A note on the theorem of NEukirch and Neumann

A Note on the Theorems of Neukirch and Neumann

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Introduction

Let k be a finite algebraic number field and $k_{\mathfrak{p}}$ be the completion at a place $\mathfrak{p}$ of k. Let S be a finite set of places of k. Denote by k_S the maximal extension of k unramified outside S, and by $\bar{k}_{\mathfrak{p}}$ the algebraic closure of $k_{\mathfrak{p}}$. Put $G_{\mathfrak{p}}=\mathrm{Gal}(\bar{k}_{\mathfrak{p}}/k_{\mathfrak{p}})$ and $G_S=\mathrm{Gal}(k_S/k)$. An embedding $\iota: k_S \to \bar{k}_{\mathfrak{p}}$ induces a homomorphism $\rho: G_{\mathfrak{p}} \to G_S$ defined by $\rho(\sigma)=\iota\cdot\sigma\cdot\iota^{-1}$. We fix one of such homomorphisms for each $\mathfrak{p} \in S$ and denote by $\rho_{\mathfrak{p}}$. Denote by $\rho_S: \bigoplus_{\mathfrak{p}\in S} G_{\mathfrak{p}} \to G_S$ the homomorphism induced from $\{\rho_{\mathfrak{p}}; \mathfrak{p}\in S\}$. Let n be a natural number and $\boldsymbol{Z}$ be the ring of rational integers. It yields a homomorphism on the second cohomology groups with coefficient $\boldsymbol{Z}/n\boldsymbol{Z}$, which we denote this by $\rho_S(n)$. Namely,

$$\rho_S(n): H^2(G_S,\boldsymbol{Z}/n\boldsymbol{Z}) \to \prod_{\mathfrak{p}\in S} H^2(G_{\mathfrak{p}},\boldsymbol{Z}/n\boldsymbol{Z}).$$

We shall determine the kernel and cokernel of $\rho_S(n)$. In particular, we shall prove that the kernel of $\rho_S(n)$ is isomorphic to the dual group of the group of (S,n)-singularity defined in the below, which is known from Neukirck[2] and Neumann[3] in the special cases.

Now we define the (S,n)-singular numbers. A number a of k is called (S,n)-singular if the principal ideal (a) is an n-th power of an ideal of k, and if there is a solution of the equation $X^n=a$ in $k_{\mathfrak{p}}$ for each $\mathfrak{p}\in S$. Let $\mathscr{S}_k(S,n)$ be the set of (S,n)-singular numbers of k which are not 0. This is a subgroup of the multiplicative group $k^{\times}$ containing k^n. We call $\mathscr{S}_k(S,n)/k^n$ the group of (S,n)-singularity of k and denote by $\text{Б}_k(S,n)$.

The results of Neukirck and Neumann mentioned above are as follows.

THEOREM A. ([2, Neukirch, Satz 7.6]). *Assume that* $n\not\equiv 0,2 \bmod 4$ *and that* k *contains a primitive* n*-th root of unity. Assume also that* S *contains all places lying over primes dividing* n*. Then we have* $\ker(\rho_S(n))\cong \text{Б}_k(S,n)$.

THEOREM B. (Neumann[4]). *Assume that* n *is a prime number and that* S *contains every places lying over* n *and all of the infinite places. Then we have* $\ker(\rho_S(n))\cong \text{Б}_k(S,n)$ *and that* $\ker(\rho_S(n))$ *is isomorphic to the multiplicative group of* n*-th roots of unity contained in* k.

