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RESULTS IN ESTIMATES FOR k-PLANE TRANSFORMS

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Abstract. This is an expository paper. We give proofs of some results of M. Christ (1984) and S. W. Drury (1984) related to k-plane transforms. Also, we give proofs for some results on interpolation appearing in estimating k-plane transforms.

1 Introduction

In this note, proofs will be presented in some detail for results related to k-plane transforms $T_{k,n}$ of M. Christ [3] in 1984 (Theorem 1.1 below) and S. W. Drury [5] in 1984 (Lemma 2.1 below; see also a result of Blaschke which can be found in [13, Chap. 12]). For a function f on $\mathbb{R}^n$ and an affine k-plane p, $1 \le k < n$, the value of $T_{k,n}f$ at p is defined to be $\int_p f$, where the integration is with respect to the k-dimensional Lebesgue measure on p (see (1.5) below for a more precise definition). We refer to [7] for a survey on a topic related to estimates for $T_{k,n}$ considered in this note and also [12] for relevant references and additional information. When $k = 1$, $T_{k,n}$ is called the X-ray transform and when $k = n-1$, it is called the Radon transform (see [4] for applications in harmonic analysis; also related results can be found in [14]).

Let $G_{k,n}$ be the Grassmannian manifold of all k-planes in $\mathbb{R}^n$ passing through the origin ($1 \le k < n$). Theorem 1.1 and Lemma 2.1 are stated in terms of the SO(n) invariant measure $d\sigma$ on $G_{k,n}$. To prove Lemma 2.1 we shall show Lemma 2.3, which is also in [5] and can be regarded as a polar coordinates expression of the Lebesgue measure on $\mathbb{R}^n$ when $k = 1$. Lemma 2.3 can be used to prove Lemma 2.4 too, which is an analogue of Lemma 2.3, where $\mathbb{R}^n$ is replaced by S^{n-1} (the unit sphere in $\mathbb{R}^n$) and the Lebesgue measure on $\mathbb{R}^n$ is replaced by the Lebesgue surface measure on S^{n-1}. Our proof of Lemma 2.3 is different from that of [5]; it is based

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on straightforward computations concerning local coordinates on the Grassmannian manifolds and the Gram determinants (see Boothby [2, pp. 63–65] and Spivak [17, Section 2 of Chapter 13] for the local coordinates).

We also consider an operator $S_{k,n}$ related to $T_{k,n}$, which maps a function on S^{n-1} to a function on $G_{k,n}$ (see (1.6) below) and prove in Theorem 1.2 estimates for $S_{k,n}$ between the Lorentz space $L^{n/k,n}(S^{n-1})$ and the Lebesgue space $L^n(G_{k,n})$ with respect to the Lebesgue surface measure on S^{n-1} and the measure $d\sigma$ on $G_{k,n}$, respectively. Theorem 1.1 follows from a more general multilinear estimates in Proposition 3.1. We shall prove Proposition 3.1 by using Theorem 1.2 together with Lemmas 2.1 and 2.4. Theorem 1.2 follows from Proposition 4.1 and from more general results in Proposition 4.2 for $k \geq 2$, which are also stated as multilinear estimates.

Proposition 3.1 and Proposition 4.2 will be shown by using the estimates in (3.5) and (4.1) below, respectively, by applying interpolation arguments. In this note, we give proofs of interpolation results required in the arguments for the multilinear estimates (Sections 5 and 6). For basic results on interpolation, we mainly refer to the book Bergh-Lofstrom [1] but we reproduce proofs of some results important for this note. To prove (4.1) we also apply induction arguments using Lemma 2.4 and Lemma 4.4 on the Gram determinants.

For $\theta \in G_{k,n}$, define

$$\theta^\perp = \{x \in \mathbb{R}^n : x \perp \theta\},$$

where $x \perp \theta$ means that $\langle x, y\rangle = 0$ for all $y \in \theta$; $\langle x, y\rangle$ denotes the inner product in $\mathbb{R}^n$:

$$\langle x, y\rangle = \sum_{j=1}^{n} x_j y_j; \quad x = (x_1, \ldots, x_n), \quad y = (y_1, \ldots, y_n).$$

Let

$$\mathbb{S} = \{(x, \theta) : x \in \theta^\perp, \theta \in G_{k,n}\}. \tag{1.1}$$

This can be regarded as a parameterization of all affine k-planes in $\mathbb{R}^n$. We write $\pi = (x, \theta)$ for $(x, \theta) \in \mathbb{S}$. We define a measure $d\nu$ on $\mathbb{S}$ by

$$d\nu(\pi) = d\nu(x, \theta) = d\lambda_{\theta^\perp}(x)\, d\sigma(\theta), \tag{1.2}$$

where

(1) $d\lambda_{\theta^\perp}$ is the $n-k$ dimensional Lebesgue measure on the hyperplane $\theta^\perp$, which is considered as a singular measure on $\mathbb{R}^n$;

(2) $d\sigma$ denotes the SO(n)-invariant measure on $G_{k,n}$, where SO(n) denotes the special orthogonal group on $\mathbb{R}^n$ (see Proposition 14.1 and Remark 14.2 in [15], which is the first version of this note with extended sections on invariant measures on homogeneous manifolds of Lie groups).

Let $d\lambda_{P(x,\theta)}$ be a measure supported on $P(x,\theta) = \{x + y : y \in \theta\}$, $x \in \mathbb{R}^n$, $\theta \in G_{k,n}$, defined as

$$\int_{\mathbb{R}^n} f(z)\, d\lambda_{P(x,\theta)}(z) = \int_\theta f(x+y)\, d\lambda_\theta(y), \tag{1.3}$$

where $d\lambda_\theta(y)$ is the k-dimensional Lebesgue measure on θ considered as a singular measure on $\mathbb{R}^n$: $d\lambda_\theta = d\lambda_{P(0,\theta)}$. If we decompose

$$x = x_\theta + x_{\theta^\perp}, \quad \text{where} \quad x_\theta \in \theta, \quad x_{\theta^\perp} \in \theta^\perp,$$

then it is easy to see that

$$\int f(x+y)\, d\lambda_\theta(y) = \int f(x_{\theta^\perp} + y)\, d\lambda_\theta(y). \tag{1.4}$$

For $(x,\theta) \in \mathbb{S}$, let

$$T_{k,n}(f)(x,\theta) = \int f(z)\, d\lambda_{P(x,\theta)}(z) = \int f(x+y)\, d\lambda_\theta(y). \tag{1.5}$$

For $\theta \in G_{k,n}$, let S_θ^{k-1} be the unit sphere in θ:

$$S_\theta^{k-1} = S^{n-1} \cap \theta = \left\{ y \in \theta : |y| = \langle y, y\rangle^{1/2} = 1 \right\}.$$

Define

$$S_{k,n}(f)(\theta) = \int_{S_\theta^{k-1}} f(\omega)\, d\lambda_\theta(\omega), \tag{1.6}$$

where $d\lambda_\theta(\omega)$ is the unique probability measure on S_θ^{k-1} invariant under the action of $\mathrm{SO}(k)$ on θ; here we note that $\mathrm{SO}(k) = \mathrm{SO}_\theta(k) = \{O \in \mathrm{SO}(n) : O(\theta) \subset \theta\}$.

Let $C_0(\mathbb{R}^n)$ be the set of all continuous functions on $\mathbb{R}^n$ with compact support. Let $C(S^{n-1})$ be the set of all continuous functions on S^{n-1}. We have the following results (see [3]).

Theorem 1.1. *Let $1 \le k < n$, $k \in \mathbb{Z}$ (the set of integers). For $f \in C_0(\mathbb{R}^n)$ we have*

$$\|T_{k,n} f\|_{L^{n+1}(\mathbb{S})} \le C \|f\|_{L^{\frac{n+1}{k+1}, n+1}(\mathbb{R}^n)},$$

where

$$\|T_{k,n} f\|_{L^{n+1}(\mathbb{S})} = \left(\int_{\mathbb{S}} |T_{k,n}(f)(x,\theta)|^{n+1}\, d\nu(x,\theta) \right)^{\frac{1}{n+1}}$$

and $L^{p,q}(\mathbb{R}^n)$ denotes the Lorentz space on $\mathbb{R}^n$ (see [8], [18, Chap. V]).

Theorem 1.2. *For $f \in C(S^{n-1})$ we have*

$$\|S_{k,n}f\|_{L^n(G_{k,n})} = \left(\int_{G_{k,n}} |S_{k,n}(f)(\theta)|^n \, d\sigma(\theta)\right)^{\frac{1}{n}} \le C\|f\|_{L^{\frac{n}{k},n}(S^{n-1})},$$

where $1 \le k < n$, $k \in \mathbb{Z}$ and the Lorentz space $L^{\frac{n}{k},n}(S^{n-1})$ is defined with respect to the Lebesgue surface measure on S^{n-1}.

We also write $\|f\|_{p,q}$ for $\|f\|_{L^{p,q}}$.

For a function g on $\mathbb{S}$, we consider a mixed norm

$$\|g\|_{L^q(L^r)} = \left(\int_{G_{k,n}} \left(\int |g(x,\theta)|^r \, d\lambda_{\theta^\perp}(x)\right)^{q/r} d\sigma(\theta)\right)^{1/q}.$$

Then the norm on the left hand side of the inequality in the conclusion of Theorem 1.1 can be expressed as $\|T_{k,n}f\|_{L^{n+1}(L^{n+1})}$. We recall two more theorems from [3].

Theorem 1.3 (Drury's conjecture)**.** *Suppose that $np^{-1} - (n-k)r^{-1} = k, 1 \le p \le (n+1)/(k+1)$ and $q \le (n-k)p'$, where p' denotes the exponent conjugate to p. Then*

$$\|T_{k,n}f\|_{L^q(L^r)} \le C\|f\|_p,$$

where $\|f\|_p$ denotes the norm of f in $L^p(\mathbb{R}^n)$.

Theorem 1.4. *Suppose that $np^{-1} - (n-k)r^{-1} = k, p \le 2, p < n/k$ and $q \le (n-k)p'$. Then*

$$\|T_{k,n}f\|_{L^q(L^r)} \le C\|f\|_p.$$

Theorem 1.1 implies Theorem 1.3 for $p = (n+1)/(k+1)$, $r = n+1$ and $q = n+1$, from which and the obvious L^1 result the other estimates of Theorem 1.3 follow by interpolation. It is known that the following conditions are necessary.

(1) $np^{-1} - (n-k)r^{-1} = k$ in Theorems 1.3 and 1.4;

(2) $q \le (n-k)p'$ in Theorems 1.3 and 1.4;

(3) $p < n/k$ in Theorem 1.4.

Here we give proofs for the necessity.

Proof for part (1)*.* Let $f^{(\delta)}(x) = f(\delta x)$ for $\delta > 0$. Then $T_{k,n}f^{(\delta)}(x,\theta) = \delta^{-k}T_{k,n}f(\delta x,\theta)$, and so $\|T_{k,n}f^{(\delta)}\|_{L^q(L^r)} = \delta^{-k}\delta^{-(n-k)/r}\|T_{k,n}f\|_{L^q(L^r)}$. On the other hand, $\|f^{(\delta)}\|_p = \delta^{-n/p}\|f\|_p$. Thus, if $\|T_{k,n}f^{(\delta)}\|_{L^q(L^r)} \lesssim \|f^{(\delta)}\|_p$, we have

$$\delta^{-k}\delta^{-(n-k)/r}\|T_{k,n}f\|_{L^q(L^r)} \lesssim \delta^{-n/p}\|f\|_p$$

for all $\delta > 0$, which implies $np^{-1} - (n-k)r^{-1} = k$. □

Proof for part (2). We write $x = (x^{(1)}, x^{(2)})$, $x^{(1)} \in \mathbb{R}^k$, $x^{(2)} \in \mathbb{R}^{n-k}$ for $x \in \mathbb{R}^n$. Define

$$B^{(1)} = \{x^{(1)} \in \mathbb{R}^k : |x^{(1)}| \le 1\}, \quad B_\epsilon^{(2)} = \{x^{(2)} \in \mathbb{R}^{n-k} : |x^{(2)}| \le \epsilon\},$$

where ϵ is a small positive number less than 1. Let $T_\epsilon = B^{(1)} \times B_\epsilon^{(2)}$. Following [3], we consider the characteristic function

$$\chi_{T_\epsilon}(x^{(1)}, x^{(2)}) = \chi_{B^{(1)}}(x^{(1)})\chi_{B_\epsilon^{(2)}}(x^{(2)}).$$

Let

$$P_0 = \{(x^{(1)}, 0) : x^{(1)} \in \mathbb{R}^k\}.$$

We can see that there exists an ϵ-neighborhood N_ϵ of P_0 in $G_{k,n}$, which is a $k(n-k)$ dimensional manifold, such that if $\theta \in N_\epsilon$, $y \in \theta$ and $|y| \le 1/2$, then $|y^{(1)}| \le 1/2$, $y^{(2)} \in B_{\epsilon/2}^{(2)}$ and $\sigma(N_\epsilon) \gtrsim \epsilon^{k(n-k)}$ (see (2.1) below in Section 2 for the dimensionality of $G_{k,n}$; also, the formula in (2.5) below is helpful). For any $\theta \in G_{k,n}$, it is obvious that if $x \in \theta^\perp$ and $|x| \le \epsilon/2$, then $|x^{(1)}| \le 1/2$ and $x^{(2)} \in B_{\epsilon/2}^{(2)}$. Therefore, if $\theta \in N_\epsilon$, $y \in \theta$, $|y| \le 1/2$, $x \in \theta^\perp$ and $|x| \le \epsilon/2$, then $x^{(1)} + y^{(1)} \in B^{(1)}$ and $x^{(2)} + y^{(2)} \in B_\epsilon^{(2)}$, and so $x + y \in T_\epsilon$. Thus

$$\int \left| \int \chi_{T_\epsilon}(x+y)\, d\lambda_\theta(y) \right|^r d\lambda_{\theta^\perp}(x) \ge \int_{|x| \le \epsilon/2} \left| \int_{|y| \le 1/2} d\lambda_\theta(y) \right|^r d\lambda_{\theta^\perp}(x) \gtrsim \epsilon^{(n-k)}$$

for all $\theta \in N_\epsilon$, and hence

$$\begin{aligned} \|T_{k,n}\chi_{T_\epsilon}\|_{L^q(L^r)} &\ge \left(\int_{N_\epsilon} \left(\left| \int \chi_{T_\epsilon}(x+y)\, d\lambda_\theta(y) \right|^r d\lambda_{\theta^\perp}(x) \right)^{q/r} d\sigma(\theta) \right)^{1/q} \\ &\gtrsim \epsilon^{(n-k)/r} \sigma(N_\epsilon)^{1/q} \gtrsim \epsilon^{(n-k)/r} \epsilon^{k(n-k)/q}. \end{aligned} \tag{1.7}$$

On the other hand, we see that

$$\|\chi_{T_\epsilon}\|_p \sim \epsilon^{(n-k)/p}. \tag{1.8}$$

Thus if $\|T_{k,n}\chi_{T_\epsilon}\|_{L^q(L^r)} \lesssim \|\chi_{T_\epsilon}\|_p$, by (1.7) and (1.8) the quantity

$$\epsilon^{\frac{n-k}{r} + \frac{k(n-k)}{q} - \frac{n-k}{p}} = \epsilon^{\frac{k(n-k)}{q} - \frac{k}{p'}}$$

remains bounded as $\epsilon \to 0$, where we have used the equation in part (1). This implies that $k(n-k)q^{-1} - k(p')^{-1} \ge 0$, which is equivalent to what we need. □

Proof for part (3). Let

$$f(x) = (1 + |x|)^{-k} \left(\log(2 + |x|)\right)^{-\delta},$$

where $1/p \le k/n < \delta < 1$. Then, $f \in L^p(\mathbb{R}^n)$, since $kp \ge n$ and $p\delta > 1$. Also, we have $T_{k,n} f(x, \theta) = \infty$ for all x, θ, since $\delta < 1$. So we need to have $p < n/k$ if $T_{k,n}$ is bounded from L^p to the space $L^q(L^r)$. □

We shall prove Theorem 1.1 in Section 3 assuming Theorem 1.2, which will be shown in Section 4. In Section 2, we give some lemmas for the proofs of the theorems including a formula in Lemma 2.3 related to a result in [5]. Theorems 1.1 and 1.2 follow from multilinear estimates in Propositions 3.1 and 4.1 below, respectively. Results in interpolation arguments needed to prove Propositions 3.1 and 4.1 will be provided in Sections 5 and 6.

2 Some results related to the proofs of Theorems 1.1 and 1.2

Let $v_1, \dots, v_k$ be vectors in $\mathbb{R}^n$. The Gram matrix $G_0(v_1, v_2, \dots, v_k)$ is defined to be the $k \times k$ matrix whose (ℓ, m) component is given by $\langle v_\ell, v_m \rangle$:

$$G_0(v_1, v_2, \dots, v_k) = \begin{pmatrix} \langle v_1, v_1 \rangle & \dots & \langle v_1, v_k \rangle \\ \dots\dots & \dots\dots & \dots\dots \\ \dots\dots & \dots\dots & \dots\dots \\ \langle v_k, v_1 \rangle & \dots & \langle v_k, v_k \rangle \end{pmatrix}.$$

Also, the Gram determinant $G(v_1, v_2, \dots, v_k)$ is defined as

$$G(v_1, v_2 \dots, v_k) = \det G_0(v_1, v_2 \dots, v_k).$$

Then, $G \ge 0$ and $G^{1/2}$ is the k-dimensional Lebesgue measure of the parallelepiped determined by $v_1, \dots, v_k$ (see [9, 1.4, 1.5]). It is known that

$$G(v_1, v_2, \dots, v_k) = G(v_{\tau(1)}, v_{\tau(2)}, \dots, v_{\tau(k)})$$

for every permutation τ of $\{1, 2, \dots, k\}$ and

$$G(\alpha_1 v_1, \alpha_2 v_2, \dots, \alpha_k v_k) = \alpha_1^2 \alpha_2^2 \dots \alpha_k^2 G(v_1, v_2, \dots, v_k)$$

for any $\alpha_1, \alpha_2, \dots, \alpha_k \in \mathbb{R}$. Also,

$$G(Uv_1, Uv_2, \dots, Uv_k) = G(v_1, v_2, \dots, v_k)$$

for every $U \in \mathrm{O}(n)$ (the orthogonal group). If $v_i = (v_{i1}, v_{i2} \ldots, v_{in})$, $1 \le i \le k$, it is known that

$$G(v_1, \ldots, v_k) = \sum_{1 \le j_1 < j_2 < \cdots < j_k \le n} \begin{vmatrix} v_{1,j_1} & v_{1,j_2} & \ldots & v_{1,j_k} \\ v_{2,j_1} & v_{2,j_2} & \ldots & v_{2,j_k} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ v_{k,j_1} & v_{k,j_2} & \ldots & v_{k,j_k} \end{vmatrix}^2,$$

where the summation is over all $J = \{j_1, j_2, \ldots, j_k\}$ in the set $\{J : J \subset \{1, 2, \ldots, n\}, \mathrm{card}\, J = k\}$ of cardinality $\binom{n}{k} = n!/(k!(n-k)!)$.

