博士論文

非圧縮粘性流れに対する圧力境界条件を含む
圧力 Poisson 法と射影法の数学解析

Mathematical analysis of pressure Poisson methods and projection methods involving pressure boundary conditions for incompressible viscous flows

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Abstract

We consider pressure Poisson equations for stationary incompressible Stokes problems and time-dependent incompressible Navier–Stokes problems. The pressure Poisson equation is an elliptic partial differential equation of second order and is used in various numerical methods for incompressible viscous flows. Since there are many mechanisms that generate flow by creating pressure differences, a Dirichlet boundary condition is often set for the pressure Poisson equation. However, in general, the pressure of the boundary condition for the numerical methods differs from the exact pressure solution of the original problem.

This thesis aims to provide a mathematical analysis for the pressure Poisson equation from the viewpoint of additional boundary conditions. We establish error estimates in suitable norms between solutions to a stationary Stokes problem and the corresponding pressure Poisson problem in terms of the additional boundary condition. As boundary conditions for the Stokes problem, we use a traction boundary condition and a Dirichlet-type pressure boundary condition with no tangent flow. In addition, for a pseudo-compressibility problem that interpolates the Stokes and pressure Poisson problems, we also give error estimates in suitable norms between the solutions to the pseudo-compressibility problem, the pressure Poisson problem, and the Stokes problem for several additional boundary condition cases.

Moreover, we propose a new additional boundary condition for the projection method of the time-dependent Navier–Stokes problem with a Dirichlet-type pressure boundary condition and no tangent flow. We demonstrate stability for the scheme and establish error estimates for the velocity and pressure under suitable norms. A numerical experiment verifies the theoretical convergence results. Furthermore, the existence of a weak solution to the original Navier–Stokes problem is proven by using stability.

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Chapter 1

Introduction

1.1 Motivations

There are various numerical methods for incompressible viscous flows described by the Navier–Stokes equation and the continuity equation (incompressible condition). For example, if we use the explicit Euler method for time, we can calculate the velocity of the next step from the Navier–Stokes equation. However, we cannot calculate the pressure from the incompressible condition. Hence, some strategy is needed for numerical methods for incompressible viscous flows. Thus, there are many numerical methods that solve a pressure Poisson equation instead of the incompressibility condition, such as the marker and cell (MAC) method [42, 64, 85], simplified MAC (SMAC) method [4, 5, 82], projection method [21, 79, 38], moving-particle semi-implicit (MPS) method [57, 58, 74], and incompressible smoothed particle hydrodynamics (ISPH) method [49, 61, 75]. Numerical methods are effective for the finite difference method (FDM) [30, 78, 81], finite element method (FEM) [17, 32, 87], finite volume method (FVM) [29, 65, 83], and particle methods [46, 58, 62] and separately solve the velocity and pressure, which are different from other numerical schemes such as the Hood–Taylor finite element method [17, 32], the pressure stabilization method [26, 33, 48], and the pseudo-compressibility method [20, 70, 86].

Additional boundary conditions are required for numerical methods using the pressure Poisson equation since the pressure Poisson equation is an elliptic partial differential equation of second order. In general, the Neumann boundary condition is imposed on the pressure when the Dirichlet boundary condition is applied for the flow velocity. In [35, 73], the authors show that the pressure Poisson and Navier–Stokes equations with appropriate boundary conditions are equivalent to the original incompressible Navier–Stokes problem. The error estimate for the projection method is first given in [76, 71]. In particular, in [71], the proof is based on the fact that the projection method can be interpreted as a pseudo-compressibility method, such as the pressure stabilization method (cf. [31, 70]). Many boundary conditions have been proposed to improve the order of the error [50, 52, 59].

On the other hand, there are many mechanisms that generate flow by creating pressure differences, such as water distribution systems, hydraulic systems, and blood circulation. Hence, there is a motivation to impose pressure as a boundary condition in engineering. Since one can naturally set Dirichlet boundary conditions on the pressure Poisson equation, the traction boundary condition or do-nothing boundary condition is often used in numerical methods using the pressure Poisson equation [37, 38, 47]. However, the pressure of the boundary condition for the numerical method differs from the exact pressure solution of the original problem. Although there are many good numerical results and error estimates for the time step size and mesh size, there is no error estimate in terms of the additional boundary condition.

We also note well-posed boundary conditions, including pressure, for the Stokes and Navier–Stokes equations introduced in [7, 23]. There are many mathematical analyses and discretization approaches; for example, the unique existence of the weak solution in the steady case [24, 9, 10], extended to L^p-theory [3], and unsteady nonlinear case [63, 56]. On the other hand, applying this type of boundary condition to the pressure Poisson method is limited; for example, the projection method [39, 40]. However, the authors assume an outflow condition for stability and the pressure to be stationary for implementation reasons.

1.2 Synopsis of the thesis

This thesis aims to provide a mathematical analysis for the pressure Poisson equation from the viewpoint of additional boundary conditions. We establish error estimates in suitable norms between solutions to a stationary Stokes problem and the corresponding pressure Poisson problem in terms of the additional boundary condition. In addition, for a pseudo-compressibility problem that interpolates the Stokes and pressure Poisson problems, we also give error estimates in suitable norms between the solutions to the pseudo-compressibility problem, the pressure Poisson problem, and the Stokes problem for several additional boundary condition cases. Moreover, we propose a new additional boundary condition for the projection method for the time-dependent Navier–Stokes problem with a Dirichlet-type pressure boundary condition and no tangent flow.

In Chapter 2, we prepare notations, function spaces, and their properties used in the thesis. Chapters 3, 4, and 5 contain all our mathematical results.

In Chapter 3, we introduce a stationary Stokes problem and the corresponding pressure Poisson equation. We establish error estimates between solutions to the Stokes problem and the pressure Poisson problem in terms of the additional boundary condition. As boundary conditions for the Stokes problem, we use a traction boundary condition and the boundary condition including pressure introduced in [7, 23].

In Chapter 4, we introduce an ε-Stokes problem as a pseudo-compressibility problem that interpolates the Stokes and pressure Poisson problems. The boundary conditions for the velocity are full Dirichlet boundary conditions, and those for the pressure are Dirichlet, mixed, and Neumann boundary conditions. We give error estimates in suitable norms between the solutions to the ε-Stokes problem, the pressure Poisson problem, and the Stokes problem. Several numerical examples show that several such error estimates are optimal in ε. In addition, we show that the solution to the ε-Stokes problem has a nice asymptotic structure.

In Chapter 5, we propose a new additional boundary condition for the projection method with a Dirichlet-type total pressure boundary condition and no tangent flow. We demonstrate stability for the scheme and establish error estimates for the velocity and pressure under suitable norms. A numerical experiment verifies the theoretical convergence results. Furthermore, the existence of a weak solution to the original Navier–Stokes problem is proven by using stability.

In Appendix A, we define the standard Lipschitz boundary and prove the Nečas inequality and its corollary. These results have already been proven, but we provide careful proof.

Chapter 2

Preliminaries

In this chapter, we provide notations, function spaces, and their properties used in the thesis.

2.1 Notations and function spaces

We provide a list of notation and function spaces used in the thesis.

General used symbols. If not stated otherwise, the symbols listed below have the following meaning:

$\mathbb{N}$	:	the set of positive integers.
$\mathbb{Z}_{\geq 0}$	:	the set of non-negative integers.
T	:	a positive real number representing the final time.
$\mathbb{R}^m$	:	m-dimensional Euclidean space for $m \in \mathbb{N}$.
Ω	:	a bounded Lipschitz domain in $\mathbb{R}^d$ for $d = 2$ or $d = 3$, corresponding to the spatial region, where the equation is solved (see Definition A.1.1 for the precise definition of Lipschitz domain).
$\overline{\Omega}$	:	the closure of domain Ω.
Γ	:	the boundary $\partial\Omega$ of domain Ω.
n	:	the outer normal vector for the boundary Γ.
$\lvert \cdot \rvert$	:	the Euclidean norm on $\mathbb{R}^d$.
$a \cdot b$	:	the inner product on $\mathbb{R}^d$.
$a \times b$	:	the cross product on $\mathbb{R}^2$ or $\mathbb{R}^3$. For three-dimensional vectors $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3)$, the cross product of a and b is defined by $$a \times b := (a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1)\,.$$ For two-dimensional vectors $a = (a_1, a_2)$ and $b = (b_1, b_2)$, the cross product of a and b is defined by $$a \times b := a_1 b_2 - a_2 b_1.$$

$A : B$: the componentwise inner product of two matrices $A = (a_{ij}), B = (b_{ij}) \in \mathbb{R}^{d\times d}$ and defined by

$$A : B \coloneqq \sum_{i,j=1}^{d} a_{ij} b_{ij}.$$

p : unknown real-valued function.

u : unknown real d-dimensional vector-valued function.

∇p : the gradient of $p = p(x)$ with respect to spatial variables and defined by

$$(\nabla p)(x) \coloneqq \left(\frac{\partial p}{\partial x_1}(x), \frac{\partial p}{\partial x_2}(x), \dots, \frac{\partial p}{\partial x_d}(x) \right)^{\mathrm{T}} = \begin{pmatrix} \dfrac{\partial p}{\partial x_1}(x) \\ \dfrac{\partial p}{\partial x_2}(x) \\ \vdots \\ \dfrac{\partial p}{\partial x_d}(x) \end{pmatrix},$$

where $^{\mathrm{T}}$ is the transpose of the vector or matrix.

∇u : the gradient of $u = (u_1(x), u_2(x), \dots, u_d(x))$ with respect to spatial variables and the square matrix of order d defined by

$$(\nabla u)(x) \coloneqq \begin{pmatrix} \dfrac{\partial u_1}{\partial x_1}(x) & \dfrac{\partial u_2}{\partial x_1}(x) & \cdots & \dfrac{\partial u_d}{\partial x_1}(x) \\ \dfrac{\partial u_1}{\partial x_2}(x) & \dfrac{\partial u_2}{\partial x_2}(x) & \cdots & \dfrac{\partial u_d}{\partial x_2}(x) \\ \vdots & \vdots & & \vdots \\ \dfrac{\partial u_1}{\partial x_d}(x) & \dfrac{\partial u_2}{\partial x_d}(x) & \cdots & \dfrac{\partial u_d}{\partial x_d}(x) \end{pmatrix}.$$

$S(u)$: the matrix defined by

$$S(u) \coloneqq \nabla u + (\nabla u)^{\mathrm{T}},$$

which is twice the symmetric part of the matrix ∇u.

$\operatorname{div} u$: the divergence of $u = (u_1(x), u_2(x), \dots, u_d(x))$ with respect to spatial variables and defined by

$$(\operatorname{div} u)(x) \coloneqq \sum_{i=1}^{d} \frac{\partial u_i}{\partial x_i}(x).$$

The divergence of $\sigma = (\sigma_{ij}(x)) \in \mathbb{R}^{d\times d}$ with respect to spatial variables and defined by

$$(\operatorname{div} \sigma)(x) \coloneqq \left(\sum_{j=1}^{d} \frac{\partial \sigma_{1j}}{\partial x_j}(x), \sum_{j=1}^{d} \frac{\partial \sigma_{2j}}{\partial x_j}(x), \cdots, \sum_{j=1}^{d} \frac{\partial \sigma_{dj}}{\partial x_j}(x) \right)^{\mathrm{T}}.$$

$\nabla\times u$: the rotation of u with respect to spatial variables. For a three-dimensional vector-valued function $u = (u_1(x), u_2(x), u_3(x))$ on three-dimensional real Euclidean space, $\nabla\times u$ is defined by

$$\nabla\times u := \left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}, \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}, \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right).$$

For a two-dimensional vector-valued function $u = (u_1(x), u_2(x))$ on two-dimensional real Euclidean space, by regarding u as the three-dimensional vector-valued function $(x_1, x_2, x_3) \mapsto (u_1(x_1, x_2), u_2(x_1, x_2), 0)$, we apply the above definition and pick up meaningful parts, i.e.,

$$\nabla\times u := \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2},$$
$$\nabla\times(\nabla\times u) := \left(\frac{\partial}{\partial x_2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right), -\frac{\partial}{\partial x_1}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right)\right).$$

Δp : the Laplacian of p with respect to spatial variables, which is defined by

$$\Delta p := \sum_{i=1}^{d} \frac{\partial^2 p}{\partial x_i^2},$$

Δu : the Laplacian of u with respect to spatial variables, which is defined by

$$\Delta u := \left(\sum_{i=1}^{d} \frac{\partial^2 u_1}{\partial x_i^2}, \sum_{i=1}^{d} \frac{\partial^2 u_2}{\partial x_i^2}, \ldots, \sum_{i=1}^{d} \frac{\partial^2 u_d}{\partial x_i^2}\right)^{\mathrm{T}},$$

For the boundary Γ, we assume that there exist two relatively open subsets Γ_1, Γ_2 of Γ satisfying

$$|\Gamma \setminus (\Gamma_1 \cup \Gamma_2)| = 0, \quad |\Gamma_1|, |\Gamma_2| > 0, \quad \Gamma_1 \cap \Gamma_2 = \emptyset, \quad \mathring{\overline{\Gamma_1}} = \Gamma_1, \quad \mathring{\overline{\Gamma_2}} = \Gamma_2, \qquad (2.1.1)$$

where $\overline{A}$ is the closure of $A \subset \Gamma$ with respect to Γ, $\mathring{A}$ is the interior of A with respect to Γ, and $|A|$ is the $(d-1)$-dimensional Hausdorff measure of A.

Function spaces. The following function spaces and their corresponding norms and inner products are used in the thesis. For a Banach space E, we denote its dual space E^* and the dual product between E^* and E by $\langle \cdot, \cdot \rangle_E$. Let Ω be an open domain in $\mathbb{R}^d$, and let $k \in \mathbb{N}, p \geq 1, T > 0$.

$C^k(\overline{\Omega})$: the set of all functions $f : \Omega \to \mathbb{R}$ such that all derivatives up to and including k-th order exist and are continuous and can be extended to the closure $\overline{\Omega}$.

$C^\infty(\overline{\Omega})$: the set of all infinitely differentiable functions $f : \Omega \to \mathbb{R}$ that can be continuously extended with all their derivatives to the closure $\overline{\Omega}$, i.e., $C^\infty(\overline{\Omega}) := \cap_{k=0}^{\infty} C^k(\overline{\Omega})$.

$C_0^\infty(\Omega)$: the set of all functions $f \in C^\infty(\overline{\Omega})$ such that there exists a compact set $K \subset \Omega$ such that f vanishes on $\Omega \setminus K$.

$\mathscr{D}'(\Omega)$: the space of all distributions on Ω.

$L^p(\Omega)$: the set of all functions $f : \Omega \to \mathbb{R}$ such that the p-th power of the absolute value is Lebesgue integrable and with the identification of functions that only differ on null sets. The Lebesgue space $L^p(\Omega)$ is a Banach space with respect to the norm

$$\|f\|_{L^p(\Omega)} := \left(\int_\Omega |f(x)|^p dx \right)^{1/p}.$$

In particular, $L^2(\Omega)$ is also a Hilbert space with respect to the inner product

$$(f, g) := \int_\Omega f(x)g(x)dx.$$

$L^\infty(\Omega)$: the set of all functions $f : \Omega \to \mathbb{R}$ such that the norm

$$\|f\|_{L^\infty(\Omega)} := \operatorname*{ess\,sup}_{x\in\Omega} |f(x)|$$

exists and is finite and with the identification of functions that only differ on null sets, which is a Banach space with respect to $\|\cdot\|_{L^\infty(\Omega)}$.

$L^2(\Omega)/\mathbb{R}$: the space of all functions $f \in L^2(\Omega)$ with the average of 0, i.e.,

$$L^2(\Omega)/\mathbb{R} := \left\{ [f] := f - \frac{1}{|\Omega|} \int_\Omega f dx \;\middle|\; f \in L^2(\Omega) \right\},$$

where $|\Omega| := \int_\Omega 1 dx$.

$W^{k,p}(\Omega)$: the set of all functions $f \in L^p(\Omega)$ such that for each multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbb{Z}^d_{\geq 0}$ with $|\alpha|_1 := \sum_{i=1}^d |\alpha_i| \leq k$, $\partial^\alpha f := (\partial/\partial x_1)^{\alpha_1} (\partial/\partial x_2)^{\alpha_2} \cdots (\partial/\partial x_d)^{\alpha_d} f$ exists in the distribution sense and belongs to $L^p(\Omega)$. The Sobolev space $W^{k,p}(\Omega)$ is a Banach space with respect to the norm

$$\|f\|_{W^{k,p}(\Omega)} := \left(\sum_{\alpha\in\mathbb{Z}^d_{\geq 0},\, |\alpha|_1 \leq k} \|\partial^\alpha f\|^p_{L^p(\Omega)} \right)^{1/p}.$$

$H^k(\Omega)$: the Sobolev space $W^{k,2}(\Omega)$, which is a Hilbert space with respect to the inner product

$$(f, g)_{H^k(\Omega)} := \sum_{\alpha\in\mathbb{Z}^d_{\geq 0},\, |\alpha|_1 \leq k} (\partial^\alpha f, \partial^\alpha g).$$

$H_{\mathrm{div}}(\Omega)$: the function space defined by

$$H_{\mathrm{div}}(\Omega) := \{v \in L^2(\Omega)^d \mid \operatorname{div} v \in L^2(\Omega)\},$$

with the norm

$$\|v\|_{H_{\mathrm{div}}(\Omega)} := \sqrt{\|v\|_{L^2(\Omega)^d} + \|\operatorname{div} v\|_{L^2(\Omega)}}.$$

$H^1(\Omega)/\mathbb{R}$: the space of all functions $f \in H^1(\Omega)$ with the average of 0, i.e., $H^1(\Omega)/\mathbb{R} = H^1(\Omega) \cap (L^2(\Omega)/\mathbb{R})$.

$H_0^1(\Omega)$: the closure of $C_0^\infty(\Omega)$ in $H^1(\Omega)$.

For $m = 1$ or $m = d$, the dual space $H^{-1}(\Omega)^m = (H_0^1(\Omega)^m)^*$ is equipped with the norm

$$\|f\|_{H^{-1}(\Omega)^m} := \sup_{\varphi \in S_m} \langle f, \varphi \rangle$$

for $f \in H^{-1}(\Omega)^m$, where

$$S_m := \{\varphi \in H_0^1(\Omega)^m \mid \|\nabla\varphi\|_{L^2(\Omega)^{n\times m}} = 1\}.$$

For $q \in L^2(\Omega)$, we set

$$\langle \nabla q, \varphi \rangle_{H_0^1(\Omega)^d} := -\int_\Omega q \operatorname{div} \varphi dx \qquad \text{for all } \varphi \in H_0^1(\Omega)^d.$$

We remark that $q \in H^1(\Omega)$ satisfies that for all $\varphi \in H_0^1(\Omega)^d$,

$$\langle \nabla q, \varphi \rangle_{H_0^1(\Omega)^d} = (\nabla q, \varphi).$$

We also use the following Lebesgue and Sobolev spaces defined on the open subset $\tilde{\Gamma} \in \{\Gamma, \Gamma_1, \Gamma_2\}$ of the boundary Γ.

$L^2(\tilde{\Gamma})$: the Lebesgue space $L^2(\tilde{\Gamma})$ with the inner product

$$(\eta, \zeta)_{L^2(\tilde{\Gamma})} := \int_{\tilde{\Gamma}} \eta(s)\zeta(s)ds,$$

where ds denotes the surface measure of Γ.

$H^{1/2}(\tilde{\Gamma})$: the set of all functions $\lambda \in L^2(\tilde{\Gamma})$ such that the norm

$$\|\eta\|_{H^{1/2}(\tilde{\Gamma})} := \left(\|\eta\|_{L^2(\tilde{\Gamma})}^2 + \int_{\tilde{\Gamma}} \int_{\tilde{\Gamma}} \frac{|\eta(s_1) - \eta(s_2)|^2}{|s_1 - s_2|^d} ds_1 ds_2 \right)^{1/2}$$

exists and is finite, which is a Banach space with respect to $\|\cdot\|_{H^{1/2}(\tilde{\Gamma})}$.

$H^{-1/2}(\Gamma)$: the dual space $(H^{1/2}(\Gamma))^*$.

We remark that $\eta^* \in L^2(\Gamma)$ can be identified with an element of the dual space $H^{-1/2}(\Gamma)$ by

$$\langle \eta^*, \eta \rangle_{H^{1/2}(\Gamma)} = \int_\Gamma \eta^* \eta \, ds \qquad \text{for all } \eta \in H^{1/2}(\Gamma).$$

Let $\gamma_0 : H^1(\Omega) \to H^{1/2}(\Gamma)$ be the standard trace operator. The trace operator γ_0 is a surjective continuous linear operator and $\mathrm{Ker}(\gamma_0) = H_0^1(\Omega)$ [32, Theorem 1.5]. For $i = 1$ or $i = 2$, the composition of the trace operator γ_0 and the restriction $H^{1/2}(\Gamma) \to H^{1/2}(\Gamma_i)$ is a continuous map from $H^1(\Omega)$ to $H^{1/2}(\Gamma_i)$. By using the map $H^1(\Omega) \ni \psi \mapsto \psi|_{\Gamma_i} \in H^{1/2}(\Gamma_i)$, we define

$$H^1_{\Gamma_i}(\Omega) \coloneqq \{\psi \in H^1(\Omega) \mid \psi|_{\Gamma_i} = 0\}.$$

and then, there exists a constant $c > 0$ such that for all $p \in H^1(\Omega)$,

$$\|p\|_{H^1(\Omega)/H^1_{\Gamma_i}(\Omega)} \leq c\|p|_{\Gamma_i}\|_{H^{1/2}(\Gamma_i)},$$

where $\|p\|_{H^1(\Omega)/H^1_{\Gamma_i}(\Omega)} \coloneqq \inf_{\psi \in H^1_{\Gamma_i}(\Omega)} \|p+\psi\|_{H^1(\Omega)}$. We simply write ψ instead of $\psi|_{\Gamma_i}$ when there is no ambiguity. Since n is a unit vector, the maps $H^1(\Omega)^d \ni u \mapsto u \cdot n \in L^2(\Gamma)$ and $H^1(\Omega)^d \ni u \mapsto u \times n \in L^2(\Gamma)^{d(d-1)/2}$ are linear and continuous. We also set

$$H \coloneqq \{\varphi \in H^1(\Omega) \mid \varphi = 0 \text{ on } L^2(\Gamma_1), \varphi \times n = 0 \text{ on } L^2(\Gamma_2)^{d(d-1)/2}\}.$$

For the open subsets Γ_1, Γ_2 of the boundary Γ, we define the following subspaces of $H^{1/2}(\Gamma)$:

$$H^{1/2}_{\gamma_0}(\Gamma_1) \coloneqq \gamma_0(H^1_{\Gamma_2}(\Omega)),$$
$$H^{1/2}_{\gamma_0}(\Gamma_2) \coloneqq \gamma_0(H^1_{\Gamma_1}(\Omega)).$$

For $i = 1$ or $i = 2$, the space $H^{1/2}_{\gamma_0}(\Gamma_i)$ is continuously embedding in $H^{1/2}(\Gamma_i)$ and equivalent to the Lions–Magenes space $H^{1/2}_{00}(\Gamma_i)$, for example, if Γ_i is a line segment with $d = 2$ [36, 72]. We remark that $\eta^* \in L^2(\Gamma_i)$ can be identified with an element of the dual space $(H^{1/2}_{\gamma_0}(\Gamma_i))^*$ by

$$\langle \eta^*, \eta \rangle_{H^{1/2}_{\gamma_0}(\Gamma_i)} = \int_{\Gamma_i} \eta^* \eta \, ds \qquad \text{for all } \eta \in H^{1/2}_{\gamma_0}(\Gamma_i).$$

2.2 Preliminary results

We use the following lemmas and theorems. These results can be found in [14, 16, 32, 80].

Proposition 2.2.1 (Cauchy–Schwarz)**.** *Let Ω be an open subset of $\mathbb{R}^d$. For $f, g \in L^2(\Omega)$, we have the following inequality:*

$$\left| \int_\Omega f g dx \right| \leq \|f\|_{L^2(\Omega)} \|g\|_{L^2(\Omega)}.$$

Proposition 2.2.2 (Young)**.** *Let $n \geq 2$, and $a_1, a_2, \ldots, a_n$ be non-negative real numbers. Additionally, let $p_1, p_2, \ldots, p_n$ be positive real numbers such that*

$$\sum_{i=1}^n \frac{1}{p_i} = 1.$$

We then have

$$\prod_{i=1}^n a_i \leq \sum_{i=1}^n \frac{a_i^{p_i}}{p_i}.$$

In particular, when $n = 2$, we have for all $c > 0$,

$$a_1 a_2 \leq \frac{c a_1^{p_1}}{p_1} + \frac{a_2^{p_2}}{c p_2}.$$

Proposition 2.2.3 (Hölder). *Let Ω be an open subset of $\mathbb{R}^d$ and let $p_1, p_2, \ldots, p_n$ be positive real numbers (possibly infinite). Additionally, let $1 \leq r \leq \infty$ such that*

$$\sum_{i=1}^{n} \frac{1}{p_i} = \frac{1}{r}.$$

For all functions $f_1, f_2, \ldots, f_n$ with $f_i \in L^{p_i}(\Omega)$, the product $\prod_{i=1}^n f_i$ belongs to $L^r(\Omega)$ and we have

$$\left\| \prod_{i=1}^{n} f_i \right\|_{L^r(\Omega)} \leq \prod_{i=1}^{n} \|f_i\|_{L^{p_i}(\Omega)}.$$

Theorem 2.2.4 (Gauss divergence formula). *There exists a continuous linear operator $\gamma_n : H_{\mathrm{div}}(\Omega) \to H^{-1/2}(\Gamma)$ such that $\gamma_n(v) = v \cdot n$ for all $v \in C^\infty(\overline{\Omega})$. Moreover, it holds that for all $v \in H_{\mathrm{div}}(\Omega), \psi \in H^1(\Omega)$,*

$$\int_\Omega v \cdot \nabla q dx + \int_\Omega (\operatorname{div} v) q dx = \langle \gamma_n(v), q \rangle_{H^{1/2}(\Gamma)}.$$

In particular, it holds that for all $v \in H^1(\Omega)^d, \psi \in H^1(\Omega)$,

$$\int_\Omega v \cdot \nabla q dx + \int_\Omega (\operatorname{div} v) q dx = \int_\Gamma (v \cdot n) q ds.$$

Theorem 2.2.5 (Sobolev embeddings). *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$.*

(i) If $1 \leq p < d$, then we have

$$W^{1,p}(\Omega) \subset L^q(\Omega)$$

with continuous embedding for all $1 \leq q \leq p^*$, where p^* is the critical exponent associated with p:

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{d}.$$

(ii) If $p = d$, then we have

$$W^{1,p}(\Omega) \subset L^q(\Omega)$$

with continuous embedding for all $1 \leq q < \infty$.

(iii) If $d < p \leq \infty$, then we have

$$W^{1,p}(\Omega) \subset C^0(\overline{\Omega})$$

with continuous embedding.

Theorem 2.2.6 (Rellich–Kondrachov). *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$.*

(i) If $1 \leq p < d$, then the embedding $W^{1,p}(\Omega) \subset L^q(\Omega)$ is compact for all $1 \leq q \leq p^*(= dp/(d-p))$.

(ii) If $p = d$, then the embedding $W^{1,p}(\Omega) \subset L^q(\Omega)$ is compact for all $1 \leq q < \infty$.

(iii) If $d < p \leq \infty$, then the embedding $W^{1,p}(\Omega) \subset C^0(\overline{\Omega})$ is compact.

Theorem 2.2.7 (Generalized Poincaré inequality)**.** *Let Ω be a bounded Lipschitz domain and let $\tilde{\Gamma}$ be a subset of the boundary Γ with nonzero surface measure (e.g., $\tilde{\Gamma} \in \{\Gamma_1, \Gamma_2, \Gamma\}$). There exists a constant $c > 0$ such that for all $q \in H^1_{\tilde{\Gamma}}(\Omega)$,*

$$\|q\|_{L^2(\Omega)} \leq c\|\nabla q\|_{L^2(\Omega)^d},$$

which implies that

$$\|q\|_{H^1(\Omega)} \leq \tilde{c}\|\nabla q\|_{L^2(\Omega)^d},$$

where $\tilde{c} := \sqrt{1+c^2}$.

Theorem 2.2.8 (Poincaré–Wirtinger)**.** *There exists a constant $c > 0$ such that*

$$\|q\|_{L^2(\Omega)} \leq c\|\nabla q\|_{L^2(\Omega)^d}$$

for all $q \in H^1(\Omega)/\mathbb{R}$.

Theorem 2.2.9. *Assume that E is a reflexive Banach space and let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence in E. Then, there exist $x \in E$ and a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that*

$$x_{n_k} \rightharpoonup x \text{ weakly in } E \text{ as } k \to \infty.$$

Theorem 2.2.10. *Assume that E is a reflexive Banach space and let $M \subset E$ be a closed linear subspace of E. Then, M is reflexive.*

Theorem 2.2.11 (Lax–Milgram)**.** *Assume that $a(\cdot,\cdot) : H \times H \to \mathbb{R}$ is a continuous coercive bilinear form on a Hilbert space H. Then, given any $f \in H^*$, there exists a unique element $u \in H$ such that*

$$a(u,v) = \langle f, v\rangle$$

for all $v \in H$.

The following Theorem 2.2.12 is necessary for the existence and uniqueness of a solution to the Stokes problem.

Theorem 2.2.12. [32, Corollary 4.1] *Let $(X, \|\cdot\|_X)$ and $(Q, \|\cdot\|_Q)$ be two real Hilbert spaces. Let $a : X \times X \to \mathbb{R}$ and $b : X \times Q \to \mathbb{R}$ be bilinear and continuous maps and let $f \in X^*$. If there exist two constants $\alpha > 0$ and $\beta > 0$ such that*

$$\begin{aligned} a(v,v) &\geq \alpha\|v\|_X^2 \quad \text{for all } v \in V, \\ \sup_{0\neq v\in X} \frac{b(v,q)}{\|v\|_X} &\geq \beta\|q\|_Q \quad \text{for all } q \in Q, \end{aligned}$$

where $V = \{v \in X \mid b(v,q) = 0 \text{ for all } q \in Q\}$, then there exists a unique solution $(u,p) \in X \times Q$ to the following problem:

$$\begin{cases} a(u,v) + b(v,p) = f(v) & \text{for all } v \in X, \\ b(u,q) = 0 & \text{for all } q \in Q. \end{cases}$$

We recall the following Theorem 2.2.13, which plays an important role in the proof of the existence of the pressure solution of the Stokes problem; see Appendix A for the proof.

Theorem 2.2.13. [66, Lemma 7.1] *There exists a constant $c > 0$ such that for all $q \in L^2(\Omega)$,*

$$\|q\|_{L^2(\Omega)} \leq c(\|q\|_{H^{-1}(\Omega)} + \|\nabla q\|_{H^{-1}(\Omega)}).$$

The following two results follow from Theorem 2.2.13.