Let G be a group. $\boldsymbol{Q}$ denotes the field of rational numbers. For a prime number p, $\boldsymbol{Q}_p$ and $\boldsymbol{Z}_p$ denote the field of p-adic numbers and the ring of p-adic integers, respectively. We call G p-multiplicator free provided $H^2(G,\boldsymbol{Q}_p/\boldsymbol{Z}_p)$ vanishes, and also call it multiplicator free provided

$H^2(G,\mathbf{Q}_p/\mathbf{Z}_p)=\{0\}$ for all primes p. Uning the development to partial fractions of rational numbers, we observe that G is multiplicator free if and only if $H^2(G,\mathbf{Q}/\mathbf{Z})=\{0\}$. We note that the Leopoldt conjecture is true for an odd prime p if and only if G_S is p-multiplicator free for S containing all places lying over p, and that, for $p=2$, the same statement is obtained when k is totally imaginary. We need assume that G_S is multiplicator free for all primes dividing n to prove our result. It is interesting that in the above two theorems there is no such mention to G_S.

For an abelian group A, we denote by $t_n(A)$ the subgroups of n-torsion points, by $t_n^{(m)}(A)$ those of n^m-torsion points and by $t_n^{(\infty)}(A)$ the unions of $t_n^{(m)}(A)$, $m=1,\cdots$. Now we state the main result.

THEOREM. *Let $n\geqq 2$ be a natural number. Let S be a finite set of places of k containig all places lying over every primes dividing n. Assume that G_S is p-multiplicator free for every prime numbers dividing n. When n is even, we assume for $p=2$, not only that G_S is 2-multiplicator free, but also that the Leopoldt conjecture is true for 2 in k. Then we have* $\ker(\rho_S)\cong\mathrm{Hom}(Б_k(S,m),\mathbf{Q}/\mathbf{Z})$ *and* $\mathrm{coker}(\rho_S)\cong\mathrm{Hom}(t_n(k^\times),\mathbf{Q}/\mathbf{Z})$.

In §1, we shall show that the theorem is obtained if it is proved in the case where n is a power of prime number. In §2, we study the torsion subgroup of $G_S/[G_S,G_S]$, where $[G_S,G_S]$ denotes the closure in G_S of the algebraic commutator subgroup. In §3, we shall prove the theorem by virtue of the reduction and the results in §2.

1. The reduction to the cases of powers of a prime number.

We recall that pro-finite group is a projective limit of finite groups, which is totally disconnected and compact, and that a projective limit of finite p-groups is called a pro-p-group. Let G be a pro-finite group. Let $\mathscr{U}$ be the set of open normal subgroups of G, and $\mathscr{U}_p=\{U\in\mathscr{U}\ ;\ G/U$ are p-groups$\}$. We note that $\mathscr{U}$ is a base of the system of neighborhoods of 1 in G. Let G_p denote the projective limit of $\{G/U\ ;\ U\in\mathscr{U}_p\}$. Let $f:G\to H$ be a continuous homomorphism to a pro-p-group H. Let $\mathscr{V}$ be the set of open subgroups of H. We have $f^{-1}(V)\in\mathscr{U}_p$ for $V\in\mathscr{V}$. There is the canonical surjection from G_p onte $G/f^{-1}(V)$. Thus there is a homomorphism $i_V:G_p\in H/V$. Taking projective limit, we obtain a homomorphism $i:G_p\to H$ whose image is equal to Image(f).

LEMMA 1. *Let G be an abelian pro-finite group. Then $t_n(G)$ is a closed subgroup and $G\cong\Pi G_p$.*

PROOF. Let g be an element contianed in the closure of $t_n(G)$. We see $t_n(G)\cap g\cdot U\neq\phi$ for $U\in\mathscr{U}$. Hence there is $h\in t_n(G)$ such that $h=g\cdot u$, $u\in U$. This means that $g^n\in U$ for every $U\in\mathscr{U}$. Namely, $g^n=1$. This shows that $t_n(G)$ is closed. Since G is abelian, we have G/U are isomorphic to a direct product $\prod_p G/U_p$, where U_p are suitable elements of $\mathscr{U}_p$ for each p. Taking projective limits in these isomorphisms, we have the latter half of the lemma. Q. E. D.