We recall the following result, which I have learned from Drury [5]; a closely related result of Blaschke can be found in [13, Chap. 12] (see [6, pp. 371–372]). We also refer to [11].

Lemma 2.1. *We have*

$$d\lambda_{P(\pi)}(x_0) \ldots d\lambda_{P(\pi)}(x_k)\, d\nu(\pi) = c|G(x_1 - x_0, \ldots, x_k - x_0)|^{(k-n)/2}\, dx_0\, dx_1 \ldots dx_k$$

with a positive constant c, where each of $dx_0, \ldots, dx_k$ is the Lebesgue measure on $\mathbb{R}^n$, and the equation means that

$$\begin{aligned} &\int_{\mathbb{R}^{n(k+1)} \times \mathbb{S}} F(x_0, \ldots, x_k, x, \theta)\, d\lambda_{P(x,\theta)}(x_0) \ldots d\lambda_{P(x,\theta)}(x_k)\, d\nu(x, \theta) \\ &\quad = c \int_{\mathbb{R}^{n(k+1)}} F(x_0, \ldots, x_k, (x_0)_{\theta(x_1 - x_0, \ldots, x_k - x_0)^\perp}, \theta(x_1 - x_0, \ldots, x_k - x_0)) \\ &\qquad\qquad \times |G(x_1 - x_0, \ldots, x_k - x_0)|^{(k-n)/2}\, dx_0\, dx_1 \ldots dx_k, \end{aligned}$$

where F is an appropriate function and $\theta(v_1, \ldots, v_k)$ is the element in $G_{k,n}$ which contains $v_1, \ldots, v_k$.

To prove this, we first show the following (see [16, Theorem 3.4]).

Lemma 2.2. *Let $\pi = (y, \theta) \in \mathbb{S}$, $x \in \mathbb{R}^n$. Define a measure $d\mu_x(\pi)$ on $\mathbb{S}$ by*

$$\int_{\mathbb{S}} F(\pi)\, d\mu_x(\pi) = \int_{\mathbb{S}} F(y, \theta)\, d\mu_x(y, \theta) = \int_{G_{k,n}} F(x_{\theta^\perp}, \theta)\, d\sigma(\theta).$$

Then $d\mu_x(\pi)\, dx = d\lambda_{P(\pi)}(x)\, d\nu(\pi)$ on $\mathbb{R}^n \times \mathbb{S}$, in the sense that

$$\int_{\mathbb{S}} \int_{\mathbb{R}^n} F(x, y, \theta)\, d\lambda_{P(y,\theta)}(x)\, d\nu(y, \theta) = \int_{\mathbb{R}^n} \int_{\mathbb{S}} F(x, y, \theta)\, d\mu_x(y, \theta)\, dx.$$

Proof. Since $y = (y+z)_{\theta^\perp}$ if $y \in \theta^\perp$ and $z \in \theta$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^n\times\mathbb{S}} F(x,y,\theta)\,d\lambda_{P(y,\theta)}(x)\,d\nu(y,\theta) = \int_{\mathbb{S}}\int_{z\in\theta} F(y+z,y,\theta)\,d\lambda_\theta(z)\,d\nu(y,\theta)\\
&= \int_{G_{k,n}}\int_{y\in\theta^\perp}\int_{z\in\theta} F(y+z,(y+z)_{\theta^\perp},\theta)\,d\lambda_\theta(z)\,d\lambda_{\theta^\perp}(y)\,d\sigma(\theta)\\
&= \int_{G_{k,n}}\int_{\mathbb{R}^n} F(x,x_{\theta^\perp},\theta)\;dx\,d\sigma(\theta)\\
&= \int_{\mathbb{R}^n}\int_{G_{k,n}} F(x,x_{\theta^\perp},\theta)\,d\sigma(\theta)\,dx\\
&= \int_{\mathbb{R}^n}\int_{\mathbb{S}} F(x,y,\theta)\,d\mu_x(y,\theta)\,dx,
\end{aligned}$$

where the third equality holds since the Lebesgue measure dx on $\mathbb{R}^n$ can be decomposed as $dx = d\lambda_\theta(z)\,d\lambda_{\theta^\perp}(y)$ for every $\theta \in G_{k,n}$:

$$\int f(z+y)\,d\lambda_\theta(z)\,d\lambda_{\theta^\perp}(y) = \int_{\mathbb{R}^n} f(x)\,dx.$$

□

Proof of Lemma 2.1. Let $(x,\theta) \in \mathbb{S}$.

$$\begin{aligned}
I &:= \int F(x_0,x_1,\dots,x_k,x,\theta)d\lambda_{P(x,\theta)}(x_1)\dots d\lambda_{P(x,\theta)}(x_k)\\
&= \int_{y_1\in\theta}\cdots\int_{y_k\in\theta} F(x_0,x+y_1,\dots,x+y_k,x,\theta)\,d\lambda_\theta(y_1)\dots d\lambda_\theta(y_k)
\end{aligned}$$

Therefore, by Lemma 2.2, we have

$$\begin{aligned}
&\int I\,d\lambda_{P(x,\theta)}(x_0)\,d\nu(x,\theta) = \int I\,d\mu_{x_0}(x,\theta)\,dx_0\\
&= \int F(x_0,(x_0)_{\theta^\perp}+y_1,\dots,(x_0)_{\theta^\perp}+y_k,(x_0)_{\theta^\perp},\theta)\\
&\qquad\qquad d\lambda_\theta(y_1)\dots d\lambda_\theta(y_k)\,d\sigma(\theta)dx_0\\
&= \int F(x_0,x_0+y_1,\dots,x_0+y_k,(x_0)_{\theta^\perp},\theta)\,d\lambda_\theta(y_1)\dots d\lambda_\theta(y_k)\,d\sigma(\theta)dx_0\\
&=: I_1,
\end{aligned}$$

where the penultimate equality follows from (1.4).

To complete the proof of Lemma 2.1, we apply the following result.

Lemma 2.3. *Let $1 \le k < n$. Then we have*

$$\int_{G_{k,n}} \int_{\theta^k} f(y_1, y_2, \dots, y_k, \theta)\, d\lambda_\theta(y_1)\, d\lambda_\theta(y_2) \dots \lambda_\theta(y_k)\, d\sigma(\theta)$$
$$= c \int_{\mathbb{R}^{nk}} f(x_1, x_2, \dots, x_k, \theta(x_1, x_2, \dots, x_k))\, |G(x_1, x_2, \dots, x_k)|^{(k-n)/2}\, dx_1\, dx_2 \dots dx_k.$$

By Lemma 2.3,

$$\begin{aligned} I_1 &= c \int_{\mathbb{R}^{n(k+1)}} F(x_0, x_0 + x_1, \dots, x_0 + x_k, (x_0)_{\theta(x_1,\dots,x_k)^\perp}, \theta(x_1, \dots, x_k)) \\ &\quad \times |G(x_1, x_2, \dots, x_k)|^{(k-n)/2}\, dx_1 \dots dx_k\, dx_0 \\ &= c \int_{\mathbb{R}^{n(k+1)}} F(x_0, x_1, \dots, x_k, (x_0)_{\theta(x_1-x_0,\dots,x_k-x_0)^\perp}, \theta(x_1 - x_0, \dots, x_k - x_0)) \\ &\quad \times |G(x_1 - x_0, x_2 - x_0, \dots, x_k - x_0)|^{(k-n)/2}\, dx_0\, dx_1 \dots dx_k, \end{aligned}$$

which completes the proof. □

Proof of Lemma 2.3. Here we consider the Grassmann manifold $G_{k,n}$ defined as in [2, pp. 63–65] (see also [17, Section 2 of Chapter 13]). The dimension of $G_{k,n}$ is $k(n-k)$ and the invariant measure on $G_{k,n}$ is realized as a measure based on the absolute value of a $k(n-k)$ local differential form on $G_{k,n}$ (see Remark 14.2 in [15]). By a suitable partition of unity we can write $d\sigma = \sum d\mu_j$, where $d\mu_j$ can be expressed by using local coordinates as

$$d\mu_j(\theta) = \rho_j(x_1(\theta), \dots, x_{k(n-k)}(\theta)) \left|(dx_1)_\theta \wedge \dots \wedge (dx_{k(n-k)})_\theta\right|,$$

where ρ_j is compactly supported, non-negative, continuous function. We also write

$$d\mu_j = \rho_j(x_1, \dots, x_{k(n-k)})\, dx_1 \dots dx_{k(n-k)}.$$

We rewrite this using $a_{1,k+1}, \dots, a_{k,k+1}, a_{1,k+2}, \dots, a_{k,k+2}, \dots, a_{1,n}, \dots, a_{k,n}$ as

$$d\mu_j = \rho_j(a_{1,k+1}, \dots, a_{k,n})\, da_{1,k+1} \dots da_{k,n}.$$

Fix j_0 and consider $d\mu_{j_0}$. Let

$$\begin{aligned} v_1 &= (1, 0, \dots, 0, a_{1,k+1}, \dots, a_{1,n}), \\ v_2 &= (0, 1, \dots, 0, a_{2,k+1}, \dots, a_{2,n}), \\ &\ \vdots \\ v_k &= (0, \dots, 0, 1, a_{k,k+1}, \dots, a_{k,n}). \end{aligned}$$

We now assume that $\mathbb{R}^{k(n-k)}$ and the local coordinates on $G_{k,n}$ are related by the injection

$$(a_{1,k+1},\dots,a_{k,k+1},a_{1,k+2},\dots,a_{k,k+2},\dots,a_{1,n},\dots,a_{k,n}) \longmapsto \theta = \mathrm{sp}\{v_1,v_2,\dots,v_k\} \in G_{k,n}, \quad (2.1)$$

where $\mathrm{sp}\{v_1,v_2,\dots,v_k\}$ denotes the subspace of $\mathbb{R}^n$ generated by $\{v_1,v_2,\dots,v_k\}$. Put

$$y_\ell = \sum_{i=1}^k s_i^\ell v_i = \left(s_1^\ell, s_2^\ell, \dots, s_k^\ell, \sum_{i=1}^k s_i^\ell a_{i,k+1}, \dots, \sum_{i=1}^k s_i^\ell a_{i,n}\right)$$

for $\ell = 1,2,\dots,k$. Let

$$E_k = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}$$

be the $k \times k$ unit matrix and define a $k \times (n-k)$ matrix $A_{k,n-k}$ by

$$A_{k,n-k} = \begin{pmatrix} a_{1,k+1} & \cdots & a_{1,n} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ a_{k,k+1} & \cdots & a_{k,n} \end{pmatrix}.$$

We note that

$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_k \end{pmatrix} = (E_k, A_{k,n-k}).$$

Let

$$u^\ell = (s_1^\ell, s_2^\ell, \dots, s_k^\ell)$$

be the k dimensional row vector and

$$0_k = (0,0,\dots,0)$$

be the k dimensional row zero vector. We define $n-k$ numbers of $k(n-k)$ row vectors by

$$\begin{aligned} S_1^\ell &= (u^\ell, 0_k, \dots, 0_k), \\ S_2^\ell &= (0_k, u^\ell, 0_k, \dots, 0_k), \\ &\vdots \\ S_{n-k}^\ell &= (0_k, 0_k, \dots, 0_k, u^\ell). \end{aligned}$$

Define the $(n-k) \times k(n-k)$ matrix by

$$B^\ell = \begin{pmatrix} S_1^\ell \\ S_2^\ell \\ \vdots \\ S_{(n-k)}^\ell \end{pmatrix}$$

for $\ell = 1, 2, \ldots, k$. Let

$$a^j = (a_{1,k+j}, \ldots, a_{k,k+j}), \quad j = 1, 2, \ldots, n-k.$$

Consider the Jacobian

$$A = \frac{\partial(y_1, y_2, \ldots, y_k)}{\partial(u^1, u^2, \ldots, u^k, a^1, a^2, \ldots, a^{n-k})}. \tag{2.2}$$

Then A is equal to the determinant of the $nk \times nk$ matrix:

$$\begin{pmatrix} E_k & 0_{k,k} & 0_{k,k} & \cdots\cdots & 0_{k,k} & 0_{k,k(n-k)} \\ A^t_{k,n-k} & 0_{n-k,k} & 0_{n-k,k} & \cdots\cdots & 0_{n-k,k} & B^1 \\ 0_{k,k} & E_k & 0_{k,k} & \cdots\cdots & 0_{k,k} & 0_{k,k(n-k)} \\ 0_{n-k,k} & A^t_{k,n-k} & 0_{n-k,k} & \cdots\cdots & 0_{n-k,k} & B^2 \\ \vdots & \vdots & & \ddots & & \vdots \\ 0_{k,k} & 0_{k,k} & 0_{k,k} & \cdots\cdots & E_k & 0_{k,k(n-k)} \\ 0_{n-k,k} & 0_{n-k,k} & 0_{n-k,k} & \cdots\cdots & A^t_{k,n-k} & B^k \end{pmatrix},$$

where $0_{j,m}$ denotes the $j \times m$ zero matrix and $A^t_{k,n-k}$ denotes the transpose. We can see easily that $|A|$ is equal to the absolute value of the determinant of the matrix

$$\begin{pmatrix} E_k & & & & \\ & E_k & & & \\ & & \ddots & & \\ & & & E_k & \\ A^t_{k,n-k} & & & & B^1 \\ & A^t_{k,n-k} & & & B^2 \\ & & \ddots & & \vdots \\ & & & A^t_{k,n-k} & B^k \end{pmatrix}$$

(where the components not expressed explicitly are all 0), which equals the absolute value of the determinant of the $k(n-k) \times k(n-k)$ matrix:

$$B = \begin{pmatrix} B^1 \\ B^2 \\ \vdots \\ B^k \end{pmatrix}.$$

By inspection $|\det B|$ is equal to $|\det S|^{n-k}$, where

$$S = \begin{pmatrix} s_1^1 & s_2^1 & \dots & s_k^1 \\ s_1^2 & s_2^2 & \dots & s_k^2 \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ s_1^k & s_2^k & \dots & s_k^k \end{pmatrix}. \tag{2.3}$$

We write

$$y_\ell = \left(\sum_{i=1}^k s_i^\ell a_{i1}, \sum_{i=1}^k s_i^\ell a_{i2}, \dots, \sum_{i=1}^k s_i^\ell a_{in} \right),$$

where $a_{ij} = \delta_{ij}$ for $1 \leq j \leq k$; $\delta_{ii} = 1$, $\delta_{ij} = 0$ if $i \neq j$.

Let $1 \leq j_1 < j_2 < \dots < j_k \leq n$. To compute $G(y_1, \dots, y_k)$, we note that

$$\begin{aligned} &\begin{vmatrix} \sum_{i_1=1}^k s_{i_1}^1 a_{i_1,j_1} & \sum_{i_2=1}^k s_{i_2}^1 a_{i_2,j_2} & \dots & \sum_{i_k=1}^k s_{i_k}^1 a_{i_k,j_k} \\ \sum_{i_1=1}^k s_{i_1}^2 a_{i_1,j_1} & \sum_{i_2=1}^k s_{i_2}^2 a_{i_2,j_2} & \dots & \sum_{i_k=1}^k s_{i_k}^2 a_{i_k,j_k} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \sum_{i_1=1}^k s_{i_1}^k a_{i_1,j_1} & \sum_{i_2=1}^k s_{i_2}^k a_{i_2,j_2} & \dots & \sum_{i_k=1}^k s_{i_k}^k a_{i_k,j_k} \end{vmatrix} \\ &= \sum_{i_1,i_2,\dots,i_k=1}^k a_{i_1,j_1} a_{i_2,j_2} \dots a_{i_k,j_k} \begin{vmatrix} s_{i_1}^1 & s_{i_2}^1 & \dots & s_{i_k}^1 \\ s_{i_1}^2 & s_{i_2}^2 & \dots & s_{i_k}^2 \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ s_{i_1}^k & s_{i_2}^k & \dots & s_{i_k}^k \end{vmatrix} \\ &= \begin{vmatrix} a_{1,j_1} & a_{1,j_2} & \dots & a_{1,j_k} \\ a_{2,j_1} & a_{2,j_2} & \dots & a_{2,j_k} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a_{k,j_1} & a_{k,j_2} & \dots & a_{k,j_k} \end{vmatrix} \begin{vmatrix} s_1^1 & s_2^1 & \dots & s_k^1 \\ s_1^2 & s_2^2 & \dots & s_k^2 \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ s_1^k & s_2^k & \dots & s_k^k \end{vmatrix}. \end{aligned}$$

Thus

$$\begin{aligned} G(y_1, \dots, y_k) &= \sum_{1 \leq j_1 < j_2 < \dots < j_k \leq n} \begin{vmatrix} a_{1,j_1} & a_{1,j_2} & \dots & a_{1,j_k} \\ a_{2,j_1} & a_{2,j_2} & \dots & a_{2,j_k} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a_{k,j_1} & a_{k,j_2} & \dots & a_{k,j_k} \end{vmatrix}^2 \begin{vmatrix} s_1^1 & s_2^1 & \dots & s_k^1 \\ s_1^2 & s_2^2 & \dots & s_k^2 \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ s_1^k & s_2^k & \dots & s_k^k \end{vmatrix}^2 \\ &= G(v_1, \dots v_k) \left| \det S \right|^2. \end{aligned}$$

Therefore

$$
\begin{aligned}
& ds_1^1 \dots ds_k^1 \dots ds_1^k \dots ds_k^k d\mu_{j_0} \\
&= ds_1^1 \dots ds_k^1 \dots ds_1^k \dots ds_k^k \rho_{j_0}(a_{1,k+1}, \dots, a_{k,n})\, da_{1,k+1} \dots da_{k,n} \\
&= \rho_{j_0}(a_{1,k+1}, \dots, a_{k,n})\, |\det S|^{k-n}\ dy_1\ \dots\ dy_k \\
&= \rho_{j_0}(a_{1,k+1}, \dots, a_{k,n}) N(a_{ij})^{n-k}\, |G(y_1, \dots, y_k)|^{(k-n)/2}\ dy_1\ \dots\ dy_k,
\end{aligned}
$$

where

$$N(a_{ij}) = G(v_1, \dots v_k)^{1/2}.$$

Let $y_\ell = \sum_{i=1}^k s_i^\ell v_i$, where $v_1, \dots, v_k$ are as above. It can be shown that

$$
\begin{aligned}
& \int_\theta \cdots \int_\theta g(z_1, z_2, \dots, z_k)\, d\lambda_\theta(z_1)\, d\lambda_\theta(z_2) \dots\ d\lambda_\theta(z_k) \\
&= |G(v_1, v_2, \dots, v_k)|^{k/2} \int_{\mathbb{R}^k} \cdots \int_{\mathbb{R}^k} g(y_1, \dots, y_k) \prod_{\ell=1}^k ds_1^\ell \dots\ ds_k^\ell \\
&= N(a_{ij})^k \int_{\mathbb{R}^k} \cdots \int_{\mathbb{R}^k} g(y_1, \dots, y_k) \prod_{\ell=1}^k ds_1^\ell \dots\ ds_k^\ell
\end{aligned}
$$

for an appropriate function g, where $\theta \in G_{k,n}$ is spanned by $v_1, v_2, \dots, v_k$. Therefore

$$
\begin{aligned}
& \int \left(\int \cdots \int f(z_1, \dots, z_k, \theta)\, d\lambda_\theta(z_1)\ \dots\ d\lambda_\theta(z_k) \right) \qquad (2.4) \\
& \qquad \rho_{j_0}(x_1(\theta), \dots, x_{k(n-k)}(\theta))\ \left| (dx_1)_\theta \wedge \dots \wedge (dx_{k(n-k)})_\theta \right| \\
&= \int \int_{\theta(a_{1,k+1},\dots,a_{k,n})^k} f(z_1, \dots, z_k, \theta(a_{1,k+1}, \dots, a_{k,n})) \\
& \qquad \prod_{j=1}^k d\lambda_{\theta(a_{1,k+1},\dots,a_{k,n})}(z_j) \rho_{j_0}(a_{1,k+1}, \dots, a_{k,n})\, da_{1,k+1}\ \dots\ da_{k,n} \\
&= \int N(a_{ij})^k \int_{\mathbb{R}^k} \cdots \int_{\mathbb{R}^k} f(y_1, \dots, y_k, \theta(a_{1,k+1}, \dots, a_{k,n})) \prod_{\ell=1}^k ds_1^\ell \dots\ ds_k^\ell \\
& \quad \times \rho_{j_0}(a_{1,k+1}, \dots, a_{k,n})\, da_{1,k+1}\ \dots\ da_{k,n} \\
&= \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} f(y_1, \dots, y_k, \theta(a_{1,k+1}, \dots, a_{k,n})) \\
& \quad \times \rho_{j_0}(a_{1,k+1}, \dots, a_{k,n}) N(a_{ij})^n\, |G(y_1, \dots, y_k)|^{(k-n)/2}\ dy_1\ \dots\ dy_k,
\end{aligned}
$$

where $\theta(a_{1,k+1}, \dots, a_{k,n}) = \theta(v_1, \dots, v_k)$.

