}$, we have an isomorphism

$$e_{\omega\bar\varphi}(F/F^p\mu_p)\cong e_{\omega\bar\varphi}(F^{p^t}/F^{p^{t+1}}\mu_p^{p^t}).$$

Let ε be a unit such that $H=<\varepsilon^\sigma;\sigma\in\mathrm{Gal}(k_1/\boldsymbol{Q})>$ is a $\boldsymbol{Z}$-free subgroup of rank $[k_1:\boldsymbol{Q}]/2-1$. Then, there is $\eta\in H$ such that $e_{\omega\bar\varphi}(1\otimes\eta)\neq 1$ in F. Let t be the maximal natural number such that $e_{\omega\bar\varphi}(1\otimes\eta)\in F^{p^t}$. We see $e_{\omega\bar\varphi}(F^{p^t}/F^{p^{t+1}})\neq\{1\}$ and hence $e_{\omega\bar\varphi}(F/F^p)\neq\{1\}$. This means

$$\dim\, e_{\omega\bar\varphi}(F/F^p\mu_p)\geqq\omega\bar\varphi(1). \tag{4.3}$$

Therefore, we have

$$\dim\, F/F^p\mu_p\geqq\sum_{\substack{\varphi:imaginary\\ \varphi\neq\omega}}\omega\bar\varphi(1)=|\Delta|/2-1$$

This implier equality holds in (4.3), namely $e_{\omega\bar\varphi}(F/F^p\mu_p)$ is $\boldsymbol{F}_p\Delta$-cyclic. Since $Б^{(1)}$ is a subgroup of E/E^p, we have

$$\dim\, e_{\omega\bar\varphi}Б^{(1)}\leqq\omega\bar\varphi(1).$$

We have $e_\varphi A_{k_1}/A_{k_1}^p\cong e_\varphi A_{k_1}D/A_{k_1}^pD\cong e_{\omega\bar\varphi}Б$ if $e_\varphi \boldsymbol{Z}_p\Delta/\Delta_p=\{0\}$. Hence, by (4.2), the proof is completed.

Proof of the main Theorem. Let φ be an $\boldsymbol{Q}_p$-irreducible character of Δ. Since the conductor of k_0 is primt to p and k_0 is real, we have $\chi=\chi_0\omega^t$, $1\leqq t<p-1$, for a C_p-irreducible factor of φ. Hence $e_\varphi Z_p\Delta/\Delta_p=\{0\}$. By Lemma 13, we have dim $A_k\leqq 1$ if $e_{\omega\bar{\varphi}}A_{k_1}=\{1\}$. Since $e_{\omega\bar{\varphi}}A_{k_1}\cong e_{\omega\bar{\varphi}}A_{k_1^+}$, we have $e_\varphi A_k$ is Z_pG-cyclic by virtue of Lemma 11 and 13. By Theorem 10, we obtain the statement (1). The statement (2) is a direct consequence from Theorem 6.

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