Recall that $[G,G]$ is the closure of the algebraic commutator group of G. Put $G^*=G/[G,G]$. It is well-known that if there is a continuous surjective homomorphism from G onto an abelian pro-finite group, it induces one from G^* onto the abelian group. We denote by $\hat{G}$ the abelian group $\mathrm{Hom}(G^*,\boldsymbol{Q}/\boldsymbol{Z})$ if no confusion occurs. We adopt the additive notation to describe this abelian group.

LEMMA 2. *Let p be a prime number. Then* $(G_p)^* \cong (G^*)_p$.

PROOF. Set $U(p) = \bigcap_{U \in \mathscr{N}_p} U$. By Lemma 1 of Serre[4], $G_p \cong G/U(p)$. Since $(G_p)^*$ is isomorphic to a factor group of G and is abelian, there is the canonical surjection $f: G^* \to (G_p)^*$. Hence f induces a surjection $\bar{f}: (G^*)_p \to (G_p)^*$. On the other hand, since $(G^*)_p$ is a pro-p-group which is also a factor group of G, we have the canonical surjection $g: G_p \to (G^*)_p$. Since $(G^*)_p$ is abelian, this induces $\bar{g}: (G_p)^* \to (G^*)_p$. We see that $\bar{f}$ and $\bar{g}$ are the inverse maps to each other. Q. E. D.

By this lemma, we use the notation G_p^* to denote both of $(G_p)^*$ and $(G^*)_p$.

Let $T_n: \boldsymbol{Q}/\boldsymbol{Z} \to \boldsymbol{Q}/\boldsymbol{Z}$ be the endomorphism defined by $T_n(a+\boldsymbol{Z}) = na + \boldsymbol{Z}$ for an integer n. We have a cohomology long exact sequence

$$(1.1) \longrightarrow H^1(G,\boldsymbol{Q}/\boldsymbol{Z}) \xrightarrow{T_n^*} H^1(G,\boldsymbol{Q}/\boldsymbol{Z}) \longrightarrow H^2(G,\tfrac{1}{n}\boldsymbol{Z}/\boldsymbol{Z}) \longrightarrow H^2(G,\boldsymbol{Q}/\boldsymbol{Z}) \longrightarrow .$$

Since $H^1(G,\boldsymbol{Q}/\boldsymbol{Z})$ is isomorphic to $\hat{G}$, we have $\mathrm{coker}\ T_n^* \cong \hat{G}/n\hat{G} \cong \mathrm{Hom}(t_n(G^*),\boldsymbol{Q}/\boldsymbol{Z})$, because $t_n(G^*)$ is closed from Lemma 1.

PROPOSITION 3. *Let p^e be the maximal power of a prime number p dividing a natural number n. Assume that G is p-multiplicator free. Then we have*

$$t_p^{(e)}(H^2(G,\boldsymbol{Z}/n\boldsymbol{Z})) \cong \mathrm{Hom}(t_p^{(e)}(G^*),\boldsymbol{Q}/\boldsymbol{Z}) \cong \mathrm{Hom}(t_p^{(e)}(G_p^*),\boldsymbol{Q}/\boldsymbol{Z}).$$

PRROF. Set $m = p^e$ in the above cohomology exact sequence. Since $t_p^{(\infty)}(H^2(G,\boldsymbol{Q}/\boldsymbol{Z})) \cong \{0\}$ and $\mathrm{coker}(T_m^*) \cong \mathrm{Hom}(t_m(G^*),\boldsymbol{Q}/\boldsymbol{Z})$, we have $t_m(H^2(G,\boldsymbol{Z}/n\boldsymbol{Z})) \cong \mathrm{Hom}(t_m(G^*),\boldsymbol{Q}/\boldsymbol{Z})$ from (1.1). Hence, by Lemma 2, $\mathrm{Hom}(t_m(G^*),\boldsymbol{Q}/\boldsymbol{Z}) \cong \mathrm{Hom}(t_m(G_p^*),\boldsymbol{Q}/\boldsymbol{Z})$. Q. E. D.