Here we note the following. Let S be as in (2.3). Then

$$[y] := \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_k \end{pmatrix} = (S, SA_{k,n-k}) = S(E_k, A_{k,n-k}). \tag{2.5}$$

Thus, writing $y_\ell = (y_1^\ell, \dots, y_n^\ell)$, we have

$$S = S(y_1, \dots, y_k) = \begin{pmatrix} y_1^1 & y_2^1 & \dots & y_k^1 \\ y_1^2 & y_2^2 & \dots & y_k^2 \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ y_1^k & y_2^k & \dots & y_k^k \end{pmatrix},$$

$$T = T(y_1, \dots, y_k) := SA_{k,n-k} = \begin{pmatrix} y_{k+1}^1 & y_{k+2}^1 & \dots & y_n^1 \\ y_{k+1}^2 & y_{k+2}^2 & \dots & y_n^2 \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ y_{k+1}^k & y_{k+2}^k & \dots & y_n^k \end{pmatrix},$$

and hence

$$A_{k,n-k} = \begin{pmatrix} a_{1,k+1} & \dots & a_{1,n} \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ a_{k,k+1} & \dots & a_{k,n} \end{pmatrix} = S^{-1}T.$$

Therefore a_{ij} can be expressed by y_m^ℓ:

$$a_{ij} = a_{ij}(y_1, \dots, y_k) = \left(S(y_1, \dots, y_k)^{-1} T(y_1, \dots, y_k)\right)_{i,j-k}$$

for $1 \le i \le k$, $k+1 \le j \le n$. So we can write

$$\begin{aligned} &\int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} f(y_1, \dots, y_k, \theta(a_{1,k+1}, \dots, a_{k,n})) \\ &\qquad \rho_{j_0}(a_{1,k+1}, \dots, a_{k,n}) N(a_{ij})^n \left|G(y_1, \dots, y_k)\right|^{(k-n)/2} dy_1 \dots dy_k \\ &\qquad\qquad = \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} f(y_1, \dots, y_k, \theta(y_1, \dots, y_k)) \widetilde{\rho}_{j_0}(y_1, \dots, y_k) \\ &\qquad\qquad\qquad \times \left|G(y_1, \dots, y_k)\right|^{(k-n)/2} dy_1 \dots dy_k, \end{aligned} \tag{2.6}$$

where

$$\begin{aligned} &\widetilde{\rho}_{j_0}(y_1, \dots, y_k) \\ &\qquad = \rho_{j_0}\left(a_{1,k+1}(y_1, \dots, y_k), \dots, a_{k,n}(y_1, \dots, y_k)\right) N\left(a_{ij}(y_1, \dots, y_k)\right)^n \end{aligned}$$

and we have used the relation:

$$\theta\left(a_{1,k+1}(y_1,\dots,y_k),\dots,a_{k,n}(y_1,\dots,y_k)\right)=\theta(v_1,\dots,v_k)=\theta(y_1,\dots,y_k),$$

which follows by (2.5). We note that each a_{ij} is homogeneous of degree 0:

$$a_{ij}(\tau y_1,\dots,\tau y_k)=a_{ij}(y_1,\dots,y_k) \qquad \text{for all } \tau>0.$$

Let $[y]$ be the $k\times n$ matrix as in (2.5). We also write $S(y_1,\dots,y_k)=S([y])$, $T(y_1,\dots,y_k)=T([y])$, $a_{ij}(y_1,\dots,y_k)=a_{ij}([y])$ and $\widetilde{\rho}_{j_0}(y_1,\dots,y_k)=\widetilde{\rho}_{j_0}([y])$. Then if α is a $k\times k$ matrix,

$$S(\alpha[y])=\alpha S([y]), \quad T(\alpha[y])=\alpha T([y]).$$

Thus if α is non-singular, we have

$$S(\alpha[y])^{-1}T(\alpha[y])=(\alpha S([y]))^{-1}\alpha T([y])=S([y])^{-1}\alpha^{-1}\alpha T([y])=S([y])^{-1}T([y]),$$

and hence $a_{ij}(\alpha[y])=a_{ij}([y])$ and

$$\widetilde{\rho}_{j_0}(\alpha[y])=\widetilde{\rho}_{j_0}([y]). \tag{2.7}$$

Summing up in j_0, by (2.4) and (2.6), we have

$$\begin{aligned}&\int_{G_{k,n}}\int_{\theta^k} f(z_1,z_2,\dots,z_k,\theta)\,d\lambda_\theta(z_1)\,d\lambda_\theta(z_2)\dots d\lambda_\theta(z_k)\,d\sigma(\theta) \qquad (2.8)\\ &=\int_{\mathbb{R}^n}\cdots\int_{\mathbb{R}^n} f(y_1,\dots,y_k,\theta(y_1,\dots,y_k))\rho(y_1,\dots,y_k)\\ &\quad\times|G(y_1,\dots,y_k)|^{(k-n)/2}\,dy_1\,\dots\,dy_k,\end{aligned}$$

where

$$\rho=\sum_{j_0}\widetilde{\rho}_{j_0}.$$

We can see that ρ is a positive constant function as follows. Let $y_1,\dots,y_k$ be linearly independent in $\mathbb{R}^n$ and $O_1,\dots,O_k$ be orthonormal in the space spanned by $y_1,\dots,y_k$. Then there exists a non-singular $k\times k$ matrix α such that

$$\alpha[y]=\begin{pmatrix}O_1\\O_2\\\vdots\\O_k\end{pmatrix}.$$

By (2.7), it follows that

$$\rho(y_1,\dots,y_k)=\rho(O_1,\dots,O_k). \tag{2.9}$$

Also (2.8) implies that ρ is invariant under the action of $\mathrm{SO}(n)$:

$$\rho(y_1, \dots, y_k) = \rho(Uy_1, \dots, Uy_k) \tag{2.10}$$

for $U \in \mathrm{SO}(n)$. This can be seen as follows. Let

$$I = \int_{G_{k,n}} \int_{\theta^k} f(Uz_1, Uz_2, \dots, Uz_k, U\theta)\, d\lambda_\theta(z_1)\, d\lambda_\theta(z_2) \dots d\lambda_\theta(z_k)\, d\sigma(\theta),$$

where $U\theta = \theta(Ux_1, \dots, Ux_k)$ if $\theta = \theta(x_1, \dots, x_k)$. Then by $\mathrm{SO}(n)$ invariance of $d\sigma$ and (2.8), we have

$$\begin{aligned} I &= \int_{G_{k,n}} \int_{(U\theta)^k} f(z_1, z_2, \dots, z_k, U\theta)\, d\lambda_{U\theta}(z_1)\, d\lambda_{U\theta}(z_2) \dots d\lambda_{U\theta}(z_k)\, d\sigma(\theta) \\ &= \int_{G_{k,n}} \int_{\theta^k} f(z_1, z_2, \dots, z_k, \theta)\, d\lambda_\theta(z_1)\, d\lambda_\theta(z_2) \dots d\lambda_\theta(z_k)\, d\sigma(\theta) \\ &= \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} f(y_1, \dots, y_k, \theta(y_1, \dots, y_k))\rho(y_1, \dots, y_k) \\ &\qquad \times |G(y_1, \dots, y_k)|^{(k-n)/2}\, dy_1 \dots dy_k. \end{aligned}$$

On the other hand, using (2.8), we see that

$$\begin{aligned} I &= \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} f(Uy_1, \dots, Uy_k, U\theta(y_1, \dots, y_k))\rho(y_1, \dots, y_k) \\ &\qquad \times |G(y_1, \dots, y_k)|^{(k-n)/2}\, dy_1 \dots dy_k \\ &= \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} f(y_1, \dots, y_k, U\theta(U^{-1}y_1, \dots, U^{-1}y_k))\rho(U^{-1}y_1, \dots, U^{-1}y_k) \\ &\qquad \times \left|G(U^{-1}y_1, \dots, U^{-1}y_k)\right|^{(k-n)/2}\, dy_1 \dots dy_k, \\ &= \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} f(y_1, \dots, y_k, \theta(y_1, \dots, y_k))\rho(U^{-1}y_1, \dots, U^{-1}y_k) \\ &\qquad \times |G(y_1, \dots, y_k)|^{(k-n)/2}\, dy_1 \dots dy_k. \end{aligned}$$

Comparing the two expressions of I above, we can see that (2.10) holds true.

Rearranging $O_1, \dots, O_k$, if necessary, we can find $U \in \mathrm{SO}(n)$ so that $UO_1 = e_1$, $\dots$, $UO_k = e_k$, where $\{e_1, \dots, e_n\}$ denotes the standard basis of $\mathbb{R}^n$. Then by (2.9) and (2.10) we have

$$\rho(y_1, \dots, y_k) = \rho(e_1, \dots, e_k).$$

This completes the proof of Lemma 2.3. □

We state a result analogous to Lemma 2.3, which will be used in proving Theorem 1.2.

Lemma 2.4. *We have*

$$d\lambda_\theta(\omega_1)\dots d\lambda_\theta(\omega_k)\, d\sigma(\theta) = c|\det G(\omega_1,\dots,\omega_k)|^{(k-n)/2}\, d\omega_1\dots d\omega_k,$$

$1 \le k < n$, *which means* (2.11) *below, where* $d\omega_1,\dots,d\omega_k$ *are the Lebesgue surface measure on the unit sphere* S^{n-1} *and* $d\lambda_\theta(\omega_1),\dots,d\lambda_\theta(\omega_k)$ *are as in* (1.6).

Proof. By Lemma 2.3, we have

$$\int_{G_{k,n}}\int_{y_1,\dots,y_k\in\theta} F(y_1,\dots,y_k,\theta)\, d\lambda_\theta(y_1)\dots\, d\lambda_\theta(y_k)\, d\sigma(\theta)$$
$$= c\int_{\mathbb{R}^{nk}} F(x_1,\dots,x_k,\theta(x_1,\dots,x_k))\,|G(x_1,\dots,x_k)|^{(k-n)/2}\, dx_1\dots dx_k.$$

Thus, using polar coordinates, we see that

$$\int_{G_{k,n}}\int_{(0,\infty)^k}\int_{(S_\theta^{k-1})^k} F(r_1\omega_1,\dots,r_k\omega_k,\theta)(r_1\dots r_k)^{k-1}\prod_{j=1}^k dr_j\, d\lambda_\theta(\omega_j)\, d\sigma(\theta)$$
$$= c\int_{(0,\infty)^k}\int_{(S^{n-1})^k} F(r_1\omega_1,\dots,r_k\omega_k,\theta(\omega_1,\dots,\omega_k))|G(\omega_1,\dots,\omega_k)|^{(k-n)/2}$$
$$\times\,(r_1\dots r_k)^{k-1}\, dr_1\dots\, dr_k\, d\omega_1\dots\, d\omega_k.$$

Taking a function of the form

$$H(|x_1|,\dots,|x_k|)F_0(x_1',\dots,x_k',\theta), \quad x_j' = x_j/|x_j| \quad (1\le j\le k)$$

as F and factoring out $\int_{(0,\infty)^k} H(r_1,\dots,r_k)(r_1\dots r_k)^{k-1}\, dr_1\dots dr_k$, we have

$$\int_{G_{k,n}}\int_{(S_\theta^{k-1})^k} F_0(\omega_1,\dots,\omega_k,\theta)\, d\lambda_\theta(\omega_1)\dots\, d\lambda_\theta(\omega_k)\, d\sigma(\theta) \tag{2.11}$$
$$= c\int_{(S^{n-1})^k} F_0(\omega_1,\dots,\omega_k,\theta(\omega_1,\dots,\omega_k))|G(\omega_1,\dots,\omega_k)|^{(k-n)/2}\, d\omega_1\dots\, d\omega_k.$$

This implies what we need. □

3 Proof of Theorem 1.1

In this section we prove Theorem 1.1 assuming Theorem 1.2; also, we assume interpolation arguments needed in the proof, the proof of which will be given in Section 5. Let

$$A_{k,n}(f_0,f_1,\dots,f_n) = \int_{\mathbb{S}}\left(\prod_{j=0}^n T_{k,n}f_j(x,\theta)\right) d\nu(x,\theta).$$

We note that
$$\int_{\mathbb{S}} (T_{k,n}f(x,\theta))^{n+1}\, d\nu(x,\theta) = A_{k,n}(f,f,\dots,f).$$
The following result obviously implies Theorem 1.1.

Proposition 3.1. *Let k be an integer such that $1 \le k < n$. We have*
$$|A_{k,n}(f_0,f_1,\dots,f_n)| \le C \prod_{j=0}^{n} \|f_j\|_{(n+1)/(k+1),n+1}.$$
In fact, we have more general estimates:
$$|A_{k,n}(f_0,f_1,\dots,f_n)| \le C(p_0,\dots,p_n) \prod_{j=0}^{n} \|f_j\|_{p_j,n+1},$$
where $\sum_{j=0}^{n} \frac{1}{p_j} = k+1$ and $1 < p_0,\dots,p_n < \frac{n}{k}$.

Proof. We have
$$\begin{aligned}
&A_{k,n}(f_0,f_1,\dots,f_n)\\
&= c\int f_0(x_0)\dots f_k(x_k)\left(\prod_{j=k+1}^{n}\int_{\theta(x_1-x_0,\dots,x_k-x_0)} f_j(x_0+y)\, d\lambda_{\theta(x_1-x_0,\dots,x_k-x_0)}(y)\right)\\
&\qquad\qquad \times |G(x_1-x_0,\dots,x_k-x_0)|^{\frac{k-n}{2}}\, dx_0\dots dx_k. \qquad (3.1)
\end{aligned}$$
We can see this as follows. Applying Lemma 2.1, we have
$$\begin{aligned}
&A_{k,n}(f_0,f_1,\dots,f_n)\\
&= \int_{\mathbb{S}}\int\cdots\int f_0(x_0)\dots f_k(x_k)\left(\prod_{j=k+1}^{n} T_{k,n}f_j(x,\theta)\right) d\lambda_{P(x,\theta)}(x_0)\dots d\lambda_{P(x,\theta)}(x_k)\, d\nu(x,\theta)\\
&= c\int\cdots\int f_0(x_0)\dots f_k(x_k)\left(\prod_{j=k+1}^{n} T_{k,n}f_j((x_0)_{\theta(x_1-x_0,\dots,x_k-x_0)^{\perp}},\theta(x_1-x_0,\dots,x_k-x_0))\right)\\
&\quad \times |G(x_1-x_0,\dots,x_k-x_0)|^{\frac{k-n}{2}}\, dx_0\dots dx_k.
\end{aligned}$$
Combining this with the observation:
$$\begin{aligned}
&T_{k,n}f_j((x_0)_{\theta(x_1-x_0,\dots,x_k-x_0)^{\perp}},\theta(x_1-x_0,\dots,x_k-x_0))\\
&= \int f_j((x_0)_{\theta(x_1-x_0,\dots,x_k-x_0)^{\perp}}+y)\, d\lambda_{\theta(x_1-x_0,\dots,x_k-x_0)}(y)\\
&= \int f_j(x_0+y)\, d\lambda_{\theta(x_1-x_0,\dots,x_k-x_0)}(y),
\end{aligned}$$

which follows by (1.4), we get (3.1).