Theorem 2.2.14. [32, Corollary 2.1, 2°] *There exists a constant $c > 0$ such that for all $q \in L^2(\Omega)$,*

$$\|[q]\|_{L^2(\Omega)} \leq c\|\nabla q\|_{H^{-1}(\Omega)^n}.$$

Theorem 2.2.15. [32, Corollary 2.4, 2°] *The operator* $\operatorname{div} : H_0^1(\Omega)^d \to L^2(\Omega)/\mathbb{R}$ *is surjective, i.e., for all $q \in L^2(\Omega)/\mathbb{R}$, there exists $v_q \in H_0^1(\Omega)^d$ such that*

$$\operatorname{div} v_q = q.$$

Furthermore, there exists a constant $c > 0$ such that for all $q \in L^2(\Omega)/\mathbb{R}$,

$$\|v_q\|_{H^1(\Omega)^d} \leq c\|q\|_{L^2(\Omega)}.$$

Theorem 2.2.15 implies the following theorem.

Theorem 2.2.16.

(i) There exists a constant $c > 0$ such that for all $q \in L^2(\Omega)/\mathbb{R}$,

$$\sup_{0 \neq v \in H_0^1(\Omega)} \frac{1}{\|v\|_{H^1(\Omega)^d}} \int_\Omega q \operatorname{div} v dx \geq c\|q\|_{L^2(\Omega)}.$$

(ii) Let $H_0^1(\Omega)^d \subset X \subset H^1(\Omega)^d$ be a subspace. If there exists a function $v_0 \in X$ such that $\int_\Omega \operatorname{div} v_0 dx \neq 0$, then there exists a constant $c > 0$ such that for all $q \in L^2(\Omega)$,

$$\sup_{0 \neq v \in X} \frac{1}{\|v\|_{H^1(\Omega)^d}} \int_\Omega q \operatorname{div} v dx \geq c\|q\|_{L^2(\Omega)}.$$

Theorem 2.2.16 (i) is well-known [2, 14, 32]. The following proof of Theorem 2.2.16 (ii) is based on [11, Proof of Theorem 2.1] and [13, Lemma 2.7] and use only Theorem 2.2.15 and existence of $v_0 \in X$. The function spaces $H_{\Gamma_1}^1(\Omega)^d$ and H satisfy the assumption of (ii).

Proof. (i) By Theorem 2.2.15, for all $q \in L^2(\Omega)/\mathbb{R}$, there exists $v_q \in H_0^1(\Omega)$ such that $\operatorname{div} v_q = q$. Then, it holds that for all $q \in L^2(\Omega)/\mathbb{R}$,

$$\int_\Omega q \operatorname{div} v_q dx = \|q\|_{L^2(\Omega)}^2 \geq \frac{1}{c}\|q\|_{L^2(\Omega)}\|v_q\|_{H^1(\Omega)^d}.$$

where the constant $c > 0$ is used in Theorem 2.2.15. Hence, we obtain for all $0 \neq q \in L^2(\Omega)/\mathbb{R}$,

$$\frac{1}{c}\|q\|_{L^2(\Omega)} \leq \frac{1}{\|v_q\|_{H^1(\Omega)^d}} \int_\Omega q \operatorname{div} v_q dx \leq \sup_{0\neq v \in H_0^1(\Omega)} \frac{1}{\|v\|_{H^1(\Omega)^d}} \int_\Omega q \operatorname{div} v dx.$$

(ii) We can assume that $v_0 \in X$ satisfies that $\int_\Omega \operatorname{div} v_0 dx = |\Omega|$ without loss of generality. For all $q \in L^2(\Omega)$, we set $q_0 := (\int_\Omega q dx)/|\Omega|$ and $q_1 := [q] = q - q_0 \in L^2(\Omega)/\mathbb{R}$.

If $q_1 = 0$, i.e., $q = q_0 \in \mathbb{R}$, then it holds that

$$\begin{aligned}
\|q\|_{L^2(\Omega)} &= \frac{1}{\sqrt{|\Omega|}}|q_0| \int_\Omega \operatorname{div} v_0 dx \\
&\leq \frac{1}{\sqrt{|\Omega|}} \left| \int_\Omega q_0 \operatorname{div} v_0 dx \right| \\
&\leq \frac{\|v_0\|_{H^1(\Omega)^d}}{\sqrt{|\Omega|}} \frac{1}{\|v_0\|_{H^1(\Omega)^d}} \left| \int_\Omega q_0 \operatorname{div} v_0 dx \right| \\
&\leq \frac{\|v_0\|_{H^1(\Omega)^d}}{\sqrt{|\Omega|}} \sup_{0\neq v \in X} \frac{1}{\|v\|_{H^1(\Omega)^d}} \int_\Omega q_0 \operatorname{div} v dx.
\end{aligned} \tag{2.2.2}$$

If $q_1 \neq 0$, then we set

$$\lambda := 1 - \frac{q_0}{\|q_1\|^2_{L^2(\Omega)}} \int_\Omega q_1 \operatorname{div} v_0 dx.$$

By Theorem 2.2.15, there exists $v_1 \in H_0^1(\Omega)$ such that $\operatorname{div} v_1 = q_1$. Let $\tilde{v}_q := q_0 v_0 + \lambda v_1$. Then, we have

$$\begin{aligned}
\int_\Omega q \operatorname{div} \tilde{v}_q dx &= \int_\Omega (q_0 + q_1) \operatorname{div}(q_0 v_0 + \lambda v_1) dx \\
&= q_0^2 \int_\Omega \operatorname{div} v_0 dx + \lambda q_0 \int_\Omega \operatorname{div} v_1 dx + q_0 \int_\Omega q_1 \operatorname{div} v_0 dx + \lambda \int_\Omega q_1 \operatorname{div} v_1 dx \\
&= \int_\Omega q_0^2 dx + q_0 \int_\Omega q_1 \operatorname{div} v_0 dx + \left(1 - \frac{q_0}{\|q_1\|^2_{L^2(\Omega)}} \int_\Omega q_1 \operatorname{div} v_0 dx \right) \|q_1\|^2_{L^2(\Omega)} \\
&= \int_\Omega (q_0^2 + q_1^2) dx \\
&= \|q\|^2_{L^2(\Omega)},
\end{aligned}$$

where we have used $\int_\Omega \operatorname{div} v_1 dx = \int_\Gamma v_1 \cdot n ds = 0$. Since it holds that

$$\begin{aligned}
\|\tilde{v}_q\|_{H^1(\Omega)^d} &\leq |q_0| \|v_0\|_{H^1(\Omega)^d} + |\lambda| \|v_1\|_{H^1(\Omega)^d} \\
&\leq |q_0| \|v_0\|_{H^1(\Omega)^d} + \left(1 + \frac{|q_0| \|\operatorname{div} v_0\|_{L^2(\Omega)}}{\|q_1\|_{L^2(\Omega)}} \right) c \|q_1\|_{L^2(\Omega)} \\
&\leq (\|v_0\|_{H^1(\Omega)^d} + c\sqrt{d} \|v_0\|_{L^2(\Omega)}) |q_0| + c \|q_1\|_{L^2(\Omega)} \\
&= \frac{1 + c\sqrt{d}}{\sqrt{|\Omega|}} \|v_0\|_{H^1(\Omega)^d} \|q_0\|_{L^2(\Omega)} + c \|q_1\|_{L^2(\Omega)} \\
&= \tilde{c} \|q\|_{L^2(\Omega)}
\end{aligned}$$

where $\tilde{c} := \sqrt{c^2 + (1 + c\sqrt{d})^2 \|v_0\|^2_{H^1(\Omega)^d}/|\Omega|}$, we obtain for all $q \in L^2(\Omega)/\mathbb{R}$ with $q_1 \neq 0$,

$$\frac{1}{\tilde{c}}\|q\|_{L^2(\Omega)} \leq \frac{\|q\|^2_{L^2(\Omega)}}{\|\tilde{v}_q\|_{H^1(\Omega)^d}} = \frac{1}{\|\tilde{v}_q\|_{H^1(\Omega)^d}} \int_\Omega q \operatorname{div} \tilde{v}_q dx \leq \sup_{0 \neq v \in X} \frac{1}{\|v\|_{H^1(\Omega)^d}} \int_\Omega q \operatorname{div} v dx,$$

and hence, by (2.2.2) and $\tilde{c} \geq \|v_0\|_{H^1(\Omega)^d}/\sqrt{|\Omega|}$, we have for all $q \in L^2(\Omega)$,

$$\frac{1}{\tilde{c}}\|q\|_{L^2(\Omega)} \leq \sup_{0 \neq v \in X} \frac{1}{\|v\|_{H^1(\Omega)^d}} \int_\Omega q \operatorname{div} v dx.$$

□

We define a bilinear form $a_0 : H^1(\Omega)^d \times H^1(\Omega)^d \to \mathbb{R}$ and a seminorm $\|\cdot\|_{a_0}$ on $H^1(\Omega)^d$, for $u, v \in H^1(\Omega)^d$,

$$\begin{aligned} a_0(u,v) &:= \int_\Omega (\operatorname{div} u)(\operatorname{div} v) dx + \int_\Omega (\nabla \times u) \cdot (\nabla \times v) dx, \\ \|u\|_{a_0} &:= \sqrt{a_0(u,u)}. \end{aligned}$$

We will assume the following condition in Chapter 3, 5:

Hypothesis 2.2.17.

(i) The open subset Γ_2 is piecewise $C^{1,1}$-class, i.e., there exist relatively open connected non-empty subsets $\Gamma_{2,1}, \Gamma_{2,2}, \dots, \Gamma_{2,N_\Gamma}$ of Γ such that for all $i, j = 1, 2, \dots, N_\Gamma$, $\Gamma_{2,i}$ is $C^{1,1}$-class and

$$\left| \Gamma_2 \setminus \left(\bigcup_{k=1}^{N_\Gamma} \Gamma_{2,k} \right) \right| = 0, \quad \Gamma_{2,i} \cap \Gamma_{2,j} = \emptyset \ (i \neq j).$$

(ii) There exists a constant $\delta > 0$ such that for all $x \in \overline{\Gamma_{2,i}} \cap \overline{\Gamma_{2,j}}$ $(i, j = 1, 2, \dots, N_\Gamma, i \neq j)$,

$$n^i(x) \cdot n^j(x) \leq 1 - \delta,$$

where $n^i(x)$ and $n^j(x)$ are the limits of the outer normal vectors when approaching x from Γ_i and Γ_j, respectively.

Remark 2.2.18. *If Γ is $C^{1,1}$-class or Ω is polygon, then Hypothesis 2.2.17 holds.*

Under Hypothesis 2.2.17, the following coercivity of the bilinear form $a_0 : H \times H \to \mathbb{R}$ holds.

Theorem 2.2.19. *Under Hypothesis 2.2.17, there exists a constant $c_a = c_a(\Omega, \Gamma_1, \Gamma_2) > 0$ such that for all $v_1, v_2, v \in H$,*

$$a_0(v_1, v_2) \leq \|v_1\|_{a_0}\|v_2\|_{a_0} \leq c_a\|v_1\|_1\|v_2\|_1, \qquad \frac{1}{c_a}\|v\|_1^2 \leq \|v\|^2_{a_0}.$$

The first inequality holds from the Cauchy–Schwarz inequality. For the proof of the second inequality, see [12, Lemma 2.11] and [51, Lemma 5].

Chapter 3

Pressure Poisson method

This chapter is based on the following published paper:

- Matsui, K.: Sharp consistency estimates for a pressure Poisson problem with Stokes boundary value problems. Discrete & Continuous Dynamical Systems - S **14** (3), 1001–1015 (2021). DOI 10.3934/dcdss.2020380

3.1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^d$ with Lipschitz continuous boundary Γ satisfying (2.1.1). The strong form of the Stokes problem is given as follows. Find $u^S : \Omega \to \mathbb{R}^d$ and $p^S : \Omega \to \mathbb{R}$ such that

$$\begin{cases} -\Delta u^S + \nabla p^S = F & \text{in } \Omega, \\ \operatorname{div} u^S = 0 & \text{in } \Omega, \\ u^S = 0 & \text{on } \Gamma_1, \\ T_n(u^S, p^S) = t^b & \text{on } \Gamma_2, \end{cases} \tag{ST}$$

holds, where $F : \Omega \to \mathbb{R}^d$, $t^b : \Gamma_2 \to \mathbb{R}^d$,

$$T(u^S, p^S)_{ij} \coloneqq \frac{\partial u_i^S}{\partial x_j} + \frac{\partial u_j^S}{\partial x_i} - p^S \delta_{ij},$$

$$T_n(u^S, p^S)_i \coloneqq \sum_{k=1}^{d} T(u^S, p^S)_{ik} n_k,$$

for all $i, j = 1, 2, \ldots, d$. Here, δ_{ij} is the Kronecker delta. The functions u^S and p^S are the velocity and the pressure of the flow governed by (ST), respectively. For the flow, $T(u^S, p^S)$ and $T_n(u^S, p^S)$ are often called the stress tensor and the normal stress on Γ, respectively. Let the fourth equation of (ST) be called the traction boundary condition. By the second equation of (ST), the first equation is equivalent to

$$-\operatorname{div} T(u^S, p^S) = F \quad \text{in } \Omega.$$

We refer to [14, 32, 80] for details on the Stokes problem (i.e., physical background and corresponding mathematical analysis). Taking the divergence of the first equation, we obtain

$$\operatorname{div} F = \operatorname{div}(-\Delta u^S + \nabla p^S) = -\Delta(\operatorname{div} u^S) + \Delta p^S = \Delta p^S. \tag{3.1.1}$$

This equation is often called the pressure Poisson equation and is used in numerical schemes, such as the MAC, SMAC, and projection method (see, e.g., [4, 21, 25, 41, 38, 40, 60, 42, 53, 64, 68]).

We need an additional boundary condition for solving equation (3.1.1). In real-world applications, the additional boundary condition is usually given by using experimental or plausible values. We consider the following boundary value problem for the pressure Poisson equation: Find $u^{PP} : \Omega \to \mathbb{R}^d$ and $p^{PP} : \Omega \to \mathbb{R}$ satisfying

$$\begin{cases} -\Delta u^{PP} - \nabla(\operatorname{div} u^{PP}) + \nabla p^{PP} = F & \text{in } \Omega, \\ -\Delta p^{PP} = -\operatorname{div} F & \text{in } \Omega, \\ u^{PP} = 0 & \text{on } \Gamma_1, \\ \dfrac{\partial p^{PP}}{\partial n} = g^b & \text{on } \Gamma_1, \\ T_n(u^{PP}, p^{PP}) = t^b & \text{on } \Gamma_2, \\ p^{PP} = p^b & \text{on } \Gamma_2, \end{cases} \tag{PPT}$$

where $g^b : \Gamma_1 \to \mathbb{R}$ and $p^b : \Gamma_2 \to \mathbb{R}$ are the data for the additional boundary conditions. We call this problem the pressure Poisson problem. The second term $-\nabla(\operatorname{div} u^{PP})$ in the first equation of (PPT) is usually omitted since $\operatorname{div} u^S = 0$, but this term is necessary to treat the traction boundary condition in a weak formulation. The idea of using (3.1.1) instead of $\operatorname{div} u^S = 0$ is useful for calculating the pressure numerically in the Navier–Stokes problem. For example, this idea is used in the MAC, SMAC, and projection methods.

As the boundary condition for the Stokes problem, we also consider the boundary condition introduced in [7, 8, 23];

$$\begin{cases} u = 0 & \text{on } \Gamma_1, \\ u \times n = 0 & \text{on } \Gamma_2, \\ p = p^b & \text{on } \Gamma_2, \end{cases} \tag{3.1.2}$$

(see also [12, 13, 24, 63]). On boundary Γ_2, the boundary value of the pressure is described, and the velocity is parallel to the normal direction. Such a situation happens at the end of pipes, such as blood vessels or pipelines (Fig. 3.1). The well-posedness is proven in [12, 13, 23, 24].

In this chapter, we establish error estimates between problems (PPT) and (ST) and between problem (PPT) and the Stokes problem with the boundary condition (3.1.2) in terms of the additional boundary conditions. In particular, since boundary conditions that contain a Dirichlet boundary condition for the pressure often appear in engineering problems, a comparison between problem (PPT) and the Stokes problem with the boundary condition (3.1.2) is important.

The organization of this paper is as follows. In Section 3.2, we introduce notations and symbols used in this work and the weak form of these problems. We also prove the well-posedness of the problems (ST) and (PPT) and show several properties of them. In Section 3.3, we establish error estimates between solutions to the problems (ST) and

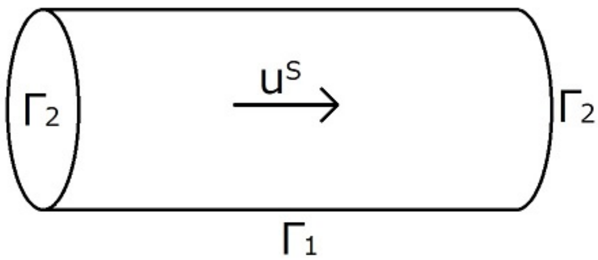

Figure 3.1: Image of a flow in a pipe

(PPT) in terms of the additional boundary conditions. Section 3.4 is devoted to the study of the Stokes problem with the boundary condition (3.1.2). We conclude this paper with several comments on future works in Section 3.5.

3.2 Weak formulation and well-posedness

3.2.1 Preliminaries

For $u \in H^1(\Omega)^d$ and $p \in H^1(\Omega)$ satisfying $\Delta u + \nabla(\operatorname{div} u)$ $(= \operatorname{div} S(u)) \in L^2(\Omega)^d$ and Δp $(= \operatorname{div}(\nabla p)) \in L^2(\Omega)$, we set

$$S(u)n \coloneqq (\gamma_n(S(u)_1), \ldots, \gamma_n(S(u)_d))^{\mathrm{T}} \in H^{-1/2}(\Gamma)^d,$$
$$\frac{\partial p}{\partial n} \coloneqq \gamma_n(\nabla p) \in H^{-1/2}(\Gamma),$$

where $S(u)_i = (S(u)_{i1}, \ldots, S(u)_{id})^{\mathrm{T}}$ for $i = 1, \ldots, d$. Since $H^{1/2}_{\gamma_0}(\Gamma_i) \subset H^{1/2}(\Gamma)$ for $i = 1, 2$, we have $S(u)n \in (H^{1/2}_{\gamma_0}(\Gamma_2)^d)^*$ and $\partial p/\partial n \in (H^{1/2}_{\gamma_0}(\Gamma_1))^*$. By Theorem 2.2.4, it holds that

$$\langle S(u)n, \varphi\rangle_{H^{1/2}_{\gamma_0}(\Gamma_2)^d} = \int_\Omega \left(\frac{1}{2}S(u) : S(\varphi) + (\Delta u + \nabla(\operatorname{div} u)) \cdot \varphi\right) dx \quad \text{for all } \varphi \in H^1_{\Gamma_1}(\Omega)^d,$$
$$\left\langle \frac{\partial p}{\partial n}, \psi\right\rangle_{H^{1/2}_{\gamma_0}(\Gamma_1)} = \int_\Omega (\nabla p \cdot \nabla\psi + (\Delta p)\psi)\, dx \quad \text{for all } \psi \in H^1_{\Gamma_2}(\Omega).$$

We remark that $u \in H^2(\Omega)^d$ and $p \in H^2(\Omega)$ satisfy

$$\langle S(u)n, \varphi\rangle_{H^{1/2}_{\gamma_0}(\Gamma_2)^d} = \int_{\Gamma_2} \left(\sum_{i,j=1}^d S_{ij}(u)\varphi_i n_j\right) ds,$$
$$\left\langle \frac{\partial p}{\partial n}, \psi\right\rangle_{H^{1/2}_{\gamma_0}(\Gamma_1)} = \int_{\Gamma_1} \frac{\partial p}{\partial n}\psi\, ds$$

for all $\varphi \in H^1_{\Gamma_1}(\Omega)^d$ and $\psi \in H^1_{\Gamma_2}(\Omega)$. For $u \in H^1(\Omega)^d$ and $p \in H^1(\Omega)$ satisfying $\Delta u + \nabla(\operatorname{div} u) \in L^2(\Omega)$, we set

$$\langle T_n(u,p)n, \varphi\rangle_{H^{1/2}_{\gamma_0}(\Gamma_2)^d} \coloneqq \langle S(u)n, \varphi\rangle_{H^{1/2}_{\gamma_0}(\Gamma_2)^d} - \int_{\Gamma_2} p\varphi \cdot n\, ds \qquad \text{for all } \varphi \in H^1_{\Gamma_1}(\Omega)^d.$$

We recall Korn's first inequality for the existence and uniqueness of a solution to the Stokes problem.

Theorem 3.2.1 (Korn's first inequality). *There exists a constant $c > 0$ such that*

$$\|\varphi\|_{H^1(\Omega)^d} \leq c\|S(\varphi)\|_{L^2(\Omega)^{d\times d}}.$$

for all $\varphi \in H^1_{\Gamma_1}(\Omega)^d$.

See [22, Theorem 6.3.4] and [69, Corollary 4.1] for the proof.

3.2.2 Weak formulations of (PPT) and (ST)

We start by defining the weak solution to (PPT). Throughout of this paper, we assume the following conditions;

$$t^b \in (H^{1/2}_{\gamma_0}(\Gamma_2)^d)^*, \qquad F \in L^2(\Omega)^d, \tag{3.2.3}$$

$$g^b \in (H^{1/2}_{\gamma_0}(\Gamma_1))^*, \quad p^b \in H^1(\Omega), \quad \operatorname{div} F \in L^2(\Omega). \tag{3.2.4}$$

Lemma 3.2.2. *For $u \in H^2(\Omega)^d, p \in H^1(\Omega)$, and $\varphi \in H^1_{\Gamma_1}(\Omega)^d$,*

$$(-\Delta u - \nabla(\operatorname{div} u) + \nabla p, \varphi) = \frac{1}{2}(S(u), S(\varphi)) - (p, \operatorname{div}\varphi) - \langle t, \varphi\rangle_{H^{1/2}_{\gamma_0}(\Gamma_2)^d}$$

holds, where $t := T_n(u,p)$.

Proof. We compute

$$\begin{aligned}
&(-\Delta u - \nabla(\operatorname{div} u) + \nabla p, \varphi)\\
&= -\int_\Omega \sum_{i,j=1}^d \left(\frac{\partial u_i}{\partial x_j \partial x_j} + \frac{\partial u_j}{\partial x_j \partial x_i}\right)\varphi_i dx + \int_\Omega \sum_{i=1}^d \frac{\partial p}{\partial x_i}\varphi_i dx\\
&= -\int_\Omega \sum_{i,j=1}^d \frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_j}\right)\varphi_i dx + \int_\Omega \sum_{i=1}^d \frac{\partial p}{\partial x_i}\varphi_i dx\\
&= \sum_{i,j=1}^d \left\{\int_\Omega \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_j}\right)\frac{\partial \varphi_i}{\partial x_j} dx - \int_\Gamma \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_j}\right)\varphi_i n_j ds\right\}\\
&\quad - \sum_{i=1}^d \left\{\int_\Omega p\frac{\partial \varphi_i}{\partial x_i} dx - \int_\Gamma p\varphi_i n_i ds\right\}\\
&= \frac{1}{2}\int_\Omega \sum_{i,j=1}^d \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_j}\right)\left(\frac{\partial \varphi_i}{\partial x_j} + \frac{\partial \varphi_j}{\partial x_j}\right) dx - \int_\Omega p\left(\sum_{i=1}^d \frac{\partial \varphi_i}{\partial x_i}\right) dx\\
&\quad - \int_\Gamma \sum_{i,j=1}^d \left\{\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_j}\right)\varphi_i n_j - p\varphi_i n_i\right\} ds\\
&= \frac{1}{2}(S(u), S(\varphi)) - (p, \operatorname{div}\varphi) - \langle t, \varphi\rangle_{H^{1/2}_{\gamma_0}(\Gamma_2)^d}
\end{aligned}$$

which completes the proof. □

For the second equation of (PPT), taking $\psi \in H^1_{\Gamma_2}(\Omega)$, we obtain

$$\begin{aligned}-(\operatorname{div} F, \psi) &= -(\Delta p^{PP}, \psi)\\ &= -\int_\Gamma \frac{\partial p^{PP}}{\partial n}\psi\, ds + (\nabla p^{PP}, \nabla\psi)\\ &= -\langle g^b, \psi\rangle_{H^{1/2}_{\gamma_0}(\Gamma_1)} + (\nabla p^{PP}, \nabla\psi).\end{aligned}$$

Therefore, the weak form of (PPT) becomes as follows. Find $u^{PP} \in H^1_{\Gamma_1}(\Omega)^d$ and $p^{PP} \in H^1(\Omega)$ such that

$$\begin{cases}\frac{1}{2}(S(u^{PP}), S(\varphi)) - (p^{PP}, \operatorname{div}\varphi) = (F, \varphi) - \langle t^b, \varphi\rangle_{H^{1/2}_{\gamma_0}(\Gamma_2)^d} & \text{for all } \varphi \in H^1_{\Gamma_1}(\Omega)^d,\\ (\nabla p^{PP}, \nabla\psi) = -(\operatorname{div} F, \psi) + \langle g^b, \psi\rangle_{H^{1/2}_{\gamma_0}(\Gamma_1)} & \text{for all } \psi \in H^1_{\Gamma_2}(\Omega),\\ p^{PP} = p^b & \text{on } \Gamma_2.\end{cases} \tag{PPT'}$$

Remark 3.2.3. *If $(u^{PP}, p^{PP}) \in H^1_{\Gamma_1}(\Omega)^d \times H^1(\Omega)$ satisfies $u^{PP} \in H^2(\Omega)^d, p^{PP} \in H^1(\Omega)$ and (PPT'), then we have for all $\varphi \in H^1_{\Gamma_1}(\Omega)^d$ and $\psi \in H^1_{\Gamma_2}(\Omega)$,*

$$\begin{cases}(-\Delta u^{PP} - \nabla(\operatorname{div} u^{PP}) + \nabla p^{PP} - F, \varphi) = \langle T_n(u^{PP}, p^{PP}) - t^b, \varphi\rangle_{H^{1/2}_{\gamma_0}(\Gamma_2)^d},\\ (-\Delta p^{PP} + \operatorname{div} F, \psi) = \left\langle -\frac{\partial p^{PP}}{\partial n} + g^b, \psi\right\rangle_{H^{1/2}_{\gamma_0}(\Gamma_1)}.\end{cases}$$

Therefore, (u^{PP}, p^{PP}) satisfies (PPT).

Next, we define the weak formulation of (ST). For all $\varphi \in H^1_{\Gamma_1}(\Omega)^d$, we obtain from the first equation of (ST),

$$\begin{aligned}(F, \varphi) &= (-\Delta u^S + \nabla p^S, \varphi)\\ &= (-\Delta u^S - \nabla(\operatorname{div} u^S) + \nabla p^S, \varphi)\\ &= \frac{1}{2}(S(u^S), S(\varphi)) - (p^S, \operatorname{div}\varphi) - \langle t^b, \varphi\rangle_{H^{1/2}_{\gamma_0}(\Gamma_2)^d}.\end{aligned}$$

Using this expression, the weak form of the Stokes problem becomes as follows: Find $(u^{S1}, p^{S1}) \in H^1_{\Gamma_1}(\Omega)^d \times L^2(\Omega)$ such that

$$\begin{cases}\frac{1}{2}(S(u^{S1}), S(\varphi)) - (p^{S1}, \operatorname{div}\varphi) = (F, \varphi) - \langle t^b, \varphi\rangle_{H^{1/2}_{\gamma_0}(\Gamma_2)^d} & \text{for all } \varphi \in H^1_{\Gamma_1}(\Omega)^d,\\ (\psi, \operatorname{div} u^{S1}) = 0 & \text{for all } \psi \in L^2(\Omega).\end{cases} \tag{ST'}$$

Remark 3.2.4. *If $(u^{S1}, p^{S1}) \in H^1_{\Gamma_1}(\Omega)^d \times L^2(\Omega)$ satisfies $u^{S1} \in H^2(\Omega)^d, p^{S1} \in H^1(\Omega)$ and (ST'), then we have*

$$\begin{cases}(-\Delta u^{S1} + \nabla p^{S1} - F, \varphi) = \langle T_n(u^{S1}, p^{S1}) - t^b, \varphi\rangle_{H^{1/2}_{\gamma_0}(\Gamma_2)^d} & \textit{for all } \varphi \in H^1_{\Gamma_1}(\Omega)^d,\\ (\psi, \operatorname{div} u^{S1}) = 0 & \textit{for all } \psi \in L^2(\Omega).\end{cases}$$

Therefore, (u^{S1}, p^{S1}) satisfies (ST).

3.2.3 Well-posedness of (PPT'), (ST')

We show the well-posedness of the problems (PPT') and (ST') in Theorems 3.2.5 and 3.2.6.

Theorem 3.2.5. *Under the conditions (3.2.3) and (3.2.4), there exists a unique solution* $(u^{PP}, p^{PP}) \in H^1_{\Gamma_1}(\Omega)^d \times H^1(\Omega)$ *satisfying (PPT').*

Proof. From the second and third equations of (PPT'), by using the Lax–Milgram theorem and Theorem 2.2.7, $p^{PP} \in H^1(\Omega)$ is uniquely determined. Then, $u^{PP} \in H^1(\Omega)^d$ is also uniquely determined from the first equation of (PPT') by the Lax–Milgram theorem, where the coercivity is guaranteed from Theorem 3.2.1. □

Theorem 3.2.6. *Under the condition (3.2.3), there exists a unique solution* $(u^{S1}, p^{S1}) \in H^1_{\Gamma_1}(\Omega)^d \times L^2(\Omega)$ *satisfying (ST').*

Proof. By Theorems 3.2.1 and 2.2.7, the continuous bilinear form $H^1_{\Gamma_1}(\Omega)^d \times H^1_{\Gamma_1}(\Omega)^d \ni (u, \varphi) \mapsto (S(u), S(\varphi)) \in \mathbb{R}$ is coercive. By Theorems 2.2.12 and 2.2.16, there exists a unique solution $(u^{S1}, p^{S1}) \in H^1_{\Gamma_1}(\Omega)^d \times L^2(\Omega)$ satisfying (ST'). □

We prove the following property of the solution to (ST').

Proposition 3.2.7. *If the weak solution* $(u^{S1}, p^{S1}) \in H^1_{\Gamma_1}(\Omega)^d \times L^2(\Omega)$ *to (ST') satisfies* $p^{S1} \in H^1(\Omega)$ *and* $\Delta p^{S1} \in L^2(\Omega)$*, then we have*

$$(\nabla p^{S1}, \nabla \psi) = -(\operatorname{div} F, \psi) + \left\langle \frac{\partial p^{S1}}{\partial n}, \psi \right\rangle_{H^{1/2}_{\gamma_0}(\Gamma_1)}$$

for all $\psi \in H^1_{\Gamma_2}(\Omega)$.