Let n be a natural number which is greater than 1 and $\{p_1,\cdots,p_t\}$ be the set of its prime divisers. Let e_i be the maximal power of p_i dividing n. Assume that G is p_i-multiplicator free for $1 \leqq i \leqq t$. We have from Proposition 3

$$H^2(G,\boldsymbol{Z}/n\boldsymbol{Z}) \cong \prod_{i=1}^{t} t_{p_i}^{(e_i)}(\mathrm{Hom}(G_{p_i}^*,\boldsymbol{Q}/\boldsymbol{Z})).$$

We know from Theorem 4 of Serre[5] that the Galois group $G_\mathfrak{p}$ of the local field for a finite place $\mathfrak{p}$ is always multiplicator free. Moreover, when $\mathfrak{p}$ is an infinite places, we prove $H^2(G_\mathfrak{p},\mathrm{Q/Z}) = \{0\}$ directely. Hence again we have from Proposition 3

$$H^2(G_\mathfrak{p},\boldsymbol{Z}/n\boldsymbol{Z}) \cong \mathrm{Hom}(t_n(G_\mathfrak{p}^*),\boldsymbol{Q}/\boldsymbol{Z}).$$

Therefore we have the following diagram.

$$
(1.2)\qquad
\begin{array}{ccc}
H^2(G,Z/nZ) & \xrightarrow{\rho_S(n)} & \prod_{\mathfrak{p}\in S} H^2(G_{\mathfrak{p}},Z/nZ) \\
\downarrow\cong & & \downarrow\cong \\
\prod_{i=1}^{t} \mathrm{Hom}(t_{p_i}^{(e_i)}(G_{p_i}^*),Q/Z) & \longrightarrow & \prod_{i=1}^{t} \mathrm{Hom}(t_{p_i}^{(e_i)}(G_{\mathfrak{p}}^*),Q/Z).
\end{array}
$$

We observe from this diagram that the kernel and cokernel of $\rho_S(n)$ are isomorphic to the diect products of those of $\rho_S(p_i^{e_i})$, respectively.

LEMMA 4. *Let* $n=\prod_{i=1}^{t} p_i^{e_i}$. *Then* $\text{Б}_k(S,n)\cong \text{Б}_k(S,p_1^{e_1})\times\cdots\times \text{Б}_k(S,p_t^{e_t})$.

PROOF. Put $n_i=n/p^{e_i}$. Let ν_i, $i=1,\cdots,t$ be integers such that $1=\nu_1\cdot n_1+\cdots+\nu_t\cdot n_t$. We see $a^{\nu_i n_i}\in\mathcal{S}_k(S,p_i^{e_i})$ for $a\in\mathcal{S}_k(n,S)$. Then $f:a\to(a^{\nu_1 n_1},\cdots,a^{\nu_t n_t})$ gives a homomorphism from $\text{Б}_k(S,n)$ to the direct product of $\text{Б}_k(S,p^{e_i})$. Conversely, set $a=a_1^{\nu_1 n_1}\cdot a_2^{\nu_2 n_2}\cdots a_t^{\nu_t n_t}$ for $a_i\in\mathcal{S}_k(S,p^{e_i})$. We see $f(a)=(a^{\nu_1 n_1},\cdots,a^{\nu_t n_t})$, because $\nu_i n_i\equiv 1 \bmod p^{e_i}$. This proves the lemma.

Clearly, $t_n(k^\times)$ is isomorphic to the direct product of $t_p^{(e_i)}(k^\times)$, $i=1,\cdots,t$. Therefore, by virtue of (1.2) and Lemma 4, we see that the theorem would be proved if we prove it when n is a power of a prime number.