Let $\bar{\omega}_1 = \frac{x_1-x_0}{|x_1-x_0|}$. When $k \geq 2$, define

$$\Omega\left(x_0, \frac{x_1-x_0}{|x_1-x_0|}\right) = \int f_2(x_2)\dots f_k(x_k)\left(\prod_{j=k+1}^{n}\int f_j(x_0+y)\,d\lambda_{\theta(x_1-x_0,\dots,x_k-x_0)}(y)\right) \times |G(\bar{\omega}_1, x_2-x_0,\dots,x_k-x_0)|^{\frac{k-n}{2}}\,dx_2\dots dx_k;$$

when $k=1$, let

$$\Omega\left(x_0, \frac{x_1-x_0}{|x_1-x_0|}\right) = \prod_{j=k+1}^{n}\int f_j(x_0+y)\,d\lambda_{\theta(x_1-x_0)}(y).$$

Let

$$K(x_0,x_1) = |x_1-x_0|^{k-n}\Omega\left(x_0, \frac{x_1-x_0}{|x_1-x_0|}\right).$$

Then, it is easy to see that

$$A_{k,n}(f_0,f_1,\dots,f_n) = \iint f_0(x_0)f_1(x_1)K(x_0,x_1)\,dx_0\,dx_1.$$

For $\omega_1 \in S^{n-1}$ and $x_0 \in \mathbb{R}^n$, when $k \geq 2$, define

$$\Omega(x_0,\omega_1) = \int f_2(x_2+x_0)\dots f_k(x_k+x_0)\left(\prod_{j=k+1}^{n}\int_{\theta(\omega_1,x_2,\dots,x_k)} f_j(x_j+x_0)\,d\lambda_{\theta(\omega_1,x_2,\dots,x_k)}(x_j)\right) \times |G(\omega_1,x_2,\dots,x_k)|^{\frac{k-n}{2}}\,dx_2\dots dx_k;$$

when $k=1$, let

$$\Omega(x_0,\omega_1) = \prod_{j=k+1}^{n}\int_{\theta(\omega_1)} f_j(x_j+x_0)\,d\lambda_{\theta(\omega_1)}(x_j).$$

We note that $\Omega(x_0,\omega_1) = |x_1-x_0|^{n-k}K(x_0,x_1)$ when $\omega_1 = \bar{\omega}_1 = (x_1-x_0)/|x_1-x_0|$. We show that

$$\sup_{x_0}\|\Omega(x_0,\cdot)\|_{L^{n/(n-k)}(S^{n-1})} \leq C\prod_{j=2}^{n}\|f_j\|_{n/k,1}. \tag{3.2}$$

Let $\omega_j = x_j/|x_j|$, $r_j = |x_j|$, $\widetilde{f}_j(x_j) = f_j(x_j+x_0)$, $j \geq 2$. Then, since $\theta(\omega_1, x_2, \ldots, x_k) = \theta(\omega_1, \ldots, \omega_k)$, $k \geq 2$, using the polar coordinates, we have

$$\begin{aligned}\int_{\theta(\omega_1,x_2,\ldots,x_k)} f_j(x_j + x_0)\, d\lambda_{\theta(\omega_1,x_2,\ldots,x_k)}(x_j) &= \int_{\theta(\omega_1,\ldots,\omega_k)} \widetilde{f}_j(x_j)\, d\lambda_{\theta(\omega_1,\ldots,\omega_k)}(x_j) \\ &= c\int_0^\infty \int_{S^{k-1}_{\theta(\omega_1,\ldots,\omega_k)}} \widetilde{f}_j(r_j\omega_j) r_j^{k-1}\, d\lambda_{\theta(\omega_1,\ldots,\omega_k)}(\omega_j)\, dr_j.\end{aligned}$$

A formula similar to this holds for $k = 1$. Let

$$F_j(\omega_j) = \int_0^\infty \widetilde{f}_j(r_j\omega_j) r_j^{k-1}\, dr_j$$

for $2 \leq j \leq n$. Then, if $k \geq 2$,

$$\begin{aligned}&\Omega\,(x_0, \omega_1) \\ &\quad= c\int \widetilde{f}_2(r_2\omega_2)\ldots\widetilde{f}_k(r_k\omega_k)\left(\prod_{j=k+1}^n \int_{S^{k-1}_{\theta(\omega_1,\ldots,\omega_k)}} F_j(\omega_j)\, d\lambda_{\theta(\omega_1,\ldots,\omega_k)}(\omega_j)\right) \\ &\qquad\quad \times (r_2\ldots r_k)^{k-1}\,|G(\omega_1,\omega_2,\ldots,\omega_k)|^{\frac{k-n}{2}}\, d\omega_2\ldots d\omega_k\, dr_2\ldots dr_k \\ &\quad= c\int_{(S^{n-1})^{k-1}} F_2(\omega_2)\ldots F_k(\omega_k)\left(\prod_{j=k+1}^n \int_{S^{k-1}_{\theta(\omega_1,\ldots,\omega_k)}} F_j(\omega_j)\, d\lambda_{\theta(\omega_1,\ldots,\omega_k)}(\omega_j)\right) \\ &\qquad\quad \times |G(\omega_1,\omega_2,\ldots,\omega_k)|^{\frac{k-n}{2}}\, d\omega_2\ldots d\omega_k;\end{aligned}$$

if $k = 1$,

$$\Omega\,(x_0, \omega_1) = \prod_{j=k+1}^n \int_{S^{k-1}_{\theta(\omega_1)}} F_j(\omega_j)\, d\lambda_{\theta(\omega_1)}(\omega_j) = \prod_{j=2}^n \left(F_j(\omega_1) + F_j(-\omega_1)\right)/2.$$

By Lemma 2.4 we see that

$$
\begin{aligned}
&\int_{S^{n-1}} F_1(\omega_1)\Omega(x_0,\omega_1)\,d\omega_1 \\
&= c\int_{G_{k,n}}\int_{(S_\theta^{k-1})^k} F_1(\omega_1)F_2(\omega_2)\dots F_k(\omega_k) \\
&\qquad\times\left(\prod_{j=k+1}^{n}\int_{S_\theta^{k-1}} F_j(\omega_j)\,d\lambda_\theta(\omega_j)\right) d\lambda_\theta(\omega_1)\dots d\lambda_\theta(\omega_k)\,d\sigma(\theta) \\
&= c\int_{G_{k,n}}\left(\prod_{j=1}^{n}\int_{S_\theta^{k-1}} F_j(\omega_j)\,d\lambda_\theta(\omega_j)\right) d\sigma(\theta) \\
&= c\int_{G_{k,n}}\left(\prod_{j=1}^{n} S_{k,n}(F_j)(\theta)\right) d\sigma(\theta).
\end{aligned}
$$

Thus by Hölder's inequality we have

$$
\left|\int_{S^{n-1}} F_1(\omega_1)\Omega(x_0,\omega_1)\,d\omega_1\right| \le c\prod_{j=1}^{n}\|S_{k,n}(F_j)\|_n .
$$

So by Theorem 1.2 we have

$$
\begin{aligned}
\left|\int_{S^{n-1}} F_1(\omega_1)\Omega(x_0,\omega_1)\,d\omega_1\right| &\le C\prod_{j=1}^{n}\|F_j\|_{n/k,n} \\
&\le C\prod_{j=1}^{n}\|F_j\|_{n/k} \\
&\le C\|F_1\|_{n/k}\prod_{j=2}^{n}\|\widetilde{f}_j\|_{n/k,1} \\
&= C\|F_1\|_{n/k}\prod_{j=2}^{n}\|f_j\|_{n/k,1}.
\end{aligned}
$$

where the last inequality follows from Lemma 3.2 below. This proves (3.2) by the converse of Hölder's inequality.

Recall that

$$
\begin{aligned}
A_{k,n}(f_0,f_1,\dots,f_n) &= \int f_0(u)f_1(v)K(u,v)\,du\,dv, \\
K(u,v) &= |v-u|^{k-n}\Omega\left(u,\frac{v-u}{|v-u|}\right).
\end{aligned}
$$

By (3.2) we can show that

$$\sup_u \|K(u,\cdot)\|_{q,\infty} \le C \prod_{j=2}^{n} \|f_j\|_{n/k,1}, \quad q^{-1} = 1 - kn^{-1} \tag{3.3}$$

as follows.

$$\begin{aligned}
&\left|\left\{ v \in \mathbb{R}^n : |u-v|^{-n+k} \left|\Omega\left(u, \frac{v-u}{|v-u|}\right)\right| > \lambda \right\}\right| \qquad (3.4)\\
&= \left|\left\{ v \in \mathbb{R}^n : |v|^{-n+k} \left|\Omega\left(u, \frac{v}{|v|}\right)\right| > \lambda \right\}\right| \\
&= \int_{\mathbb{R}^n} \chi_{[1,\infty)} \left(\lambda^{-1}|v|^{-n+k}\left|\Omega\left(u, \frac{v}{|v|}\right)\right|\right) dv \\
&= \int_{S^{n-1}} \int_0^\infty \chi_{[1,\infty)} \left(\lambda^{-1} r^{-n+k} |\Omega(u,\omega)|\right) r^{n-1}\, dr\, d\omega \\
&= \int_{S^{n-1}} \int_0^{(\lambda^{-1}|\Omega(u,\omega)|)^{1/(n-k)}} r^{n-1}\, dr\, d\omega = \int_{S^{n-1}} (\lambda^{-1}|\Omega(u,\omega)|)^{n/(n-k)} \frac{1}{n}\, d\omega \\
&= \frac{1}{n}\lambda^{-q} \int_{S^{n-1}} |\Omega(u,\omega)|^q\, d\omega,
\end{aligned}$$

where $\lambda > 0$. By this and (3.2) we have (3.3).

We note that (3.3) implies

$$\sup_u \left| \int f_1(v) K(u,v)\, dv \right| \le \|f_1\|_{n/k,1} \sup_u \|K(u,\cdot)\|_{q,\infty} \le C\|f_1\|_{n/k,1} \prod_{j=2}^{n} \|f_j\|_{n/k,1}.$$

Therefore

$$\begin{aligned}
|A_{k,n}(f_0, f_1, \dots, f_n)| &= \left| \int f_0(u) f_1(v) K(u,v)\, du\, dv \right| \qquad (3.5)\\
&= \left| \int f_0(u) \left(\int f_1(v) K(u,v)\, dv \right) du \right| \\
&\le \|f_0\|_1 \left\| \int f_1(v) K(u,v)\, dv \right\|_{L^\infty(du)} \\
&\le C \|f_0\|_1 \|f_1\|_{n/k,1} \prod_{j=2}^{n} \|f_j\|_{n/k,1}.
\end{aligned}$$

By interpolation arguments using (3.5), which will be given in Section 5, we have Proposition 3.1. □

Finally, we prove the following lemma used in the proof of (3.2).

Lemma 3.2. *Let* $1 < p < \infty, \alpha = n/p, \omega \in S^{n-1}$. *Define*

$$B_\alpha f(\omega) = \int_0^\infty f(t\omega)t^{\alpha-1}\, dt.$$

Then

$$\|B_\alpha f\|_{L^p(S^{n-1})} \le C\|f\|_{p,1}.$$

Proof. Let $g \in L^{p'}(S^{n-1})$. Then

$$\begin{aligned}\int_{S^{n-1}} g(\omega)B_\alpha f(\omega)\, d\omega &= \int_{\mathbb{R}^n} f(x)|x|^{\alpha-n}g(x')\, dx \\ &\le \|f\|_{p,1}\||x|^{\alpha-n}g(x')\|_{p',\infty} \\ &\le C\|f\|_{p,1}\|g\|_{L^{p'}(S^{n-1})},\end{aligned}$$

where the last inequality follows by arguing similarly to (3.4). This will imply the conclusion. □

4 Proof of Theorem 1.2

In this section we give a proof of Theorem 1.2 by assuming interpolation arguments needed in the proof, whose proof will be given in Section 5.

Let

$$B_{k,n}(f_1, \dots, f_n) = \int_{G_{k,n}} \left(\prod_{j=1}^{n} \int_{S_\theta^{k-1}} f_j(\omega_j)\, d\lambda_\theta(\omega_j) \right) d\sigma(\theta).$$

The following result implies Theorem 1.2.

Proposition 4.1. *Let* $1 \le k < n$, $k \in \mathbb{Z}$. *Then we have*

$$|B_{k,n}(f_1, \dots, f_n)| \le C\|f_1\|_{n/k,n}\|f_2\|_{n/k,n} \cdots \|f_n\|_{n/k,n}.$$

We note that this follows by Hölder's inequality for $k = 1$. This can be described more precisely as follows. Let $\theta \in G_{1,n}$ and $\theta \cap S^{n-1} = \{\eta, -\eta\}$. Then we have

$$S_{1,n}(f)(\theta) = \int_{S_\theta^0} f(\omega)\, d\lambda_\theta(\omega) = (f(\eta) + f(-\eta))/2.$$

Let $\beta : S^{n-1} \to G_{1,n}$ be defined by $\beta(\omega) = \theta(\omega)$. We note that $\beta^{-1}(\{\theta(\omega)\}) = \{\omega, -\omega\}$. The measure $d\sigma$ on $G_{1,n}$ is defined as (see [9, 3.2])

$$\int_{G_{1,n}} F(\theta)\, d\sigma(\theta) = \int_{S^{n-1}} F(\beta(\omega))\, d\omega.$$

Thus

$$B_{1,n}(f_1,\dots,f_n) = \int_{G_{1,n}} \left(\prod_{j=1}^{n} S_{1,n}(f_j)(\theta)\right) d\sigma(\theta) = \int_{S^{n-1}} \left(\prod_{j=1}^{n}(f_j(\omega)+f_j(-\omega))/2\right) d\omega.$$

Therefore, Hölder's inequality implies Proposition 4.1 for $k = 1$.

As in Proposition 3.1, we have more general result, when $k \geq 2$.

Proposition 4.2. *Suppose that* $2 \leq k < n$, $k \in \mathbb{Z}$. *Let*

$$\frac{k-1}{n-1} < \frac{1}{p_j} < 1, \qquad 1 \leq j \leq n, \qquad \sum_{j=1}^{n} \frac{1}{p_j} = k.$$

Then

$$|B_{k,n}(f_1,\dots,f_n)| \leq C \prod_{j=1}^{n} \|f_j\|_{p_j,n}.$$

For $k \geq 2$, we show that

$$|B_{k,n}(f_1,\dots,f_n)| \leq C\|f_1\|_1 \prod_{j=2}^{n} \|f_j\|_{(n-1)/(k-1),1}, \tag{4.1}$$

which implies Proposition 4.2 by interpolation. See Section 5 for the interpolation arguments.

Let $1 < \alpha < n$, $\widetilde{\omega} \in S^{n-2}$ and

$$C_\alpha f(\widetilde{\omega}) = \int_0^\pi f(\cos t, (\sin t)\widetilde{\omega})|\sin t|^{\alpha-2}\, dt.$$

Suppose that a function g satisfies

$$g(\cos t, (\sin t)\widetilde{\omega}) = g(0,\widetilde{\omega}), \quad 0 \leq t \leq \pi, \quad \widetilde{\omega} \in S^{n-2}. \tag{4.2}$$

Then we have

$$\begin{aligned}
&\left|\int_{S^{n-2}} g(0,\widetilde{\omega}) C_\alpha f(\widetilde{\omega})\, d\widetilde{\omega}\right| \\
&= \left|\int_{S^{n-2}} \int_0^\pi g(\cos t, (\sin t)\widetilde{\omega}) f(\cos t, (\sin t)\widetilde{\omega})|\sin t|^{\alpha-2}\, dt\, d\widetilde{\omega}\right| \\
&= c\left|\int_{S^{n-1}} g(\omega) f(\omega)|\omega'|^{\alpha-n}\, d\omega\right| \\
&\leq c\|f\|_{p,1}\|g(\omega)|\omega'|^{\alpha-n}\|_{p',\infty},
\end{aligned}$$

where $\omega = (\omega^{(1)}, \omega')$, $\omega' \in \mathbb{R}^{n-1}$, $p = (n-1)/(\alpha-1)$. We need the inequality

$$\|C_\alpha f\|_{L^{(n-1)/(\alpha-1)}(S^{n-2})} \leq C\|f\|_{L^{(n-1)/(\alpha-1),1}(S^{n-1})}, \tag{4.3}$$

which can be shown by applying the following lemma in the estimates above.

Lemma 4.3. *We have*

$$\|g(\omega)|\omega'|^{\alpha-n}\|_{(n-1)/(n-\alpha),\infty} \leq C\left(\int_{S^{n-2}} |g(0,\widetilde{\omega})|^{(n-1)/(n-\alpha)}\, d\widetilde{\omega}\right)^{(n-\alpha)/(n-1)},$$

where g is assumed to satisfy (4.2) and $1 < \alpha < n$.

Proof. Let $\lambda > 0$. We have

$$\begin{aligned}
&\left|\left\{\omega \in S^{n-1} : |\omega'|^{\alpha-n}\,|g(\omega)| > \lambda\right\}\right| = \int_{S^{n-1}} \chi_{[1,\infty)}\left(\lambda^{-1}|\omega'|^{\alpha-n}|g(\omega)|\right)\, d\omega \\
&= c\int_{S^{n-2}}\int_0^{\pi} \chi_{[1,\infty)}\left(\lambda^{-1}(\sin t)^{\alpha-n}|g(\cos t, (\sin t)\widetilde{\omega})|\right)(\sin t)^{n-2}\, dt\, d\widetilde{\omega} \\
&= c\int_{S^{n-2}}\int_0^{\pi} \chi_{[1,\infty)}\left(\lambda^{-1}(\sin t)^{\alpha-n}|g(0,\widetilde{\omega})|\right)(\sin t)^{n-2}\, dt\, d\widetilde{\omega},
\end{aligned}$$

which is equal to

$$\begin{aligned}
&c\int_{S^{n-2}}\int_0^{\pi} \chi_{\left(0,(\lambda^{-1}|g(0,\widetilde{\omega})|)^{1/(n-\alpha)}\right]}(\sin t)(\sin t)^{n-2}\, dt\, d\widetilde{\omega} \\
&= 2c\int_{S^{n-2}}\int_0^{\pi/2} \chi_{\left(0,(\lambda^{-1}|g(0,\widetilde{\omega})|)^{1/(n-\alpha)}\right]}(\sin t)(\sin t)^{n-2}\, dt\, d\widetilde{\omega} =: I.
\end{aligned}$$

Changing variables, we see that

$$\begin{aligned}
I &= 2c\int_{S^{n-2}}\int_0^{1} \chi_{\left(0,(\lambda^{-1}|g(0,\widetilde{\omega})|)^{1/(n-\alpha)}\right]}(u)u^{n-2}(1-u^2)^{-1/2}\, du\, d\widetilde{\omega} \\
&= 2c\int_{S^{n-2}}\int_0^{\min\left(1,(\lambda^{-1}|g(0,\widetilde{\omega})|)^{1/(n-\alpha)}\right)} u^{n-2}(1-u^2)^{-1/2}\, du\, d\widetilde{\omega} \\
&\leq C\int_{S^{n-2}} (\lambda^{-1}|g(0,\widetilde{\omega})|)^{(n-1)/(n-\alpha)}\, d\widetilde{\omega}.
\end{aligned}$$

This completes the proof. □

We assume Proposition 4.1 for $B_{k-1,m}$, $m > k-1$, and prove (4.1) for $B_{k,n}$, $2 \leq k < n$. This proves Proposition 4.2 for $B_{k,n}$, $2 \leq k < n$ by interpolation. Since Proposition 4.1 is true for $k = 1$, this will give the proofs of Propositions 4.1 and 4.2 by induction.