Proof. From the second equation of (ST') and $u^{S1} \in H^1(\Omega)$, $\operatorname{div} u^{S1} = 0$ holds in $L^2(\Omega)$. From the first equation of (ST'), we obtain

$$-\Delta u^{S1} + \nabla p^{S1} = -\Delta u^{S1} - \nabla(\operatorname{div} u^{S1}) + \nabla p^{S1} = F \quad \text{in } \mathscr{D}'(\Omega).$$

Taking the divergence, we get

$$\operatorname{div} F = \operatorname{div}(-\Delta u^{S1} + \nabla p^{S1}) = -\Delta(\operatorname{div} u^{S1}) + \Delta p^{S1} = \Delta p^{S1} \quad \text{in } \mathscr{D}'(\Omega).$$

By the assumptions $\Delta p^{S1} \in L^2(\Omega)$ and $\operatorname{div} F \in L^2(\Omega)$, $\Delta p^{S1} = \operatorname{div} F$ holds in $L^2(\Omega)$. Multiplying $\psi \in H^1_{\Gamma_2}(\Omega)$ and integrating over Ω, we get

$$-(\operatorname{div} F, \psi) = -(\Delta p^{S1}, \psi) = (\nabla p^{S1}, \nabla \psi) - \left\langle \frac{\partial p^{S1}}{\partial n}, \psi \right\rangle_{H^{1/2}_{\gamma_0}(\Gamma_1)},$$

which is the desired result. □

3.3 The traction boundary condition

The purpose of this paper is to give an estimate of the difference between the solutions of the Stokes problem and the pressure Poisson problem. Roughly speaking, from (3.1.1) and the second equation of (PPT), $\Delta(p^S - p^{PP}) = 0$ holds. Hence, we get

$$\|p^S - p^{PP}\|_{H^1(\Omega)} \lesssim (\text{ difference between } p^S \text{ and } p^{PP} \text{ on } \Gamma),$$

where $A \lesssim B$ means that there exists a constant $c > 0$, independent of A and B, such that $A \leq cB$. From (ST) and the second equation of (PPT), we have

$$-\Delta(u^S - u^{PP}) = -\nabla(p^S - p^{PP}).$$

We obtain

$$\|u^S - u^{PP}\|_{H^1(\Omega)^d} \lesssim \|\nabla(p^S - p^{PP})\|_{L^2(\Omega)^d} + (\text{ difference between } p^S \text{ and } p^{PP} \text{ on } \Gamma).$$

Therefore, we have

$$\begin{aligned} & \|u^S - u^{PP}\|_{H^1(\Omega)^d} + \|p^S - p^{PP}\|_{H^1(\Omega)} \\ \lesssim & \ (\text{ difference between } (u^S, p^S) \text{ and } (u^{PP}, p^{PP}) \text{ on } \Gamma). \end{aligned}$$

In other words, if we have a good prediction for the boundary data, then (PPT) is good approximation for (ST).

In this section, we prove these types of estimates for the weak solutions. Let the solutions of (PPT') and (ST') be denoted by (u^{PP}, p^{PP}) and (u^{S1}, p^{S1}), respectively. First, we establish a lemma.

Lemma 3.3.1. *If $p \in H^1(\Omega)$, $f \in L^2(\Omega)$ and $g \in (H^{1/2}_{\gamma_0}(\Gamma_1))^*$ satisfy*

$$(\nabla p, \nabla \psi) = (f, \psi) + \langle g, \psi \rangle_{H^{1/2}_{\gamma_0}(\Gamma_1)} \quad \text{for all } \psi \in H^1_{\Gamma_2}(\Omega), \tag{3.3.5}$$

then there exists a constant $c > 0$ such that

$$\|p\|_{H^1(\Omega)} \leq c\left(\|f\|_{L^2(\Omega)} + \|g\|_{(H^{1/2}_{\gamma_0}(\Gamma_1))^*} + \|p\|_{H^{1/2}(\Gamma_2)}\right).$$

Proof. Let $p_0 \in H^1(\Omega)$ such that $p_0 - p \in H^1_{\Gamma_2}(\Omega)$. Putting $\psi := p - p_0$ in (3.3.5), we have

$$\begin{aligned} & \|\nabla(p - p_0)\|^2_{L^2(\Omega)^d} \\ = & (\nabla(p - p_0), \nabla(p - p_0)) \\ = & (f, p - p_0) + \langle g, p - p_0 \rangle_{H^{1/2}_{\gamma_0}(\Gamma_1)} - (\nabla p_0, \nabla(p - p_0)) \\ \leq & \|f\|_{L^2(\Omega)} \|p - p_0\|_{L^2(\Omega)} + \|g\|_{(H^{1/2}_{\gamma_0}(\Gamma_1))^*} \|p - p_0\|_{H^{1/2}_{\gamma_0}(\Gamma_1)} + \|\nabla p_0\|_{L^2(\Omega)^d} \|\nabla(p - p_0)\|_{L^2(\Omega)^d} \\ \leq & (\|f\|_{L^2(\Omega)} + c_1 \|g\|_{(H^{1/2}_{\gamma_0}(\Gamma_1))^*} + \|p_0\|_{H^1(\Omega)}) \|p - p_0\|_{H^1(\Omega)}. \end{aligned}$$

By Theorem 2.2.7, there exists a constant $c_2 > 0$ such that

$$c_2 \|p - p_0\|^2_{H^1(\Omega)} \leq (\|f\|_{L^2(\Omega)} + c_1 \|g\|_{(H^{1/2}_{\gamma_0}(\Gamma_1))^*} + \|p_0\|_{H^1(\Omega)}) \|p - p_0\|_{H^1(\Omega)}.$$

Hence,

$$\|p - p_0\|_{H^1(\Omega)} \leq c_3 (\|f\|_{L^2(\Omega)} + \|g\|_{(H^{1/2}_{\gamma_0}(\Gamma_1))^*} + \|p_0\|_{H^1(\Omega)}).$$

Since $\|p\|_{H^1(\Omega)} - \|p_0\|_{H^1(\Omega)} \le \|p - p_0\|_{H^1(\Omega)}$, we obtain

$$\|p\|_{H^1(\Omega)} \le c_4(\|f\|_{L^2(\Omega)} + \|g\|_{(H^{1/2}_{\gamma_0}(\Gamma_1))^*} + \|p_0\|_{H^1(\Omega)}). \tag{3.3.6}$$

For all $p_0 \in H^1(\Omega)$ satisfying $p_0 - p \in H^1_{\Gamma_2}(\Omega)$, (3.3.6) holds. Therefore,

$$\begin{aligned} \|p\|_{H^1(\Omega)} &\le c_4\left(\|f\|_{L^2(\Omega)} + \|g\|_{(H^{1/2}_{\gamma_0}(\Gamma_1))^*} + \inf_{q \in H^1_{\Gamma_2}(\Omega)} \|p+q\|_{H^1(\Omega)}\right) \\ &= c_4(\|f\|_{L^2(\Omega)} + \|g\|_{(H^{1/2}_{\gamma_0}(\Gamma_1))^*} + \|p\|_{H^1(\Omega)/H^1_{\Gamma_2}(\Omega)}) \\ &\le c_5(\|f\|_{L^2(\Omega)} + \|g\|_{(H^{1/2}_{\gamma_0}(\Gamma_1))^*} + \|p\|_{H^{1/2}(\Gamma_2)}). \end{aligned}$$

□

Using Proposition 3.2.7, we prove the following theorem which is the main result of this section.

Theorem 3.3.2. *If $p^{S1} \in H^1(\Omega)$ and $\Delta p^{S1} \in L^2(\Omega)$, there exists a constant $c > 0$ such that*

$$\begin{aligned} &\|u^{S1} - u^{PP}\|_{H^1(\Omega)^d} + \|p^{S1} - p^{PP}\|_{H^1(\Omega)} \\ \le\ & c\left(\left\|\frac{\partial p^{S1}}{\partial n} - g^b\right\|_{(H^{1/2}_{\gamma_0}(\Gamma_1))^*} + \|p^{S1} - p^b\|_{H^{1/2}(\Gamma_2)}\right). \end{aligned} \tag{3.3.7}$$

Proof. Using Proposition 3.2.7, we obtain from (ST') and (PPT'),

$$\begin{cases} \frac{1}{2}(S(u^{S1} - u^{PP}), S(\varphi)) = (p^{S1} - p^{PP}, \operatorname{div}\varphi) & \text{for all } \varphi \in H^1_{\Gamma_1}(\Omega)^d, \\ (\nabla(p^{S1} - p^{PP}), \nabla\psi) = \left\langle \frac{\partial p^{S1}}{\partial n} - g^b, \psi \right\rangle_{H^{1/2}_{\gamma_0}(\Gamma_1)} & \text{for all } \psi \in H^1_{\Gamma_2}(\Omega). \end{cases} \tag{3.3.8}$$

Putting $\varphi := u^{S1} - u^{PP} \in H^1_{\Gamma_1}(\Omega)^d$ in (3.3.8), we get

$$\begin{aligned} \frac{1}{2}\|S(u^{S1} - u^{PP})\|^2_{L^2(\Omega)^{d\times d}} &= (p^{S1} - p^{PP}, \operatorname{div}(u^{S1} - u^{PP})) \\ &\le \|p^{S1} - p^{PP}\|_{L^2(\Omega)} \|\operatorname{div}(u^{S1} - u^{PP})\|_{L^2(\Omega)} \\ &\le \sqrt{d}\|p^{S1} - p^{PP}\|_{H^1(\Omega)} \|u^{S1} - u^{PP}\|_{H^1(\Omega)^d}. \end{aligned}$$

From Theorem 3.2.1,

$$\|u^{S1} - u^{PP}\|_{H^1(\Omega)^d} \le c_1\|p^{S1} - p^{PP}\|_{H^1(\Omega)}$$

holds for a constant $c_1 > 0$. By the second equation of (3.3.8) and Lemma 3.3.1, there exists a constant $c_2 > 0$ such that

$$\begin{aligned} \|p^{S1} - p^{PP}\|_{H^1(\Omega)} &\le c_2\left(\left\|\frac{\partial p^{S1}}{\partial n} - g^b\right\|_{(H^{1/2}_{\gamma_0}(\Gamma_1))^*} + \|p^{S1} - p^{PP}\|_{H^{1/2}(\Gamma_2)}\right) \\ &\le c_2\left(\left\|\frac{\partial p^{S1}}{\partial n} - g^b\right\|_{(H^{1/2}_{\gamma_0}(\Gamma_1))^*} + \|p^{S1} - p^b\|_{H^{1/2}(\Gamma_2)}\right). \end{aligned}$$

Therefore, it holds that

$$\begin{aligned}&\|u^{S1} - u^{PP}\|_{H^1(\Omega)^d} + \|p^{S1} - p^{PP}\|_{H^1(\Omega)} \\ \le\, & c_3 \left(\left\| \frac{\partial p^{S1}}{\partial n} - g^b \right\|_{(H^{1/2}_{\gamma_0}(\Gamma_1))^*} + \|p^{S1} - p^b\|_{H^{1/2}(\Gamma_2)} \right),\end{aligned}$$

for a constant $c_3 > 0$. □

3.4 Boundary condition involving pressure

Let $p^b \in H^1(\Omega)$. We consider the Stokes problem with the boundary condition (3.1.2):

$$\begin{cases} -\Delta u^S + \nabla p^S &= F \quad \text{in } \Omega, \\ \operatorname{div} u^S &= 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \Gamma_1, \\ u \times n &= 0 \quad \text{on } \Gamma_2, \\ p &= p^b \quad \text{on } \Gamma_2. \end{cases} \tag{3.4.9}$$

In this section, we evaluate the difference between the solutions to (PPT) and (3.4.9) as in (3.3.7). First, we define the weak formulation of (3.4.9) and prove the existence and the uniqueness of the weak solution. Next, we prove a proposition and a lemma as preparation for the proof of our main theorem: Theorem 3.4.6.

We define the weak formulation of (3.4.9). Multiplying the first equation of (3.4.9) by $v \in H$, integrating by parts in Ω, and using the second equation of (3.4.9), we obtain

$$(F, v) = (\nabla \times u^S, \nabla \times v) - (p^S, \operatorname{div} v) + \int_{\Gamma_2} p^b v \cdot n \, ds,$$

where we have used the following lemma.

Lemma 3.4.1. *For $u \in H^2(\Omega)^d, p \in H^1(\Omega)$ and $v \in H$, there holds*

$$(-\Delta u + \nabla(\operatorname{div} u) + \nabla p, v) = (\nabla \times u, \nabla \times v) - (p, \operatorname{div} v) + \int_{\Gamma_2} pv \cdot n \, ds.$$

Proof. We compute

$$\begin{aligned} &(-\Delta u + \nabla(\operatorname{div} u) + \nabla p, v) \\ &= (\nabla \times (\nabla \times u) + \nabla p, v) \\ &= (\nabla \times u, \nabla \times v) - \int_\Gamma ((\nabla \times u) \times n) \cdot v \, ds - (p, \operatorname{div} v) + \int_\Gamma pv \cdot n \, ds \\ &= (\nabla \times u, \nabla \times v) - \int_\Gamma (n \times v) \cdot (\nabla \times u) \, ds - (p, \operatorname{div} v) + \int_{\Gamma_2} pv \cdot n \, ds \\ &= (\nabla \times u, \nabla \times v) - (p, \operatorname{div} v) + \int_{\Gamma_2} pv \cdot n \, ds. \end{aligned}$$

□

The weak form of the Stokes problem (3.4.9) becomes as follows: Find $(u^{S2}, p^{S2}) \in H \times L^2(\Omega)$ such that

$$\begin{cases} (\nabla \times u^{S2}, \nabla \times v) - (p^{S2}, \operatorname{div} v) = (F, v) - \int_{\Gamma_2} p^b v \cdot n \, ds & \text{for all } v \in H, \\ (\psi, \operatorname{div} u^{S2}) = 0 & \text{for all } \psi \in L^2(\Omega). \end{cases} \tag{SP}$$

Remark 3.4.2. *If $(u^{S2}, p^{S2}) \in H \times L^2(\Omega)$ satisfies $u^{S2} \in H^2(\Omega)^d, p^{S2} \in H^1(\Omega)$ and (SP), then we have*

$$\begin{cases} (-\Delta u^{S2} + \nabla p^{S2} - F, v) = \int_{\Gamma_2} (p^{S2} - p^b) v \cdot n \, ds & \textit{for all } v \in H, \\ (\psi, \operatorname{div} u^{S2}) = 0 & \textit{for all } \psi \in L^2(\Omega). \end{cases}$$

Therefore, (u^{S2}, p^{S2}) satisfies (3.4.9).

We establish the well-posedness of this problem (SP) in the following theorem.

Theorem 3.4.3. [23, Theorem 1.5] *For $F \in L^2(\Omega)^d$ and $p^b \in H^1(\Omega)$, under Hypothesis 2.2.17, there exists a unique solution $(u^{S2}, p^{S2}) \in H \times L^2(\Omega)$ to (SP).*

Proof. We set

$$a(u, v) := a_0(u, v), \quad b(v, q) := -(q, \operatorname{div} v), \quad f(v) := (F, v) - \int_{\Gamma_2} p^b v \cdot n \, ds$$

for all $u, v \in H$ and $q \in L^2(\Omega)$. Clearly, a and b are continuous and bilinear forms and $f \in H^*$. By Theorem 2.2.19, a is coercive on $\{v \in H \mid b(v, q) = 0 \text{ for all } q \in L^2(\Omega)\} = \{v \in H \mid \operatorname{div} v = 0\}$. By Theorem 2.2.16, b satisfies the assumption of Theorem 2.2.12. Therefore, there exists a unique solution $(u^{S2}, p^{S2}) \in H \times L^2(\Omega)$ to (SP) by Theorem 2.2.12. □

From here on, let the solutions of (PPT') and (SP) be denoted by (u^{PP}, p^{PP}) and (u^{S2}, p^{S2}), respectively. The solution (u^{S2}, p^{S2}) to (SP) satisfies the following property.

Proposition 3.4.4. *If $\Delta u^{S2} + \nabla(\operatorname{div} u^{S2}) \in L^2(\Omega)^d$, $p^{S2} \in H^1(\Omega)$ and $\Delta p^{S2} \in L^2(\Omega)$, then*

$$\begin{cases} \frac{1}{2}(S(u^{S2}), S(\varphi)) - (p^{S2}, \operatorname{div} \varphi) = (F, \varphi) - \langle T_n(u^{S2}, p^{S2}), \varphi \rangle_{H^{1/2}_{\gamma_0}(\Gamma_2)^d} & \textit{for all } \varphi \in H^1_{\Gamma_1}(\Omega)^d, \\ (\nabla p^{S2}, \nabla \psi) = -(\operatorname{div} F, \psi) + \left\langle \frac{\partial p^{S2}}{\partial n}, \psi \right\rangle_{H^{1/2}_{\gamma_0}(\Gamma_1)} & \textit{for all } \psi \in H^1_{\Gamma_2}(\Omega), \\ p^{S2} = p^b & \textit{on } \Gamma_2. \end{cases}$$

Proof. From the second equation of (SP) and $u^{S2} \in H^1(\Omega)$, $\operatorname{div} u^{S2} = 0$ holds in $L^2(\Omega)$. From the first equation of (SP), we obtain

$$-\Delta u^{S2} - \nabla(\operatorname{div} u^{S2}) + \nabla p^{S2} = -\Delta u^{S2} + \nabla(\operatorname{div} u^{S2}) + \nabla p^{S2} = F \tag{3.4.10}$$

in $\mathscr{D}'(\Omega)$. By the assumptions $\Delta u^{S2} + \nabla(\operatorname{div} u^{S2}) \in L^2(\Omega)^d$, $p^{S1} \in H^1(\Omega)$ and $\operatorname{div} F \in L^2(\Omega)$, equation (3.4.10) holds in $L^2(\Omega)$. Multiplying $\varphi \in H^1_{\Gamma_1}(\Omega)$ and integrating over Ω, we get

$$\begin{aligned}(F, v) &= (-\Delta u^{S2} - \nabla(\operatorname{div} u^{S2}) + \nabla p^{S2}, \varphi)\\ &= \frac{1}{2}(S(u^{S2}), S(\varphi)) - (p^{S2}, \operatorname{div} \varphi) + \langle T_n(u^{S2}, p^{S2}), \varphi\rangle_{H^{1/2}_{\gamma_0}(\Gamma_2)^d}.\end{aligned}$$

Taking the divergence of (3.4.10), we have

$$\Delta p^{S2} = \operatorname{div} F \quad \text{in } \mathscr{D}'(\Omega).$$

By the assumptions $\Delta p^{S2} \in L^2(\Omega)$ and $\operatorname{div} F \in L^2(\Omega)$, $\Delta p^{S2} = \operatorname{div} F$ holds in $L^2(\Omega)$. Multiplying $\psi \in H^1_{\Gamma_2}(\Omega)$ and integrating over Ω, we get

$$-(\operatorname{div} F, \psi) = -(\Delta p^{S2}, \psi) = (\nabla p^{S2}, \nabla \psi) - \left\langle \frac{\partial p^{S2}}{\partial n}, \psi \right\rangle_{H^{1/2}_{\gamma_0}(\Gamma_1)}.$$

Multiplying (3.4.10) by $v \in H$ and integrating over Ω, we get

$$\begin{aligned}(F, v) &= (-\Delta u^{S2} + \nabla(\operatorname{div} u^{S2}) + \nabla p^{S2}, v)\\ &= (\nabla \times u^{S2}, \nabla \times v) - (p^{S2}, \operatorname{div} v) + \int_{\Gamma_2} p^{S2} v \cdot n \, ds.\end{aligned}$$

By the first equation of (SP), it holds that

$$\int_{\Gamma_2} p^{S2} v \cdot n = -(\nabla \times u^{S2}, \nabla \times v) + (p^{S2}, \operatorname{div} v) + (F, v) = \int_{\Gamma_2} p^b v \cdot n \, ds$$

for all $v \in H$. Hence, $p^{S2} = p^b$ holds in $H^{1/2}(\Gamma_2)$. □

We establish a lemma.

Lemma 3.4.5. *If $u \in H^1_{\Gamma_1}(\Omega)^d$, $p \in L^2(\Omega)$ and $t \in H^{-1/2}(\Gamma_2)$ satisfy*

$$\frac{1}{2}(S(u), S(\varphi)) = (p, \operatorname{div} \varphi) - \langle t, \varphi\rangle_{H^{1/2}_{\gamma_0}(\Gamma_2)^d} \quad \textit{for all } \varphi \in H^1_{\Gamma_1}(\Omega), \tag{3.4.11}$$

then there exists a constant $c > 0$ such that

$$\|u\|_{H^1(\Omega)^d} \le c(\|p\|_{L^2(\Omega)} + \|t\|_{H^{-1/2}(\Gamma_2)}).$$

Proof. Putting $\varphi := u$ in (3.4.11), we obtain

$$\begin{aligned}\frac{1}{2}\|S(u)\|^2_{L^2(\Omega)^{d\times d}} &= (p, \operatorname{div} u) - \langle t, u\rangle_{H^{1/2}_{\gamma_0}(\Gamma_2)^d}\\ &\le \|p\|_{L^2(\Omega)}\|\operatorname{div} u\|_{L^2(\Omega)} + \|t\|_{H^{-1/2}(\Gamma_2)}\|u\|_{H^{1/2}(\Gamma_2)}\\ &\le (\sqrt{d}\|p\|_{L^2(\Omega)} + c_1\|t\|_{H^{-1/2}(\Gamma_2)})\|u\|_{H^1(\Omega)^d},\end{aligned}$$

for a constant $c_1 > 0$. By Theorem 3.2.1, there exists a constant $c_2 > 0$ such that

$$\frac{c_2}{2}\|u\|^2_{H^1(\Omega)^d} \le (\sqrt{d}\|p\|_{L^2(\Omega)} + c_1\|t\|_{H^{-1/2}(\Gamma_2)})\|u\|_{H^1(\Omega)^d}.$$

Hence, we obtain the result with $c = (2/c_2)\max\{\sqrt{d}, c_1\}$. □

The next theorem is the main result of this section.

Theorem 3.4.6. *If* $\Delta u^{S2} + \nabla(\operatorname{div} u^{S2}) \in L^2(\Omega)^d$, $p^{S2} \in H^1(\Omega)$ *and* $\Delta p^{S2} \in L^2(\Omega)$, *then there exists a constant* $c > 0$ *such that*

$$\|p^{S2} - p^{PP}\|_{H^1(\Omega)} \leq c \left\| \frac{\partial p^{S2}}{\partial n} - g^b \right\|_{(H^{1/2}_{\gamma_0}(\Gamma_1))^*},$$

$$\|u^{S2} - u^{PP}\|_{H^1(\Omega)^d} \leq c \left(\left\| \frac{\partial p^{S2}}{\partial n} - g^b \right\|_{(H^{1/2}_{\gamma_0}(\Gamma_1))^*} + \|t^{S2} - t^b\|_{(H^{1/2}_{\gamma_0}(\Gamma_2)^d)^*} \right),$$

where $t^{S2} = T_n(u^{S2}, p^{S2})$.

Proof. Using Proposition 3.4.4, we obtain from (SP) and (PPT), for all $\varphi \in H^1_{\Gamma_1}(\Omega)^d$ and $\psi \in H^1_{\Gamma_2}(\Omega)$,

$$\begin{cases} \frac{1}{2}(S(u^{S2} - u^{PP}), S(\varphi)) = (p^{S2} - p^{PP}, \operatorname{div} \varphi) - \langle t^{S2} - t^b, \varphi \rangle_{H^{1/2}_{\gamma_0}(\Gamma_2)^d}, \\ (\nabla(p^{S2} - p^{PP}), \nabla \psi) = \left\langle \frac{\partial p^{S2}}{\partial n} - g^b, \psi \right\rangle_{H^{1/2}_{\gamma_0}(\Gamma_1)}, \\ p^{S2} - p^{PP} = 0 \qquad \text{on } \Gamma_2, \end{cases} \tag{3.4.12}$$

where $t^{S2} = T_n(u^{S2}, p^{S2})$. By the second equation of (3.4.12) and Lemma 3.3.1, there exists a constant $c_1 > 0$ such that

$$\begin{aligned} \|p^{S2} - p^{PP}\|_{H^1(\Omega)} &\leq c_1 \left(\left\| \frac{\partial p^{S2}}{\partial n} - g^b \right\|_{(H^{1/2}_{\gamma_0}(\Gamma_1))^*} + \|p^{S2} - p^{PP}\|_{H^{1/2}(\Gamma_2)} \right) \\ &\leq c_1 \left\| \frac{\partial p^{S2}}{\partial n} - g^b \right\|_{(H^{1/2}_{\gamma_0}(\Gamma_1))^*}. \end{aligned}$$

By the first equation of (3.4.12) and Lemma 3.4.5,

$$\begin{aligned} \|u^{S2} - u^{PP}\|_{H^1(\Omega)^d} &\leq c_2 \left(\|p^{S2} - p^{PP}\|_{L^2(\Omega)} + \|t^{S2} - t^b\|_{(H^{1/2}_{\gamma_0}(\Gamma_2)^d)^*} \right) \\ &\leq c_2 \left(c_1 \left\| \frac{\partial p^{S2}}{\partial n} - g^b \right\|_{(H^{1/2}_{\gamma_0}(\Gamma_1))^*} + \|t^{S2} - t^b\|_{(H^{1/2}_{\gamma_0}(\Gamma_2)^d)^*} \right). \end{aligned}$$

□

3.5 Conclusion and future works

We have proposed a new formulation for the pressure Poisson problem (PPT). We have established error estimates between the solutions to (PPT') and (ST') in Theorem 3.3.2 and between the solutions to (PPT') and (SP) in Theorem 3.4.6. Theorems 3.3.2 and 3.4.6 state that if we have a good prediction for the boundary data (g^b or p^b), then the pressure Poisson problem is a good approximation for the Stokes problem. In particular, by using Theorem 3.4.6, we propose a new viewpoint of the pressure Poisson problem and the boundary condition (3.1.2). The numerical solution to the Stokes problem with the boundary condition (3.1.2) requires delicate choices of the weak formulation and special finite element techniques [12]. On the other hand, the pressure Poisson problem was previously used as a simple numerical scheme. From our results, we can confirm that the pressure Poisson problem is also available for the Stokes problem with the boundary condition (3.1.2).

For problem (SP), a finite element scheme is proposed in [13] (under the assumption that Γ_2 is flat). On the other hand, in many practical problems, the projection method is more widely used due to its ease of implementation. Numerical comparison of (PPT') and (SP) offers an interesting direction for future works from those points of view.

As another extension of our research, generalization of our results to the Navier–Stokes problem is important but is still completely open.

Chapter 4

ε-Stokes problem

This chapter is based on the following published paper:

- M. Kimura, K. Matsui, A. Muntean, and H. Notsu: Analysis of a projection method for the Stokes problem using an ε-Stokes approach. Japan Journal of Industrial and Applied Mathematics **36**, 959–985 (2019). DOI 10.1007/s13160-019-00373-3

- K. Matsui and A. Muntean: Asymptotic analysis of an ε-Stokes problem connecting Stokes and pressure Poisson problems. Advances in Mathematical Sciences and Applications **27**, 181–191 (2018).

4.1 Introduction

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$ ($d \geq 2, d \in \mathbb{N}$) and let $F : \Omega \to \mathbb{R}^d$ be a given applied force field and $u^b : \Gamma := \partial\Omega \to \mathbb{R}^d$ be given Dirichlet boundary data satisfying $\int_\Gamma u^b \cdot n ds = 0$. A strong form of the Stokes problem is given as follows. Find $u^S : \Omega \to \mathbb{R}^d$ and $p^S : \Omega \to \mathbb{R}$ such that

$$\begin{cases} -\Delta u^S + \nabla p^S = F & \text{in } \Omega, \\ \operatorname{div} u^S = 0 & \text{in } \Omega, \\ u^S = u^b & \text{on } \Gamma, \end{cases} \tag{S}$$

where u^S and p^S are the velocity and the pressure of the flow governed by (S), respectively. We refer to [14, 32, 80] for details on the Stokes problem (i.e., physical background and corresponding mathematical analysis). Taking the divergence of the first equation, we obtain

$$\operatorname{div} F = \operatorname{div}(-\Delta u^S + \nabla p^S) = -\Delta(\operatorname{div} u^S) + \Delta p^S = \Delta p^S.$$

This equation is often called the pressure Poisson equation and is used in numerical schemes, such as the MAC, SMAC, and projection methods (see, e.g., [4, 21, 25, 40, 42, 53, 64, 68]).

We need an additional boundary condition for solving equation (3.1.1). In real-world applications, the additional boundary condition is usually given by using experimental or plausible values. We consider the following problem: Find $u^{PP} : \Omega \to \mathbb{R}^d$ and $p^{PP} : \Omega \to \mathbb{R}$

satisfying

$$\begin{cases} -\Delta u^{PP} + \nabla p^{PP} = F & \text{in } \Omega, \\ -\Delta p^{PP} = -\operatorname{div} F & \text{in } \Omega, \\ u^{PP} = u^b & \text{on } \Gamma, \\ +\text{boundary condition for } p^{PP}. \end{cases} \tag{PP}$$

We call this problem the pressure Poisson problem. The idea of using (3.1.1) instead of $\operatorname{div} u^S = 0$ is useful for calculating the pressure numerically in the Navier–Stokes equation. For example, this idea is used in the MAC, SMAC, and projection methods. The Dirichlet boundary condition for the pressure is used in an outflow boundary [18, 84]. See also [23, 24, 63].

We introduce an "interpolation" between problems (S) and (PP). For $\varepsilon > 0$, find $u^\varepsilon : \Omega \to \mathbb{R}^d$ and $p^\varepsilon : \Omega \to \mathbb{R}$ such that

$$\begin{cases} -\Delta u^\varepsilon + \nabla p^\varepsilon = F & \text{in } \Omega, \\ -\varepsilon \Delta p^\varepsilon + \operatorname{div} u^\varepsilon = -\varepsilon \operatorname{div} F & \text{in } \Omega, \\ u^\varepsilon = u^b & \text{on } \Gamma, \\ +\text{boundary condition for } p^\varepsilon. \end{cases} \tag{ES}$$

We call this problem the ε-Stokes problem (ES). In [26, 33, 48], the authors treat this problem as an approximation of the Stokes problem to avoid numerical instabilities. The ε-Stokes problem approximates the Stokes problem (S) as $\varepsilon \to 0$ and the pressure Poisson problem (PP) as $\varepsilon \to \infty$ (Fig. 4.1). As in Chapter 3, we will show that if $p^S \in H^1(\Omega)$, then there exists a constant $c > 0$ independent of ε such that

$$\|u^S - u^{PP}\|_{H^1(\Omega)^d} + \|p^S - p^{PP}\|_{H^1(\Omega)} \leq c\|p^S - p^{PP}\|_{H^{1/2}(\Gamma)},$$
$$\|u^S - u^\varepsilon\|_{H^1(\Omega)^d} + \|p^S - p^\varepsilon\|_{H^1(\Omega)} \leq c\|p^S - p^{PP}\|_{H^{1/2}(\Gamma)}.$$

From the first inequality, if we have a good predictive value for pressure on Γ, then u^{PP} is a good approximation of u^S. Moreover, u^ε is also a good approximation of u^S from the second inequality.