2. Torsion subgroups of G_S^*.

Let L be the maximal unramified abelian extension of k and $\tilde{k}_S$ be the maximal abelian p-extension of k which is unramified outside S. Put $G=\mathrm{Gal}(\tilde{k}_S L/k)$. We note $(G_S^*)_p\cong G_p$. We can describe this group by using class field theory as follows.

Let $U_{\mathfrak{p}}$ be the unit group of the $\mathfrak{p}$-adic completion of k for each finite place. Put $V_{\mathfrak{p}}=t_m(k_{\mathfrak{p}}^\times)$ and $m=N\mathfrak{p}-1$, where $N\mathfrak{p}$ denotes the absolute norm. When $\mathfrak{p}$ is an infinite place, we set $U_{\mathfrak{p}}=k_{\mathfrak{p}}^\times$ and $V_{\mathfrak{p}}=U_{\mathfrak{p}}^p$. We define

$$U_S=\prod_{\mathfrak{p}\in S} U_{\mathfrak{p}},\quad V_S=\prod_{\mathfrak{p}\in S} V_{\mathfrak{p}}\quad\text{and}\quad W_S=\prod_{\mathfrak{p}\notin S} U_{\mathfrak{p}}.$$

Let J be the idele group of k. These groups are considered as subgroups of J. Let H be the closure of $k^\times W_S V_S$ in J. By class field theory, we have

$$G=J/H.$$

This identification of G and J/H is given by means of the norm residue symbol. Let h be the class number of k. We note that $\{J^{hp^n}\cdot H/H\}_{n=1}^{\infty}$ forms a base of the system of neighborhoods of 1. We note that $J^{hp^n}\cdot H=J^{hp^n}\cdot k^\times\cdot W_S\cdot V_S$, because $J^{hp^n}\cdot W_S$ are open subgroups of J. Hence $x\in H$ is equivalent to $x\in J^{hp^n}\cdot k^\times\cdot W_S\cdot V_S$ for every natural numbers n.

Now we suppose that the Leopoldt conjecture is true for p in k and that S contains the set P of places lying over p. We have from Lemma 2 of Miki[1] the following lemma, which is fundamental to study $t_p^{(m)}(G)$.

LEMMA 5. *Let E be the unit group of k. Then for each natural number m, there is a natural number n such that E^{p^m} contains $E \cap U_P^{p^n}$.*

We denote by $\mu(m)$ the samllest naturl number n such that $E^{p^m} \supset E \cap U_P^{p^n}$.

For $\mathfrak{c} \in H$, we have the following decomposition for each n

$$\mathfrak{c} = c \cdot w \cdot v \cdot \mathfrak{b}^{hp^n},\ c \in k^\times,\ w \in W,\ v \in V,\ \mathfrak{b} \in J.$$

Since $V_S^p = V_S$, there is $v_1 \in V$ such that $v = v_1^{p^n}$. On the other hand, since $\mathfrak{b}^h \in U_S \cdot W_S \cdot k^\times$, we have $\mathfrak{b}^h = b \cdot w_1 \cdot u$, $b \in k^\times$, $w_1 \in W_S$, $u \in U_S$. Consequently, the following decomposition of $\mathfrak{c} \in H$ is obtained for each n by setting $c^{(n)} = c \cdot b^{p^n}$, $w^{(n)} = w \cdot w_1^{p^n}$, $u^{(n)} = v_1 \cdot u$.

(2.1) $\mathfrak{c} = c^{(n)} \cdot w^{(n)} \cdot u^{(n)p^n}$, $c^{(n)} \in k^\times$, $w^{(n)} \in W_S$, $u^{(n)} \in U_S$.

Let $\mathfrak{a} \cdot H \in t_p^{(m)}(G)$ for $a \in J$. It follows from this decomposition that

$$a^{p^m} = a^{(n)} \cdot w^{(n)} \cdot u^{(n)p^n},\ a^{(n)} \in k^\times,\ w^{(n)} \in W_S,\ u^{(n)} \in U_S.$$

We shall show that $a^{(n)}$ determines an uniqe element of $\mathrm{Б}_k(S,p^m)$ despite of the choice of n when $n \geqq \mu(m)$. Obviousely, $a^{(n)}$ is (S,p^m)-singular when $n \geqq m$.