We note by Lemma 2.4 that

$$\begin{aligned}
&B_{k,n}(f_1,\dots,f_n) \qquad (4.4)\\
&= \int f_1(\omega_1)\dots f_k(\omega_k)\left(\prod_{j=k+1}^{n}\int_{S_\theta^{k-1}} f_j(\omega_j)\,d\lambda_\theta(\omega_j)\right) d\lambda_\theta(\omega_1)\dots\, d\lambda_\theta(\omega_k)\,d\sigma(\theta)\\
&= c\int f_1(\omega_1)\dots f_k(\omega_k)\left(\prod_{j=k+1}^{n}\int_{S^{k-1}_{\theta(\omega_1,\dots,\omega_k)}} f_j(\omega_j)\,d\lambda_{\theta(\omega_1,\dots,\omega_k)}(\omega_j)\right)\\
&\qquad \times |G(\omega_1,\dots,\omega_k)|^{\frac{k-n}{2}}\,d\omega_1\dots\,d\omega_k.
\end{aligned}$$

Let $\omega_1 = e_1$ and write $\omega_\ell = (\cos t_\ell, (\sin t_\ell)\widetilde{\omega}_\ell)$, $0 < t_\ell \le \pi$, $\widetilde{\omega}_\ell \in S^{n-2}$ for $2 \le \ell \le k$. Then, for $j \ge k+1$,

$$\begin{aligned}
&\int_{S^{k-1}_{\theta(\omega_1,\dots,\omega_k)}} f_j(\omega_j)\,d\lambda_{\theta(\omega_1,\dots,\omega_k)}(\omega_j)\\
&\qquad = c\int_{S^{k-2}_{\theta(\tilde\omega_2,\dots,\tilde\omega_k)}}\int_0^\pi f_j(\cos t,(\sin t)\widetilde{\omega}_j)(\sin t)^{k-2}\,dt\,d\lambda_{\theta(\tilde\omega_2,\dots,\tilde\omega_k)}(\widetilde{\omega}_j).
\end{aligned}$$

Let

$$F(f_j)(\widetilde{\omega}_j) = C_k(f_j)(\widetilde{\omega}_j) = \int_0^\pi f_j(\cos t,(\sin t)\tilde\omega_j)(\sin t)^{k-2}\,dt,\quad \tilde\omega_j \in S^{n-2}$$

for $2 \le j \le n$. Define

$$\begin{aligned}
\Omega(\omega_1,\omega_2) &= \Omega(\omega_1,\omega_2)(f_3,\dots,f_n)\\
&= \int f_3(\omega_3)\dots f_k(\omega_k)\\
&\times \prod_{j=k+1}^{n}\int_{S^{k-1}_{\theta(\omega_1,\dots,\omega_k)}} f_j(\omega_j)\,d\lambda_{\theta(\omega_1,\dots,\omega_k)}(\omega_j)|G(\omega_1,\dots,\omega_k)|^{(k-n)/2}\,d\omega_3\dots\,d\omega_k
\end{aligned}$$

for $k \ge 3$; let

$$\begin{aligned}
\Omega(\omega_1,\omega_2) &= \Omega(\omega_1,\omega_2)(f_3,\dots,f_n)\\
&= \prod_{j=3}^{n}\int_{S^1_{\theta(\omega_1,\omega_2)}} f_j(\omega_j)\,d\lambda_{\theta(\omega_1,\omega_2)}(\omega_j)|G(\omega_1,\omega_2)|^{(2-n)/2}
\end{aligned}$$

when $k = 2$.

We need the following result.

Lemma 4.4. *Let* $\omega_1 = e_1 = (1, 0, \dots, 0)$ *and* $\omega_j = (\cos t_j, (\sin t_j)\widetilde{\omega}_j)$, $\widetilde{\omega}_j \in S^{n-2}$, $2 \leq j \leq k$. *Then*

$$G(\omega_1, \omega_2, \dots, \omega_k) = \sin^2 t_2 \dots \sin^2 t_k G(\widetilde{\omega}_2, \dots, \widetilde{\omega}_k).$$

Proof. We have

$$\langle \omega_j, \omega_\ell \rangle = \cos t_j \cos t_\ell + \sin t_j \sin t_\ell \langle \widetilde{\omega}_j, \widetilde{\omega}_\ell \rangle,$$

where $\langle \widetilde{\omega}_j, \widetilde{\omega}_\ell \rangle$ denotes the inner product in $\mathbb{R}^{n-1}$. Thus $G(\omega_1, \omega_2, \dots, \omega_k)$ is equal to

$$\begin{vmatrix} 1 & \cos t_2 & \cos t_3 & \dots & \cos t_k \\ \cos t_2 & \langle \omega_2, \omega_2 \rangle & \langle \omega_2, \omega_3 \rangle & \dots & \langle \omega_2, \omega_k \rangle \\ \cos t_3 & \langle \omega_3, \omega_2 \rangle & \langle \omega_3, \omega_3 \rangle & \dots & \langle \omega_3, \omega_k \rangle \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \cos t_k & \langle \omega_k, \omega_2 \rangle & \langle \omega_k, \omega_3 \rangle & \dots & \langle \omega_k, \omega_k \rangle \end{vmatrix}$$

$$= \begin{vmatrix} 1 & \cos t_2 & \cos t_3 & \dots & \cos t_k \\ 0 & 1 - \cos^2 t_2 & \langle \omega_2, \omega_3 \rangle - \cos t_2 \cos t_3 & \dots & \langle \omega_2, \omega_k \rangle - \cos t_2 \cos t_k \\ 0 & \langle \omega_3, \omega_2 \rangle - \cos t_3 \cos t_2 & 1 - \cos^2 t_3 & \dots & \langle \omega_3, \omega_k \rangle - \cos t_3 \cos t_k \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \langle \omega_k, \omega_2 \rangle - \cos t_k \cos t_2 & \langle \omega_k, \omega_3 \rangle - \cos t_k \cos t_3 & \dots & 1 - \cos^2 t_k \end{vmatrix}$$

$$= \begin{vmatrix} \sin^2 t_2 & \sin t_2 \sin t_3 \langle \widetilde{\omega}_2, \widetilde{\omega}_3 \rangle & \dots & \sin t_2 \sin t_k \langle \widetilde{\omega}_2, \widetilde{\omega}_k \rangle \\ \sin t_3 \sin t_2 \langle \widetilde{\omega}_3, \widetilde{\omega}_2 \rangle & \sin^2 t_3 & \dots & \sin t_3 \sin t_k \langle \widetilde{\omega}_3, \widetilde{\omega}_k \rangle \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \sin t_k \sin t_2 \langle \widetilde{\omega}_k, \widetilde{\omega}_2 \rangle & \sin t_k \sin t_3 \langle \widetilde{\omega}_k, \widetilde{\omega}_3 \rangle & \dots & \sin^2 t_k \end{vmatrix}$$

$$= \sin^2 t_2 \sin^2 t_3 \dots \sin^2 t_k \begin{vmatrix} 1 & \langle \widetilde{\omega}_2, \widetilde{\omega}_3 \rangle & \dots & \langle \widetilde{\omega}_2, \widetilde{\omega}_k \rangle \\ \langle \widetilde{\omega}_3, \widetilde{\omega}_2 \rangle & 1 & \dots & \langle \widetilde{\omega}_3, \widetilde{\omega}_k \rangle \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \langle \widetilde{\omega}_k, \widetilde{\omega}_2 \rangle & \langle \widetilde{\omega}_k, \widetilde{\omega}_3 \rangle & \dots & 1 \end{vmatrix}$$

$$= \sin^2 t_2 \sin^2 t_3 \dots \sin^2 t_k G(\widetilde{\omega}_2, \dots, \widetilde{\omega}_k).$$

This completes the proof. □

Choose $O \in \mathrm{SO}(n)$ so that $O^{-1}\omega_1 = e_1$. Then, by changing variables and applying Lemma 4.4, we have

$$\int f_2(\omega_2)\Omega(\omega_1,\omega_2)\, d\omega_2 = c\int_{(S^{n-2})^{k-1}} F(f_2(O\cdot))(\widetilde{\omega}_2)\dots F(f_k(O\cdot))(\widetilde{\omega}_k)$$
$$\times \prod_{j=k+1}^{n} \int_{S^{k-2}_{\theta(\widetilde{\omega}_2,\dots,\widetilde{\omega}_k)}} F(f_j(O\cdot))(\widetilde{\omega}_j)\, d\lambda_{\theta(\widetilde{\omega}_2,\dots,\widetilde{\omega}_k)}(\widetilde{\omega}_j)$$
$$\times |G(\widetilde{\omega}_2,\dots,\widetilde{\omega}_k)|^{(k-n)/2}\, d\widetilde{\omega}_2\dots d\widetilde{\omega}_k.$$

By Lemma 2.4 this is equal to

$$c\int_{G_{k-1,n-1}}\int_{(S^{k-2}_\theta)^{k-1}} F(f_2(O\cdot))(\widetilde{\omega}_2)\dots F(f_k(O\cdot))(\widetilde{\omega}_k)$$
$$\times \left(\prod_{j=k+1}^{n}\int_{S^{k-2}_\theta} F(f_j(O\cdot))(\widetilde{\omega}_j)\, d\lambda_\theta(\widetilde{\omega}_j)\right) d\lambda_\theta(\widetilde{\omega}_2)\dots d\lambda_\theta(\widetilde{\omega}_k)\, d\sigma(\theta)$$
$$= c\int_{G_{k-1,n-1}}\prod_{j=2}^{n} S_{k-1,n-1}(F(f_j(O\cdot)))(\theta)\, d\sigma(\theta).$$

We have Proposition 4.1 for $B_{k-1,m}$, $m > k-1$, as the induction hypothesis and hence we have the inequality of Theorem 1.2 for $S_{k-1,m}$. Thus by Hölder's inequality and the induction hypothesis, we have

$$\left|\int_{G_{k-1,n-1}}\prod_{j=2}^{n} S_{k-1,n-1}(F(f_j(O\cdot)))(\theta)\, d\sigma(\theta)\right| \le \prod_{j=2}^{n}\|S_{k-1,n-1}(F(f_j(O\cdot)))\|_{n-1}$$
$$\le C\prod_{j=2}^{n}\|F(f_j(O\cdot))\|_{(n-1)/(k-1),n-1}$$
$$\le C\prod_{j=2}^{n}\|F(f_j(O\cdot))\|_{(n-1)/(k-1)}.$$

Applying (4.3) with $\alpha = k$,

$$\|F(f_j(O\cdot))\|_{(n-1)/(k-1)} \le C\|f_j\|_{(n-1)/(k-1),1}.$$

This estimate is uniform in O and hence in ω_1. Thus we have

$$\left|\int f_1(\omega_1)f_2(\omega_2)\Omega(\omega_1,\omega_2)\, d\omega_1\, d\omega_2\right| \le C\|f_1\|_1\prod_{j=2}^{n}\|f_j\|_{(n-1)/(k-1),1}.$$

By (4.4) this proves (4.1) for $B_{k,n}$.

5 Interpolation arguments assumed in proofs of Proposition 3.1 and Proposition 4.2

By (3.5) we can show the following result, which will be used in proving Proposition 3.1.

Lemma 5.1. *Suppose that $1 \leq k < n$. Let $1 < p_0, p_1, \dots, p_n < \frac{n}{k}$, $\sum_{j=0}^{n} \frac{1}{p_j} = k+1$. Then*

$$|A_{k,n}(f_0, f_1, \dots, f_n)| \leq C\|f_0\|_{p_0,1}\|f_1\|_{p_1,\infty} \cdots \|f_n\|_{p_n,\infty}.$$

When $p = \infty$, we consider only the case $q = \infty$ in $L^{p,q}$. We need the following interpolation results.

Lemma 5.2. *Let $1 \leq v, w \leq \infty$ and*

$$\frac{1}{v} + \frac{1}{w} = 1.$$

Let $0 < \theta < 1$. Let $1 \leq s_i, u_i, a_i, b_i \leq \infty$, $i = 0, 1$. We assume that $a_i = 1$ if $s_i = 1$ and that $b_i = 1$ if $u_i = 1$. Define $A_i = L^{s_i,a_i}(\mathbb{R}^n)$, $B_i = L^{u_i,b_i}(\mathbb{R}^n)$, $i = 0, 1$, and $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$. Then $\bar{A}_{\theta,v} = L^{s,v}(\mathbb{R}^n)$, where

$$\frac{1}{s} = \frac{1-\theta}{s_0} + \frac{\theta}{s_1}$$

and we require

$$\frac{1}{v} = \frac{1-\theta}{a_0} + \frac{\theta}{a_1}$$

if $s_0 = s_1$ and also $\bar{B}_{\theta,w} = L^{u,w}(\mathbb{R}^n)$, where

$$\frac{1}{u} = \frac{1-\theta}{u_0} + \frac{\theta}{u_1}$$

and we assume

$$\frac{1}{w} = \frac{1-\theta}{b_0} + \frac{\theta}{b_1}$$

if $u_0 = u_1$. (See [1, Chap. 3] for $\bar{A}_{\theta,v}$.) Suppose that $T : A_i \times B_i \to \mathbb{C}$ be a bilinear operator such that

$$|T(f_1, f_2)| \leq M_i\|f_1\|_{A_i}\|f_2\|_{B_i}, \quad i = 0, 1.$$

We assume that

$$T(f_1, f_2) = A_{k,n}(g_0, g_1, \dots, g_n)$$

with $f_1 = g_j$, $f_2 = g_k$ for some fixed j, k, $j \neq k$; functions except for g_j, g_k are fixed. Also, all functions g_j are initially assumed to be continuous and compactly supported. Then

$$|T(f_1, f_2)| \leq CM_1^{1-\theta}M_2^{\theta}\|f_1\|_{\bar{A}_{\theta,v}}\|f_2\|_{\bar{B}_{\theta,w}} = CM_1^{1-\theta}M_2^{\theta}\|f_1\|_{L^{s,v}}\|f_2\|_{L^{u,w}}.$$

See [10] and [19] for results relevant to Lemma 5.2.

Here we recall $\bar{A}_{[\theta]}$, where $\bar{A} = (A_0, A_1)$ denotes a compatible pair of normed vector spaces (see 2.3 of [1]). Let $\mathcal{F}(\bar{A})$ be the space of all continuous functions f from $S = \{z \in \mathbb{C} : 0 \le \operatorname{Re} z \le 1\}$ to $\Sigma(\bar{A}) = A_0 + A_1$ which are analytic in $S_0 = \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1\}$ and functions $f_j(t) = f(j + it)$, $t \in \mathbb{R}$, are A_j-valued continuous functions on $\mathbb{R}$ with respect to A_j-norm such that $\lim_{|t|\to\infty} \|f_j(t)\|_{A_j} = 0$, $j = 0, 1$.

Let $0 < \theta < 1$ and

$$\bar{A}_{[\theta]} = \{a \in \Sigma(\bar{A}) : a = f(\theta) \quad \text{for some } f \in \mathcal{F}(\bar{A})\}.$$

We define

$$\|a\|_{\bar{A}_{[\theta]}} = \inf\{\|f\|_{\mathcal{F}(\bar{A})} : a = f(\theta), \quad f \in \mathcal{F}(\bar{A})\},$$

where

$$\|f\|_{\mathcal{F}(\bar{A})} = \max\left(\sup_{t\in\mathbb{R}} \|f(it)\|_{A_0}, \sup_{t\in\mathbb{R}} \|f(1+it)\|_{A_1}\right).$$

We also write $\|a\|_{[\theta]}$ for $\|a\|_{\bar{A}_{[\theta]}}$ when $\bar{A}$ is fixed. (See [1, Chap. 4].)

Lemma 5.3. *Let* $1 \le p, r, q, q_0, q_1 \le \infty$*,* $r \le q_0, q_1$*,* $0 < \theta, \eta < 1$*,*

$$\frac{1}{q} = \frac{1-\eta}{q_0} + \frac{\eta}{q_1},$$
$$\frac{1}{p} = \frac{1-\theta}{r}, \qquad r < p.$$

Then

$$(L^{p,q_0}, L^{p,q_1})_{\eta,q} = (L^r, L^\infty)_{\theta,q} = L^{p,q} = (L^{p,q_0}, L^{p,q_1})_{[\eta]}.$$

In the conclusion of the lemma, the equality of spaces means that the spaces are equal with equivalent norms; we also have this rule for description in what follows.

Lemma 5.4. *Let* $0 < \eta < 1$*. Let* $1 \le a_i, b_i \le \infty$*,* $i = 0, 1$ *and* $1 < s, u \le \infty$*. Define* v, w *by*

$$\frac{1}{v} = \frac{1-\eta}{a_0} + \frac{\eta}{a_1}, \qquad \frac{1}{w} = \frac{1-\eta}{b_0} + \frac{\eta}{b_1}.$$

Let $T : L^{s,a_i} \times L^{u,b_i} \to \mathbb{C}$ *be as in Lemma 5.2. Suppose that*

$$|T(f_1, f_2)| \le M_i \|f_1\|_{L^{s,a_i}} \|f_2\|_{L^{u,b_i}}, \quad i = 0, 1.$$

Then

$$|T(f_1, f_2)| \le C M_0^{1-\eta} M_1^{\eta} \|f_1\|_{L^{s,v}} \|f_2\|_{L^{u,w}}.$$

Proof. By Theorem 4.4.1 of [1], we have

$$|T(f_1, f_2)| \le M_0^{1-\eta} M_1^{\eta} \|f_1\|_{(L^{s,a_0}, L^{s,a_1})_{[\eta]}} \|f_2\|_{\left(L^{u,b_0}, L^{u,b_1}\right)_{[\eta]}}.$$

From this and Lemma 5.3, the conclusion follows. □

Remark 5.5. *We have analogues of Lemmas 5.2 and 5.4 for the Lorentz spaces over S^{n-1}, where the operator T is replaced with the one defined by using $B_{k,n}$ in Section 4.*

We shall give proofs of Lemmas 5.2 and 5.3 in Section 6.