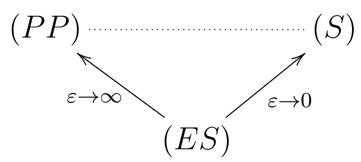

Figure 4.1: Sketch of the connections between problems (S), (PP) and (ES).

Next, we specify the boundary conditions for p^{PP} and p^ε. We consider a Neumann boundary condition (4.1.1) and a mixed boundary condition (4.1.2),

$$\frac{\partial p^{PP}}{\partial n} = g^b \text{ on } \Gamma, \quad \frac{\partial p^\varepsilon}{\partial n} = g^b \text{ on } \Gamma, \tag{4.1.1}$$

$$\begin{cases} \dfrac{\partial p^{PP}}{\partial n} = g^b & \text{on } \Gamma_1, \\ p^{PP} = p^b & \text{on } \Gamma_2, \end{cases} \quad \begin{cases} \dfrac{\partial p^\varepsilon}{\partial n} = g^b & \text{on } \Gamma_1, \\ p^\varepsilon = p^b & \text{on } \Gamma_2, \end{cases} \tag{4.1.2}$$

$$p^{PP} = p^b \text{ on } \Gamma, \quad p^\varepsilon = p^b \text{ on } \Gamma, \tag{4.1.3}$$

where $p^b : \Gamma \to \mathbb{R}$ and $g^b = \Gamma \to \mathbb{R}$ satisfying $\int_\Gamma g^b = \int_\Gamma \operatorname{div} F$ are given boundary data. The boundary condition (4.1.2) corresponds to (4.1.1) when $\Gamma_1 = \Gamma, \Gamma_2 = \emptyset$ and to (4.1.1) when $\Gamma_1 = \emptyset, \Gamma_2 = \Gamma$.

In this chapter, we introduce a weak solution $(u^\varepsilon, p^\varepsilon)$ to the ε-Stokes problem (ES) and prove that $(u^\varepsilon, p^\varepsilon)$ strongly converges in $H^1(\Omega)^d \times H^1(\Omega)$ to a weak solution to the pressure Poisson problem (PP) as $\varepsilon \to \infty$ and weakly converges in $H_0^1(\Omega)^d \times (L^2(\Omega)/\mathbb{R})$ to a weak solution (u^S, p^S) to the Stokes problem (S) as $\varepsilon \to 0$. In addition, for the Neumann boundary condition, we estimate the error between the weak solutions to (ES) and (S) provided $p^S \in H^1(\Omega)$. We also give an asymptotic expansion for the weak solution to (ES). We further check this convergence result using numerical computations.

The organization of this chapter is as follows. In Section 4.2, we introduce the weak form of these problems. We also prove the well-posedness of the problems (PP) and (ES). In Section 4.3, we establish error estimates between solutions to the problems (PP), (ES) and (S) in terms of the additional boundary conditions. In Section 4.4, we study that the solution to (ES) converges to the solution to (PP) in the strong topology as $\varepsilon \to \infty$. Here, we also explore the structure of regular perturbation asymptotics. Section 4.5 is devoted to proving that the solution to (ES) converges to the solution to (S) in the weak and strong topology as $\varepsilon \to 0$. In Section 4.6, we show several numerical examples of these problems. The numerical errors between problems (ES) and (PP), and between the problems (ES) and (S) using the P2/P1 finite element method. We conclude this chapter with several comments on future works in Section 4.7.

4.2 Weak formulation and well-posedness

In this section, we introduce the weak form of the problems (S), (PP) and (ES), and prove their well-posedness. We give estimates between these solutions by using a pressure error on the boundary Γ.

Let $Q \subset H^1(\Omega)$ be a closed subspace such that there exists a constant $c > 0$ for which $\|q\|_{L^2(\Omega)} \le c\|\nabla q\|_{L^2(\Omega)^d}$ for all $q \in Q$. The dual space Q^* is equipped with the norm

$$\|f\|_{Q^*} := \sup_{\psi \in S_Q} \langle f, \psi \rangle$$

for $f \in Q^*$, where

$$S_Q := \{\psi \in Q \mid \|\nabla \psi\|_{L^2(\Omega)^d} = 1\}.$$

4.2.1 Weak formulations of the problems (S), (PP) and (ES)

We assume the following conditions for F, u_b, g_b and p_b:

$$F \in L^2(\Omega)^d, \qquad u_b \in H^{1/2}(\Gamma), \qquad \int_\Gamma u_b \cdot n = 0, \tag{4.2.4}$$

$$g_b \in H^{-1/2}(\Gamma), \qquad \operatorname{div} F \in L^2(\Omega), \tag{4.2.5}$$

$$\langle g_b, 1\rangle_{H^{1/2}(\Gamma)} = \int_\Omega \operatorname{div} F\, dx, \tag{4.2.6}$$

$$p_b \in H^1(\Omega). \tag{4.2.7}$$

We start by defining the weak solution to (S). For all $\varphi \in H_0^1(\Omega)^d$, we obtain from the first equation of (S) that

$$\begin{aligned}(F,\varphi) &= -\int_\Gamma \frac{\partial u^S}{\partial n}\cdot\varphi\, ds + (\nabla u^S, \nabla\varphi) + \int_\Gamma p^S \varphi\cdot n\, ds - (p^S, \operatorname{div}\varphi)\\ &= (\nabla u^S, \nabla\varphi) + \langle \nabla p^S, \varphi\rangle_{H_0^1(\Omega)^d}.\end{aligned}$$

Using this expression, the weak form of the Stokes problem becomes as follows: Find $u^S \in H^1(\Omega)^d$ and $p^S \in L^2(\Omega)/\mathbb{R}$ such that

$$\begin{cases}(\nabla u^S, \nabla\varphi) + \langle \nabla p^S, \varphi\rangle_{H_0^1(\Omega)^d} = (F,\varphi) & \text{for all } \varphi \in H_0^1(\Omega)^d,\\ (\operatorname{div} u^S, \psi) = 0 & \text{for all } \psi \in L^2(\Omega)/\mathbb{R},\\ u^S = u_b & \text{in } H^{1/2}(\Gamma)^n.\end{cases} \tag{S'}$$

Remark 4.2.1. *If $(u^S, p^S) \in H^1(\Omega)^d \times L^2(\Omega)$ satisfies $u^S \in H^2(\Omega)^d, p^S \in H^1(\Omega)$ and (S'), then we have*

$$\begin{cases}(-\Delta u^S + \nabla p^S - F, \varphi) = 0 & \text{for all } \varphi \in H_0^1(\Omega)^d,\\ \operatorname{div} u^S = 0 & \text{in } L^2(\Omega),\\ u^S = u_b & \text{in } H^{1/2}(\Gamma)^n.\end{cases}$$

Therefore, (u^S, p^S) satisfies (S).

Next, we define the weak formulations of (PP) and (ES) first for the Neumann boundary condition (4.1.1) and them for the mixed boundary condition (4.1.2). After that, we define generalized weak formulations for (PP) and (ES) which cover both cases.

First, we apply the Neumann boundary condition (4.1.1) for (PP) and (ES). We take a test function $\psi \in H^1(\Omega)$. From the second equation of (PP), we obtain

$$\begin{aligned}-(\operatorname{div} F, \psi) &= -(\Delta p^{PP}, \psi)\\ &= -\int_\Gamma \frac{\partial p^{PP}}{\partial n}\psi\, ds + (\nabla p^{PP}, \nabla\psi)\\ &= -\langle g_b, \psi\rangle_{H^{1/2}(\Gamma)} + (\nabla p^{PP}, \nabla\psi).\end{aligned}$$

Hence,

$$(\nabla p^{PP}, \nabla\psi) = \langle g_b, \psi\rangle_{H^{1/2}(\Gamma)} - (\operatorname{div} F, \psi).$$

We note that $\langle g_b, \psi\rangle_{H^{1/2}(\Gamma)} - (\operatorname{div} F, \psi) = \langle g_b, [\psi]\rangle_{H^{1/2}(\Gamma)} - (\operatorname{div} F, [\psi])$ for all $\psi \in H^1(\Omega)$ by (4.2.6). Therefore, the weak form of the pressure Poisson problem with the Neumann boundary condition (4.1.1) becomes as follows. Find $u^{PP} \in H^1(\Omega)^d$ and $p^{PP} \in H^1(\Omega)/\mathbb{R}$ such that

$$\begin{cases} (\nabla u^{PP}, \nabla\varphi) + (\nabla p^{PP}, \varphi) = (F, \varphi) & \text{for all } \varphi \in H_0^1(\Omega)^d, \\ (\nabla p^{PP}, \nabla\psi) = \langle G_1, \psi\rangle_{H^1(\Omega)} & \text{for all } \psi \in H^1(\Omega)/\mathbb{R}, \\ u^{PP} = u_b & \text{in } H^{1/2}(\Gamma)^n, \end{cases} \tag{PP$_1$}$$

where $G_1 \in H^1(\Omega)^*$ defined by for all $\psi \in H^1(\Omega)$,

$$\langle G_1, \psi\rangle_{H^1(\Omega)} := \langle g_b, \psi\rangle_{H^{1/2}(\Gamma)} - (\operatorname{div} F, \psi). \tag{4.2.8}$$

The weak form of (ES) with the Neumann boundary condition can be defined similarly to that of (PP). Find $u^\varepsilon \in H^1(\Omega)^d$ and $p^\varepsilon \in H^1(\Omega)/\mathbb{R}$ such that

$$\begin{cases} (\nabla u^\varepsilon, \nabla\varphi) + (\nabla p^\varepsilon, \varphi) = (F, \varphi) & \text{for all } \varphi \in H_0^1(\Omega)^d, \\ \varepsilon(\nabla p^\varepsilon, \nabla\psi) + (\operatorname{div} u^\varepsilon, \psi) = \varepsilon\langle G_1, \psi\rangle_{H^1(\Omega)} & \text{for all } \psi \in H^1(\Omega)/\mathbb{R}, \\ u^\varepsilon = u_b & \text{in } H^{1/2}(\Gamma)^n. \end{cases} \tag{ES$_1$}$$

Remark 4.2.2. *If $(u^{PP}, p^{PP}) \in H^1(\Omega)^d \times H^1(\Omega)$ satisfies $u^{PP} \in H^2(\Omega)^d, p^{PP} \in H^1(\Omega)$ and (PP$_1$), then we have*

$$\begin{cases} (-\Delta u^{PP} + \nabla p^{PP} - F, \varphi) = 0 & \textit{for all } \varphi \in H_0^1(\Omega)^d, \\ (-\Delta p^{PP} + \operatorname{div} F, \psi) = \left\langle -\dfrac{\partial p^{PP}}{\partial n} + g_b, \psi \right\rangle_{H^{1/2}(\Gamma)} & \textit{for all } \psi \in H^1(\Omega), \\ u^{PP} = u_b & \textit{in } H^{1/2}(\Gamma)^n. \end{cases}$$

Therefore, (u^{PP}, p^{PP}) satisfies (PP) and the Neumann boundary condition (4.1.1).

In the same way, if $(u^\varepsilon, p^\varepsilon) \in H^1(\Omega)^d \times H^1(\Omega)$ satisfies $u^\varepsilon \in H^2(\Omega)^d, p^\varepsilon \in H^1(\Omega)$ and (ES$_1$), then we have

$$\begin{cases} (-\Delta u^\varepsilon + \nabla p^\varepsilon - F, \varphi) = 0 & \textit{for all } \varphi \in H_0^1(\Omega)^d, \\ (-\varepsilon\Delta p^\varepsilon + \operatorname{div} u^\varepsilon + \varepsilon \operatorname{div} F, \psi) = \varepsilon\left\langle -\dfrac{\partial p^\varepsilon}{\partial n} + g_b, \psi \right\rangle_{H^{1/2}(\Gamma)} & \textit{for all } \psi \in H^1(\Omega), \\ u^\varepsilon = u_b & \textit{in } H^{1/2}(\Gamma)^n. \end{cases}$$

Therefore, $(u^\varepsilon, p^\varepsilon)$ satisfies (ES) and the Neumann boundary condition (4.1.1).

Secondly, we apply the mixed boundary condition (4.1.2) for (PP) and (ES). We take a test function $\psi \in H^1_{\Gamma_2}(\Omega)$. From the second equation of (PP), we obtain

$$\begin{aligned} -(\operatorname{div} F, \psi) &= -(\Delta p^{PP}, \psi) \\ &= -\int_\Gamma \frac{\partial p^{PP}}{\partial n}\psi\, ds + (\nabla p^{PP}, \nabla\psi) \\ &= -\langle g_b, \psi\rangle_{H^{1/2}_{\gamma_0}(\Gamma_1)} + (\nabla p^{PP}, \nabla\psi). \end{aligned}$$

Hence,

$$(\nabla p^{PP}, \nabla\psi) = \langle g_b, \psi\rangle_{H^{1/2}_{\gamma_0}(\Gamma_1)} - (\operatorname{div} F, \psi).$$

The weak form of the pressure Poisson problem with the mixed boundary condition (4.1.2) becomes as follows. Find $u^{PP} \in H^1(\Omega)^d$ and $p^{PP} \in H^1(\Omega)$ such that

$$\begin{cases} (\nabla u^{PP}, \nabla\varphi) + (\nabla p^{PP}, \varphi) = (F, \varphi) & \text{for all } \varphi \in H^1_0(\Omega)^d, \\ (\nabla p^{PP}, \nabla\psi) = \langle G_2, \psi\rangle_{H^1_{\Gamma_2}(\Omega)} & \text{for all } \psi \in H^1_{\Gamma_2}(\Omega), \\ u^{PP} = u_b & \text{in } H^{1/2}(\Gamma)^n, \\ p^{PP} = p_b & \text{in } H^{1/2}(\Gamma_2), \end{cases} \tag{PP$_2$}$$

where $G_2 \in H^1_{\Gamma_2}(\Omega)^*$ defined by for all $\psi \in H^1_{\Gamma_2}(\Omega)$,

$$\langle G_2, \psi\rangle_{H^1_{\Gamma_2}(\Omega)} := \langle g_b, \psi\rangle_{H^{1/2}_{\gamma_0}(\Gamma_1)} - (\operatorname{div} F, \psi). \tag{4.2.9}$$

The weak form of (ES) with the mixed boundary condition (4.1.2) can be defined similarly to that of (PP). It reads as follows. Find $u^\varepsilon \in H^1(\Omega)^d$ and $p^\varepsilon \in H^1(\Omega)$ such that

$$\begin{cases} (\nabla u^\varepsilon, \nabla\varphi) + (\nabla p^\varepsilon, \varphi) = (F, \varphi) & \text{for all } \varphi \in H^1_0(\Omega)^d, \\ \varepsilon(\nabla p^\varepsilon, \nabla\psi) + (\operatorname{div} u^\varepsilon, \psi) = \varepsilon\langle G_2, \psi\rangle_{H^1_{\Gamma_2}(\Omega)} & \text{for all } \psi \in H^1_{\Gamma_2}(\Omega), \\ u^\varepsilon = u_b & \text{in } H^{1/2}(\Gamma)^n, \\ p^\varepsilon = p_b & \text{in } H^{1/2}(\Gamma_2). \end{cases} \tag{ES$_2$}$$

Remark 4.2.3. *If $(u^{PP}, p^{PP}) \in H^1(\Omega)^d \times H^1(\Omega)$ satisfies $u^{PP} \in H^2(\Omega)^d, p^{PP} \in H^1(\Omega)$ and (PP$_2$), then we have*

$$\begin{cases} (-\Delta u^{PP} + \nabla p^{PP} - F, \varphi) = 0 & \textit{for all } \varphi \in H^1_0(\Omega)^d, \\ (-\Delta p^{PP} + \operatorname{div} F, \psi) = \left\langle -\dfrac{\partial p^{PP}}{\partial n} + g_b, \psi \right\rangle_{H^1_{\Gamma_1}(\Omega)} & \textit{for all } \psi \in H^1_{\Gamma_2}(\Omega), \\ u^{PP} = u_b & \textit{in } H^{1/2}(\Gamma)^n, \\ p^{PP} = p_b & \textit{in } H^{1/2}(\Gamma_2). \end{cases}$$

Therefore, (u^{PP}, p^{PP}) satisfies (PP) and the mixed boundary condition (4.1.2).

In the same way, if $(u^\varepsilon, p^\varepsilon) \in H^1(\Omega)^d \times H^1(\Omega)$ satisfies $u^\varepsilon \in H^2(\Omega)^d, p^\varepsilon \in H^1(\Omega)$ and (ES$_2$), then we have

$$\begin{cases} (-\Delta u^\varepsilon + \nabla p^\varepsilon - F, \varphi) = 0 & \textit{for all } \varphi \in H^1_0(\Omega)^d, \\ (-\varepsilon\Delta p^\varepsilon + \operatorname{div} u^\varepsilon + \varepsilon \operatorname{div} F, \psi) = \varepsilon \left\langle -\dfrac{\partial p^\varepsilon}{\partial n} + g_b, \psi \right\rangle_{H^1_{\Gamma_1}(\Omega)} & \textit{for all } \psi \in H^1(\Omega)/\mathbb{R}, \\ u^\varepsilon = u_b & \textit{in } H^{1/2}(\Gamma)^n, \\ p^\varepsilon = p_b & \textit{in } H^{1/2}(\Gamma_2). \end{cases}$$

Therefore, $(u^\varepsilon, p^\varepsilon)$ satisfies (ES) and the mixed boundary condition (4.1.2).

When $\Gamma_1 = \emptyset$ and $\Gamma_2 = \Gamma$, the mixed boundary condition (4.1.2) becomes the full-Dirichlet boundary condition (4.1.3). Hence, the weak form of the pressure Poisson problem with the full-Dirichlet boundary condition (4.1.3) becomes as follows. Find

$u^{PP} \in H^1(\Omega)^d$ and $p^{PP} \in H^1(\Omega)$ such that

$$\begin{cases} (\nabla u^{PP}, \nabla\varphi) + (\nabla p^{PP}, \varphi) = (F, \varphi) & \text{for all } \varphi \in H_0^1(\Omega)^d, \\ (\nabla p^{PP}, \nabla\psi) = \langle G_3, \psi\rangle_{H_0^1(\Omega)} & \text{for all } \psi \in H_0^1(\Omega), \\ u^{PP} = u_b & \text{in } H^{1/2}(\Gamma)^n, \\ p^{PP} = p_b & \text{in } H^{1/2}(\Gamma), \end{cases} \tag{PP$_3$}$$

where $G_2 \in H^{-1}(\Omega)$ defined by for all $\psi \in H_0^1(\Omega)$,

$$\langle G_3, \psi\rangle_{H_0^1(\Omega)} := -(\operatorname{div} F, \psi). \tag{4.2.10}$$

In the same way, the weak form of the ε-Stokes problem with the full-Dirichlet boundary condition (4.1.3) becomes as follows. Find $u^\varepsilon \in H^1(\Omega)^d$ and $p^\varepsilon \in H^1(\Omega)$ such that

$$\begin{cases} (\nabla u^\varepsilon, \nabla\varphi) + (\nabla p^\varepsilon, \varphi) = (F, \varphi) & \text{for all } \varphi \in H_0^1(\Omega)^d, \\ \varepsilon(\nabla p^\varepsilon, \nabla\psi) + (\operatorname{div} u^\varepsilon, \psi) = \varepsilon\langle G_3, \psi\rangle_{H_0^1(\Omega)} & \text{for all } \psi \in H_0^1(\Omega), \\ u^\varepsilon = u_b & \text{in } H^{1/2}(\Gamma)^n, \\ p^\varepsilon = p_b & \text{in } H^{1/2}(\Gamma). \end{cases} \tag{ES$_3$}$$

Finally, we generalize (PP$_1$), (PP$_2$), and (PP$_3$) to an abstract pressure Poisson problem. Let $Q \subset H^1(\Omega)$ be a closed subspace as defined in Section 4.2. Find $u^{PP} \in H^1(\Omega)^d$ and $p^{PP} \in Q$ such that

$$\begin{cases} (\nabla u^{PP}, \nabla\varphi) + (\nabla p^{PP}, \varphi) = (F, \varphi) & \text{for all } \varphi \in H_0^1(\Omega)^d, \\ (\nabla p^{PP}, \nabla\psi) = \langle G, \psi\rangle_Q & \text{for all } \psi \in Q, \\ u^{PP} = u_b & \text{in } H^{1/2}(\Gamma)^n, \\ p^{PP} - p_b \in Q, \end{cases} \tag{PP'}$$

with $G \in Q^*$. Indeed, we obtain (PP$_1$) (resp. (PP$_2$), (PP$_3$)) from (PP') by putting $Q := H^1(\Omega)/\mathbb{R}$ (resp. $H^1_{\Gamma_2}(\Omega), H^1_0(\Omega)$) and $G := G_1$ (resp. G_2, G_3).

We generalize (ES$_1$), (ES$_2$), and (ES$_3$) to an abstract ε-Stokes problem. Find $u^\varepsilon \in H^1(\Omega)^d$ and $p^\varepsilon \in Q$ such that

$$\begin{cases} (\nabla u^\varepsilon, \nabla\varphi) + (\nabla p^\varepsilon, \varphi) = (F, \varphi) & \text{for all } \varphi \in H_0^1(\Omega)^d, \\ \varepsilon(\nabla p^\varepsilon, \nabla\psi) + (\operatorname{div} u^\varepsilon, \psi) = \varepsilon\langle G, \psi\rangle_Q & \text{for all } \psi \in Q, \\ u^\varepsilon - u_b \in H_0^1(\Omega)^d, \\ p^\varepsilon - p_b \in Q. \end{cases} \tag{ES'}$$

Indeed, we obtain (ES$_1$) (resp. (ES$_2$), (ES$_3$)) from (ES') by putting $Q := H^1(\Omega)/\mathbb{R}$ (resp. $H^1_{\Gamma_2}(\Omega), H^1_0(\Omega)$) and $G := G_1$ (resp. G_2, G_3).

4.2.2 Well-posedness of (S'), (PP') and (ES')

We show the well-posedness of problems (S'), (PP') and (ES') in Theorems 4.2.4, 4.2.5 and 4.2.6.

Theorem 4.2.4. *Under the condition (4.2.4), there exists a unique solution* $(u^S, p^S) \in H^1(\Omega)^d \times (L^2(\Omega)/\mathbb{R})$ *satisfying (S').*

Proof. We take arbitrary $u_1 \in H^1(\Omega)^d$ with $\gamma_0 u_1 = u_b$. By Theorem 2.2.15, there exists $u_2 \in H_0^1(\Omega)^d$ such that $\operatorname{div} u_2 = \operatorname{div} u_1$. We put $u_0 := u_1 - u_2$, and note that $\gamma_0 u_0 = u_b$ and $\operatorname{div} u_0 = 0$. The problem (S') is equivalent to the following equations:

$$\begin{cases} (\nabla(u^S - u_0), \nabla\varphi) - (p^S, \operatorname{div}\varphi) = (F, \varphi) - (\nabla u_0, \nabla\varphi) & \text{for all } \varphi \in H_0^1(\Omega)^d, \\ (\psi, \operatorname{div}(u^S - u_0)) = 0 & \text{for all } \psi \in L^2(\Omega)/\mathbb{R}, \\ u^S - u_0 \in H_0^1(\Omega)^d. \end{cases} \tag{4.2.11}$$

By Theorem 2.2.7, the continuous bilinear form $H_0^1(\Omega)^d \times H_0^1(\Omega)^d \ni (u, \varphi) \mapsto \int_\Omega \nabla u : \nabla\varphi \, dx \in \mathbb{R}$ is coercive. By Theorems 2.2.12 and 2.2.14, there exists a unique solution $(u^S, p^S) \in H^1(\Omega)^d \times (L^2(\Omega)/\mathbb{R})$ satisfying (4.2.11). □

Theorem 4.2.5. *Under the conditions (4.2.4) and (4.2.7), for* $G \in Q^*$*, there exists a unique solution* $(u^{PP}, p^{PP}) \in H^1(\Omega)^d \times Q$ *satisfying (PP').*

Proof. Using the Lax–Milgram theorem, since $Q \times Q \ni (p, \psi) \mapsto \int_\Omega \nabla p \cdot \nabla\psi \, dx \in \mathbb{R}$ is a continuous and coercive bilinear form, $p^{PP} \in H^1(\Omega)$ is uniquely determined from the second and fourth equations of (PP'). Then, $u^{PP} \in H^1(\Omega)^d$ is also uniquely determined from the first and third equations, again using the Lax–Milgram theorem. □

Theorem 4.2.6. *Under the conditions (4.2.4) and (4.2.7), for* $\varepsilon > 0$ *and* $G \in Q^*$*, there exists a unique solution* $(u^\varepsilon, p^\varepsilon) \in H^1(\Omega)^d \times H^1(\Omega)$ *satisfying (ES').*

Proof. We take arbitrary $u_1 \in H^1(\Omega)^d$ with $\gamma_0 u_1 = u_b$. Since $\operatorname{div} : H_0^1(\Omega)^d \to L^2(\Omega)/\mathbb{R}$ is surjective [32, Corollary 2.4, 2°], there exists $u_2 \in H_0^1(\Omega)^d$ such that $\operatorname{div} u_2 = \operatorname{div} u_1$. We put

$$u_0 := u_1 - u_2, \tag{4.2.12}$$

and note that $\gamma_0 u_0 = u_b$ and $\operatorname{div} u_0 = 0$. To simplify the notation, we set $u := u^\varepsilon - u_0 \in H_0^1(\Omega)^d, p := p^\varepsilon - p_b \in Q$, and define $f \in H^{-1}(\Omega)^d$ and $g \in Q^*$ by

$$\begin{aligned} \langle f, v \rangle_{H_0^1(\Omega)^d} &:= (F, v) - (\nabla u_0, \nabla v) - (\nabla p_b, v) && \text{for all } v \in H_0^1(\Omega)^d, \\ \langle g, q \rangle_Q &:= \langle G, q \rangle_Q - (\nabla p_b, \nabla q) && \text{for all } q \in Q. \end{aligned} \tag{4.2.13}$$

Then, $(u^\varepsilon, p^\varepsilon)$ satisfies (ES') if and only if (u, p) satisfies

$$\begin{cases} (\nabla u, \nabla\varphi) + (\nabla p, \varphi) = \langle f, \varphi \rangle_{H_0^1(\Omega)^d} & \text{for all } \varphi \in H_0^1(\Omega)^d, \\ \varepsilon(\nabla p, \nabla\psi) + (\operatorname{div} u, \psi) = \varepsilon\langle g, \psi \rangle_Q & \text{for all } \psi \in Q. \end{cases} \tag{4.2.14}$$

Adding the equations in (4.2.14), we get

$$\begin{aligned} \left(\begin{pmatrix} u \\ p \end{pmatrix}, \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \right)_\varepsilon &:= (\nabla u, \nabla\varphi) + \varepsilon(\nabla p, \nabla\psi) + (\nabla p, \varphi) + (\operatorname{div} u, \psi) \\ &= \langle f, \varphi \rangle_{H_0^1(\Omega)^d} + \varepsilon\langle g, \psi \rangle_Q. \end{aligned}$$

We check that $(\cdot,\cdot)_\varepsilon$ is a continuous coercive bilinear form on $H_0^1(\Omega)^d \times Q$. The bilinearity and continuity of $(\cdot,\cdot)_\varepsilon$ are obvious. The coercivity of $(\cdot,\cdot)_\varepsilon$ is obtained in the following way. Take $(v,q)^T \in H_0^1(\Omega)^d \times Q$. We have the following sequence of inequalities:

$$\begin{aligned} \left(\begin{pmatrix} v \\ q \end{pmatrix}, \begin{pmatrix} v \\ q \end{pmatrix} \right)_\varepsilon &= (\nabla v, \nabla v) + \varepsilon(\nabla q, \nabla q) + (v, \nabla q) + (\operatorname{div} v, q) \\ &= \|\nabla v\|_{L^2(\Omega)}^2 + \varepsilon \|\nabla q\|_{L^2(\Omega)}^2 \\ &\geq \min\{1,\varepsilon\} \left(\|\nabla v\|_{L^2(\Omega)}^2 + \|\nabla q\|_{L^2(\Omega)}^2 \right) \\ &\geq c \min\{1,\varepsilon\} \left(\|v\|_{H^1(\Omega)^d}^2 + \|q\|_{H^1(\Omega)}^2 \right). \end{aligned}$$

Summarizing, $(\cdot,\cdot)_\varepsilon$ is a continuous coercive bilinear form and $H_0^1(\Omega)^d \times Q$ is a Hilbert space. Therefore, the conclusion of Theorem 4.2.6 follows from the Lax–Milgram Theorem. □

4.3 Error estimates in terms of the additional boundary condition

In this section, as in Chapter 3, we give estimates of the difference between the solutions to the pressure Poisson problem, the ε-Stokes problem and the Stokes problem, respectively.

We prove the following lemma about estimates of the difference between the solutions to the ε-Stokes problem and the Stokes problem.

Lemma 4.3.1. *If $p^S \in H^1(\Omega)$, then there exists a constant $c > 0$ independent of ε such that*

$$\|u^S - u^\varepsilon\|_{H^1(\Omega)^d} \leq c \|\nabla(p^S - p^{PP})\|_{L^2(\Omega)^d}.$$

Proof. Let $w^\varepsilon := u^S - u^\varepsilon \in H_0^1(\Omega)^d$ and $r^\varepsilon := p^{PP} - p^\varepsilon \in Q$. By (S'), (PP') and (ES'), we obtain

$$\begin{cases} (\nabla w^\varepsilon, \nabla \varphi) + (\nabla r^\varepsilon, \varphi) = -(\nabla(p^S - p^{PP}), \varphi) & \text{for all } \varphi \in H_0^1(\Omega)^d, \\ \varepsilon(\nabla r^\varepsilon, \nabla \psi) + (\operatorname{div} w^\varepsilon, \psi) = 0 & \text{for all } \psi \in Q. \end{cases} \tag{4.3.15}$$

Putting $\varphi := w^\varepsilon$ and $\psi := r^\varepsilon$ and adding the two equations of (4.3.15), we get

$$\|\nabla w^\varepsilon\|_{L^2(\Omega)^{d\times d}}^2 + \varepsilon \|\nabla r^\varepsilon\|_{L^2(\Omega)^d}^2 \leq \|\nabla(p^S - p^{PP})\|_{L^2(\Omega)^d} \|w^\varepsilon\|_{L^2(\Omega)^d}$$

from $\int_\Omega (\nabla r^\varepsilon) \cdot w^\varepsilon \, dx = -\int_\Omega (\operatorname{div} w^\varepsilon) r^\varepsilon \, dx$. Thus we find

$$\|w^\varepsilon\|_{H^1(\Omega)^d} \leq c \|\nabla(p^S - p^{PP})\|_{L^2(\Omega)^d}$$

for a constant $c > 0$ independent of ε. □

By Lemma 4.3.1, if we have a good prediction for the pressure boundary data, then (ES) is also good approximation for (S). In this section, we prove these types of estimates for the weak solutions.