Assume $n \geqq \mu(m)$. Let $\mathfrak{a}^{p^m} = \tilde{a}^{(n)} \cdot \tilde{w}^{(n)} \cdot \tilde{u}^{(n)p^n}$ be another decompositiom. We have

$$a^{(n)} \cdot (\tilde{a}^{(n)})^{-1} = (w^{(n)})^{-1} \cdot \tilde{w}^{(n)} \cdot ((u^{(n)})^{-1} \cdot \tilde{u}^{(n)})^{p^n}.$$

This shows that $\varepsilon = a^{(n)} (\tilde{a}^{(n)})^{-1}$ is an unit contained in $U_S^{p^n}$. By Lemma 5, we have $\varepsilon \in E^{p^m}$, because $S \supset P$. Hence $a^{(n)} k^{p^m} = \tilde{a}^{(n)} k^{p^m}$. We can show that $a^{(l)} k^{p^m} = a^{(n)} k^{p^m}$ for every natural numbers $l \geqq n$, similarly. Therefore, $a^{(n)} k^{p^m}$ does not depend on the choice of n and $a^{(n)}$. Next, we change the represetative of the coset $\mathfrak{a} \cdot H$ from $\mathfrak{a}$ to $\mathfrak{a}' = \mathfrak{a} \cdot \mathfrak{c}$, $\mathfrak{c} \in H$. By (2.1), we have

$$\mathfrak{c} = c^{(n)} \cdot y^{(n)} \cdot z^{(n)p^n},\ b^{(n)} \in k^\times,\ y^{(n)} \in W_S,\ z^{(n)} \in U_S.$$

Thus

$$\mathfrak{a}'^{p^m} = (a^{(n)} \cdot c^{(n)p^m}) \cdot (w^{(n)} \cdot y^{(n)p^m}) \cdot (u^{(n)} \cdot z^{(n)})^{p^n}.$$

Since $a^{(n)} k^{p^m} = (a^{(n)} \cdot b^{(n)p^m}) k^{p^m}$, we have that $a^{(n)} k^{p^m}$ dose not depend on the choice of the representative. It followes from the uniqueness of $a^{(n)} k^{p^m}$ that there is a homomorphism $s_p^{(m)} : t_p^{(m)}(G) \to \mathrm{Б}_k(S,p^m)$ defined by $s_p^{(m)}(\mathfrak{a} \cdot H) = a^{(n)} \cdot k^{p^m}$, where n is an arbitrary integer such shat $n \geqq \mu(m)$.

PROPOSITION 6. *$s_p^{(m)}$ is surjective and the kernel is $t_p^{(m)}(U_S) \cdot H/H$.*

PROOF. Let $a \in \mathscr{Y}_k(S,p^m)$. There are $v \in W_S$, $u \in U_S$ and $\mathfrak{a} \in J$ such that $a = v \cdot u^{p^m} \cdot \mathfrak{a}^{p^m}$. Since $s_p^{(m)}(\mathfrak{a} \cdot H) = a \cdot k^{p^m}$, we have $s_p^{(m)}$ is surjective. Assume $s_p^{(m)}(\mathfrak{a} \cdot H) = 1$. For each natural number n which is greater than m, we have

$$\mathfrak{a}^{p^m} = a^{(n)} \cdot w^{(n)} \cdot u^{(n)p^n},\ a^{(n)} \in k^{p^m},\ w^{(n)} \in W_S,\ u^{(n)} \in U_S.$$

Set $b = \sqrt[p^m]{a^{(n)}} \in k$. Since $(\mathfrak{a} \cdot b^{-1} \cdot (u^{(n)})^{-p^{n-m}})^{p^m} = w^{(n)}$, there is $y \in W_S$ such that $w^{(n)} = y^{p^m}$. Put $c^{(n)} =$