Proof of Lemma 5.1. We may assume

$$\frac{k}{n} < \frac{1}{p_0} \le \frac{1}{p_1} \le \cdots \le \frac{1}{p_n} < 1, \qquad \sum_{j=0}^{n} \frac{1}{p_j} = k+1.$$

Define $\theta_0 \in (0,1)$ and u_1 by

$$\frac{1}{p_0} = (1-\theta_0)\frac{k}{n} + \theta_0, \qquad \frac{1}{u_1} = (1-\theta_0)\ + \theta_0 \frac{k}{n}.$$

Then

$$\frac{1}{p_0} + \frac{1}{u_1} = 1 + \frac{k}{n}$$

and

$$\frac{1}{p_n} < \frac{1}{u_1},$$

since if $\frac{1}{p_n} \ge \frac{1}{u_1}$, then

$$\frac{1}{p_0} + \frac{1}{p_n} \ge \frac{1}{p_0} + \frac{1}{u_1} = \frac{n+k}{n},$$

and so

$$\frac{1}{p_0} + \frac{1}{p_n} + \frac{1}{p_1} + \cdots + \frac{1}{p_{n-1}} > \frac{n+k}{n} + \frac{k}{n}(n-1) = k+1,$$

which contradicts our assumption. Next, define $\theta_1 \in (0,1)$ and u_2 by

$$\frac{1}{p_1} = (1-\theta_1)\frac{k}{n} + \theta_1 \frac{1}{u_1}, \qquad \frac{1}{u_2} = (1-\theta_1)\frac{1}{u_1}\ + \theta_1 \frac{k}{n}.$$

Then

$$\frac{1}{p_1} + \frac{1}{u_2} = \frac{k}{n} + \frac{1}{u_1}$$

and if $n \ge 3$,

$$\frac{1}{p_n} < \frac{1}{u_2},$$

since if we would have $\frac{1}{p_n} \geq \frac{1}{u_2}$, then using

$$\frac{1}{p_0}+\frac{1}{p_1}+\frac{1}{u_2}=\frac{1}{p_0}+\frac{1}{u_1}+\frac{k}{n}=\frac{n+2k}{n},$$

we would have

$$\frac{1}{p_0}+\frac{1}{p_1}+\frac{1}{p_n}\geq\frac{1}{p_0}+\frac{1}{p_1}+\frac{1}{u_2}=\frac{n+2k}{n},$$

and hence

$$\frac{1}{p_0}+\frac{1}{p_1}+\frac{1}{p_n}+\frac{1}{p_2}+\cdots+\frac{1}{p_{n-1}}>\frac{n+2k}{n}+\frac{k}{n}(n-2)=k+1,$$

which violates our assumption. Also, if $n=2$, $1/p_2=1/u_2$.

After defining θ_{j-1} and u_j, similarly, we can define $\theta_j\in(0,1)$ and u_{j+1} by

$$\frac{1}{p_j}=(1-\theta_j)\frac{k}{n}+\theta_j\frac{1}{u_j}, \qquad \frac{1}{u_{j+1}}=(1-\theta_j)\frac{1}{u_j}+\theta_j\frac{k}{n}$$

for $j=1,2,\dots,n-1$. Then

$$\frac{1}{p_j}+\frac{1}{u_{j+1}}=\frac{k}{n}+\frac{1}{u_j}.$$

We have

$$\frac{1}{p_n}<\frac{1}{u_{j+1}}, \qquad 0\leq j\leq n-2.$$

To see this we observe

$$\begin{aligned}\frac{1}{p_0}+\frac{1}{p_1}+\cdots+\frac{1}{p_j}+\frac{1}{u_{j+1}}&=\frac{1}{p_0}+\cdots+\frac{1}{p_{j-1}}+\frac{1}{u_j}+\frac{k}{n}\\&=\frac{1}{p_0}+\cdots+\frac{1}{p_{j-2}}+\frac{1}{u_{j-1}}+\frac{2k}{n}\\&=\dots\\&=\frac{1}{p_0}+\frac{1}{u_1}+\frac{jk}{n}.\end{aligned}$$

Thus if $\frac{1}{p_n}\geq\frac{1}{u_{j+1}}$, then

$$\begin{aligned}\frac{1}{p_0}+\frac{1}{p_1}+\cdots+\frac{1}{p_j}+\frac{1}{p_n}&\geq\frac{1}{p_0}+\frac{1}{p_1}+\cdots+\frac{1}{p_j}+\frac{1}{u_{j+1}}\\&=\frac{1}{p_0}+\frac{1}{u_1}+\frac{jk}{n}\\&=\frac{n+(j+1)k}{n}\end{aligned}$$

and hence

$$\begin{aligned} &\frac{1}{p_0} + \frac{1}{p_1} + \cdots + \frac{1}{p_j} + \frac{1}{p_n} + \frac{1}{p_{j+1}} + \cdots + \frac{1}{p_{n-1}} \\ &> \frac{n + (j+1)k}{n} + \frac{k}{n}(n - j - 1) \\ &= k + 1, \end{aligned}$$

which contradicts our assumption. Also, we have

$$\frac{1}{u_n} = \frac{1}{p_n}$$

since

$$\frac{1}{p_0} + \frac{1}{p_1} + \cdots + \frac{1}{p_{n-1}} + \frac{1}{u_n} = \frac{n + nk}{n} = k + 1.$$

We write $A = |A_{k,n}(f_0, f_1, \ldots, f_n)|$. By (3.5) and symmetry, we have

$$A \le C\|f_0\|_{n/k,1}\|f_1\|_1\|f_2\|_{n/k,1}\|f_3\|_{n/k,1} \cdots \|f_n\|_{n/k,1},$$

$$A \le C\|f_0\|_1\|f_1\|_{n/k,1}\|f_2\|_{n/k,1}\|f_3\|_{n/k,1} \cdots \|f_n\|_{n/k,1}.$$

Interpolating by using Lemma 5.2 and applying symmetry,

$$A \le C\|f_0\|_{p_0,\infty}\|f_1\|_{u_1,1}\|f_2\|_{n/k,1}\|f_3\|_{n/k,1} \cdots \|f_n\|_{n/k,1},$$

$$A \le C\|f_0\|_{p_0,\infty}\|f_1\|_{n/k,1}\|f_2\|_{u_1,1}\|f_3\|_{n/k,1} \cdots \|f_n\|_{n/k,1}.$$

Using Lemma 5.2 and symmetry,

$$A \le C\|f_0\|_{p_0,\infty}\|f_1\|_{p_1,\infty}\|f_2\|_{u_2,1}\|f_3\|_{n/k,1} \cdots \|f_n\|_{n/k,1},$$

$$A \le C\|f_0\|_{p_0,\infty}\|f_1\|_{p_1,\infty}\|f_2\|_{n/k,1}\|f_3\|_{u_2,1} \cdots \|f_n\|_{n/k,1}.$$

Continuing this procedure,

$$A \le C\|f_0\|_{p_0,\infty}\|f_1\|_{p_1,\infty} \cdots \|f_{n-2}\|_{p_{n-2},\infty}\|f_{n-1}\|_{u_{n-1},1}\|f_n\|_{n/k,1},$$

$$A \le C\|f_0\|_{p_0,\infty}\|f_1\|_{p_1,\infty} \cdots \|f_{n-2}\|_{p_{n-2},\infty}\|f_{n-1}\|_{n/k,1}\|f_n\|_{u_{n-1},1}.$$

Interpolating between these estimates, we have

$$A \le C\|f_0\|_{p_0,\infty}\|f_1\|_{p_1,\infty} \cdots \|f_{n-2}\|_{p_{n-2},\infty}\|f_{n-1}\|_{p_{n-1},\infty}\|f_n\|_{u_n,1}.$$

This and symmetry complete the proof of Lemma 5.1, since $u_n = p_n$. □

Interpolation arguments deriving Proposition 3.1 from Lemma 5.1. We also write $A = |A_{k,n}(f_0, f_1, \dots, f_n)|$. By Lemma 5.1,

$$A \le C\|f_0\|_{p_0,1}\|f_1\|_{p_1,\infty}\|f_1\|_{p_2,\infty}\cdots\|f_{n-2}\|_{p_{n-2},\infty}\|f_{n-1}\|_{p_{n-1},\infty}\|f_n\|_{p_n,\infty},$$

$$A \le C\|f_0\|_{p_0,\infty}\|f_1\|_{p_1,1}\|f_1\|_{p_2,\infty}\cdots\|f_{n-2}\|_{p_{n-2},\infty}\|f_{n-1}\|_{p_{n-1},\infty}\|f_n\|_{p_n,\infty}.$$

Since

$$\frac{1}{n+1} = \frac{1}{1}\frac{1}{n+1} + \frac{1}{\infty}\frac{n}{n+1}, \quad \frac{n}{n+1} = \frac{1}{\infty}\frac{1}{n+1} + \frac{1}{1}\frac{n}{n+1}, \quad \frac{1}{n+1} + \frac{n}{n+1} = 1,$$

by Lemma 5.2 in the case $s_0 = s_1$ and $u_0 = u_1$ or Lemma 5.4 and symmetry, we have

$$A \le C\|f_0\|_{p_0,n+1}\|f_1\|_{p_1,\frac{n+1}{n}}\|f_2\|_{p_2,\infty}\|f_3\|_{p_3,\infty}\cdots\|f_{n-2}\|_{p_{n-2},\infty}\|f_{n-1}\|_{p_{n-1},\infty}\|f_n\|_{p_n,\infty},$$

$$A \le C\|f_0\|_{p_0,n+1}\|f_1\|_{p_1,\infty}\|f_2\|_{p_2,\frac{n+1}{n}}\|f_3\|_{p_3,\infty}\cdots\|f_{n-2}\|_{p_{n-2},\infty}\|f_{n-1}\|_{p_{n-1},\infty}\|f_n\|_{p_n,\infty}.$$

Since

$$\frac{1}{n+1} = \frac{n}{n+1}\frac{1}{n} + \frac{1}{\infty}\frac{n-1}{n}, \quad \frac{n-1}{n+1} = \frac{1}{\infty}\frac{1}{n} + \frac{n}{n+1}\frac{n-1}{n},$$

by Lemma 5.4 and symmetry, we have

$$A \le C\|f_0\|_{p_0,n+1}\|f_1\|_{p_1,n+1}\|f_2\|_{p_2,\frac{n+1}{n-1}}\|f_3\|_{p_3,\infty}\cdots\|f_{n-1}\|_{p_{n-1},\infty}\|f_n\|_{p_n,\infty},$$

$$A \le C\|f_0\|_{p_0,n+1}\|f_1\|_{p_1,n+1}\|f_2\|_{p_2,\infty}\|f_3\|_{p_3,\frac{n+1}{n-1}}\cdots\|f_{n-1}\|_{p_{n-1},\infty}\|f_n\|_{p_n,\infty}.$$

(Note that $1/(n+1) + (n-1)/(n+1) = n/(n+1) \neq 1$.) In general, since

$$\frac{1}{n+1} = \frac{n-j+1}{n+1}\frac{1}{n-j+1} + \frac{1}{\infty}\frac{n-j}{n-j+1},$$
$$\frac{n-j}{n+1} = \frac{1}{\infty}\frac{1}{n-j+1} + \frac{n-j+1}{n+1}\frac{n-j}{n-j+1},$$
$$\frac{1}{n-j+1} + \frac{n-j}{n-j+1} = 1,$$

interpolating by using Lemma 5.4 between the estimates:

$$A \le C\|f_0\|_{p_0,n+1}\cdots\|f_{j-1}\|_{p_{j-1},n+1} \times \|f_j\|_{p_j,\frac{n+1}{n-j+1}}\|f_{j+1}\|_{p_{j+1},\infty}\|f_{j+2}\|_{p_{j+2},\infty}\cdots\|f_n\|_{p_n,\infty},$$

$$A \le C\|f_0\|_{p_0,n+1}\cdots\|f_{j-1}\|_{p_{j-1},n+1} \times \|f_j\|_{p_j,\infty}\|f_{j+1}\|_{p_{j+1},\frac{n+1}{n-j+1}}\|f_{j+2}\|_{p_{j+2},\infty}\cdots\|f_n\|_{p_n,\infty}$$

for $j \le n-1$, we have

$$A \le C\|f_0\|_{p_0,n+1} \dots \|f_{j-1}\|_{p_{j-1},n+1} \times \|f_j\|_{p_j,n+1} \|f_{j+1}\|_{p_{j+1},\frac{n+1}{n-j}} \|f_{j+2}\|_{p_{j+2},\infty} \dots \|f_n\|_{p_n,\infty}$$

for $j \le n-2$, and for $j = n-1$ this becomes

$$A \le C\|f_0\|_{p_0,n+1} \dots \|f_n\|_{p_n,n+1},$$

which completes the proof of Proposition 3.1 by induction. □

Next, we see that Proposition 4.2 follows from (4.1) by interpolation arguments as above. Recall (4.1):

$$|B_{k,n}(f_1, \dots, f_n)| \le C\|f_1\|_1 \prod_{j=2}^{n} \|f_j\|_{(n-1)/(k-1),1}.$$

Let

$$\frac{k-1}{n-1} < \frac{1}{p_j} < 1, \quad 1 \le j \le n, \qquad \sum_{j=1}^{n} \frac{1}{p_j} = k.$$

Then using Remark 5.5 with (4.1) and arguing as in the proof of Lemma 5.1 from (3.5), by taking $n-1$ and $k-1$ for n and k, respectively, we have

$$|B_{k,n}(f_1, \dots, f_n)| \le C\|f_1\|_{p_0,\infty} \dots \|f_{n-1}\|_{p_{n-1},\infty} \|f_n\|_{p_n,1}. \tag{5.1}$$

Similarly, as Proposition 3.1 follows from Lemma 5.1, by (5.1) and Remark 5.5 it follows that

$$|B_{k,n}(f_1, \dots, f_n)| \le C \prod_{j=1}^{n} \|f_j\|_{p_j,n}.$$

This completes the proof of Proposition 4.2.

6 Appendix

In this section we give proofs of Lemma 5.2 and Lemma 5.3.

6.1 Proof of Lemma 5.2

Let $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$ and let $S(\bar{A}, v, \theta) = S(\bar{A}, (v, v), \theta)$, $S(\bar{B}, w, \theta) = S(\bar{B}, (w, w), \theta)$. Here $S(\bar{A}, (r_0, r_1), \theta)$ is the subspace of $\Sigma(\bar{A})$ consisting of all $a \in \Sigma(\bar{A})$ such that

$$a = \int_0^\infty u(t)\, \frac{dt}{t}, \tag{6.1}$$

with $u(t) \in \Delta(\bar{A}) = A_0 \cap A_1$ for all $t > 0$ (see [1, 2.3]), and the integral is taken in $\Sigma(\bar{A})$; further it is assumed that

$$\max\left(\left\|t^{-\theta}u(t)\right\|_{L^{r_0}(A_0,dt/t)}, \left\|t^{1-\theta}u(t)\right\|_{L^{r_1}(A_1,dt/t)}\right) < \infty,$$

where

$$\|U(t)\|_{L^{r_j}(A_j,dt/t)} = \left(\int_0^\infty \|U(t)\|_{A_j}^{r_j}\,\frac{dt}{t}\right)^{1/r_j},$$

with the usual modification when $r_j = \infty$. (See [1, 3.12]).) The norm is defined as

$$\|a\|_{S(\bar{A},(r_0,r_1),\theta)} = \inf\left\{\max\left(\left\|t^{-\theta}u(t)\right\|_{L^{r_0}(A_0,dt/t)}, \left\|t^{1-\theta}u(t)\right\|_{L^{r_1}(A_1,dt/t)}\right) : u \text{ is as in } (6.1)\right\}.$$

We assume that u has the form

$$u(t)(x) = \sum_{j=1}^{M} F_j(x)m_j(t), \tag{6.2}$$

where $F_j \in A_0 \cap A_1$ and m_j is a bounded measurable function on $(0,\infty)$ supported on a compact subinterval of $(0,\infty)$.

We assume that $a \in \Delta(\bar{A})$ is expressed as in (6.1) with u as in (6.2):

$$a = \int_0^\infty u(t)\,\frac{dt}{t} = \sum_{j=1}^{M} c_jF_j, \quad c_j = \int_0^\infty m_j(t)\,\frac{dt}{t}.$$

We note that $\|u(t)\|_{A_j}$ is measurable in t for $j = 0, 1$ and define

$$\|a\|_{S^*(\bar{A},q,\theta)} = \inf\Big\{\max\left(\left\|t^{-\theta}u(t)\right\|_{L^{q}(A_0,dt/t)}, \left\|t^{1-\theta}u(t)\right\|_{L^{q}(A_1,dt/t)}\right) : a \text{ and } u \text{ are as in } (6.1) \text{ and } u \text{ is as in } (6.2)\Big\}.$$

Let f_1, f_2 be continuous functions on $\mathbb{R}^n$ with compact support. Let u_1, u_2 be as in (6.2) such that

$$f_i = \int_0^\infty u_i(t)\,\frac{dt}{t}, \quad i = 1, 2.$$

We choose an infinitely differentiable non-negative function φ on $\mathbb{R}^n$ with compact support and with integral 1. Put $\varphi_\epsilon(x) = \epsilon^{-n}\varphi(\epsilon^{-1}x)$ with $\epsilon > 0$. Let $f_i^{(\epsilon)} = f_i * \varphi_\epsilon$ and

$$u_i^{(\epsilon)}(t)(x) = \sum_{j=1}^{M} \widetilde{F}_j^{(i)} * \varphi_\epsilon(x)m_j(t),$$

where $\widetilde{F}_j^{(i)} = F_j^{(i)}\chi_K$ with χ_K denoting the characteristic function of a compact set K in $\mathbb{R}^n$, if

$$u_i(t)(x) = \sum_{j=1}^{M} F_j^{(i)}(x) m_j(t).$$

Then if K is sufficiently large according to the supports of f_1, f_2, we see that

$$f_i^{(\epsilon)} = \int_0^\infty u_i^{(\epsilon)}(t)\,\frac{dt}{t}, \quad i = 1, 2.$$

Further, we see that $\widetilde{F}_j^{(i)} * \varphi_\epsilon$ is compactly supported and continuous. So we can define

$$w_\epsilon(t) = \int_0^\infty T\left(u_1^{(\epsilon)}\left(\frac{t}{t_1}\right), u_2^{(\epsilon)}(t_1)\right)\frac{dt_1}{t_1},$$

which satisfies

$$T(f_1^{(\epsilon)}, f_2^{(\epsilon)}) = \int_0^\infty w_\epsilon(t)\,\frac{dt}{t}. \tag{6.3}$$

For $\theta \in (0,1)$, we have

$$\left|\int_0^\infty w_\epsilon(t)\,\frac{dt}{t}\right| \le C_\theta \|t^{-\theta} w_\epsilon(t)\|_\infty^{1-\theta} \|t^{1-\theta} w_\epsilon(t)\|_\infty^{\theta}. \tag{6.4}$$