Theorem 4.3.2. *Suppose that $p^S \in H^1(\Omega)$, $H_0^1(\Omega) \subset Q$ and $\langle G, \psi\rangle_Q = -(\operatorname{div} F, \psi)$ for all $\psi \in H_0^1(\Omega)$. Then, there exists a constant $c > 0$ independent of ε such that*

$$\begin{aligned} \|u^S - u^{PP}\|_{H^1(\Omega)^d} &\leq c\|p^S - p^{PP}\|_{H^{1/2}(\Gamma)}, \\ \|u^S - u^{\varepsilon}\|_{H^1(\Omega)^d} &\leq c\|p^S - p^{PP}\|_{H^{1/2}(\Gamma)}. \end{aligned} \tag{4.3.16}$$

In particular, if $p^S = p^{PP}$, then $(u^S, p^S) = (u^{PP}, p^{PP}) = (u^{\varepsilon}, p^{\varepsilon})$ holds for all $\varepsilon > 0$.

Proof. First, we prove that there exists a constant $c > 0$ independent of ε such that $\|u^S - u^{PP}\|_{H^1(\Omega)^d} \leq c\|p^S - p^{PP}\|_{H^{1/2}(\Gamma)}$, and if $(p^S - p^{PP}) = 0$, then $p^{PP} = p^S$. Taking the divergence of the first equation of (S'), we obtain

$$\operatorname{div} F = \operatorname{div}(-\Delta u^S + \nabla p^S) = -\Delta(\operatorname{div} u^S) + \Delta p^S = \Delta p^S.$$

in distributions sense. Since $p^S \in H^1(\Omega)$ and $C_0^{\infty}(\Omega)$ is dense in $H_0^1(\Omega)$, it follows that

$$(\nabla p^S, \nabla \psi) = -(\operatorname{div} F, \psi)$$

for all $\psi \in H_0^1(\Omega)$. Together with (S'), (PP') and $H_0^1(\Omega) \subset Q$, we obtain

$$\begin{cases} (\nabla(u^S - u^{PP}), \nabla\varphi) = -(\nabla(p^S - p^{PP}), \varphi) & \text{for all } \varphi \in H_0^1(\Omega)^d, \\ (\nabla(p^S - p^{PP}), \nabla\psi) = 0 & \text{for all } \psi \in H_0^1(\Omega) \end{cases} \tag{4.3.17}$$

from the assumption $\langle G, \psi\rangle_Q = (\nabla F, \psi)$. Putting $\varphi := u^S - u^{PP} \in H_0^1(\Omega)^d$ in (4.3.17), we get

$$\begin{aligned} \|\nabla(u^S - u^{PP})\|^2_{L^2(\Omega)^{d\times d}} &= -(\nabla(p^S - p^{PP}), u^S - u^{PP}) \\ &\leq \|\nabla(p^S - p^{PP})\|_{L^2(\Omega)^d}\|u^S - u^{PP}\|_{L^2(\Omega)^d}. \end{aligned}$$

Hence,

$$\|u^S - u^{PP}\|_{H^1(\Omega)^d} \leq c_1\|\nabla(p^S - p^{PP})\|_{L^2(\Omega)^d}. \tag{4.3.18}$$

From the second equation of (4.3.17) and Lemma 3.3.1 (with $\Gamma_1 = \emptyset$ and $\Gamma_2 = \Gamma$ i.e., $H^1_{\Gamma_2}(\Omega) = H_0^1(\Omega)$), we obtain

$$\|p^S - p^{PP}\|_{H^1(\Omega)} \leq c_2\|p^S - p^{PP}\|_{H^{1/2}(\Gamma)}. \tag{4.3.19}$$

Together with (4.3.18), we obtain $\|u^S - u^{PP}\|_{H^1(\Omega)^d} \leq c_1c_2\|p^S - p^{PP}\|_{H^{1/2}(\Gamma)}$. Moreover, if $\gamma_0(p^S - p^{PP}) = 0$ then $p^{PP} = p^S$.

Next, we prove that there exists a constant $c > 0$ independent of ε such that $\|u^S - u^{\varepsilon}\|_{H^1(\Omega)^d} \leq c\|p^S - p^{\varepsilon}\|_{H^{1/2}(\Gamma)}$, and if $\gamma_0(p^S - p^{PP}) = 0$, then $p^{PP} = p^{\varepsilon}$. Let $w^{\varepsilon} := u^S - u^{\varepsilon} \in H_0^1(\Omega)^d$ and $r^{\varepsilon} := p^{PP} - p^{\varepsilon} \in Q$. By (S'), (PP') and (ES'), we obtain

$$\begin{cases} (\nabla w^{\varepsilon}, \nabla\varphi) + (\nabla r^{\varepsilon}, \varphi) = -(\nabla(p^S - p^{PP}), \varphi) & \text{for all } \varphi \in H_0^1(\Omega)^d, \\ \varepsilon(\nabla r^{\varepsilon}, \nabla\psi) + (\operatorname{div} w^{\varepsilon}, \psi) = 0 & \text{for all } \psi \in Q. \end{cases} \tag{4.3.20}$$

Putting $\varphi := w^\varepsilon$ and $\psi := r^\varepsilon$ and adding the two equations of (4.3.20), we get

$$\|\nabla w^\varepsilon\|^2_{L^2(\Omega)^{d\times d}} + \varepsilon\|\nabla r^\varepsilon\|^2_{L^2(\Omega)^d} \le \|\nabla(p^S - p^{PP})\|_{L^2(\Omega)^d}\|w^\varepsilon\|_{L^2(\Omega)^d} \tag{4.3.21}$$

from $\int_\Omega(\nabla r^\varepsilon)\cdot w^\varepsilon\, dx = -\int_\Omega(\operatorname{div} w^\varepsilon) r^\varepsilon\, dx$. Thus we find

$$\|w^\varepsilon\|_{H^1(\Omega)^d} \le c_3\|\nabla(p^S - p^{PP})\|_{L^2(\Omega)^d}.$$

Together with (4.3.19), we obtain

$$\|u^S - u^\varepsilon\|_{H^1(\Omega)^d} = \|w^\varepsilon\|_{H^1(\Omega)^d} \le c_2c_3\|p^S - p^{PP}\|_{H^{1/2}(\Gamma)}.$$

Moreover, by (4.3.21), we obtain

$$\varepsilon\|p^{PP} - p^\varepsilon\|^2_{L^2(\Omega)} = \varepsilon\|r^\varepsilon\|^2_{L^2(\Omega)} \le c_4\|\nabla(p^S - p^{PP})\|_{L^2(\Omega)^d}\|w^\varepsilon\|_{L^2(\Omega)^d}.$$

Hence, if $\gamma_0(p^S - p^{PP}) = 0$, then $p^{PP} = p^\varepsilon$. □

Since $H^1_0(\Omega) \not\subset H^1(\Omega)/\mathbb{R}$, Theorem 4.3.2 does not apply directly for the case of the Neumann boundary condition (4.1.1). However, we add natural assumptions, then it leads to (4.3.16).

Corollary 4.3.3. *Suppose that $p^S \in H^1(\Omega)$ and $Q = H^1(\Omega)/\mathbb{R}$. If $G = G_1$ defined by (4.2.8), then we have (4.3.16).*

Proof. By (4.2.8), it holds that

$$(\nabla p^{PP}, \nabla\psi) = -(\operatorname{div} F, \psi)$$

for all $\psi \in H^1_0(\Omega)$ from the second equation of (PP'). Hence, it leads the second equation of (4.3.17). Using the proof of Theorem 4.3.2, we obtain (4.3.16). □

We focus on the mixed boundary conditions (4.1.2), i.e., (PP_2) and (ES_2).

Proposition 4.3.4. *If (u^S, p^S) satisfies $p^S \in H^1(\Omega)$ and $\Delta p^S \in L^2(\Omega)$, then we have*

$$(\nabla p^S, \nabla\psi) = -(\operatorname{div} F, \psi) + \left\langle \frac{\partial p^S}{\partial n}, \psi\right\rangle_{H^{1/2}_{\gamma_0}(\Gamma_1)}$$

for all $\psi \in H^1_{\Gamma_2}(\Omega)$.

Proof. From the first equation of (S'), we obtain

$$-\Delta u^S + \nabla p^S = F \qquad \text{in } \mathscr{D}'(\Omega).$$

Taking the divergence, we get

$$\operatorname{div} F = \operatorname{div}(-\Delta u^S + \nabla p^S) = -\Delta(\operatorname{div} u^S) + \Delta p^S = \Delta p^S \qquad \text{in } \mathscr{D}'(\Omega).$$

By the assumptions $\Delta p^S \in L^2(\Omega)$ and $\operatorname{div} F \in L^2(\Omega)$, $\Delta p^S = \operatorname{div} F$ holds in $L^2(\Omega)$. Multiplying $\psi \in H^1_{\Gamma_2}(\Omega)$ and integrating over Ω, we get

$$-(\operatorname{div} F, \psi) = -(\Delta p^S, \psi) = (\nabla p^S, \nabla\psi) - \left\langle \frac{\partial p^S}{\partial n}, \psi\right\rangle_{H^{1/2}_{\gamma_0}(\Gamma_1)},$$

which is the desired result. □

Using Proposition 4.3.4, we prove the following theorem.

Theorem 4.3.5. *Let* $Q = H^1_{\Gamma_2}(\Omega)$ *and* $G = G_2$ *defined by (4.2.9). If* $p^S \in H^1(\Omega)$ *and* $\Delta p^S \in L^2(\Omega)$*, there exists a constant* $c > 0$ *such that*

$$\begin{aligned} \|u^S - u^{PP}\|_{H^1(\Omega)^d} &\leq c\left(\left\|\frac{\partial p^S}{\partial n} - g_b\right\|_{(H^{1/2}_{\gamma_0}(\Gamma_1))^*} + \|p^S - p_b\|_{H^{1/2}(\Gamma_2)}\right), \\ \|u^S - u^{\varepsilon}\|_{H^1(\Omega)^d} &\leq c\left(\left\|\frac{\partial p^S}{\partial n} - g_b\right\|_{(H^{1/2}_{\gamma_0}(\Gamma_1))^*} + \|p^S - p_b\|_{H^{1/2}(\Gamma_2)}\right). \end{aligned} \tag{4.3.22}$$

Proof. Using Proposition 4.3.4, we obtain from (S') and (PP'),

$$\begin{cases} (\nabla(u^S - u^{PP}), \nabla\varphi) = (p^S - p^{PP}, \varphi) & \text{for all } \varphi \in H^1_0(\Omega)^d, \\ (\nabla(p^S - p^{PP}), \nabla\psi) = \left\langle \dfrac{\partial p^S}{\partial n} - g_b, \psi \right\rangle_{H^{1/2}_{\gamma_0}(\Gamma_1)} & \text{for all } \psi \in H^1_{\Gamma_2}(\Omega). \end{cases} \tag{4.3.23}$$

Putting $\varphi := u^S - u^{PP} \in H^1_0(\Omega)^d$ in (4.3.23), we get

$$\begin{aligned} \|\nabla(u^S - u^{PP})\|^2_{L^2(\Omega)^{d\times d}} &= (p^S - p^{PP}, \operatorname{div}(u^S - u^{PP})) \\ &\leq \|p^S - p^{PP}\|_{L^2(\Omega)}\|\operatorname{div}(u^S - u^{PP})\|_{L^2(\Omega)} \\ &\leq \sqrt{d}\|p^S - p^{PP}\|_{H^1(\Omega)}\|u^S - u^{PP}\|_{H^1(\Omega)^d}. \end{aligned}$$

From Theorem 2.2.7, it follows that

$$\|u^S - u^{PP}\|_{H^1(\Omega)^d} \leq c_1\|p^S - p^{PP}\|_{H^1(\Omega)}$$

for a constant $c_1 > 0$. By the second equation of (4.3.23) and Lemma 3.3.1, there exists a constant $c_2 > 0$ such that

$$\begin{aligned} \|p^S - p^{PP}\|_{H^1(\Omega)} &\leq c_2\left(\left\|\frac{\partial p^S}{\partial n} - g_b\right\|_{(H^{1/2}_{\gamma_0}(\Gamma_1))^*} + \|p^S - p^{PP}\|_{H^{1/2}(\Gamma_2)}\right) \\ &= c_2\left(\left\|\frac{\partial p^S}{\partial n} - g_b\right\|_{(H^{1/2}_{\gamma_0}(\Gamma_1))^*} + \|p^S - p_b\|_{H^{1/2}(\Gamma_2)}\right). \end{aligned}$$

Hence, we obtain the first inequality of (4.3.22) with $c = c_1c_2$. By Lemma 4.3.1, it holds that

$$\begin{aligned} \|u^S - u^{PP}\|_{H^1(\Omega)^d} &\leq c_3\|\nabla(p^S - p^{PP})\|_{L^2(\Omega)^d} \\ &\leq c_2c_3\left(\left\|\frac{\partial p^S}{\partial n} - g_b\right\|_{(H^{1/2}_{\gamma_0}(\Gamma_1))^*} + \|p^S - p_b\|_{H^{1/2}(\Gamma_2)}\right). \end{aligned}$$

□

In the same way as above, we also obtain estimates of the difference between the solutions to (S'), (PP_1) and (ES_1), respectively.

Corollary 4.3.6. *Let* $Q = H^1(\Omega)/\mathbb{R}$ *and* $G = G_1$ *defined by (4.2.8). If* $p^S \in H^1(\Omega)$ *and* $\Delta p^S \in L^2(\Omega)$*, there exists a constant* $c > 0$ *such that*

$$\begin{aligned} \|u^S - u^{PP}\|_{H^1(\Omega)^d} &\leq c\left\|\frac{\partial p^S}{\partial n} - g_b\right\|_{H^{-1/2}(\Gamma)}, \\ \|u^S - u^{\varepsilon}\|_{H^1(\Omega)^d} &\leq c\left\|\frac{\partial p^S}{\partial n} - g_b\right\|_{H^{-1/2}(\Gamma)}. \end{aligned}$$

4.4 Links between (ES) and (PP)

In this section, we show that $(u^\varepsilon, p^\varepsilon)$ converges to (u^{PP}, p^{PP}) strongly in $H^1(\Omega)^d \times H^1(\Omega)$ as $\varepsilon \to \infty$. We also treat the case of the regular perturbation asymptotics by exploring the structure of the lower order terms and their effect on the convergence rate.

4.4.1 Convergence as $\varepsilon \to \infty$

We use the following Lemma 4.4.1 for the proofs of the theorems in this section.

Lemma 4.4.1. *Let $h \in Q^*$ and $(v^\varepsilon, q^\varepsilon) \in H_0^1(\Omega)^d \times Q$ satisfy*

$$\begin{cases} (\nabla v^\varepsilon, \nabla \varphi) + (\nabla q^\varepsilon, \varphi) = 0 & \text{for all } \varphi \in H_0^1(\Omega)^d, \\ \varepsilon(\nabla q^\varepsilon, \nabla \psi) + (\operatorname{div} v^\varepsilon, \psi) = \langle h, \psi \rangle_Q & \text{for all } \psi \in Q \end{cases} \tag{4.4.24}$$

for an arbitrarily fixed $\varepsilon > 0$. Then, there exists a constant $c > 0$ such that

$$\|v^\varepsilon\|_{H^1(\Omega)^d} + \|q^\varepsilon\|_{H^1(\Omega)} \le \frac{c}{\varepsilon}\|h\|_{Q^*}.$$

Proof. Putting $\varphi := v^\varepsilon$ and $\psi := q^\varepsilon$ and adding two equations of (4.4.24), we obtain

$$\|\nabla v^\varepsilon\|^2_{L^2(\Omega)^{d\times d}} + \varepsilon\|\nabla q^\varepsilon\|^2_{L^2(\Omega)^d} \le \|h\|_{Q^*}\|\nabla q^\varepsilon\|_{L^2(\Omega)^d}.$$

where we have used $(\nabla q^\varepsilon, v^\varepsilon) = -(\operatorname{div} v^\varepsilon, q^\varepsilon)$. Thus

$$\|\nabla q^\varepsilon\|_{L^2(\Omega)^d} \le \frac{1}{\varepsilon}\|h\|_{Q^*}.$$

In addition, from the first equation of (4.4.24) by putting $\varphi := v^\varepsilon$, we have

$$\begin{aligned} \|\nabla v^\varepsilon\|^2_{L^2(\Omega)^d} = \ & (\nabla v^\varepsilon, \nabla v^\varepsilon) = -(\nabla q^\varepsilon, v^\varepsilon) \le \|\nabla q^\varepsilon\|_{L^2(\Omega)^d}\|v^\varepsilon\|_{L^2(\Omega)^d} \\ \le \ & c\|\nabla q^\varepsilon\|_{L^2(\Omega)^d}\|\nabla v^\varepsilon\|_{L^2(\Omega)^{d\times d}} \end{aligned}$$

for a constant $c > 0$, and then

$$\|\nabla v^\varepsilon\|_{L^2(\Omega)^d} \le c\|\nabla q^\varepsilon\|_{L^2(\Omega)^d} \le \frac{c}{\varepsilon}\|h\|_{Q^*}.$$

□

Using Lemma 4.4.1, we obtain the following theorem.

Theorem 4.4.2. *There exists a constant $c > 0$ independent of $\varepsilon > 0$ such that*

$$\|u^\varepsilon - u^{PP}\|_{H^1(\Omega)^d} + \|p^\varepsilon - p^{PP}\|_{H^1(\Omega)} \le \frac{c}{\varepsilon}\|\operatorname{div} u^{PP}\|_{Q^*}.$$

for all $\varepsilon > 0$. In particular, we have

$$u^\varepsilon \to u^{PP} \text{ strongly in } H^1(\Omega)^d,\ p^\varepsilon \to p^{PP} \text{ strongly in } H^1(\Omega) \quad \text{as } \varepsilon \to \infty.$$

Proof. Combining (PP') and (ES'), we obtain

$$\begin{cases} (\nabla v^\varepsilon, \nabla \varphi) + (\nabla q^\varepsilon, \varphi) = 0 & \text{for all } \varphi \in H_0^1(\Omega)^d, \\ \varepsilon(\nabla q^\varepsilon, \nabla \psi) + (\operatorname{div} v^\varepsilon, \psi) = -(\operatorname{div} u^{PP}, \psi) & \text{for all } \psi \in Q, \end{cases} \tag{4.4.25}$$

where $v^\varepsilon := u^\varepsilon - u^{PP}$ and $q^\varepsilon := p^\varepsilon - p^{PP}$. By Lemma 4.4.1, we conclude the proof. □

Corollary 4.4.3. *If u^{PP} satisfies $\operatorname{div} u^{PP} = 0$, then $u^\varepsilon = u^{PP}$ and $p^\varepsilon = p^{PP}$ hold for all $\varepsilon > 0$. Furthermore, $u^S = u^\varepsilon = u^{PP}$ and $p^S = [p^\varepsilon] = [p^{PP}]$ hold for all $\varepsilon > 0$.*

4.4.2 Regular Perturbation Asymptotics

By Theorem 4.4.2, there exists a constant $c > 0$ such that $\|\varepsilon(u^\varepsilon - u^{PP})\|_{H^1(\Omega)^d} \leq c$ and $\|\varepsilon(p^\varepsilon - p^{PP})\|_{H^1(\Omega)} \leq c$ for all $\varepsilon > 0$. It implies that there exists a subsequence of $(\varepsilon(u^\varepsilon - u^{PP}), \varepsilon(p^\varepsilon - p^{PP}))$ which converges weakly to $(v_{(1)}, q_{(1)}) \in H_0^1(\Omega)^d \times Q$ if $\varepsilon \to \infty$. The next theorem states properties of the limit functions $v_{(1)}$ and $q_{(1)}$.

Theorem 4.4.4. *Let $v_{(1)}^\varepsilon := \varepsilon(u^\varepsilon - u^{PP}) \in H_0^1(\Omega)^d, q_{(1)}^\varepsilon := \varepsilon(p^\varepsilon - p^{PP}) \in Q$ and let $(v_{(1)}, q_{(1)}) \in H_0^1(\Omega)^d \times Q$ satisfy*

$$\begin{cases} (\nabla v_{(1)}, \nabla\varphi) + (\nabla q_{(1)}, \varphi) = 0 & \text{for all } \varphi \in H_0^1(\Omega)^d, \\ (\nabla q_{(1)}, \nabla\psi) = -(\operatorname{div} u^{PP}, \psi) & \text{for all } \psi \in Q. \end{cases} \tag{4.4.26}$$

Then, there exists a constant $c > 0$ independent of ε such that

$$\|v_{(1)}^\varepsilon - v_{(1)}\|_{H^1(\Omega)^d} + \|q_{(1)}^\varepsilon - q_{(1)}\|_{H^1(\Omega)} \leq \frac{c}{\varepsilon}\|\operatorname{div} v_{(1)}\|_{Q^*}.$$

Proof. The existence and uniqueness of the pair $(v_{(1)}, q_{(1)}) \in H_0^1(\Omega)^d \times Q$ as a solution to (4.4.26) follows from Theorem 4.2.5. As in (4.4.25), we have

$$\begin{cases} (\nabla v_{(1)}^\varepsilon, \nabla\varphi) + (\nabla q_{(1)}^\varepsilon, \varphi) = 0 & \text{for all } \varphi \in H_0^1(\Omega)^d, \\ (\nabla q_{(1)}^\varepsilon, \nabla\psi) + \frac{1}{\varepsilon}(\operatorname{div} v_{(1)}^\varepsilon, \psi) = -(\operatorname{div} u^{PP}, \psi) & \text{for all } \psi \in Q. \end{cases} \tag{4.4.27}$$

Subtracting (4.4.26) from (4.4.27), it holds that

$$\begin{cases} (\nabla(v_{(1)}^\varepsilon - v_{(1)}), \nabla\varphi) + (\nabla(q_{(1)}^\varepsilon - q_{(1)}), \varphi) = 0 & \text{for all } \varphi \in H_0^1(\Omega)^d, \\ (\nabla(q_{(1)}^\varepsilon - q_{(1)}), \nabla\psi) + \frac{1}{\varepsilon}(\operatorname{div} v_{(1)}^\varepsilon, \psi) = 0 & \text{for all } \psi \in Q. \end{cases}$$

Hence,

$$\begin{cases} (\nabla v^\varepsilon, \nabla\varphi) + (\nabla q^\varepsilon, \varphi) = 0 & \text{for all } \varphi \in H_0^1(\Omega)^d, \\ (\nabla q^\varepsilon, \nabla\psi) + \frac{1}{\varepsilon}(\operatorname{div} v^\varepsilon, \psi) = -(\operatorname{div} v_{(1)}, \psi) & \text{for all } \psi \in Q. \end{cases}$$

where $v^\varepsilon := v_{(1)}^\varepsilon - v_{(1)}$ and $q^\varepsilon := q_{(1)}^\varepsilon - q_{(1)}$. By Lemma 4.4.1 , there exists a constant $c > 0$ independent of ε such that

$$\|v_{(1)}^\varepsilon - v_{(1)}\|_{H^1(\Omega)^d} + \|q_{(1)}^\varepsilon - q_{(1)}\|_{H^1(\Omega)} \leq \frac{c}{\varepsilon}\|\operatorname{div} v_{(1)}\|_{Q^*}$$

for all $\varepsilon > 0$. □

Next, we generalize Theorem 4.4.4 to the following theorem:

Theorem 4.4.5. *Let $k \in \mathbb{N}$ be arbitrary ($k \geq 1$) and let $v_{(0)} := u^{PP}$. If functions $v_{(1)}$, $v_{(2)}, \cdots, v_{(k)} \in H_0^1(\Omega)^d$ and $q_{(1)}, q_{(2)}, \cdots, q_{(k)} \in Q$ satisfy*

$$\begin{cases} (\nabla v_{(i)}, \nabla\varphi) + (\nabla q_{(i)}, \varphi) = 0 & \text{for all } \varphi \in H_0^1(\Omega)^d, \\ (\nabla q_{(i)}, \nabla\psi) = -(\operatorname{div} v^{(i-1)}, \psi) & \text{for all } \psi \in Q, \end{cases} \tag{4.4.28}$$

for all $1 \le i \le k$*, then there exists a constant* $c > 0$ *independent of* ε *satisfying*

$$\left\| u^\varepsilon - \left(u^{PP} + \frac{1}{\varepsilon} v_{(1)} + \cdots + \left(\frac{1}{\varepsilon}\right)^k v_{(k)} \right) \right\|_{H^1(\Omega)^d} \le \frac{c}{\varepsilon^{k+1}} \| \operatorname{div} v_{(k)} \|_{Q^*},$$

$$\left\| p^\varepsilon - \left(p^{PP} + \frac{1}{\varepsilon} q_{(1)} + \cdots + \left(\frac{1}{\varepsilon}\right)^k q_{(k)} \right) \right\|_{H^1(\Omega)} \le \frac{c}{\varepsilon^{k+1}} \| \operatorname{div} v_{(k)} \|_{Q^*}.$$

Proof. Let $(v^\varepsilon_{(i)}, q^\varepsilon_{(i)}) \in H^1_0(\Omega)^d \times Q$ $(1 \le i \le k)$ satisfy

$$\begin{cases} (\nabla v^\varepsilon_{(i)}, \nabla \varphi) + (\nabla q^\varepsilon_{(i)}, \varphi) = 0 & \text{for all } \varphi \in H^1_0(\Omega)^d, \\ (\nabla q^\varepsilon_{(i)}, \nabla \psi) + \dfrac{1}{\varepsilon}(\operatorname{div} v^\varepsilon_{(i)}, \psi) = -(\operatorname{div} v^{(i-1)}, \psi) & \text{for all } \psi \in Q. \end{cases} \tag{4.4.29}$$

Subtracting (4.4.28) from (4.4.29), it holds that

$$\begin{cases} (\nabla (v^\varepsilon_{(i)} - v_{(i)}), \nabla \varphi) + (\nabla (q^\varepsilon_{(i)} - q_{(i)}), \varphi) = 0 & \text{for all } \varphi \in H^1_0(\Omega)^d, \\ (\nabla (q^\varepsilon_{(i)} - q_{(i)}), \nabla \psi) + \dfrac{1}{\varepsilon}(\operatorname{div} v^\varepsilon_{(i)}, \psi) = 0 & \text{for all } \psi \in Q. \end{cases}$$

Setting $v^\varepsilon := v^\varepsilon_{(i)} - v_{(i)}, q^\varepsilon := q^\varepsilon_{(i)} - q_{(i)}$ and $h := -\operatorname{div} v_{(i)}$, we obtain from Lemma 4.4.1 that the estimates

$$\| v^\varepsilon_{(i)} - v_{(i)} \|_{H^1(\Omega)^d} + \| q^\varepsilon_{(i)} - q_{(i)} \|_{H^1(\Omega)} \le \frac{c}{\varepsilon} \| \operatorname{div} v_{(i)} \|_{Q^*}$$

hold for all $\varepsilon > 0$. In particular, putting $i := k$, we obtain

$$\| v^\varepsilon_{(k)} - v_{(k)} \|_{H^1(\Omega)^d} + \| q^\varepsilon_{(k)} - q_{(k)} \|_{H^1(\Omega)} \le \frac{c}{\varepsilon} \| \operatorname{div} v_{(k)} \|_{Q^*}$$

for all $\varepsilon > 0$. By the uniqueness of the solution to (ES') in Theorem 4.2.6, it leads that $v^{(\varepsilon,i+1)} = \varepsilon(v^\varepsilon_{(i)} - v_{(i)}), q^{(\varepsilon,i+1)} = \varepsilon(q^\varepsilon_{(i)} - q_{(i)})$ for all $i = 1, \cdots, k-1$, and thus

$$\begin{aligned} v^\varepsilon_{(k)} - v_{(k)} = {} & \varepsilon(v^\varepsilon_{(k-1)} - v_{(k-1)}) - v_{(k)} \\ = {} & \varepsilon \left(v^\varepsilon_{(k-1)} - \left(v_{(k-1)} + \left(\frac{1}{\varepsilon}\right) v_{(k)} \right) \right) \\ = {} & \cdots \\ = {} & \varepsilon^{k-1} \left(v^\varepsilon_{(1)} - \left(v_{(1)} + \cdots + \left(\frac{1}{\varepsilon}\right)^{k-2} v_{(k-1)} + \left(\frac{1}{\varepsilon}\right)^{k-1} v_{(k)} \right) \right) \\ = {} & \varepsilon^k \left(u^\varepsilon - \left(u^{PP} + \frac{1}{\varepsilon} v_{(1)} + \cdots + \left(\frac{1}{\varepsilon}\right)^{k-1} v_{(k-1)} + \left(\frac{1}{\varepsilon}\right)^k v_{(k)} \right) \right), \end{aligned}$$

$$\begin{aligned}
q^\varepsilon_{(k)} - q_{(k)} &= \varepsilon(q^\varepsilon_{(k-1)} - q_{(k-1)}) - q_{(k)} \\
&= \varepsilon\left(q^\varepsilon_{(k-1)} - \left(q_{(k-1)} + \left(\frac{1}{\varepsilon}\right) q_{(k)}\right)\right) \\
&= \cdots \\
&= \varepsilon^{k-1}\left(q^\varepsilon_{(1)} - \left(q_{(1)} + \cdots + \left(\frac{1}{\varepsilon}\right)^{k-2} q_{(k-1)} + \left(\frac{1}{\varepsilon}\right)^{k-1} q_{(k)}\right)\right) \\
&= \varepsilon^{k}\left(p^\varepsilon - \left(p^{PP} + \frac{1}{\varepsilon} q_{(1)} + \cdots + \left(\frac{1}{\varepsilon}\right)^{k-1} q_{(k-1)} + \left(\frac{1}{\varepsilon}\right)^{k} q_{(k)}\right)\right).
\end{aligned}$$

Hence it holds that

$$\left\| u^\varepsilon - \left(u^{PP} + \frac{1}{\varepsilon} v_{(1)} + \cdots + \left(\frac{1}{\varepsilon}\right)^k v_{(k)}\right)\right\|_{H^1(\Omega)^d} \le \frac{c}{\varepsilon^{k+1}} \|\operatorname{div} v_{(k)}\|_{Q^*},$$

$$\left\| p^\varepsilon - \left(p^{PP} + \frac{1}{\varepsilon} q_{(1)} + \cdots + \left(\frac{1}{\varepsilon}\right)^k q_{(k)}\right)\right\|_{H^1(\Omega)} \le \frac{c}{\varepsilon^{k+1}} \|\operatorname{div} v_{(k)}\|_{Q^*}.$$

□

Remark 4.4.6. *Theorem 4.4.5 can be interpreted from the operator theory.*

Let $t \ge 0, X := H^1_0(\Omega)^d \times Q, Y := H^{-1}(\Omega)^d \times Q^*$ *be equipped with norms*

$$\begin{aligned}
\|(u,p)\|_X^2 &:= \|u\|^2_{H^1(\Omega)^d} + \|p\|^2_{H^1(\Omega)}, \\
\|(f,g)\|_Y^2 &:= \|f\|^2_{H^{-1}(\Omega)^d} + \|g\|^2_{Q^*}
\end{aligned}$$

for $(u,p) \in X, (f,g) \in Y$*, and let* A *and* B *be*

$$\begin{array}{cccc} A: & X & \longrightarrow & Y \\ & \Psi & & \Psi \\ & (u,p) & \longmapsto & (-\Delta u + \nabla p, \Delta p), \end{array} \qquad \begin{array}{cccc} B: & X & \longrightarrow & Y \\ & \Psi & & \Psi \\ & (u,p) & \longmapsto & (0, \operatorname{div} u). \end{array}$$

Then, (u^{PP}, p^{PP}) *and* $(u^\varepsilon, p^\varepsilon)$ *satisfy*

$$A(u^{PP}, p^{PP}) = f, \quad \left(A + \frac{1}{\varepsilon} B\right)(u^\varepsilon, p^\varepsilon) = f,$$

where $f = (F, G)$*. We have* $A + tB \in \operatorname{Isom}(X, Y)$ *for an arbitrary* $t \ge 0$ *by the analogy of Theorem 4.2.5 (*$t = 0$*) and Theorem 4.2.6 (*$t = 1/\varepsilon$*). Equation (4.4.28) states that*

$$A(v_{(i)}, q_{(i)}) = -B(v^{(i-1)}, q^{(i-1)})$$

for $i = 1, \cdots, k$*, i.e.,*

$$\begin{aligned}
(v_{(i)}, q_{(i)}) &= -A^{-1}B(v^{(i-1)}, q^{(i-1)}) = \cdots = (-A^{-1}B)^i (u^{PP}, p^{PP}) \\
&= A^{-1}(-BA^{-1})^i f.
\end{aligned}$$

By Theorem 4.4.5, there exists a constant $c > 0$ *such that*

$$\left\| \left(A + \frac{1}{\varepsilon} B\right)^{-1} f - A^{-1} \sum_{i=0}^{k} \left(-\frac{1}{\varepsilon} BA^{-1}\right)^i f \right\|_X \le \frac{c}{\varepsilon^{k+1}} \|(BA^{-1})^{k+1} f\|_Y$$

for all $\varepsilon > 0, f \in Y$.