$b \cdot y \cdot u^{(n)p^{n-m}}$. We have $\mathfrak{a} \cdot (c^{(n)})^{-1} \in t_p^{(m)}(U_S)$, and hence $\mathfrak{a} \in t_p^{(m)}(U_S) \cdot H \cdot J^{hp^{n-m}}$. This implies that $t_p^{(m)}(U_S) \cdot H$ contains $\mathfrak{a}$. Since $\ker(s_p^{(m)})$ contains $t_p^{(m)}(U_S) \cdot H/H$, we have the proposition.

LEMMA 7. *We have* $t_p^{(m)}(k^\times) = t_p^{(m)}(U_S) \cap H$ *as subgroups of* U_S.

PROOF. Let $x \in t_p^{(m)}(U_S) \cap H$. We consider x is an element $(1,x) \in W_S \times U_S \cap H$. Let l be a natural number greater than m. Set $n = \mu(l) + l - m$. By (2.1), we have $x = a^{(n)} \cdot w^{(n)} \cdot u^{(n)p^n}$, $a^{(n)} \in k^\times$, $w^{(n)} \in W_S$, $u^{(n)} \in U_S$. Here, we see $a^{(n)p^m}$is an unit contained in $U_S^{p^{n+m}}$, because $x^{p^m} = 1$. Thus there is an unit $\varepsilon \in E$ such that $\varepsilon^{p^l} = a^{(n)p^m}$. Hence $a^{(n)} \in E^{p^{l-m}} \cdot t_p(k^\times)$. It follows $x \in E^{p^{l-m}} \cdot t_p(k^\times) \cdot W_S \cdot U_S^{p^n}$, and we have

$$x \in \bigcap_{l>m} E^{p^{l-m}} \cdot t_p(k^\times) \cdot W_S \cdot U_S^{p^n}.$$

Let π_S denote the projection from $W_S \cdot U_S$ onto U_S. Let $t_p'(U_S)$ be the subgroup of U_S generated by all torsion elements whose order are prime to p. Since $t_p(k^\times) \cdot W_S \cdot t_p'(U_S) = \bigcap E^{p^{l-m}} \cdot t_p(k^\times) \cdot W_S \cdot U_S^{p^n}$, we have $\pi(x) \in t_p(k^\times) \cdot t_p'(U_S)$, where we identify $t_p(k^\times)$ to its image by the map π. We obtain $x \in t_p^{(m)}(k^\times)$. This proves the lemma, because the inclusion $t_p(k^\times) \subset t_p(U_S) \cap H$ is obvious.

3. The proof of the theorem.

By virtue of Proposition 6 and Lemma 7, we obtain the exact sequence

$$1 \to t_p^{(m)}(k^\times) \to t_p^{(m)}(U_S) \to t_p^{(m)}(G) \overset{s_p^{(m)}}{\to} \text{Б}_k(S,p^m) \to 1.$$

We consider the dual sequence. Since $\prod_{\mathfrak{p} \in S} t_p^{(m)}((G_\mathfrak{p})^*) \cong t_p^{(m)}(U_S)$ by local class field theory, we have the following exact sequence by means of Proposition 3.

$$1 \to \mathrm{Hom}(\text{Б}_k(S,p^m), \boldsymbol{Q}/\boldsymbol{Z}) \to H^2(G, \boldsymbol{Z}/p^m\boldsymbol{Z}) \to \prod_{\mathfrak{p} \in S} H^2(G_\mathfrak{p}, \boldsymbol{Z}/p^m\boldsymbol{Z}) \to \mathrm{Hom}(t_p^{(m)}(k^\times), \boldsymbol{Q}/\boldsymbol{Z}) \to 1.$$

This proves the theorem when n is a power of a prime number, and completes the proof according to the reduction in §2.

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