To see this we evaluate the integral on the left hand side as follows:

$$\begin{aligned}
\left|\int_0^\infty w_\epsilon(t)\,\frac{dt}{t}\right| &\le \left|\int_0^A w_\epsilon(t)\,\frac{dt}{t}\right| + \left|\int_A^\infty w_\epsilon(t)\,\frac{dt}{t}\right| \\
&\le \|t^{-\theta} w_\epsilon(t)\|_\infty \left|\int_0^A t^{\theta-1}\,dt\right| + \|t^{1-\theta} w_\epsilon(t)\|_\infty \left|\int_A^\infty t^{-2+\theta}\,dt\right| \\
&= \frac{1}{\theta} A^\theta \|t^{-\theta} w_\epsilon(t)\|_\infty + \frac{1}{1-\theta} A^{\theta-1} \|t^{1-\theta} w_\epsilon(t)\|_\infty \\
&= 2\theta^{\theta-1}(1-\theta)^{-\theta} \|t^{-\theta} w_\epsilon(t)\|_\infty^{1-\theta} \|t^{1-\theta} w_\epsilon(t)\|_\infty^{\theta},
\end{aligned}$$

where $A = \theta(1-\theta)^{-1} \|t^{-\theta} w_\epsilon(t)\|_\infty^{-1} \|t^{1-\theta} w_\epsilon(t)\|_\infty$. Now, by Hölder's inequality,

$$\begin{aligned}
|t^{-\theta} w_\epsilon(t)| &\le M_0 \int_0^\infty t^{-\theta} \|u_1^{(\epsilon)}(t t_1^{-1})\|_{A_0} \|u_2^{(\epsilon)}(t_1)\|_{B_0}\,\frac{dt_1}{t_1} \\
&\le M_0 \|t^{-\theta} u_1^{(\epsilon)}(t)\|_{L^v(A_0, dt/t)} \|t^{-\theta} u_2^{(\epsilon)}(t)\|_{L^w(B_0, dt/t)}.
\end{aligned}$$

Similarly,

$$|t^{1-\theta} w_\epsilon(t)| \le M_1 \|t^{1-\theta} u_1^{(\epsilon)}(t)\|_{L^v(A_1, dt/t)} \|t^{1-\theta} u_2^{(\epsilon)}(t)\|_{L^w(B_1, dt/t)}.$$

Thus by (6.3) and (6.4) we have

$$\begin{aligned}|T(f_1^{(\epsilon)}, f_2^{(\epsilon)})| &\le C_\theta M_0^{1-\theta} M_1^\theta \|t^{-\theta} u_1^{(\epsilon)}(t)\|_{L^v(A_0,dt/t)}^{1-\theta} \|t^{1-\theta} u_1^{(\epsilon)}(t)\|_{L^v(A_1,dt/t)}^{\theta} \\ &\quad \times \|t^{-\theta} u_2^{(\epsilon)}(t)\|_{L^w(B_0,dt/t)}^{1-\theta} \|t^{1-\theta} u_2^{(\epsilon)}(t)\|_{L^w(B_1,dt/t)}^{\theta} \\ &\le C_\theta M_0^{1-\theta} M_1^\theta \|t^{-\theta} u_1(t)\|_{L^v(A_0,dt/t)}^{1-\theta} \|t^{1-\theta} u_1(t)\|_{L^v(A_1,dt/t)}^{\theta} \\ &\quad \times \|t^{-\theta} u_2(t)\|_{L^w(B_0,dt/t)}^{1-\theta} \|t^{1-\theta} u_2(t)\|_{L^w(B_1,dt/t)}^{\theta},\end{aligned}$$

where the last inequality follows since

$$\|u_1^{(\epsilon)}(t)\|_{A_i} \le C\|u_1(t)\|_{A_i}, \|u_2^{(\epsilon)}(t)\|_{B_i} \le C\|u_2(t)\|_{B_i}, \quad i = 1, 2,$$

with a constant C independent of $\epsilon > 0$. Letting $\epsilon \to 0$, we have

$$\begin{aligned}|T(f_1, f_2)| &\le C_\theta M_0^{1-\theta} M_1^\theta \|t^{-\theta} u_1(t)\|_{L^v(A_0,dt/t)}^{1-\theta} \|t^{1-\theta} u_1(t)\|_{L^v(A_1,dt/t)}^{\theta} \\ &\quad \times \|t^{-\theta} u_2(t)\|_{L^w(B_0,dt/t)}^{1-\theta} \|t^{1-\theta} u_2(t)\|_{L^w(B_1,dt/t)}^{\theta},\end{aligned}$$

and hence, taking u_1, u_2 suitably, we see that

$$|T(f_1, f_2)| \le C_\theta M_0^{1-\theta} M_1^\theta \|f_1\|_{S^*(\bar{A},v,\theta)} \|f_2\|_{S^*(\bar{B},w,\theta)}.$$

This and Lemma 6.1 below imply

$$|T(f_1, f_2)| \le C_\theta M_0^{1-\theta} M_1^\theta \|f_1\|_{\bar{A}_{\theta,v}} \|f_2\|_{\bar{B}_{\theta,w}}.$$

The relation $\bar{A}_{\theta,v} = L^{s,v}$, $\bar{B}_{\theta,w} = L^{u,w}$ claimed in the lemma follows from Theorem 5.3.1 of [1].

Lemma 6.1. *Let $1 \le q \le \infty$ and $0 < \theta < 1$. Then $\|a\|_{S^*(\bar{A},q,\theta)} \sim \|a\|_{\bar{A}_{\theta,q}}$ for $a \in \Delta(\bar{A})$.*

See [1, Theorem 3.12.1] for the case $q < \infty$ and [1, 3.14.12] for the case $q = \infty$. We give a proof of Lemma 6.1 in the following section.

6.1.1 Proof of Lemma 6.1

Let $\underline{S}(\bar{A}, (r_0, r_1), \theta)$ be the subspace of $\Sigma(\bar{A})$ of all $a \in \Sigma(\bar{A})$ such that

$$a = a_0(t) + a_1(t) \quad \text{for every } t > 0, \text{ with}$$
$$\left\|t^{-\theta} a_0(t)\right\|_{L^{r_0}(A_0,dt/t)} < \infty, \quad \left\|t^{1-\theta} a_1(t)\right\|_{L^{r_1}(A_1,dt/t)} < \infty. \tag{6.5}$$

The norm is defined as

$$\|a\|_{\underline{S}(\bar{A},(r_0,r_1),\theta)} = \inf\left\{\left\|t^{-\theta} a_0(t)\right\|_{L^{r_0}(A_0,dt/t)} + \left\|t^{1-\theta} a_1(t)\right\|_{L^{r_1}(A_1,dt/t)} : a_0,\ a_1 \text{ are as in (6.5)}\right\}.$$

Let $\underline{S}(\bar{A}, q, \theta) = \underline{S}(\bar{A}, (q, q), \theta)$.

We assume that

$$a_0(t)(x) = \sum_{j=1}^{M} G_j(x) g_j(t), \tag{6.6}$$
$$a_1(t)(x) = \sum_{j=1}^{M} H_j(x) h_j(t),$$

where $G_j, H_j \in A_0 \cap A_1$ and g_j, h_j are bounded measurable functions supported on $[\epsilon, \infty)$ and $(0, \tau]$, respectively, for some $\epsilon, \tau > 0$.

Let $a \in \Delta(\bar{A})$ and

$$\|a\|_{\underline{S}^*(\bar{A},q,\theta)} = \inf \left\{ \left\| t^{-\theta} a_0(t) \right\|_{L^q(A_0, dt/t)} + \left\| t^{1-\theta} a_1(t) \right\|_{L^q(A_1, dt/t)} : (a_0, a_1) \in \mathcal{G}(a, \bar{A}) \right\},$$

where

$$\mathcal{G}(a, \bar{A}) = \{(a_0, a_1) : a_0,\ a_1 \text{ are as in (6.6) and } a = a_0(s) + a_1(s) \text{ for all } s > 0\}.$$

The conclusion of Lemma 6.1 follows from the next two results.

$$\|a\|_{S^*(\bar{A},q,\theta)} \sim \|a\|_{\underline{S}^*(\bar{A},q,\theta)}, \tag{6.7}$$
$$\|a\|_{\underline{S}^*(\bar{A},q,\theta)} \sim \|a\|_{\bar{A}_{\theta,q}}, \tag{6.8}$$

where $a \in \Delta(\bar{A})$.

We note that $\bar{A}_{\theta,q}$ is as in [1, Chap. 3], although the norms of $S^*(\bar{A}, q, \theta)$ and $\underline{S}^*(\bar{A}, q, \theta)$ are stated in expressions slightly different from those of $S(\bar{A}, q, \theta)$ and $\underline{S}(\bar{A}, q, \theta)$, respectively.

Proof of (6.7). We first prove $\|a\|_{S^*(\bar{A},q,\theta)} \gtrsim \|a\|_{\underline{S}^*(\bar{A},q,\theta)}$. Let $a = \int_0^\infty u(s) ds/s$ with u satisfying (6.2). If we define

$$a_0(t) = \int_0^1 u(ts) \frac{ds}{s}, \quad a_1(t) = \int_1^\infty u(ts) \frac{ds}{s},$$

then $(a_0, a_1) \in \mathcal{G}(a, \bar{A})$ and

$$\left\| t^{-\theta} a_0(t) \right\|_{L^q(A_0, dt/t)} \le \theta^{-1} \left\| t^{-\theta} u(t) \right\|_{L^q(A_0, dt/t)},$$
$$\left\| t^{1-\theta} a_1(t) \right\|_{L^q(A_1, dt/t)} \le (1-\theta)^{-1} \left\| t^{1-\theta} u(t) \right\|_{L^q(A_1, dt/t)}.$$

This implies that $\|a\|_{\underline{S}^*(\bar{A},q,\theta)} \le C_\theta \|a\|_{S^*(\bar{A},q,\theta)}$.

Next, let $(a_0, a_1) \in \mathcal{G}(a, \bar{A})$. Take $\varphi \in C_0^\infty(\mathbb{R})$ such that $\operatorname{supp}(\varphi) \subset [1, 2]$ and $\int_0^\infty \varphi(s)\, ds/s = 1$. Define

$$b_j(t) = \int_0^\infty \varphi(s) a_j(ts^{-1}) \frac{ds}{s} = \int_0^\infty \varphi(ts^{-1}) a_j(s) \frac{ds}{s}.$$

Then $a = b_0(t) + b_1(t)$, $b_0(t) = 0$ if t is small, $b_1(t) = 0$ if t is large. Also we have

$$\|t^{-\theta} t b_0'(t)\|_{L^q(A_0, dt/t)} \le C \|t^{-\theta} a_0(t)\|_{L^q(A_0, dt/t)}, \tag{6.9}$$

$$\|t^{1-\theta} t b_1'(t)\|_{L^q(A_1, dt/t)} \le C \|t^{1-\theta} a_1(t)\|_{L^q(A_1, dt/t)}. \tag{6.10}$$

Let

$$u(t) = t b_0'(t) = -t b_1'(t) \in \Delta(\bar{A}).$$

Then, u is supported in a compact subinterval of $(0, \infty)$ and u is as in (6.2). We note that

$$\int_0^\infty u(t) \frac{dt}{t} = \int_0^1 b_0'(t)\, dt - \int_1^\infty b_1'(t)\, dt = b_0(1) + b_1(1) = a.$$

Thus

$$\|a\|_{S^*(\bar{A},q,\theta)} \le C \max\left(\|t^{-\theta} t b_0'(t)\|_{L^q(A_0, dt/t)}, \|t^{1-\theta} t b_1'(t)\|_{L^q(A_1, dt/t)} \right).$$

By this and (6.9), (6.10) we have $\|a\|_{S^*(\bar{A},q,\theta)} \le C \|a\|_{\underline{S}^*(\bar{A},q,\theta)}$. This completes the proof of (6.7). □

Proof of (6.8). We first consider the case $q < \infty$. We easily see that

$$\|a\|_{\underline{S}^*(\bar{A},q,\theta)} \sim \inf_{(a_0,a_1)\in\mathcal{G}(a,\bar{A})} \left(\left\| t^{-\theta} a_0(t) \right\|_{L^q(A_0, dt/t)}^q + \left\| t^{1-\theta} a_1(t) \right\|_{L^q(A_1, dt/t)}^q \right)^{1/q}.$$

This implies

$$\begin{aligned} \|a\|_{\underline{S}^*(\bar{A},q,\theta)}^q &\gtrsim \int_0^\infty \inf_{(a_0,a_1)\in\mathcal{G}(a,\bar{A})} \left(t^{-q\theta} \|a_0(t)\|_{A_0}^q + t^{q(1-\theta)} \|a_1(t)\|_{A_1}^q \right) \frac{dt}{t} \qquad (6.11) \\ &\gtrsim \int_0^\infty \left(t^{-\theta} K(t, a; A_0, A_1) \right)^q \frac{dt}{t}, \end{aligned}$$

where K is the functional as in [1, Chap. 3]. It follows that

$$\|a\|_{(A_0,A_1)_{\theta,q}} \le C \|a\|_{\underline{S}^*(\bar{A},q,\theta)}. \tag{6.12}$$

Next we prove the reverse inequality. We note the following.

$$\int_0^\infty t^{-q\theta} K(t,a;A_0,A_1)^q \,\frac{dt}{t} \sim \sum_{k=-\infty}^{\infty} 2^{-kq\theta} K(2^k,a;A_0,A_1)^q$$
$$\gtrsim \sum_{k=-\infty}^{\infty} 2^{-kq\theta}\left(\|a_0(2^k)\|_{A_0}^q + 2^{kq}\|a_1(2^k)\|_{A_1}^q\right) - \epsilon$$
$$\gtrsim \sum_k \int_0^\infty t^{-q\theta}\left(\left\|a_0(2^k)\chi_{[2^k,2^{k+1})}(t)\right\|_{A_0}^q + t^q\left\|a_1(2^k)\chi_{[2^k,2^{k+1})}(t)\right\|_{A_1}^q\right)\frac{dt}{t} - c\epsilon$$
$$\gtrsim \int_0^\infty t^{-q\theta}\left(\left\|\sum_k a_0(2^k)\chi_{[2^k,2^{k+1})}(t)\right\|_{A_0}^q + t^q\left\|\sum_k a_1(2^k)\chi_{[2^k,2^{k+1})}(t)\right\|_{A_1}^q\right)\frac{dt}{t} - c\epsilon,$$

for any $\epsilon > 0$ with some $a_0(2^k) \in A_0$, $a_1(2^k) \in A_1$ such that $a = a_0(2^k) + a_1(2^k)$. We note that $a_0(2^k), a_1(2^k) \in A_0 \cap A_1$ since $a \in A_0 \cap A_1$. Thus we have

$$\int_0^\infty t^{-q\theta} K(t,a;A_0,A_1)^q \,\frac{dt}{t}$$
$$\gtrsim \int_0^\infty \left(\left\|t^{-\theta}\sum_k a_0(2^k)\chi_{[2^k,2^{k+1})}(t)\right\|_{A_0}^q + \left\|t^{1-\theta}\sum_k a_1(2^k)\chi_{[2^k,2^{k+1})}(t)\right\|_{A_1}^q\right)\frac{dt}{t} - c\epsilon$$
$$= \lim_{M\to\infty}\int_0^\infty \left(\left\|t^{-\theta}\sum_{k=-M}^{M} a_0(2^k)\chi_{[2^k,2^{k+1})}(t)\right\|_{A_0}^q \right.$$
$$\left. + \left\|t^{1-\theta}\sum_{k=-M}^{M} a_1(2^k)\chi_{[2^k,2^{k+1})}(t)\right\|_{A_1}^q\right)\frac{dt}{t} - c\epsilon$$
$$=: I.$$

To comply with the definition of the norm of $\underline{S}^*(\bar{A},q,\theta)$ in (6.8) (see (6.6)), this may be modified as follows.

$$I = \lim_{M\to\infty}\int_0^\infty \left(\left\|t^{-\theta}\left(\sum_{k=-M}^{M} a_0(2^k)\chi_{[2^k,2^{k+1})}(t) + a\chi_{[2^{M+1},\infty)}(t)\right)\right\|_{A_0}^q \right.$$
$$\left. + \left\|t^{1-\theta}\left(\sum_{k=-M}^{M} a_1(2^k)\chi_{[2^k,2^{k+1})}(t) + a\chi_{(0,2^{-M})}(t)\right)\right\|_{A_1}^q\right)\frac{dt}{t} - c\epsilon.$$
$$\gtrsim \inf_{(a_0,a_1)\in\mathcal{G}(a,\bar{A})}\left(\left\|t^{-\theta}a_0(t)\right\|_{L^q(A_0,dt/t)}^q + \left\|t^{1-\theta}a_1(t)\right\|_{L^q(A_1,dt/t)}^q\right) - c\epsilon$$
$$\gtrsim \|a\|_{\underline{S}^*(\bar{A},q,\theta)}^q - c\epsilon,$$

which implies that

$$\int_0^\infty t^{-q\theta} K(t,a;A_0,A_1)^q \frac{dt}{t} \gtrsim \|a\|^q_{\underline{S}^*(\bar{A},q,\theta)}. \tag{6.13}$$

Combining this with (6.12), we have

$$\|a\|_{\underline{S}^*(\bar{A},q,\theta)} \sim \|a\|_{(A_0,A_1)_{\theta,q}}.$$

The case $q=\infty$ can be handled as follows by an obvious modification of the arguments for the case $q<\infty$. As in (6.11), we have

$$\begin{aligned}
\|a\|_{\underline{S}^*(\bar{A},\infty,\theta)} &= \inf_{(a_0,a_1)\in \mathcal{G}(a,\bar{A})} \left(\sup_{t>0} t^{-\theta} \|a_0(t)\|_{A_0} + \sup_{t>0} t^{1-\theta} \|a_1(t)\|_{A_1} \right) \\
&\geq \inf_{(a_0,a_1)\in \mathcal{G}(a,\bar{A})} \sup_{t>0} t^{-\theta} \left(\|a_0(t)\|_{A_0} + t\|a_1(t)\|_{A_1} \right) \\
&\geq \sup_{t>0} t^{-\theta} K(t,a;A_0,A_1) \\
&= \|a\|_{(A_0,A_1)_{\theta,\infty}}.
\end{aligned} \tag{6.14}$$

Next, we prove the reverse inequality. As in the proof of (6.13) we have

$$\begin{aligned}
&\sup_{t>0} t^{-\theta} K(t,a;A_0,A_1) \sim \sup_{k\in\mathbb{Z}} 2^{-k\theta} K(2^k,a;A_0,A_1) \\
&\gtrsim \sup_{k\in\mathbb{Z}} 2^{-k\theta} \left(\|a_0(2^k)\|_{A_0} + 2^k \|a_1(2^k)\|_{A_1} \right) - \epsilon \\
&\gtrsim \sup_{t>0} t^{-\theta} \left(\left\| \sum_{k\in\mathbb{Z}} a_0(2^k)\chi_{[2^k,2^{k+1})}(t) \right\|_{A_0} + t \left\| \sum_{k\in\mathbb{Z}} a_1(2^k)\chi_{[2^k,2^{k+1})}(t) \right\|_{A_1} \right) - c\epsilon.
\end{aligned}$$

We modify this as follows to comply with the definition of the norm of $\underline{S}^*(\bar{A},q,\theta)$

in (6.8).

$$\begin{aligned}
&\sup_{t>0} t^{-\theta} K(t, a; A_0, A_1) \\
&\gtrsim \lim_{M\to\infty} \sup_{t>0} \Bigg(\left\| t^{-\theta} \left(\sum_{k=-M}^{M} a_0(2^k)\chi_{[2^k,2^{k+1})}(t) + a\chi_{[2^{M+1},\infty)}(t) \right) \right\|_{A_0} \\
&\quad + \left\| t^{1-\theta} \left(\sum_{k=-M}^{M} a_1(2^k)\chi_{[2^k,2^{k+1})}(t) + a\chi_{(0,2^{-M})}(t) \right) \right\|_{A_1} \Bigg) - c\epsilon \\
&\geq \frac{1}{2} \lim_{M\to\infty} \Bigg(\sup_{t>0} \left\| t^{-\theta} \left(\sum_{k=-M}^{M} a_0(2^k)\chi_{[2^k,2^{k+1})}(t) + a\chi_{[2^{M+1},\infty)}(t) \right) \right\|_{A_0} \\
&\quad + \sup_{t>0} \left\| t^{1-\theta} \left(\sum_{k=-M}^{M} a_1(2^k)\chi_{[2^k,2^{k+1})}(t) + a\chi_{(0,2^{-M})}(t) \right) \right\|_{A_1} \Bigg) - c\epsilon \\
&\gtrsim \inf_{(a_0,a_1)\in\mathcal{G}(a,\bar{A})} \left(\left\| t^{-\theta} a_0(t) \right\|_{L^\infty(A_0,dt/t)} + \left\| t^{1-\theta} a_1(t) \right\|_{L^\infty(A_1,dt/t)} \right) - c\epsilon \\
&= \|a\|_{\underline{S}^*(\bar{A},\infty,\theta)} - c\epsilon
\end{aligned}$$

for all $\epsilon > 0$, where the first inequality holds since $0 < \theta < 1$. Thus it follows that

$$\left\| t^{-\theta} K(t, a; A_0, A_1) \right\|_{L^\infty(dt/t)} \gtrsim \|a\|_{\underline{S}^*(\bar{A},\infty,\theta)}. \tag{6.15}$$

By (6.14) and (6.15) we have

$$\|a\|_{\underline{S}^*(\bar{A},\infty,\theta)} \sim \|a\|_{(A_0,A_1)_{\theta,\infty}}.$$

This proves (6.8) for $q = \infty$. □

This completes the proof of Lemma 6.1. We refer to [10] and [1, 3.12] for relevant results.