4.5 Links between (ES) and (S)

In this section, we show that $(u^\varepsilon, p^\varepsilon)$ converges to (u^S, p^S) strongly in $H^1(\Omega)^d \times (L^2(\Omega)/\mathbb{R})$ as $\varepsilon \to 0$. The outline of the proof of our convergence results (Theorems 4.5.2 and 4.5.3) is as follows. First, we prove the boundedness of the sequence $((u^\varepsilon, p^\varepsilon))_{\varepsilon>0}$ in $H^1(\Omega)^d \times (L^2(\Omega)/\mathbb{R})$. By the reflexivity of $H^1(\Omega)^d \times (L^2(\Omega)/\mathbb{R})$, the sequence has a subsequence converging weakly in $H^1(\Omega)^d \times (L^2(\Omega)/\mathbb{R})$. Next, we show that the limit pair of functions satisfies (S'). Finally, we prove the strong convergence in $H^1(\Omega)^d \times (L^2(\Omega)/\mathbb{R})$.

We start this section with a useful lemma.

Lemma 4.5.1. *If $v \in H^1(\Omega)^d, q \in L^2(\Omega)$ and $f \in H^{-1}(\Omega)^d$ satisfy*

$$(\nabla v, \nabla \varphi) + \langle \nabla q, \varphi \rangle_{H_0^1(\Omega)^d} = \langle f, \varphi \rangle_{H_0^1(\Omega)^d} \quad \text{for all } \varphi \in H_0^1(\Omega)^d,$$

then there exists a constant $c > 0$ such that

$$\|[q]\|_{L^2(\Omega)} \leq c(\|\nabla v\|_{L^2(\Omega)^{d\times d}} + \|f\|_{H^{-1}(\Omega)^d}).$$

Proof. Let c be the constant from Theorem 2.2.14. Then, we obtain

$$\begin{aligned}
\|[q]\|_{L^2(\Omega)} \leq\ & c\|\nabla q\|_{H^{-1}(\Omega)^d} = c \sup_{\varphi\in S_n} |\langle \nabla q, \varphi \rangle_{H_0^1(\Omega)^d}| \\
\leq\ & c \sup_{\varphi\in S_n} \Big(|(\nabla v, \nabla\varphi)| + |\langle f, \varphi\rangle_{H_0^1(\Omega)^d}| \Big) \\
\leq\ & c(\|\nabla v\|_{L^2(\Omega)^{d\times d}} + \|f\|_{H^{-1}(\Omega)^d}).
\end{aligned}$$

□

Theorem 4.5.2. *There exists a constant $c > 0$ independent of ε such that*

$$\|u^\varepsilon\|_{H^1(\Omega)^d} + \|[p^\varepsilon]\|_{L^2(\Omega)} \leq c \quad \text{for all } \varepsilon > 0.$$

Furthermore, if the range of Q under the map $[\cdot]$ is dense in $L^2(\Omega)/\mathbb{R}$, then we obtain

$$u^\varepsilon \to u^S \text{ strongly in } H^1(\Omega)^d,\ [p^\varepsilon] \to p^S \text{ strongly in } L^2(\Omega)/\mathbb{R} \quad \text{as } \varepsilon \to 0.$$

Proof. We take $u_0 \in H^1(\Omega)^d, f \in H^{-1}(\Omega)^d$ and $g \in Q^*$ as (4.2.12) and (4.2.13) in the proof of Theorem 4.2.6. We put $\tilde{u}^\varepsilon := u^\varepsilon - u_0 \in H_0^1(\Omega)^d, \tilde{p}^\varepsilon := p^\varepsilon - p_b \in Q$. Then, we obtain

$$\begin{cases} (\nabla\tilde{u}^\varepsilon, \nabla\varphi) + (\nabla\tilde{p}^\varepsilon, \varphi) = \langle f, \varphi\rangle_{H_0^1(\Omega)^d} & \text{for all } \varphi \in H_0^1(\Omega)^d, \\ \varepsilon(\nabla\tilde{p}^\varepsilon, \nabla\psi) + (\operatorname{div}\tilde{u}^\varepsilon, \psi) = \varepsilon\langle g, \psi\rangle_Q & \text{for all } \psi \in Q. \end{cases} \tag{4.5.30}$$

Putting $\varphi := \tilde{u}^\varepsilon, \psi := \tilde{p}^\varepsilon$ and adding the two equations of (4.5.30), we get

$$\|\nabla\tilde{u}^\varepsilon\|^2_{L^2(\Omega)^{d\times d}} + \varepsilon\|\nabla\tilde{p}^\varepsilon\|^2_{L^2(\Omega)^d} \leq \|f\|_{H^{-1}(\Omega)^d}\|\nabla\tilde{u}^\varepsilon\|_{L^2(\Omega)^{d\times d}} + \varepsilon\|g\|_{Q^*}\|\nabla\tilde{p}^\varepsilon\|_{L^2(\Omega)^d}$$

since $(\nabla\tilde{p}^\varepsilon, \tilde{u}^\varepsilon) = -(\operatorname{div}\tilde{u}^\varepsilon, \tilde{p}^\varepsilon)$. Hence, $(\|\tilde{u}^\varepsilon\|_{H^1(\Omega)^d})_{0<\varepsilon<1}$ and $(\|\sqrt{\varepsilon}\tilde{p}^\varepsilon\|_{H^1(\Omega)})_{0<\varepsilon<1}$ are bounded. Moreover, by Lemma 4.5.1, we obtain

$$\|[\tilde{p}^\varepsilon]\|_{L^2(\Omega)} \leq c(\|\nabla\tilde{u}^\varepsilon\|_{L^2(\Omega)^{d\times d}} + \|f\|_{H^{-1}(\Omega)^d}),$$

i.e., $(\|[\tilde{p}^\varepsilon]\|_{L^2(\Omega)})_{0<\varepsilon<1}$ is bounded. By Theorem 4.4.2, $(\|u^\varepsilon\|_{H^1(\Omega)^d})_{\varepsilon\geq 1}$ and $(\|[\tilde{p}^\varepsilon]\|_{L^2(\Omega)})_{\varepsilon\geq 1}$ are bounded, and thus $(\|u^\varepsilon\|_{H^1(\Omega)^d})_{\varepsilon>0}$ and $(\|[\tilde{p}^\varepsilon]\|_{L^2(\Omega)})_{\varepsilon>0}$ are bounded.

Since $H^1(\Omega)^d \times (L^2(\Omega)/\mathbb{R})$ is reflexive and $(\tilde{u}^\varepsilon, [\tilde{p}^\varepsilon])_{0<\varepsilon<1}$ is bounded in $H^1(\Omega)^d \times (L^2(\Omega)/\mathbb{R})$, there exist $(u,p) \in H^1(\Omega)^d \times (L^2(\Omega)/\mathbb{R})$ and a subsequence of pairs $(\tilde{u}^{\varepsilon_k}, \tilde{p}^{\varepsilon_k})_{k\in\mathbb{N}} \subset H_0^1(\Omega)^d \times Q$ such that

$$\tilde{u}^{\varepsilon_k} \rightharpoonup u \text{ weakly in } H^1(\Omega)^d,\ [\tilde{p}^{\varepsilon_k}] \rightharpoonup p \text{ weakly in } L^2(\Omega)/\mathbb{R} \quad \text{as } k \to \infty.$$

Hence, from (4.5.30) with $\varepsilon := \varepsilon_k$, taking $k \to \infty$, we obtain

$$\begin{cases} (\nabla u, \nabla\varphi) + \langle \nabla p, \varphi\rangle_{H_0^1(\Omega)^d} = \langle f, \varphi\rangle_{H_0^1(\Omega)^d} & \text{for all } \varphi \in H_0^1(\Omega)^d \\ (\operatorname{div} u, [\psi]) = 0 & \text{for all } \psi \in Q, \end{cases} \tag{4.5.31}$$

where we have used that

$$|\varepsilon_k(\nabla \tilde{p}^{\varepsilon_k}, \nabla\psi)| \leq \sqrt{\varepsilon_k}\|\sqrt{\varepsilon_k}\tilde{p}^{\varepsilon_k}\|_{H^1(\Omega)}\|\psi\|_{H^1(\Omega)} \to 0,$$

$$(\nabla \tilde{p}^{\varepsilon_k}, \varphi) = -([\tilde{p}^{\varepsilon_k}], \operatorname{div}\varphi) \to -(p, \operatorname{div}\varphi) = \langle \nabla p, \varphi\rangle_{H_0^1(\Omega)^d}$$

as $k \to \infty$. By (4.2.13), the first equation of (4.5.31) implies that for all $\varphi \in H_0^1(\Omega)^d$,

$$(\nabla(u+u_0), \nabla\varphi) + \langle \nabla(p+p_b), \varphi\rangle_{H_0^1(\Omega)^d} = (F, \varphi).$$

By the second equation of (4.5.31), if the range of Q under the map $[\cdot]$ is dense in $L^2(\Omega)/\mathbb{R}$, then it holds that for all $\psi \in L^2(\Omega)/\mathbb{R}$,

$$(\operatorname{div}(u+u_0), \psi) = (\operatorname{div} u, \psi) = 0.$$

Hence, we obtain that $(u+u_0, p+[p_b])$ satisfies (S'), i.e., $u^S = u + u_0$ and $p^S = p + [p_b]$. Then, we have

$$u^{\varepsilon_k} - u^S = u^{\varepsilon_k} - u - u_0 = \tilde{u}^{\varepsilon_k} - u \rightharpoonup 0 \text{ weakly in } H^1(\Omega)^d,$$

$$[p^{\varepsilon_k}] - p^S = [p^{\varepsilon_k} - p - p_b] = [\tilde{p}^{\varepsilon_k}] - p \rightharpoonup 0 \text{ weakly in } L^2(\Omega)/\mathbb{R}$$

as $k \to \infty$. Since any arbitrarily chosen subsequence of $((u^\varepsilon, [p^\varepsilon]))_{0<\varepsilon<1}$ has a subsequence which converges to (u^S, p^S), we obtain

$$u^\varepsilon \rightharpoonup u^S \textit{ weakly} \text{ in } H^1(\Omega)^d,\ [p^\varepsilon] \rightharpoonup p^S \textit{ weakly} \text{ in } L^2(\Omega)/\mathbb{R} \quad \text{as } \varepsilon \to 0.$$

Finally, we show the strong convergences. We have from (ES') and (S') that

$$\begin{cases} (\nabla(u^\varepsilon - u^S), \nabla\varphi) - (p^\varepsilon - p^S, \operatorname{div}\varphi) = 0 & \text{for all } \varphi \in H_0^1(\Omega)^d, \\ \varepsilon(\nabla(p^\varepsilon - p_b), \nabla\psi) + (\operatorname{div}(u^\varepsilon - u^S), \psi) = \varepsilon\langle G, \psi\rangle_Q - \varepsilon(\nabla p_b, \nabla\psi) & \text{for all } \psi \in Q. \end{cases}$$

Putting $\varphi := u^\varepsilon - u^S \in H_0^1(\Omega)^d$, $\psi := p^\varepsilon - p_b \in Q$ and adding two equations, we get

$$\begin{aligned} &\|\nabla(u^\varepsilon - u^S)\|_{L^2(\Omega)^{d\times d}}^2 + \varepsilon\|\nabla(p^\varepsilon - p_b)\|_{L^2(\Omega)^d}^2 \\ &= \varepsilon\langle G, p^\varepsilon - p_b\rangle_Q - \varepsilon(\nabla p_b, \nabla(p^\varepsilon - p_b)) - (p^S - p_b, \operatorname{div}(u^\varepsilon - u^S)). \end{aligned}$$

Hence, we have

$$\begin{aligned}&\|\nabla(u^\varepsilon - u^S)\|^2_{L^2(\Omega)^{d\times d}}\\ &\le \varepsilon(\|G\|_{Q^*} + \|\nabla p_b\|_{L^2(\Omega)^d})\|\nabla(p^\varepsilon - p_b)\|_{L^2(\Omega)^d} - (p^S - p_b, \operatorname{div}(u^\varepsilon - u^S))\\ &\to 0\end{aligned}$$

as $\varepsilon \to 0$, which implies that

$$\|[p^\varepsilon] - p^S\|_{L^2(\Omega)} = \|[p^\varepsilon - p^S]\|_{L^2(\Omega)} \le c\|\nabla(u^\varepsilon - u^S)\|_{L^2(\Omega)^{d\times d}} \to 0 \text{ as } k \to \infty$$

by Lemma 4.5.1. □

Theorem 4.5.2 does not give the convergence rate. If $Q = H^1(\Omega)/\mathbb{R}$ (corresponding to the Neumann boundary condition (4.1.1)), then the convergence rate becomes $\sqrt{\varepsilon}$.

Theorem 4.5.3. *Suppose that $Q = H^1(\Omega)/\mathbb{R}$ and $p^S \in H^1(\Omega)$. Then, there exists a constant $c > 0$ independent of ε such that*

$$\|u^\varepsilon - u^S\|_{H^1(\Omega)^d} + \|p^\varepsilon - p^S\|_{L^2(\Omega)} \le c\sqrt{\varepsilon}.$$

Proof. We obtain from (ES') and (S') that

$$\begin{cases} (\nabla(u^\varepsilon - u^S), \nabla\varphi) + (\nabla(p^\varepsilon - p^S), \varphi) = 0 & \text{for all } \varphi \in H^1_0(\Omega)^d,\\ \varepsilon(\nabla p^\varepsilon, \nabla\psi) + (\operatorname{div} u^\varepsilon, \psi) = \varepsilon\langle G, \psi\rangle_{H^1(\Omega)/\mathbb{R}} & \text{for all } \psi \in H^1(\Omega)/\mathbb{R}.\end{cases}$$

Putting $\varphi := u^\varepsilon - u^S \in H^1_0(\Omega)^d$ and $\psi := p^\varepsilon - p^S \in H^1(\Omega)/\mathbb{R}$, we get

$$\begin{aligned}&\|\nabla(u^\varepsilon - u^S)\|^2_{L^2(\Omega)^{d\times d}} + \varepsilon(\nabla p^\varepsilon, \nabla(p^\varepsilon - p^S))\\ &= -(\nabla(p^\varepsilon - p^S), u^\varepsilon - u^S) - (\operatorname{div} u^\varepsilon, p^\varepsilon - p^S) + \varepsilon\langle G, p^\varepsilon - p^S\rangle_{H^1(\Omega)/\mathbb{R}}\\ &= (\operatorname{div} u^\varepsilon - \operatorname{div} u^S, p^\varepsilon - p^S) - (\operatorname{div} u^\varepsilon, p^\varepsilon - p^S) + \varepsilon\langle G, p^\varepsilon - p^S\rangle_{H^1(\Omega)/\mathbb{R}}\\ &= \varepsilon\langle G, p^\varepsilon - p^S\rangle_{H^1(\Omega)/\mathbb{R}}.\end{aligned} \tag{4.5.32}$$

Subtracting $\varepsilon(\nabla p^S, \nabla(p^\varepsilon - p^S)$ from both sides of (4.5.32), we obtain

$$\begin{aligned}&\|\nabla(u^\varepsilon - u^S)\|^2_{L^2(\Omega)^{d\times d}} + \varepsilon\|\nabla(p^\varepsilon - p^S)\|^2_{L^2(\Omega)^d}\\ &= -\varepsilon(\nabla p^S, \nabla(p^\varepsilon - p^S)) + \varepsilon\langle G, p^\varepsilon - p^S\rangle_{H^1(\Omega)/\mathbb{R}}\\ &\le \varepsilon(\|\nabla p^S\|_{L^2(\Omega)^d} + \|G\|_{(H^1(\Omega)/\mathbb{R})^*})\|\nabla(p^\varepsilon - p^S)\|_{L^2(\Omega)^d}.\end{aligned} \tag{4.5.33}$$

To clarify the following estimates, we set $\alpha := \|\nabla(u^\varepsilon - u^S)\|_{L^2(\Omega)^{d\times d}}, \beta := \|\nabla(p^\varepsilon - p^S)\|_{L^2(\Omega)^d}, a := \|\nabla p^S\|_{L^2(\Omega)^d} + \|G\|_{(H^1(\Omega)/\mathbb{R})^*}$. The estimate (4.5.33) reads as

$$\alpha^2 + \varepsilon\beta^2 \le \varepsilon a\beta, \qquad \left(\frac{\alpha}{\sqrt{\varepsilon}}\right)^2 + \left(\beta - \frac{a}{2}\right)^2 \le \left(\frac{a}{2}\right)^2.$$

Hence, $\alpha \le a\sqrt{\varepsilon}/2$, i.e., $\|\nabla(u^\varepsilon - u^S)\|_{L^2(\Omega)^{d\times d}} \le (\sqrt{\varepsilon}/2)(\|\nabla p^S\|_{L^2(\Omega)^d} + \|G\|_{(H^1(\Omega)/\mathbb{R})^*})$. By Lemma 4.5.1, we obtain

$$\|p^\varepsilon - p^S\|_{L^2(\Omega)} \le c\|\nabla(u^\varepsilon - u^S)\|_{L^2(\Omega)^{d\times d}} \le c\frac{\sqrt{\varepsilon}}{2}(\|\nabla p^S\|_{L^2(\Omega)^d} + \|G\|_{(H^1(\Omega)/\mathbb{R})^*})$$

for a constant $c > 0$ independent of ε. □

4.6 Numerical examples

For our simulations, we consider $\Omega = (0,1) \times (0,1)$. We take the following boundary conditions:

$$u_b = (x(x-1), y(y-1))^T, \;\; g_b = (2,2)^T \cdot n$$

on Γ. The exact solutions for (PP_1) are $u^{PP} = (x(x-1), y(y-1))^T$ and $p^{PP} = 2x+2y-2$. We solve the problems (PP_1), (ES_1) and (S') numerically by using the finite element method with P2/P1 elements by the FreeFEM software [43]. The numerical solutions $(u^{PP}, p^{PP}), (u^\varepsilon, p^\varepsilon)$ ($\varepsilon = 1, 10^{-2}$ or 10^{-4}) and (u^S, p^S) to problems (PP_1), (ES_1) and (S'), respectively, are illustrated in Fig. 4.2–4.4. From these images, we observe that $(u^\varepsilon, p^\varepsilon)$ seems to converge to (u^{PP}, p^{PP}) as $\varepsilon \to \infty$ and to (u^S, p^S) as $\varepsilon \to 0$ (as expected from Theorems 4.4.2 and 4.5.2.)

Next, we compute the error estimate between the numerical solutions of (ES_1) and (PP_1). The numerical errors $\|u^\varepsilon - u^{PP}\|_{L^2(\Omega)^d}$, $\|\nabla(u^\varepsilon - u^{PP})\|_{L^2(\Omega)^{d\times d}}$, $\|p^\varepsilon - p^{PP}\|_{L^2(\Omega)}$ and $\|\nabla(p^\varepsilon - p^{PP})\|_{L^2(\Omega)^d}$ are shown in Fig. 4.5 and Fig. 4.6. Based on these values, we fitted a constant c such that $\|u^\varepsilon - u^{PP}\|_{H^1(\Omega)^d} \sim c/\varepsilon$ and $\|p^\varepsilon - p^{PP}\|_{H^1(\Omega)} \sim c/\varepsilon$ for large ε. Fig. 4.5 and Fig. 4.6 indicate that there exists a constant c such that $\|u^\varepsilon - u^{PP}\|_{H^1(\Omega)^d} \le c/\varepsilon$ and $\|p^\varepsilon - p^{PP}\|_{H^1(\Omega)} \le c/\varepsilon$, as expected from Theorem 4.4.2.

We also compute the error estimate between problems (ES_1) and (S') by numerical calculation. The numerical error estimates $\|u^\varepsilon - u^S\|_{L^2(\Omega)^d}$, $\|\nabla(u^\varepsilon - u^S)\|_{L^2(\Omega)^{d\times d}}$, $\|p^\varepsilon - p^S\|_{L^2(\Omega)}$ and $\|\nabla(p^\varepsilon - p^S)\|_{L^2(\Omega)^d}$ are shown in Fig. 4.7 and Fig. 4.8. Based on these values, we fitted a constant c such that $\|u^\varepsilon - u^S\|_{H^1(\Omega)^d} \sim c\varepsilon$ and $\|p^\varepsilon - p^S\|_{L^2(\Omega)} \sim c\varepsilon$ for small ε. Fig. 4.7 and Fig. 4.8 indicate that there exists a constant $\tilde{c}$ such that $\|u^\varepsilon - u^S\|_{H^1(\Omega)^d} \le \tilde{c}\sqrt{\varepsilon}$ and $\|p^\varepsilon - p^S\|_{L^2(\Omega)} \le \tilde{c}\sqrt{\varepsilon}$, as expected from Theorem 4.5.3.

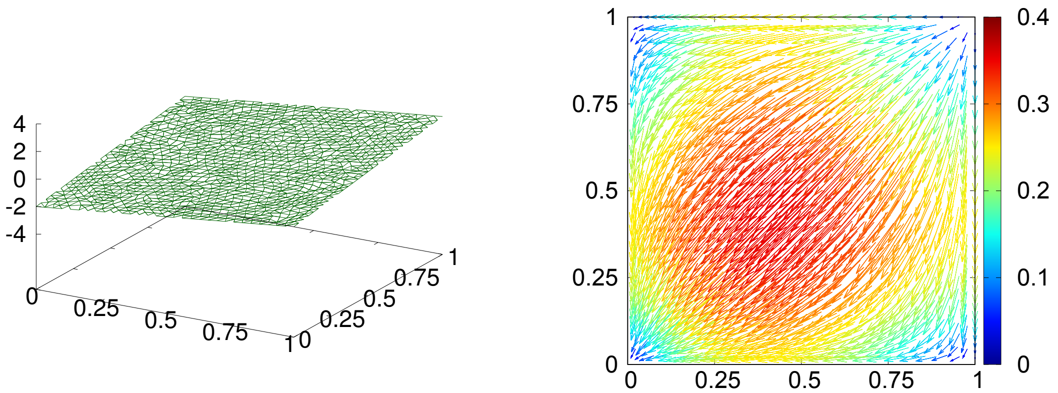

Figure 4.2: p^{PP} (left) and u^{PP} (right). The color scale indicates the length of $|u^{PP}(\xi)|$ at each node ξ.

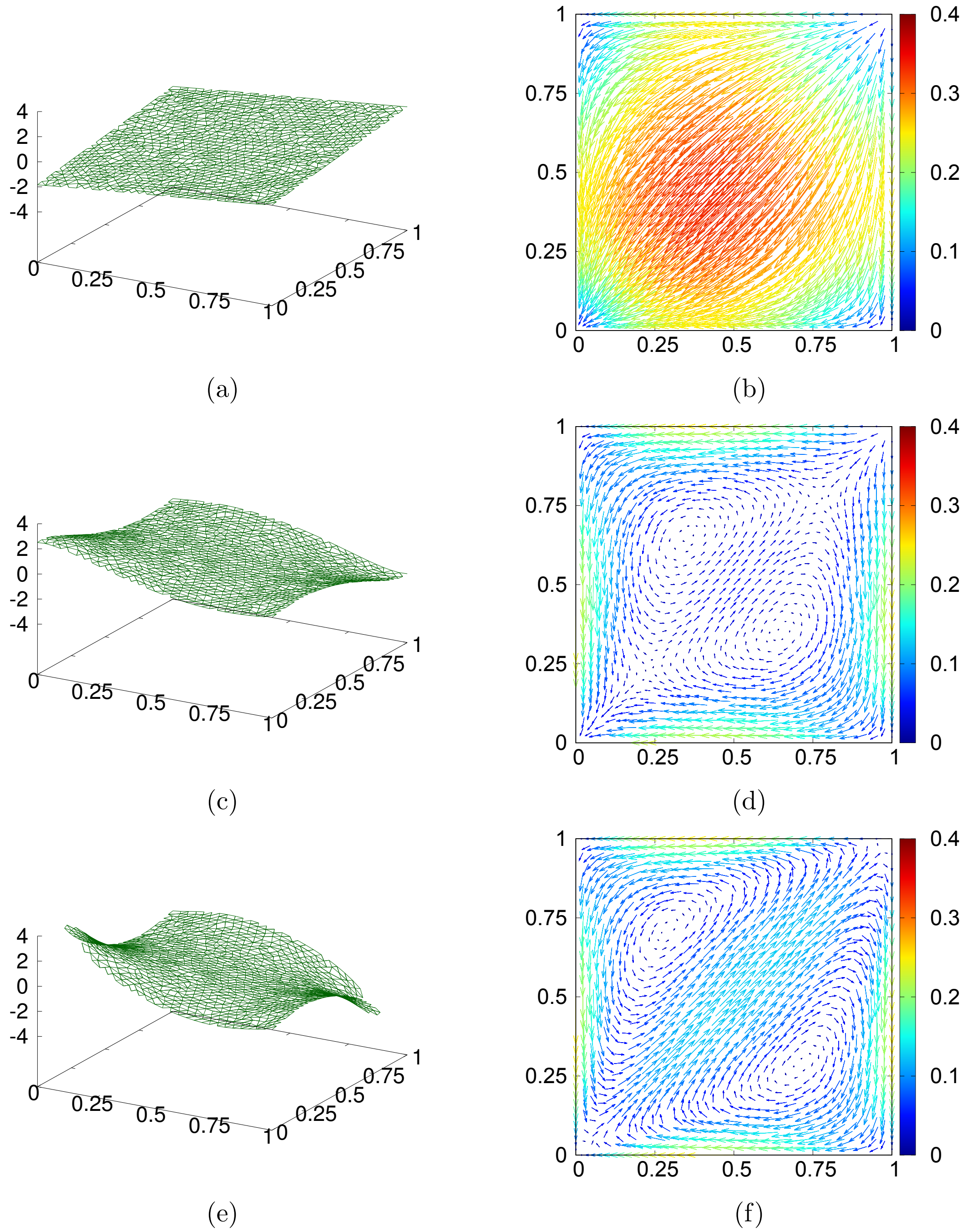

Figure 4.3: p^ε (a) and u^ε (b) with $\varepsilon = 1$. p^ε (c) and u^ε (d) with $\varepsilon = 10^{-2}$. p^ε (e) and u^ε (f) with $\varepsilon = 10^{-4}$. The color scales indicate the length of $|u^\varepsilon(\xi)|$ at each node ξ.

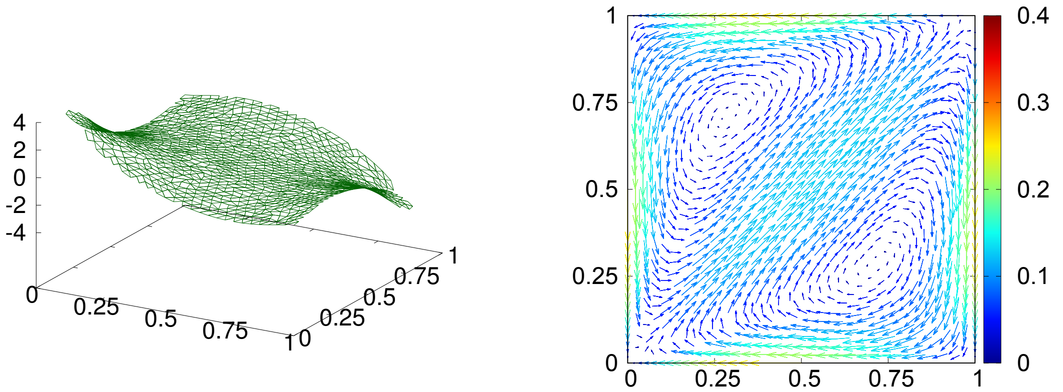

Figure 4.4: p^S (left) and u^S (right). The color scale indicates the length of $|u^S(\xi)|$ at each node ξ.

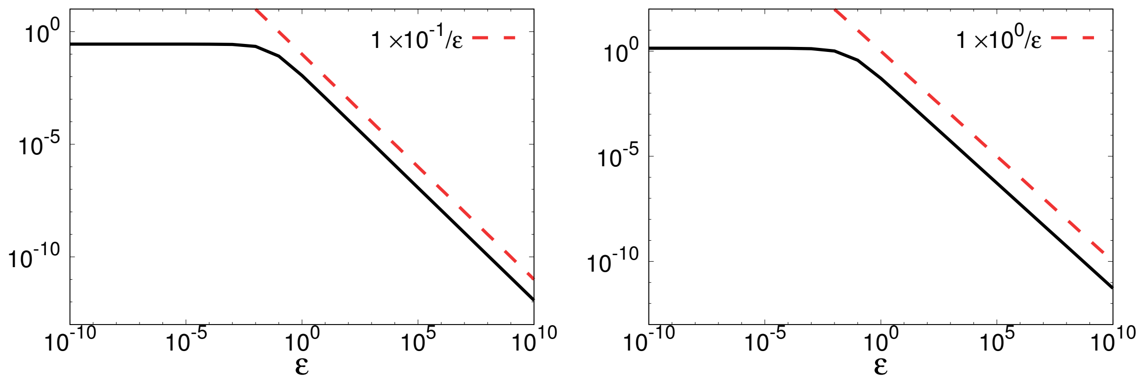

Figure 4.5: $\|u^\varepsilon - u^{PP}\|_{L^2(\Omega)^d}$ (left, solid line) and $\|\nabla(u^\varepsilon - u^{PP})\|_{L^2(\Omega)^{d\times d}}$ (right, solid line) as functions of ε.

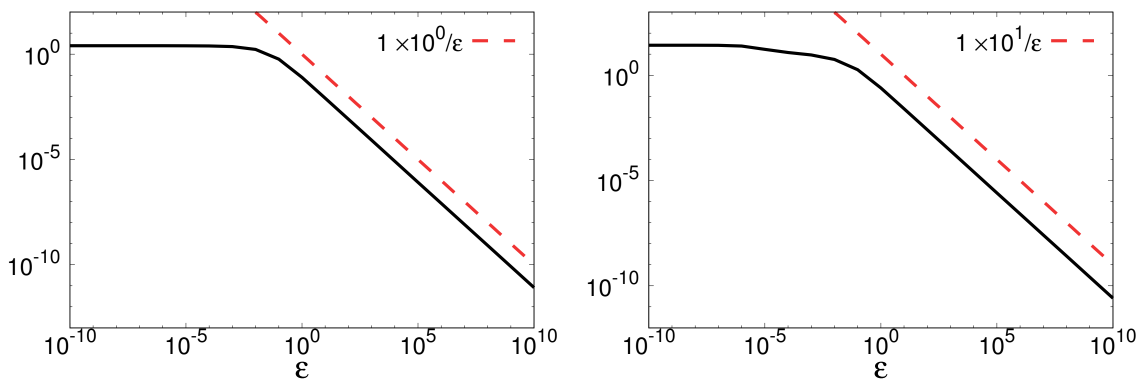

Figure 4.6: $\|p^\varepsilon - p^{PP}\|_{L^2(\Omega)}$ (left, solid line) and $\|\nabla(p^\varepsilon - p^{PP})\|_{L^2(\Omega)^d}$ (right, solid line) as functions of ε.

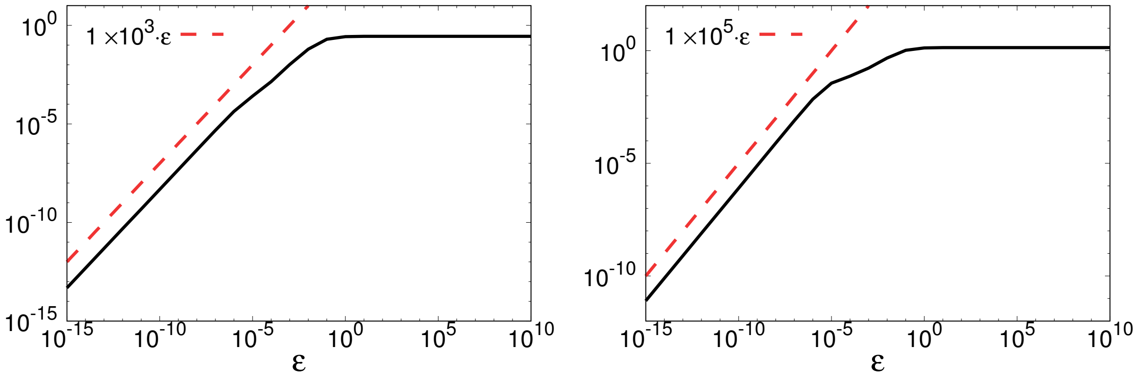

Figure 4.7: $\|u^\varepsilon - u^S\|_{L^2(\Omega)^d}$ (left, solid line) and $\|\nabla(u^\varepsilon - u^S)\|_{L^2(\Omega)^{d\times d}}$ (right, solid line) as functions of ε.

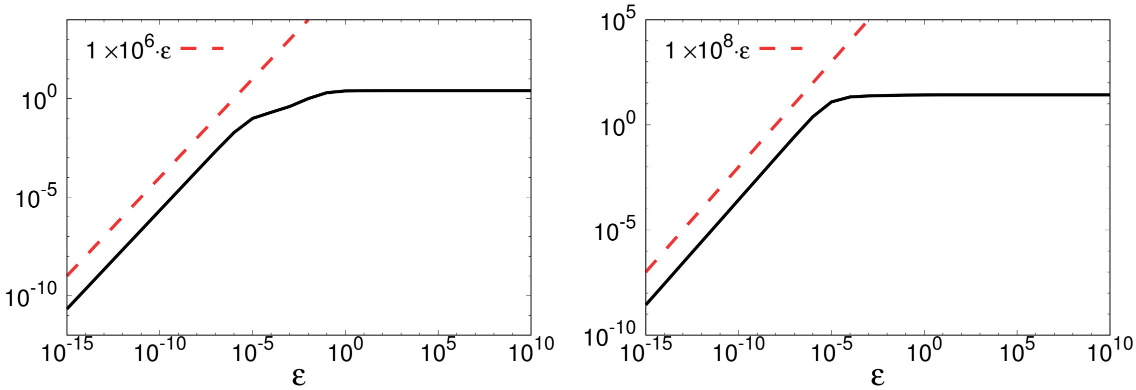

Figure 4.8: $\|p^\varepsilon - p^S\|_{L^2(\Omega)}$ (left, solid line) and $\|\nabla(p^\varepsilon - p^S)\|_{L^2(\Omega)^d}$ (right, solid line) as functions of ε.

4.7 Conclusion

We introduced the ε-Stokes problem (ES) connecting the Stokes problem (S) and the corresponding pressure Poisson problem (PP). For any fixed $\varepsilon > 0$, the ε-Stokes problem has a unique weak solution $(u^\varepsilon, p^\varepsilon)$ (Theorem 4.2.6) and u^ε is a good approximation as the solution to (S), while the solutions to (S) and (PP) are close in the following sense;

$$\|u^S - u^{PP}\|_{H^1(\Omega)^d} + \|p^S - p^{PP}\|_{H^1(\Omega)} \leq c\|p^S - p^{PP}\|_{H^{1/2}(\Gamma)},$$
$$\|u^S - u^\varepsilon\|_{H^1(\Omega)^d} + \|p^S - p^\varepsilon\|_{H^1(\Omega)} \leq c\|p^S - p^{PP}\|_{H^{1/2}(\Gamma)},$$

see Theorems 4.3.2 and 4.3.5 and Corollary 4.3.6 for details. In other words, if we have a good prediction for the boundary data, then (PP) and (ES) are good approximations for (S).

We proved in Theorem 4.4.2 that a sequence $((u^\varepsilon, p^\varepsilon))_{\varepsilon>0}$ converges strongly in $H^1(\Omega)^d \times H^1(\Omega)$ to the solution to (PP) as $\varepsilon \to \infty$ with convergence rate $O(1/\varepsilon)$. We also treated the case of regular perturbation asymptotics by exploring the structure of the lower order terms and their effect on the convergence rate.

We proved in Theorem 4.5.2 that $((u^\varepsilon, p^\varepsilon))_{\varepsilon>0}$ converges strongly in $H^1(\Omega)^d \times (L^2(\Omega)/\mathbb{R})$ to the solution (u^S, p^S) to (S) as $\varepsilon \to 0$. By numerical examples, we observed the expected convergences as $\varepsilon \to \infty$ or $\varepsilon \to 0$.

We summarize our results as follows:

- We introduce the ε-Stokes problem (ES) as an interpolation between the Stokes problem (S) and the pressure Poisson problem (PP).
- The solution $(u^\varepsilon, p^\varepsilon)$ to (ES) strongly converges in $H^1(\Omega)^d \times H^1(\Omega)$ to (u^{PP}, p^{PP}) as $\varepsilon \to \infty$ with convergence rate $O(1/\varepsilon)$.
- The solution $(u^\varepsilon, p^\varepsilon)$ to (ES) weakly converges in $H_0^1(\Omega)^d \times (L^2(\Omega)/\mathbb{R})$ to (u^S, p^S) as $\varepsilon \to 0$. If $p^S \in H^1(\Omega)$, then strong convergence of $(u^\varepsilon, p^\varepsilon)$ to (u^S, p^S) as $\varepsilon \to 0$ holds. Furthermore, if $Q = H^1(\Omega)/\mathbb{R}$ and $p^S \in H^1(\Omega)$, then the convergence rate is $O(\sqrt{\varepsilon})$.

In this chapter, the domain of the numerical examples is in $\mathbb{R}^2$. Numerical comparison of (ES), (PP) and (S) in 3D is one of our interesting future works, for example the convergence rates and numerical instability. As another extension of our research, generalization of our results to the Navier–Stokes problem is important but still remains unknown.

Chapter 5

Projection method

This chapter is based on the following paper:

- K. Matsui: A projection method for Navier–Stokes equations with a boundary condition including the total pressure. arXiv:2105.13014 (submitted), 2021.

5.1 Introduction

Let $T > 0$ and let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$ ($d = 2, 3$) with the boundary Γ satisfying (2.1.1) and Hypothesis 2.2.17. We consider the following Navier–Stokes problem: Find two functions $u : \Omega \times [0, T] \to \mathbb{R}^d$ and $p : \Omega \times [0, T] \to \mathbb{R}$ such that

$$\begin{cases} \dfrac{\partial u}{\partial t} + (u \cdot \nabla)u - \nu\Delta u + \dfrac{1}{\rho}\nabla p = f & \text{in } \Omega \times (0, T), \\ \operatorname{div} u = 0 & \text{in } \Omega \times (0, T), \\ u = 0 & \text{on } \Gamma_1 \times (0, T), \\ u \times n = 0 & \text{on } \Gamma_2 \times (0, T), \\ p + \dfrac{\rho}{2}|u|^2 = p^b & \text{on } \Gamma_2 \times (0, T), \\ u(0) = u_0 & \text{in } \Omega, \end{cases} \tag{5.1.1}$$

where $\nu, \rho > 0$, $f : \Omega \times (0, T) \to \mathbb{R}^d$, $p^b : \Gamma_2 \times (0, T) \to \mathbb{R}$, and $u_0 : \Omega \to \mathbb{R}^d$. The functions u and p are the velocity and the pressure of the flow governed by (5.1.1), respectively. For Γ_2, we assume a boundary condition including a pressure value $p + \frac{\rho}{2}|u|^2$, which is called the total pressure, stagnation pressure, or Bernoulli pressure. Usual pressure is often called static pressure to distinguish it from the total pressure. In an experimental measurement of the total and static pressure using a Pitot tube, the boss measurement is dependent on the yaw angle of the Pitot tube. Then, the effect on the total pressure $p+\frac{\rho}{2}|u|^2$ is smaller than the effect on the usual pressure p [45, Section 7.15]. The boundary condition on Γ_2 in (5.1.1) is introduced in [7], and the existence of a weak velocity solution is proven in [10, 55, 56]. We will show the existence in a different way (Corollary 5.3.10). The stationary case has been studied in [8, 9, 12, 23, 55, 56]. In [12, 13], the finite element discretization problems with this type of boundary condition are proposed.

Next, we introduce a projection method for (5.1.1). The projection method is one of the numerical schemes for Navier–Stokes equations [21, 79]. Error analysis in the case of the full Dirichlet boundary condition for the velocity is carried out in [6, 70, 71, 76, 77].

In the case of a boundary condition for the static pressure, the finite element analysis of a projection method is proposed in [39, 40]. For the nonlinear term in the first equation of (5.1.1), it holds that

$$(u \cdot \nabla)u = (\nabla \times u) \times u + \frac{1}{2}\nabla|u|^2$$

(cf. [34]). Hence, if we set $D(v, w) := (\nabla \times v) \times w$ and $P = p + \frac{\rho}{2}|u|^2$, then (5.1.1) is equivalent to the following[1]:

$$\begin{cases} \dfrac{\partial u}{\partial t} + D(u, u) - \nu\Delta u + \dfrac{1}{\rho}\nabla P = f & \text{in } \Omega \times (0, T), \\ \operatorname{div} u = 0 & \text{in } \Omega \times (0, T), \\ u = 0 & \text{on } \Gamma_1 \times (0, T), \\ u \times n = 0 & \text{on } \Gamma_2 \times (0, T), \\ P = p^b & \text{on } \Gamma_2 \times (0, T), \\ u(0) = u_0 & \text{in } \Omega \end{cases} \tag{5.1.2}$$

The first equation of (5.1.2) is called the rotation form of the Navier–Stokes equation [19, 67]. In [19], a projection method for the rotation form using the total pressure is introduced to avoid checkerboard oscillation of pressure in the finite difference method.

Let $\tau(:= T/N < 1, N \in \mathbb{N})$ be a time increment and let $t_k := k\tau$ $(k = 0, 1, \ldots, N)$. We set $u_0^* := u_0$ and calculate u_k^*, u_k, p_k $(k = 1, 2, \ldots, N)$ by repeatedly solving the following problems (Step 1) and (Step 2).

(Step 1) Find $u_k^* : \Omega \to \mathbb{R}^d$ such that

$$\begin{cases} \dfrac{u_k^* - u_{k-1}}{\tau} + D(u_{k-1}^*, u_k^*) - \nu\Delta u_k^* = f(t_k) & \text{in } \Omega, \\ u_k^* = 0 & \text{on } \Gamma_1, \\ u_k^* \times n = 0 & \text{on } \Gamma_2, \\ \operatorname{div} u_k^* = 0 & \text{on } \Gamma_2. \end{cases} \tag{5.1.3}$$

(Step 2) Find $P_k :\to \mathbb{R}$ and $u_k :\to \mathbb{R}^d$ such that

$$\begin{cases} -\dfrac{\tau}{\rho}\Delta P_k = -\operatorname{div} u_k^* & \text{in } \Omega, \\ \dfrac{\partial P_k}{\partial n} = 0 & \text{on } \Gamma_1, \\ P_k = p^b(t_k) & \text{on } \Gamma_2, \end{cases} \tag{5.1.4}$$

$$u_k = u_k^* - \frac{\tau}{\rho}\nabla P_k \quad \text{in } \Omega. \tag{5.1.5}$$

For the velocity boundary condition on Γ_2, we can rewrite the third and fourth equations of (5.1.3) by using $\kappa := \operatorname{div} n = (d-1)\times(\text{mean curvature})$ as stated in the following remark.

[1] If $d = 2$, then $\nabla \times v$ and $(\nabla \times v) \times w$ denote the scalar and vector functions, respectively, defined as follows: for all $v = (v_x, v_y), w = (w_x, w_y) \in \mathbb{R}^2$,

$$\nabla \times v := \partial_x v_y - \partial_y v_x, \quad (\nabla \times v) \times w := (w_y(\partial_y v_x - \partial_x v_y), w_x(\partial_x v_y - \partial_y v_x)).$$

Remark 5.1.1. *If $v \in C^1(\overline{\Omega})$ satisfies that $v \times n = 0$ on Γ_2, then we have*

$$\frac{\partial v}{\partial n} \cdot n + \kappa v \cdot n = \operatorname{div} v \qquad \text{on } \Gamma_2.$$

For the proof, see [51, Lemma 7]. Hence, the third and fourth equations of (5.1.3) are equivalent to the following equations:

$$u_k^* \times n = 0, \quad \frac{\partial u_k^*}{\partial n} \cdot n + \kappa u_k^* \cdot n = 0 \qquad \text{on } \Gamma_2.$$

In particular, if Γ_2 is flat, then it holds that

$$u_k^* \times n = 0, \quad \frac{\partial u_k^*}{\partial n} \cdot n = 0 \qquad \text{on } \Gamma_2.$$

Remark 5.1.2. *By replacing u_{k-1} in the first equation of (5.1.3) with (5.1.5) at the previous step, it holds that for all $k = 1, 2, \ldots, N$,*

$$\frac{u_k^* - u_{k-1}^*}{\tau} + D(u_{k-1}^*, u_k^*) - \nu \Delta u_k^* + \frac{1}{\rho} \nabla P_{k-1} = f(t_k) \qquad \text{in } \Omega.$$

It follows from (5.1.4) and (5.1.5) that $\operatorname{div} u_k = 0$ in Ω, $u_k \cdot n = 0$ on Γ_1. Hence, by (5.1.3), (5.1.4), and (5.1.5), it holds that for all $k = 1, 2, \ldots, N$,

$$\begin{cases} \dfrac{u_k^* - u_{k-1}^*}{\tau} + D(u_{k-1}^*, u_k^*) - \nu \Delta u_k^* + \dfrac{1}{\rho} \nabla P_{k-1} = f(t_k) & \text{in } \Omega, \\ \operatorname{div} u_k = 0 & \text{in } \Omega, \\ u_k^* = 0 & \text{on } \Gamma_1, \\ u_k^* \times n = 0 & \text{on } \Gamma_2, \\ P_k = p^b(t_k) & \text{on } \Gamma_2, \end{cases}$$

where $P_0 := 0$. Compare with (5.1.2).

In this chapter, we demonstrate the solvability (Proposition 5.2.6) and stability (Theorem 5.3.1) of the projection method and establish error estimates in suitable norms (Theorems 5.3.3 and 5.3.8). Furthermore, we prove the existence of a weak solution of (5.1.1) with a different approach than [10, 55] by using the stability result (Corollary 5.3.10).

The organization of this paper is as follows. In Section 5.2, we introduce the notations used in this work, the weak formulations of the Navier–Stokes equations (5.1.2), and the projection method (5.1.3), (5.1.4), and (5.1.5). We also prove the existence of the weak solution to the scheme. In Section 5.3, we provide the main results. Section 5.4 is devoted to proving that the solution to the scheme is bounded in suitable norms and converges to the solution to (5.1.2) in a strong topology as $\tau \to 0$. We also establish error estimates in suitable norms between the solutions to the Navier–Stokes equations and the projection method. In Section 5.5, we show a numerical example of the projection method and the numerical errors between the Navier–Stokes equations and the projection method using the P2/P1 finite element method. We conclude this paper with several comments on future works in Section 5.6.

5.2 Preliminaries

In this section, we introduce the notations used in this work and the weak formulations of the Navier–Stokes equations (5.1.2) and the projection method (5.1.3), (5.1.4), and (5.1.5).

5.2.1 Notation

Let p_d be

$$p_d := \begin{cases} 2+\varepsilon & \text{if } d=2, \\ 3 & \text{if } d=3, \end{cases}$$

where $\varepsilon > 0$ is arbitrarily small. It follows from the Sobolev embeddings that $H^1(\Omega) \subset L^{p_d}(\Omega)$ and the embedding is continuous [14, Theorem III.2.34]. We define a trilinear operator $a_1 : L^{p_d}(\Omega)^d \times H \times H \to \mathbb{R}$ for $u \in L^{p_d}(\Omega)^d$ and $v, w \in H$,

$$a_1(u,v,w) := \int_\Omega u \cdot (\nabla \times (v \times w))dx.$$

We note that for all $u \in H^1(\Omega)^d$ and $v, w \in H$,

$$a_1(u,v,w) = -\int_\Gamma (u \times n)\cdot(v \times w)ds + \int_\Omega ((\nabla \times u) \times v)\cdot w dx = \int_\Omega D(u,v)\cdot w dx.$$

For a Banach space E, we employ the standard notation of Bochner spaces such as $L^2(0,T;X), H^1(0,T;X)$ and we denote $L^2(0,T;X)$ and $H^1(0,T;X)$ by $L^2(X)$ and $H^1(X)$, respectively. In this chapter, we write the norm $\|\cdot\|_{H^m(\Omega)}$ as $\|\cdot\|_m$.

For two sequences $(x_k)_{k=0}^N$ and $(y_k)_{k=1}^N$ in a Banach space E, we define a piecewise linear interpolant $\hat{x}_\tau \in W^{1,\infty}(0,T;E)$ of $(x_k)_{k=0}^N$ and a piecewise constant interpolant $\bar{y}_\tau \in L^\infty(0,T;E)$ of $(y_k)_{k=1}^N$, respectively, by

$$\begin{aligned} \hat{x}_\tau(t) &:= x_{k-1} + \frac{t-t_{k-1}}{\tau}(x_k - x_{k-1}) && \text{for } t \in [t_{k-1}, t_k] \text{ and } k = 1,2,\ldots,N, \\ \bar{y}_\tau(t) &:= y_k && \text{for } t \in (t_{k-1}, t_k] \text{ and } k = 1,2,\ldots,N. \end{aligned}$$

We define a backward difference operator by

$$D_\tau x_k := \frac{x_k - x_{k-1}}{\tau}, \qquad D_\tau y_l := \frac{y_l - y_{l-1}}{\tau}$$

for $k = 1,2,\ldots,N$ and $l = 2,3,\ldots,N$. Then, the sequence $(D_\tau x)_k := D_\tau x_k$ satisfies $\frac{\partial \hat{x}_\tau}{\partial t} = \overline{(D_\tau x)}_\tau$ on (t_{k-1}, t_k) for all $k = 1,2,\ldots,N$. For a function $F \in C([0,T];E)$, we define $F_\tau \in L^\infty(0,T;E)$ as the piecewise constant interpolant of $(F(t_k))_{k=1}^N$, i.e.,

$$F_\tau(t) := F(t_k) \qquad \text{for } t \in (t_{k-1}, t_k] \text{ and } k = 1,2,\ldots,N.$$

5.2.2 Preliminary results

Lemma 5.2.1. [11, proof of Theorem 2.1] *There exists a constant $c = c(\Omega, \Gamma_1, \Gamma_2) > 0$ such that for all $q \in L^2(\Omega)$,*

$$\|q\|_0 \le c \sup_{0 \neq \varphi \in H} \frac{|(q, \operatorname{div}\varphi)|}{\|\varphi\|_1}.$$

We prepare the following lemma to use the Aubin–Nitsche trick.

Lemma 5.2.2. *We define an operator* $T : L^2(\Omega)^d \ni e \mapsto (w, r) \in H \times L^2(\Omega)$ *as follows:*

$$\begin{cases} a_0(w,\varphi) - (r, \operatorname{div}\varphi) = (e,\varphi) & \text{for all } \varphi \in H, \\ \operatorname{div} w = 0 & \text{in } L^2(\Omega). \end{cases} \tag{5.2.6}$$

Then, T is a linear and continuous operator and there exists a constant $c = c(\Omega, \Gamma_1, \Gamma_2) > 0$ such that for all $e \in L^2(\Omega)^d$ and $(w, r) = T(e)$,

$$\|w\|_1 + \|r\|_0 \le c\|e\|_{H^*}, \qquad \frac{1}{c}\|e\|_{V^*} \le \|w\|_1 \le c\|e\|_{V^*}.$$

By Lemmas 2.2.12, 5.2.1, and Theorem 2.2.19, the operator T is well-posed and continuous. See the Appendix for the proof of the inequalities. Next, we show the following two lemmas for the operator d.

Lemma 5.2.3. *It holds that for all $u \in L^{p_d}(\Omega)^d, v, v_1, v_2 \in H^1(\Omega)^d$*

$$a_1(u, v, v) = 0, \qquad a_1(u, v_1, v_2) = -a_1(u, v_2, v_1).$$

By the definition of the operator a_1, it is easy to check Lemma 5.2.3.

Lemma 5.2.4. *There exists a constant $c_d = c_d(\Omega, \Gamma_1, \Gamma_2) > 0$ such that*

$$a_1(u,v,w) \le \begin{cases} c_d\|u\|_{L^{p_d}}\|v\|_1\|w\|_1 & \text{for all } u \in L^{p_d}(\Omega)^d, v, w \in H, \\ c_d\|u\|_0\|v\|_1\|w\|_2 & \text{for all } u \in L^{p_d}(\Omega)^d, v \in H, w \in H \cap H^2(\Omega)^d, \\ c_d\|u\|_1\|v\|_1\|w\|_1 & \text{for all } u \in H^1(\Omega)^d, v, w \in H, \\ c_d\|u\|_1\|v\|_2\|w\|_0 & \text{for all } u \in H^1(\Omega)^d, v \in H \cap H^2(\Omega)^d, w \in H \\ c_d\|u\|_2\|v\|_1\|w\|_0 & \text{for all } u \in H^2(\Omega)^d, v, w \in H. \end{cases}$$

Proof.
(i) For all $u \in L^{p_d}(\Omega)^d, v, w \in H$, we have

$$\begin{aligned} &|a_1(u,v,w)| \\ &\le \int_\Omega |u \cdot ((w\cdot\nabla)v - (v\cdot\nabla)w + v \operatorname{div} w - w \operatorname{div} v)|\, dx \\ &\le c_1\|u\|_{L^{p_d}}(\|w\|_{L^{\tilde q_d}}\|\nabla v\|_0 + \|v\|_{L^{\tilde q_d}}\|\nabla w\|_0 + \|v\|_{L^{\tilde q_d}}\|\operatorname{div} w\|_0 + \|w\|_{L^{\tilde q_d}}\|\operatorname{div} v\|_0) \\ &\le \tilde c_1\|u\|_{L^{p_d}}\|v\|_1\|w\|_1 \end{aligned}$$

for two constants $c_1, \tilde c_1 > 0$, which implies the third inequality of Lemma 5.2.4.
(ii) For all $u \in L^{p_d}(\Omega)^d, v \in H, w \in H \cap H^2(\Omega)^d$, we have

$$\begin{aligned} &|a_1(u,v,w)| \\ &\le c_2\|u\|_0(\|w\|_{L^\infty}\|\nabla v\|_0 + \|v\|_{L^{p_d}}\|\nabla w\|_{L^{\tilde q_d}} + \|v\|_{L^{p_d}}\|\operatorname{div} w\|_{L^{\tilde q_d}} + \|w\|_{L^\infty}\|\operatorname{div} v\|_0) \\ &\le \tilde c_2\|u\|_0\|v\|_1\|w\|_2 \end{aligned}$$

for two constants $c_2, \tilde c_2 > 0$.
(iii) For all $u \in H^1(\Omega)^d, v \in H \cap H^2(\Omega)^d, w \in H$, we have

$$|a_1(u,v,w)| \le \int_\Omega |((\nabla \times u) \times v)\cdot w|dx \le c_3\|\nabla \times u\|_0\|v\|_{L^\infty}\|w\|_0 \le \tilde c_3\|u\|_1\|v\|_2\|w\|_0$$

for two constants $c_3, \tilde{c}_3 > 0$.

(ix) For all $u \in H^2(\Omega)^d, v, w \in H$, we have

$$|a_1(u,v,w)| \leq c_4 \|\nabla \times u\|_{L^{p_d}} \|v\|_{L^{\tilde{q}_d}} \|w\|_0 \leq \tilde{c}_4 \|u\|_2 \|v\|_1 \|w\|_0$$

for two constants $c_4, \tilde{c}_4 > 0$. □

Finally, we recall the discrete Gronwall inequality.

Lemma 5.2.5. [44, Lemma 5.1] *Let $\tau, \beta > 0$ and let non-negative sequences $(a_k)_{k=0}^N$, $(b_k)_{k=0}^N$, $(c_k)_{k=0}^N$, $(\alpha_k)_{k=0}^N \subset \{x \in \mathbb{R} \mid x \geq 0\}$ satisfy that*

$$a_n + \tau \sum_{k=0}^m b_k \leq \tau \sum_{k=0}^m \alpha_k a_k + \tau \sum_{k=0}^m c_k + \beta \qquad \text{for all } m = 0, 1, \ldots, N.$$

If $\tau\alpha_k < 1$ for all $k = 0, 1, \ldots, N$, then we have

$$a_n + \tau \sum_{k=0}^m b_k \leq e^C \left(\tau \sum_{k=0}^m c_k + \beta \right) \qquad \text{for all } m = 0, 1, \ldots, N,$$

where $C := \tau \sum_{k=0}^N \frac{\alpha_k}{1-\tau\alpha_k}$.

5.2.3 Weak formulations of (5.1.2), (5.1.3), (5.1.4), and (5.1.5)

We assume $\nu = \rho = 1$ and the following conditions for f, p^b, and u_0:

$$f \in L^2(0,T;H^*), \quad p^b \in L^2(0,T;H^1(\Omega)), \quad u_0 \in L^{p_d}(\Omega)^d. \tag{5.2.7}$$

To define weak formulations of the Navier–Stokes equations (5.1.2) and the projection method (5.1.3), (5.1.4), and (5.1.5), we prepare the following equation:

Proposition 5.2.6. *It holds that for all $u \in H^2(\Omega)$ and $\varphi \in H$,*

$$-(\Delta u, \varphi) = a_0(u, \varphi) - \int_{\Gamma_2} (\operatorname{div} u)\varphi \cdot n ds. \tag{5.2.8}$$

Proof. It holds that $-\Delta u = \nabla\times(\nabla\times u) - \nabla(\operatorname{div} u)$ for all $u \in C^2(\overline{\Omega})^d$. Hence, we have for all $u \in C^2(\overline{\Omega})^d$ and $\varphi \in C^1(\overline{\Omega})^d$,

$$(-\Delta u, \varphi) = a_0(u, \varphi) + \int_\Gamma (\nabla\times u) \cdot (\varphi \times n) ds - \int_\Gamma (\operatorname{div} u)\varphi \cdot n ds,$$

which also holds for all $\varphi \in H^2(\Omega)$ and $\psi \in H^1(\Omega)$ since the two spaces $C^2(\overline{\Omega})$ and $C^1(\overline{\Omega})$ are dense in $H^2(\Omega)$ and $H^1(\Omega)$, respectively. By the definition of H, equation (5.2.8) holds for all $u \in H^2(\Omega)$ and $\varphi \in H$. □

By Proposition 5.2.6 and the Gauss divergence formula, it holds that for all $u \in H \cap H^2(\Omega)^d, P \in H^1(\Omega)$, and $\varphi \in V$ with $\operatorname{div} u = 0$ in $H^1(\Omega)$,

$$(D(u,u) - \Delta u + \nabla P, \varphi) = a_0(u, \varphi) + a_1(u,u,\varphi) - (P, \operatorname{div} \varphi) + \int_{\Gamma_2} P\varphi \cdot n ds.$$

Hence, a weak formulation of (5.1.2) is as follows: Find $u \in L^2(0,T;V)$ and $P \in L^1(0,T;L^2(\Omega))$ such that $\frac{\partial u}{\partial t} \in L^1(0,T;H^*)$, $u(0)=u_0$, and for all $\varphi \in H$,

$$\left\langle \frac{\partial u}{\partial t}, \varphi \right\rangle_H + a(u,\varphi) + d(u,u,\varphi) - (P, \operatorname{div}\varphi) = \langle f, \varphi \rangle_H - \int_{\Gamma_2} p^b \varphi \cdot n ds \tag{NS}$$

in $L^1(0,T)$. In main convergence theorems (Theorems 5.3.3 and 5.3.8), we assume that (NS) has a unique solution and that the solution is as smooth as needed.