6.2 Proof of Lemma 5.3

To prove Lemma 5.3, we need the next two results.

Lemma 6.2. *Let $f \in L^p + L^\infty, 1 \leq p < \infty$. Then*

$$K(t, f; L^p, L^\infty) \sim \left(\int_0^{t^p} (f^*(s))^p \, ds \right)^{1/p}, \tag{6.16}$$

where f^* denotes the nonincreasing rearrangement of f.

Lemma 6.3. *Suppose that $1 \le p \le q \le \infty$ and $p < \infty$. Let $1/r = (1-\theta)/p$, $0 < \theta < 1$. Then*

$$(L^p, L^\infty)_{\theta,q} = L^{r,q}, \quad \textit{with equivalent norms.} \tag{6.17}$$

Lemma 6.2 is found in Theorem 5.2.1 of [1]. Lemma 6.3 is also almost in the same theorem (the case $p = q < \infty$ is not stated there). Here we give a proof for completeness.

Proof of Lemma 6.3. We first consider the case $q < \infty$. Since

$$\|f\|_{(L^p,L^\infty)_{\theta,q}} = \left(\int_0^\infty \left(t^{-\theta} K(t, f; L^p, L^\infty)\right)^q \frac{dt}{t}\right)^{1/q}, \quad p \le q < \infty,$$

by Lemma 6.2 we have

$$\begin{aligned}\|f\|_{(L^p,L^\infty)_{\theta,q}} &\sim \left(\int_0^\infty \left(t^{-\theta p} \int_0^{t^p} (f^*(s))^p \, ds\right)^{q/p} \frac{dt}{t}\right)^{1/q} \\ &= \left(\int_0^\infty \left(t^{-\theta p + p} \int_0^1 (f^*(st^p))^p \, ds\right)^{q/p} \frac{dt}{t}\right)^{1/q}.\end{aligned}$$

Since $q/p \ge 1$, Minkowski's inequality with changing variables implies that

$$\begin{aligned}\|f\|_{(L^p,L^\infty)_{\theta,q}} &\le C \left(\int_0^1 \left(\int_0^\infty t^{(-\theta+1)q} (f^*(st^p))^q \frac{dt}{t}\right)^{p/q} ds\right)^{1/p} \\ &= C \left(\int_0^1 p^{-p/q} s^{-p/r} \left(\int_0^\infty t^{q/r} (f^*(t))^q \frac{dt}{t}\right)^{p/q} ds\right)^{1/p} \\ &= Cp^{-1/q} \left(\int_0^1 s^{\theta-1} \, ds\right)^{1/p} \left(\int_0^\infty t^{q/r} (f^*(t))^q \frac{dt}{t}\right)^{1/q} \\ &= Cp^{-1/q} \theta^{-1/p} \|f\|_{r,q},\end{aligned}$$

where we have used the relation $1/r = (1-\theta)/p$.

To prove the reverse inequality, we simply apply that $0 \le f^*(t^p) \le f^*(s)$, if $0 < s \le t^p$, to get

$$\begin{aligned}\|f\|_{(L^p,L^\infty)_{\theta,q}} &\ge C \left(\int_0^\infty \left(t^{-\theta p} \int_0^{t^p} (f^*(s))^p \, ds\right)^{q/p} \frac{dt}{t}\right)^{1/q} \\ &\ge C \left(\int_0^\infty \left(t^{-\theta p} t^p (f^*(t^p))^p\right)^{q/p} \frac{dt}{t}\right)^{1/q} \\ &\ge C \|f\|_{r,q}.\end{aligned}$$

Next we consider the case $q=\infty$. By Lemma 6.2 we see that

$$\begin{aligned}\|f\|_{(L^p,L^\infty)_{\theta,\infty}} &= \sup_{t>0} t^{-\theta} K(t,f;L^p,L^\infty)\\ &\le C \sup_{t>0} t^{-\theta}\left(\int_0^{t^p} (f^*(s))^p\,ds\right)^{1/p}\\ &\le C \sup_{s>0} s^{1/r} f^*(s) \sup_{t>0} t^{-\theta}\left(\int_0^{t^p} s^{-p/r}\,ds\right)^{1/p} =: I.\end{aligned}$$

Using the relation $1/r=(1-\theta)/p$, we have

$$\begin{aligned}I &= C \sup_{s>0} s^{1/r} f^*(s) \sup_{t>0} t^{-\theta}\left(\int_0^{t^p} s^{\theta-1}\,ds\right)^{1/p}\\ &= C\theta^{-1/p} \sup_{s>0} s^{1/r} f^*(s)\\ &= C\theta^{-1/p}\|f\|_{r,\infty}.\end{aligned}$$

Also,

$$\begin{aligned}\|f\|_{(L^p,L^\infty)_{\theta,\infty}} &\ge C \sup_{t>0} t^{-\theta}\left(\int_0^{t^p} (f^*(s))^p\,ds\right)^{1/p}\\ &\ge C \sup_{t>0} t^{-\theta} t f^*(t^p)\\ &= C \sup_{t>0} t^{1/r} f^*(t)\\ &= C\|f\|_{r,\infty}.\end{aligned}$$

This completes the proof of the case $q=\infty$. □

Proof of Lemma 5.3. By Lemma 6.3, since $r \le q_j$, we have

$$L^{p,q_j} = (L^r, L^\infty)_{\theta,q_j}, \quad j=0,1. \tag{6.18}$$

By Theorem 5.3.1 of [1] and Lemma 6.3, we have

$$(L^{p,q_0}, L^{p,q_1})_{\eta,q} = L^{p,q} = (L^r, L^\infty)_{\theta,q}. \tag{6.19}$$

Also, by Theorem 4.7.2 of [1],

$$((L^r, L^\infty)_{\theta,q_0}, (L^r, L^\infty)_{\theta,q_1})_{[\eta]} = (L^r, L^\infty)_{\theta,q}. \tag{6.20}$$

Although Theorem 4.7.2 is stated with $\theta_0, \theta_1 \in (0,1)$ such that $\theta_0 \ne \theta_1$, we easily see that we may assume that $\theta_0=\theta_1=\theta$ to get the result above. Combining (6.18), (6.19) and (6.20), we see that

$$(L^{p,q_0}, L^{p,q_1})_{\eta,q} = (L^{p,q_0}, L^{p,q_1})_{[\eta]} = L^{p,q}.$$

This completes the proof of Lemma 5.3. □

We give a proof of (6.20) in Section 6.2.1 below.

6.2.1 Proof of (6.20)

Let

$$\ell^q_\eta = \left\{ (\alpha_k)_{k\in\mathbb{Z}} : \alpha_k \in \mathbb{C}, k \in \mathbb{Z}, \|(\alpha_k)\|_{\ell^q_\eta} = \left(\sum_{k=-\infty}^{\infty} \left(2^{-\eta k}|\alpha_k|\right)^q \right)^{1/q} < \infty \right\},$$

where $1 \le q \le \infty$, $0 < \eta < 1$, with the usual modification when $q = \infty$. Let $1 \le q, q_0, q_1 \le \infty$, $1 \le r < \infty$, $0 < \theta, \eta < 1$,

$$\frac{1}{q} = \frac{1-\eta}{q_0} + \frac{\eta}{q_1}.$$

Let $X_j = \bar{A}_{\theta,q_j}$, $j = 0, 1$, with $\bar{A} = (A_0, A_1)$, $A_0 = L^r$, $A_1 = L^\infty$. Let $\bar{X} = (X_0, X_1)$.

Let $a \in \bar{A}_{\theta,q}$, $a \ne 0$. We show that $a \in \bar{X}_{[\eta]}$. By Lemma 3.2.3 and Theorem 3.3.1 of [1], there exists a sequence $(u_\nu)_{\nu\in\mathbb{Z}}$ in $\Delta(\bar{A})$ such that $a = \sum_\nu u_\nu$ in $\Sigma(\bar{A})$ and

$$\left\| \left(J(2^\nu, u_\nu; \bar{A})\right)_\nu \right\|_{\ell^q_\theta} \le C\|a\|_{\bar{A}_{\theta,q}}.$$

We first assume that $q < \infty$. For $\delta > 0$ and $z \in \mathbb{C}$ with $0 \le \operatorname{Re} z \le 1$, let

$$f_\nu(z) = \left(2^{-\theta\nu} J(2^\nu, u_\nu; \bar{A})\|a\|^{-1}_{\bar{A}_{\theta,q}}\right)^{q(1/q_1 - 1/q_0)(z-\eta)} u_\nu,$$

where $1/q_i = 0$ if $q_i = \infty$, and

$$f(z) = \exp(\delta(z-\eta)^2) \sum_\nu f_\nu(z).$$

Then

$$\left|\exp(-\delta(it-\eta)^2)\right| \|f(it)\|_{\bar{A}_{\theta,q_0}} \le C \left\| \left(J(2^\nu, f_\nu(it); \bar{A})\right)_\nu \right\|_{\ell^{q_0}_\theta} \le C\|a\|_{\bar{A}_{\theta,q}}.$$

Lemma 3.2.3 of [1] implies the first inequality. The second inequality can be seen as follows. First note that $-\eta q(1/q_1 - 1/q_0) = q/q_0 - 1$. Thus, if $q_0 < \infty$,

$$\begin{aligned}
&\left\| \left(J(2^\nu, f_\nu(it); \bar{A})\right)_\nu \right\|^{q_0}_{\ell^{q_0}_\theta} \\
&= \sum_\nu 2^{-\theta\nu q_0} 2^{\theta\nu(q_0-q)} J(2^\nu, u_\nu; \bar{A})^{q-q_o} \|a\|^{q_0-q}_{\bar{A}_{\theta,q}} J(2^\nu, u_\nu; \bar{A})^{q_o} \\
&= \sum_\nu 2^{-\theta q\nu} J(2^\nu, u_\nu; \bar{A})^q \|a\|^{q_0-q}_{\bar{A}_{\theta,q}} \\
&\le C\|a\|^{q_0}_{\bar{A}_{\theta,q}}.
\end{aligned}$$

If $q_0 = \infty$, then $-\eta q(1/q_1 - 1/q_0) = -1$ and

$$\left\| \left(J(2^\nu, f_\nu(it); \bar{A})\right)_\nu \right\|_{\ell^{q_0}_\theta} = \sup_\nu 2^{-\theta\nu} 2^{\theta\nu} J(2^\nu, u_\nu; \bar{A})^{-1} \|a\|_{\bar{A}_{\theta,q}} J(2^\nu, u_\nu; \bar{A}) = \|a\|_{\bar{A}_{\theta,q}}.$$

Also, we have

$$\left|\exp(-\delta(1+it-\eta)^2)\right| \|f(1+it)\|_{\bar{A}_{\theta,q_1}} \leq C \left\| \left(J(2^\nu, f_\nu(1+it); \bar{A})\right)_\nu \right\|_{\ell_\theta^{q_1}} \leq C\|a\|_{\bar{A}_{\theta,q}},$$

since $(1-\eta)q(1/q_1 - 1/q_0) = q/q_1 - 1$ and

(a) if $q_1 < \infty$,

$$\begin{aligned}
&\left\| \left(J(2^\nu, f_\nu(1+it); \bar{A})\right)_\nu \right\|_{\ell_\theta^{q_1}}^{q_1} \\
&= \sum_\nu 2^{-\theta\nu q_1} 2^{\theta\nu(q_1-q)} J(2^\nu, u_\nu; \bar{A})^{q-q_1} \|a\|_{\bar{A}_{\theta,q}}^{q_1-q} J(2^\nu, u_\nu; \bar{A})^{q_1} \\
&= \sum_\nu 2^{-\theta q\nu} J(2^\nu, u_\nu; \bar{A})^q \|a\|_{\bar{A}_{\theta,q}}^{q_1-q} \\
&\leq C\|a\|_{\bar{A}_{\theta,q}}^{q_1};
\end{aligned}$$

(b) if $q_1 = \infty$, then $(1-\eta)q(1/q_1 - 1/q_0) = -1$ and

$$\begin{aligned}
\left\| \left(J(2^\nu, f_\nu(1+it); \bar{A})\right)_\nu \right\|_{\ell_\theta^{q_1}} &= \sup_\nu 2^{-\theta\nu} 2^{\theta\nu} J(2^\nu, u_\nu; \bar{A})^{-1} \|a\|_{\bar{A}_{\theta,q}} J(2^\nu, u_\nu; \bar{A}) \\
&= \|a\|_{\bar{A}_{\theta,q}}.
\end{aligned}$$

Thus $f \in \mathcal{F}(\bar{A}_{\theta,q_0}, \bar{A}_{\theta,q_1})$ and $f(\eta) = a$; also $\|a\|_{\bar{X}_{[\eta]}} \leq C\|a\|_{\bar{A}_{\theta,q}}$ by letting $\delta \to 0$.

If $q = \infty$, then $q_0 = q_1 = \infty$. Let $f_\nu(z) = u_\nu$. Then we can argue similarly (more directly) to the case $q < \infty$ to have the same conclusion.

Next, we show that $a \in \bar{A}_{\theta,q}$ assuming $a \in \bar{X}_{[\eta]}$. Take $f \in \mathcal{F}(\bar{A}_{\theta,q_0}, \bar{A}_{\theta,q_1})$ such that $f(\eta) = a$. Define

$$g(z) = 2^{(z-\eta)\gamma} f(z).$$

Then $g \in \mathcal{F}(\bar{A}_{\theta,q_0}, \bar{A}_{\theta,q_1})$ and $g(\eta) = a$. Thus a can be expressed by the Poisson integral in $\Sigma(\bar{A})$ (see [1, Chap. 4]):

$$a = \int_{-\infty}^{\infty} P_0(\eta, t) g(it)\, dt + \int_{-\infty}^{\infty} P_1(\eta, t) g(1+it)\, dt.$$

This proves

$$\begin{aligned}
&K(2^\nu, a; \bar{A}) \qquad (6.21)\\
&\leq 2^{-\eta\gamma} \int P_0(\eta, t) K(2^\nu, f(it); \bar{A})\, dt + 2^{(1-\eta)\gamma} \int P_1(\eta, t) K(2^\nu, f(1+it); \bar{A})\, dt \\
&= 2 \left(\int P_0(\eta, t) K(2^\nu, f(it); \bar{A})\, dt \right)^{1-\eta} \left(\int P_1(\eta, t) K(2^\nu, f(1+it); \bar{A})\, dt \right)^{\eta},
\end{aligned}$$

if γ is chosen to satisfy

$$2^{\gamma} = \left(\int P_0(\eta, t) K(2^{\nu}, f(it); \bar{A})\, dt \right) \left(\int P_1(\eta, t) K(2^{\nu}, f(1+it); \bar{A})\, dt \right)^{-1}.$$

Putting

$$C_{\nu} = 2^{-\nu\theta} \int P_0(\eta, t) K(2^{\nu}, f(it); \bar{A})\, dt,$$
$$D_{\nu} = 2^{-\nu\theta} \int P_1(\eta, t) K(2^{\nu}, f(1+it); \bar{A})\, dt,$$

by Lemma 3.1.3 of [1], (6.21), Hölder's inequality and Minkowski's inequality, we have

$$\begin{aligned}
\|a\|_{\bar{A}_{\theta,q}} &\le C \left\| \left(K(2^{\nu}, a; \bar{A}) \right)_{\nu} \right\|_{\ell^q_\theta} \\
&\le C \left(\sum_{\nu} C_{\nu}^{q_0} \right)^{(1-\eta)/q_0} \left(\sum_{\nu} D_{\nu}^{q_1} \right)^{\eta/q_1} \\
&\le C \left(\int P_0(\eta, t) \left\| \left(K(2^{\nu}, f(it); \bar{A}) \right)_{\nu} \right\|_{\ell^{q_0}_\theta} dt \right)^{1-\eta} \\
&\quad \times \left(\int P_1(\eta, t) \left\| \left(K(2^{\nu}, f(1+it); \bar{A}) \right)_{\nu} \right\|_{\ell^{q_1}_\theta} dt \right)^{\eta} \\
&\le C \left(\int P_0(\eta, t) \|f(it)\|_{\bar{A}_{\theta,q_0}} dt \right)^{1-\eta} \left(\int P_1(\eta, t) \|f(1+it)\|_{\bar{A}_{\theta,q_1}} dt \right)^{\eta} \\
&\le C \|f\|_{\mathcal{F}(\bar{A}_{\theta,q_0}, \bar{A}_{\theta,q_1})}.
\end{aligned}$$

This implies $\|a\|_{\bar{A}_{\theta,q}} \le C\|a\|_{\bar{X}_{[\eta]}}$, completing the proof of (6.20).

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