On the other hand, by Proposition 5.2.6, we have for all $u^*_{k-1} \in H^1(\Omega)^d, u^*_k \in H \cap H^2(\Omega)^d$, and $\varphi \in H$,

$$(D(u^*_{k-1}, u^*_k) - \Delta u^*_k, \varphi) = a_0(u^*_k, \varphi) + a_1(u^*_{k-1}, u^*_k, \varphi) - \int_{\Gamma_2} (\operatorname{div} u^*_k)\varphi \cdot n ds.$$

Hence, a weak formulation of (5.1.3), (5.1.4), and (5.1.5) with the initial datum $u_0 (=: u^*_0)$ is as follows:

Problem 5.2.7. *Let $(f_k)_{k=1}^N \subset H^*$ and $(p^b_k)_{k=1}^N \subset H^1(\Omega)$. For all $k = 1, 2, \ldots, N$, find $(u^*_k, P_k,$
$u_k) \in H \times H^1(\Omega) \times L^2(\Omega)^d$ such that $P_k - p^b_k \in H^1_{\Gamma_2}(\Omega)$ and for all $\varphi \in H$ and $\psi \in H^1_{\Gamma_2}(\Omega)$,*

$$\begin{cases} \frac{1}{\tau}(u^*_k - u_{k-1}, \varphi) + a_0(u^*_k, \varphi) + a_1(u^*_{k-1}, u^*_k, \varphi) = \langle f_k, \varphi \rangle_H \\ \tau(\nabla P_k, \nabla \psi) = -(\operatorname{div} u^*_k, \psi) \\ u_k = u^*_k - \tau \nabla P_k \ \textit{in } L^2(\Omega)^d. \end{cases} \tag{PM}$$

Remark 5.2.8. *For $f \in L^2(0,T;H^*)$ and $p^b \in L^2(0,T;H^1(\Omega))$, we set for all $k = 1, 2, \ldots, N$,*

$$f_k := \frac{1}{\tau}\int_{t_{k-1}}^{t_k} f(s) ds, \qquad p^b_k := \frac{1}{\tau}\int_{t_{k-1}}^{t_k} p^b(s) ds. \tag{5.2.9}$$

Here, it holds that $\bar{f}_\tau \in L^2(0,T;H^*)$ and $\bar{p}^b_\tau \in L^2(0,T;H^1(\Omega))$:

$$\|\bar{f}_\tau\|_{L^2(H^*)} \le \|f\|_{L^2(H^*)}, \qquad \|\bar{p}^b_\tau\|_{L^2(H^1)} \le \|p^b\|_{L^2(H^1)}.$$

In Theorems 5.3.3 and 5.3.8, we assume $f \in C([0,T];H^*), p^b \in C([0,T];H^1(\Omega))$ to use $f(t_k)$ and $p^b(t_k)$ for all $k = 1, 2, \ldots, N$ (Hypothesis 5.3.2). Then, we set for all $k = 1, 2, \ldots, N$,

$$f_k := f(t_k), \qquad p^b_k := p^b(t_k),$$

which implies that $\bar{f}_\tau = f_\tau \in L^2(H^*)$ and $\bar{p}^b_\tau = p^b_\tau \in L^2(H^1(\Omega))$. From Hypothesis 5.3.2 used in Theorems 5.3.3 and 5.3.8, the regularity assumption of f and p^b is natural.

We show the existence and uniqueness of the solution to (PM) in the following proposition.

Proposition 5.2.9. *For all $(f_k)_{k=1}^N \subset H^*$, $(p^b_k)_{k=1}^N \subset H^1(\Omega)^d$, and $u_0 \in L^{p_d}(\Omega)^d$, Problem 5.2.7 has a unique solution.*

Proof. By Theorem 2.2.19 and Lemmas 5.2.3 and 5.2.4, if $u_{k-1}^* \in L^{p_d}(\Omega)^d$ are known, then it holds that for all $v, \varphi \in H$,

$$\frac{1}{\tau}(v,\varphi) + a_0(v,\varphi) + a_1(u_{k-1}^*, v, \varphi) \leq \left(\frac{1}{\tau} + c_a + c_d\|u_{k-1}^*\|_{L^{p_d}}\right)\|v\|_1\|\varphi\|_1,$$
$$\frac{1}{\tau}(v,v) + a_0(v,v) + a_1(u_{k-1}^*, v, v) \geq \frac{1}{c_a}\|v\|_1^2,$$

which implies that the mapping $H \times H \ni (v,\varphi) \mapsto \frac{1}{\tau}(v,\varphi) + a_0(v,\varphi) + a_1(u_{k-1}^*, v, \varphi) \in \mathbb{R}$ is a continuous and coercive bilinear form. On the other hand, if $u_{k-1} \in L^2(\Omega)^d$, then the mapping $H \ni \varphi \mapsto \langle f(t_k), \varphi\rangle_H + \tau^{-1}(u_{k-1}, \varphi) \in \mathbb{R}$ is a functional on H. By the Lax–Milgram theorem, there exists a unique solution $u_k^* \in H \subset L^{p_d}(\Omega)^d$ to the first equation of (PM). Since $\operatorname{div} u_k^* \in L^2(\Omega)$, by the Poincaré inequality and the Lax–Milgram theorem, the second equation of (PM) also has a unique solution $P_k \in H^1(\Omega)$. Furthermore, we obtain $u_k := u_k^* - \tau\nabla P_k \in L^2(\Omega)^d$. Therefore, since $u_0 (= u_0^*) \in L^{p_d}(\Omega)^d$, (PM) has a unique solution $(u_k^*, P_k, u_k)_{k=1}^N \subset H \times H^1(\Omega) \times L^2(\Omega)^d$. □

Remark 5.2.10. *The function space $L^2(\Omega)^d$ has the following orthogonal decomposition:*

$$L^2(\Omega)^d = U \oplus \nabla(H_{\Gamma_2}^1(\Omega)),$$

where $U := \{\varphi \in L^2(\Omega)^d \mid \operatorname{div}\varphi = 0$ in $L^2(\Omega), \langle\gamma_n\varphi, \psi\rangle_{H^{1/2}(\Gamma)} = 0$ for all $\psi \in H_{\Gamma_2}^1(\Omega)\}$ [40, Proposition 4.1]. By the second and third equation of (PM) and the Gauss divergence formula, it holds that for all $k = 1, 2, \ldots, N$ and $\psi \in H_{\Gamma_2}^1(\Omega)$,

$$(u_k, \nabla\psi) = (u_k^*, \nabla\psi) - \tau(\nabla P_k, \nabla\psi) = -(\operatorname{div} u_k^*, \psi) - \tau(\nabla P_k, \nabla\psi) = 0,$$

which implies that $u_k \in U$. Since the third equation of (PM) is equivalent to

$$u_k^* - \tau\nabla p^b(t_k) = u_k + \tau\nabla(P_k - p^b(t_k)) \qquad \text{in } L^2(\Omega)^d,$$

Step 2 ((5.1.4) and (5.1.5)) is the projection of $u_k^* - \tau\nabla p^b(t_k)$ to the divergence-free space U.

Remark 5.2.11. *By replacing u_{k-1} in the first equation of (PM) with the third equation of (PM) at the previous step, it holds that for all $k = 1, 2, \ldots, N$, $\varphi \in H$, and $\psi \in H_{\Gamma_2}^1(\Omega)$,*

$$\begin{cases} \frac{1}{\tau}(u_k^* - u_{k-1}^*, \varphi) + a_0(u_k^*, \varphi) + a_1(u_{k-1}^*, u_k^*, \varphi) + (\nabla P_{k-1}, \varphi) = \langle f_k, \varphi\rangle_H \\ \tau(\nabla P_k, \nabla\psi) = -(\operatorname{div} u_k^*, \psi) \end{cases}$$

where $P_0 := 0$. Ones can calculate $(u_k^*, P_k)_{k=1}^N$ without the velocity $(u_k)_{k=1}^M$. Since the calculation $u_k = u_k^* - \tau\nabla P_k$ is not used, this formulation is suitable for numerical calculations such as the finite element method (see Section 5.5).

On the other hand, by replacing u_k^* in the first term of the first equation of (PM) with the third equation of (PM) at the same step, it holds that for all $k = 1, 2, \ldots, N$, $\varphi \in H$, and $\psi \in H_{\Gamma_2}^1(\Omega)$,

$$\begin{cases} \frac{1}{\tau}(u_k - u_{k-1}, \varphi) + a_0(u_k^*, \varphi) + (\nabla P_k, \varphi) = \langle f_k, \varphi\rangle_H - a_1(u_{k-1}^*, u_k^*, \varphi) \\ \tau(\nabla P_k, \nabla\psi) + (\operatorname{div} u_k^*, \psi) = 0 \\ u_k = u_k^* - \tau\nabla P_k \ \text{ in } L^2(\Omega)^d. \end{cases} \tag{5.2.10}$$

This formulation is helpful to prove stability and convergence results.

5.3 Main theorems

5.3.1 Stability and convergence

We show the stability of the projection method (PM) and establish error estimates in suitable norms between the solutions to the Navier–Stokes equations (NS) and the projection method (PM).

Theorem 5.3.1. *Under the condition (5.2.7), we set $f_k \in H^*$ and $p_k^b \in H^1(\Omega)^d$ as (5.2.9) for all $k = 1, 2, \dots, N$. Then, there exists a constant $c > 0$ independent of τ such that*

$$\begin{aligned}&\|\bar{u}_\tau\|_{L^\infty(L^2)} + \|\bar{u}_\tau^*\|_{L^\infty(L^2)} + \|\bar{u}_\tau^*\|_{L^2(H^1)} + \frac{1}{\sqrt{\tau}}\|\bar{u}_\tau - \bar{u}_\tau^*\|_{L^2(L^2)} \\ &\le c\left(\|u_0\|_0 + \|f\|_{L^2(H^*)} + \|p^b\|_{L^2(H^1)}\right).\end{aligned}$$

For a convergence theorem, we assume:

Hypothesis 5.3.2. *The solution (u, P) to (NS) satisfies*

$$\begin{aligned}&u \in C([0,T]; H \cap H^2(\Omega)^d) \cap H^1(0,T; L^2(\Omega)^d) \cap H^2(0,T; H^*), \\ &P \in C([0,T]; H^1(\Omega)).\end{aligned}$$

We also assume $f \in C([0,T]; H^*)$ and $p^b \in C([0,T]; H^1(\Omega))$ and set in Problem 5.2.7 for all $k = 1, 2, \dots, N$,

$$f_k := f(t_k), \qquad p_k^b := p^b(t_k).$$

Theorem 5.3.3. *Under Hypothesis 5.3.2, there exist two constants $c, \tau_0 > 0$ independent of τ such that for all $0 < \tau < \tau_0$,*

$$\begin{aligned}\|u - \bar{u}_\tau\|_{L^\infty(L^2)} + \|u - \bar{u}_\tau^*\|_{L^\infty(L^2)} + \|u - \bar{u}_\tau^*\|_{L^2(H^1)} &\le c\sqrt{\tau}, \\ \|\bar{u}_\tau - \bar{u}_\tau^*\|_{L^2(L^2)} &\le c\tau.\end{aligned}$$

Remark 5.3.4. *For regularity of the solution (u, P) to (NS), see [10, Theorem 1.3] and [54, Theorems 4.2 and 4.3]. In the case of the homogeneous Dirichlet boundary condition on the whole boundary Γ, high regularity properties of the solution to the Navier–Stokes equations are proven in [14, Theorem V.2.10].*

Remark 5.3.5. *If $u \in C([0,T]; H \cap H^2(\Omega)^d)$, then $|u|^2 \in C([0,T]; H^1(\Omega))$, and hence, $p \in C([0,T]; H^1(\Omega))$ is equivalent to $P = p + \frac{1}{2}|u|^2 \in C([0,T]; H^1(\Omega))$.*

Furthermore, we assume the following regularity assumptions:

Hypothesis 5.3.6 (Regularity of the Stokes problem)**.** *There exists a constant $c = c(\Omega, \Gamma_1,$
$\Gamma_2) > 0$ such that*

$$\|w\|_2 + \|r\|_1 \le c\|e\|_0.$$

for all $e \in L^2(\Omega)^d$ and $(w, r) = T(e)$.

Hypothesis 5.3.7. *The solution (u, P) to (NS) satisfies*

$$u \in H^1(0,T;H^1(\Omega)^d) \cap H^2(0,T;L^2(\Omega)^d) \cap H^3(0,T;H^*),$$
$$P \in H^1(0,T;H^1(\Omega)).$$

Then, we can improve the convergence rate:

Theorem 5.3.8. *Under Hypotheses 5.3.2 and 5.3.6, there exist two constants $\tau_1, c > 0$ independent of τ such that for all $0 < \tau < \tau_1$,*

$$\|u - \bar{u}_\tau\|_{L^2(L^2)} + \|u - \bar{u}_\tau^*\|_{L^2(L^2)} \leq c\tau.$$

Furthermore, if we also assume Hypothesis 5.3.7, then there exist two constants $\tau_2, c > 0$ independent of τ such that for all $0 < \tau < \tau_2 (\leq \tau_1)$,

$$\|P - \bar{P}_\tau\|_{L^2(L^2)} \leq c\sqrt{\tau}.$$

Remark 5.3.9. *Hypothesis 5.3.6 holds, e.g., if Ω is of class $C^{2,1}$ [9, Theorem 1.2].*

5.3.2 Main result for existence of a weak solution to (5.1.2)

Using Theorem 5.3.1, we prove that there exists a solution to a weak formulation of (5.1.2) weaker than (NS). Putting $\varphi := v \in V$ in the first equation of (NS), we obtain the following equation: for all $v \in V$,

$$\left\langle \frac{\partial u}{\partial t}, v \right\rangle_V + a_0(u,v) + a_1(u,u,v) = \langle f, v\rangle_H - \int_{\Gamma_2} p^b v \cdot n ds \tag{5.3.11}$$

in $L^1(0,T)$.

Corollary 5.3.10. *Under the condition (5.2.7), there exists a solution $u \in L^2(0,T;V) \cap L^\infty(0,T;L^2(\Omega)^d) \cap C([0,T];V^*)$ to (5.3.11) with $u(0) = u_0$ such that $\frac{\partial u}{\partial t} \in L^{4/p_d}(0,T;V^*)$.*

Remark 5.3.11. *For $f \in L^2(0,T;L^2(\Omega)^d)$, local existence and uniqueness of a weak solution u to (5.3.11) with $u_0 \in H$ are proven in [10, Theorem 1.3]. By [51, Lemma 4]:*

$$a_0(u,v) = \sum_{i,j=1}^d \int_\Omega \nabla u : \nabla v dx + \int_{\Gamma_2} \kappa u \cdot v ds \quad \text{for all } u, v \in H,$$

where $\kappa := \operatorname{div} n = (d-1)\times$(mean curvature) (cf. Remark 5.1.1), (5.3.11) is equivalent to

$$\begin{aligned} &\left\langle \frac{\partial u}{\partial t}, v \right\rangle_V + \sum_{i,j=1}^d \int_\Omega \nabla u : \nabla v dx + \int_{\Gamma_2} \kappa u \cdot v ds + a_1(u,u,v) \\ =&\langle f, v\rangle_H - \int_{\Gamma_2} p^b v \cdot n ds \quad \text{for all } v \in V \end{aligned} \tag{5.3.12}$$

in $L^1(0,T)$. It is known [55, Theorem 5.1] that there exists a weak solution u to (5.3.12) with $u_0 \in U$, where U is defined in Remark 5.2.10. We demonstrate the existence of a weak solution $u \in L^2(0,T;V) \cap L^\infty(0,T;L^2(\Omega)^d)$ to (5.3.11) with a different approach than [10, 55] (Corollary 5.3.10).

5.4 Proofs

In this section, we prove that the solution to (PM) is bounded in suitable norms (Theorem 5.3.1) and error estimates (Theorems 5.3.3 and 5.3.8) in suitable norms between the solutions to (NS) and (PM).

5.4.1 Stability

We prepare the following useful lemma for the proofs of Theorems 5.3.1, 5.3.3, and 5.3.8.

Lemma 5.4.1. *Let* $v_0 \in L^2(\Omega)^d, (F_k, G_k, Q_k)_{k=1}^N \subset H^* \times H^* \times H^1(\Omega)$ *and let* $(v_k^*, v_k, q_k)_{k=1}^N \in H \times L^2(\Omega)^d \times H^1(\Omega)$ *satisfy that for all* $k = 1, 2, \ldots, N$, $\varphi \in H$, *and* $\psi \in H^1_{\Gamma_2}(\Omega)$,

$$\begin{cases} \frac{1}{\tau}(v_k - v_{k-1}, \varphi) + a_0(v_k^*, \varphi) - (q_k, \operatorname{div}\varphi) = \langle F_k + G_k, \varphi\rangle_H, \\ \tau(\nabla q_k, \nabla\psi) + (\operatorname{div} v_k^*, \psi) = -\tau(\nabla Q_k, \nabla\psi), \\ v_k = v_k^* - \tau\nabla(q_k + Q_k) \text{ in } L^2(\Omega)^d. \end{cases} \tag{5.4.13}$$

If we assume that for all $\delta > 0$ *there exist a constant* $A_\delta > 0$ *independent of* k *and* τ, *and a sequence* $(\beta_k)_{k=1}^N \subset \mathbb{R}$ *such that*

$$\langle G_k, v_k^*\rangle_H \le \delta\|v_k^*\|_1^2 + A_\delta(\|v_{k-1}^*\|_0^2 + \beta_k^2) \qquad \text{for all } k = 1, 2, \ldots, N, \tag{5.4.14}$$

where $v_0^* := v_0$, *then there exist two constants* $\tau_0, c > 0$ *independent of* τ *such that for all* $0 < \tau < \tau_0$,

$$\begin{aligned} &\|\bar{v}_\tau\|_{L^\infty(L^2)}^2 + \|\bar{v}_\tau^*\|_{L^\infty(L^2)}^2 + \|\bar{v}_\tau^*\|_{L^2(H^1)}^2 + \tau\left\|\frac{\partial\hat{v}_\tau}{\partial t}\right\|_{L^2(L^2)}^2 + \frac{1}{\tau}\|\bar{v}_\tau - \bar{v}_\tau^*\|_{L^2(L^2)}^2 \\ &\le c\left(\|v_0\|_0^2 + \|\bar{F}_\tau\|_{L^2(H^*)}^2 + \tau\|\bar{Q}_\tau^b\|_{L^2(H^1)}^2 + \|\bar{\beta}_\tau\|_{L^2(0,T)}^2\right). \end{aligned} \tag{5.4.15}$$

In particular, if $\langle G_k, v_k^*\rangle_H \le 0$ *for all* $k = 1, 2, \ldots, N$, *then* $\tau_0 = T$.

Proof. Putting $\varphi := v_k^*$ and $\psi := q_k$ and adding the two equations, we obtain for all $k = 1, 2, \ldots, N$,

$$\begin{aligned} &\frac{1}{\tau}(v_k - v_{k-1}, v_k^*) + \|v_k^*\|_{a_0}^2 + \tau\|\nabla q_k\|_0^2 + \tau(\nabla Q_k, \nabla q_k) \\ &= \langle F_k + G_k, v_k^*\rangle_H \le \frac{c_a}{2}\|F_k\|_{H^*}^2 + \frac{1}{2c_a}\|v_k^*\|_1 + \langle G_k, v_k^*\rangle_H. \end{aligned}$$

Here, by Theorem 2.2.19 and the third equation of (5.4.13), it holds that

$$
\begin{aligned}
&\frac{1}{\tau}(v_k - v_{k-1}, v_k^*) + \|v_k^*\|_{a_0}^2 + \tau\|\nabla q_k\|_0^2 + \tau(\nabla Q_k, \nabla q_k) \\
&= \frac{1}{\tau}(v_k - v_{k-1}, v_k) + \frac{1}{\tau}(v_k - v_{k-1}, v_k^* - v_k) + \|v_k^*\|_{a_0}^2 \\
&\quad + \tau\|\nabla(q_k + Q_k)\|_0^2 - \tau(\nabla Q_k, \nabla(q_k + Q_k)) \\
&\geq \frac{1}{2\tau}(\|v_k\|_0^2 - \|v_{k-1}\|_0^2 + \|v_k - v_{k-1}\|_0^2) - \frac{3}{8\tau}\|v_k - v_{k-1}\|_0^2 - \frac{2}{3\tau}\|v_k^* - v_k\|_0^2 + \frac{1}{c_a}\|v_k^*\|_1^2 \\
&\quad + \tau\|\nabla(q_k + Q_k)\|_0^2 - 3\tau\|\nabla Q_k\|_0^2 - \frac{\tau}{12}\|\nabla(q_k + Q_k)\|_0^2 \\
&= \frac{1}{2\tau}\left(\|v_k\|_0^2 - \|v_{k-1}\|_0^2 + \frac{\tau^2}{4}\|D_\tau v_k\|_0^2 + \frac{1}{2}\|v_k^* - v_k\|_0^2\right) + \frac{1}{c_a}\|v_k^*\|_1^2 - 3\tau\|\nabla Q_k\|_0^2.
\end{aligned}
$$

Hence, we have for all $k = 1, 2, \ldots, N$,

$$
\begin{aligned}
&\|v_k\|_0^2 - \|v_{k-1}\|_0^2 + \frac{\tau^2}{4}\|D_\tau v_k\|_0^2 + \frac{1}{2}\|v_k^* - v_k\|_0^2 + \frac{\tau}{c_a}\|v_k^*\|_1^2 \\
&\leq c_a\tau\|F_k\|_{H^*}^2 + 6\tau^2\|\nabla Q_k\|_0^2 + 2\tau\langle G_k, v_k^*\rangle_H.
\end{aligned} \tag{5.4.16}
$$

By summing up (5.4.16) for $k = 1, 2, \ldots, m$ with an arbitrary natural number $m \leq N$, it holds that

$$
\begin{aligned}
&\|v_m\|_0^2 - \|v_0\|_0^2 + \tau\sum_{k=1}^{m}\left(\frac{\tau}{4}\|D_\tau v_k\|_0^2 + \frac{1}{2\tau}\|v_k^* - v_k\|_0^2 + \frac{1}{c_a}\|v_k^*\|_1^2\right) \\
&\leq \tau\sum_{k=1}^{m}\left(c_a\|F_k\|_{H^*}^2 + 6\tau\|\nabla Q_k\|_0^2 + 2\langle G_k, v_k^*\rangle_H\right).
\end{aligned} \tag{5.4.17}
$$

From the assumption (5.4.14) with $\delta := \frac{1}{4c_a}$;

$$
\langle G_k, v_k^*\rangle_H \leq \frac{\|v_k^*\|_1^2}{4c_a} + A_{\frac{1}{4c_a}}(\|v_{k-1}^*\|_0^2 + \beta_k^2) \leq \frac{\|v_k^*\|_1^2}{4c_a} + A_{\frac{1}{4c_a}}(2\|v_{k-1}\|_0^2 + 2\|v_{k-1} - v_{k-1}^*\|_0^2 + \beta_k^2),
$$

we obtain

$$
\begin{aligned}
&\|v_m\|_0^2 - \|v_0\|_0^2 + \tau\sum_{k=1}^{m}\left(\frac{\tau}{4}\|D_\tau v_k\|_0^2 + \frac{1 - 8\tau A_{\frac{1}{4c_a}}}{2\tau}\|v_k - v_k^*\|_0^2 + \frac{1}{2c_a}\|v_k^*\|_1^2\right) \\
&\leq \tau\sum_{k=0}^{m-1} 4A_{\frac{1}{4c_a}}\|v_k\|_0^2 + \tau\sum_{k=1}^{m}\left(c_a\|F_k\|_{H^*}^2 + 6\tau\|\nabla Q_k\|_0^2 + 2A_{\frac{1}{4c_a}}\beta_k^2\right),
\end{aligned}
$$

where we have used $v_0 - v_0^* = 0$. By the discrete Gronwall inequality, if $\tau \leq \tau_0 := 1/(16A_{\frac{1}{4c_a}})$, then it holds that for all $m = 0, 1, \ldots, N$,

$$
\begin{aligned}
&\|v_m\|_0^2 + \tau\sum_{k=1}^{m}\left(\frac{\tau}{4}\|D_\tau v_k\|_0^2 + \frac{1}{4\tau}\|v_k - v_k^*\|_0^2 + \frac{1}{2c_a}\|v_k^*\|_1^2\right) \\
&\leq \exp\left(\frac{16}{3}A_{\frac{1}{4c_a}}\right)\left\{\|v_0\|_0^2 + \tau\sum_{k=1}^{m}\left(c_a\|F_k\|_{H^*}^2 + 6\tau\|\nabla Q_k\|_0^2 + 2A_{\frac{1}{4c_a}}\beta_k^2\right)\right\},
\end{aligned}
$$

which implies that

$$\begin{aligned}
&\|\bar{v}_\tau(t)\|_0^2 + \int_0^t \left(\tau \left\| \frac{\partial \hat{v}_\tau}{\partial t}(s) \right\|_0^2 + \frac{1}{\tau}\|\bar{v}_\tau(s) - \bar{v}_\tau^*(s)\|_0^2 + \|\bar{v}_\tau^*(s)\|_1^2 \right) ds \\
\leq\ & c_1 \left\{ \|v_0\|_0^2 + \int_0^t (\|\bar{F}_\tau(s)\|_{H^*}^2 + \tau\|\bar{Q}_\tau(s)\|_1^2 + \bar{\beta}_\tau^2(s)) ds \right\}
\end{aligned}$$

for all $t \in (0, T]$, where $c_1 := \exp(16A_{\frac{1}{4c_a}}/3) \times \max\{c_a, 6, 2A_{\frac{1}{4c_a}}\} \times \max\{4, 2c_a\}$. Hence,

$$\|\bar{v}_\tau\|_{L^\infty(L^2)}^2 \leq M, \qquad \tau \left\| \frac{\partial \hat{v}_\tau}{\partial t} \right\|_{L^2(L^2)}^2 + \frac{1}{\tau}\|\bar{v}_\tau - \bar{v}_\tau^*\|_{L^2(L^2)}^2 + \|\bar{v}_\tau^*\|_{L^2(H^1)}^2 \leq M, \tag{5.4.18}$$

where $M := c_1(\|v_0\|_0^2 + \|\bar{F}_\tau\|_{L^2(H^*)}^2 + \tau\|\bar{Q}_\tau\|_{L^2(H^1)}^2 + \|\bar{\beta}_\tau\|_{L^2(0,T)}^2)$. If $\langle G_k, v_k^* \rangle_H \leq 0$ for all $k = 1, 2, \ldots, N$, then we immediately obtain (5.4.15) for all $0 < \tau < T$ from (5.4.17).

Since it holds that for all $m = 1, 2, \ldots, N$,

$$\|v_m^*\|_0^2 \leq 2(\|v_m\|_0^2 + \|v_m - v_m^*\|_0^2) \leq 2 \max_{k=1,\ldots,N} \|v_k\|_0^2 + \tau \sum_{k=1}^N \frac{2}{\tau}\|v_k - v_k^*\|_0^2,$$

we obtain for all $0 < \tau < \tau_0$,

$$\|\bar{v}_\tau^*\|_{L^\infty(L^2)}^2 \leq 2\|\bar{v}_\tau\|_{L^\infty(L^2)}^2 + \frac{2}{\tau}\|\bar{v}_\tau - \bar{v}_\tau^*\|_{L^2(L^2)}^2 \leq 4M.$$

□

By using Lemma 5.4.1, we prove Theorem 5.3.1.

Proof of Theorem 5.3.1. We set $(F_k)_{k=1}^N, (G_k)_{k=1}^N \subset H^*$ defined by

$$\langle F_k, \varphi \rangle_H := \langle f_k, \varphi \rangle_H - (\nabla p_k^b, \varphi), \qquad \langle G_k, \varphi \rangle_H := -a_1(u_{k-1}^*, u_k^*, \varphi)$$

for all $k = 1, 2, \ldots, N$ and $\varphi \in H$. From Problem 5.2.7 and the condition (5.2.7), if we set $q_k := P_k - p_k^b$, then $(u_k^*, u_k, q_k)_{k=1}^N \subset H \times L^2(\Omega)^d \times H_{\Gamma_2}^1(\Omega)$ satisfies that for all $k = 1, 2, \ldots, N$,

$$\begin{cases}
\frac{1}{\tau}(u_k - u_{k-1}, \varphi) + a_0(u_k^*, \varphi) - (q_k, \operatorname{div} \varphi) = \langle F_k + G_k, \varphi \rangle, \\
\tau(\nabla q_k, \nabla \psi) + (\operatorname{div} u_k^*, \psi) = -(\nabla p_k^b, \nabla \psi), \\
u_k = u_k^* - \tau \nabla(q_k + p_k^b) \text{ in } L^2(\Omega)^d,
\end{cases}$$

with $u_0 \in L^{p_d}(\Omega)^d (\subset L^2(\Omega)^d)$. By Lemma 5.2.3, it holds that

$$\langle G_k, u_k^* \rangle_H = -a_1(u_{k-1}^*, u_k^*, u_k^*) = 0 \qquad \text{for all } k = 1, 2, \ldots, N.$$

Therefore, by Lemma 5.4.1 and Remark 5.2.8, we conclude the proof. □

5.4.2 Convergence

In this section, we assume Hypothesis 5.3.2. We calculate the error estimates in suitable norms between the solutions to (NS) and (PM). By Hypothesis 5.3.2 and the first equation of (NS), it holds that $\frac{\partial u}{\partial t} \in C([0,T]; H^*)$ and, for all $\varphi \in H$ and $k = 1, 2, \ldots, N$,

$$\begin{aligned}
&\frac{1}{\tau}(u(t_k) - u(t_{k-1}), \varphi) + a_0(u(t_k), \varphi) + a_1(u^*_{k-1}, u^*_k, \varphi) + (\nabla P(t_k), \varphi) \\
=&\langle f(t_k) - R_k - R_k^{\text{n.l.}}, \varphi\rangle_H,
\end{aligned}$$

where $R_k, R_k^{\text{n.l.}} \in H^*$ defined by

$$\begin{aligned}
\langle R_k, \varphi\rangle_H &:= \left\langle \frac{\partial u}{\partial t}(t_k) - \frac{u(t_k) - u(t_{k-1})}{\tau}, \varphi \right\rangle_H, \\
\langle R_k^{\text{n.l.}}, \varphi\rangle_H &:= a_1(u(t_k), u(t_k), \varphi) - a_1(u^*_{k-1}, u^*_k, \varphi)
\end{aligned}$$

for all $\varphi \in H$. If we put $e_0 = 0$, $e_k := u_k - u(t_k) \in L^2(\Omega)^d$, $e^*_k := u^*_k - u(t_k) \in H$, and $q_k := P_k - P(t_k) \in H^1_{\Gamma_2}(\Omega)$ for $k = 1, 2, \ldots, N$, by (5.2.10), then it holds that for all $k = 1, 2, \ldots, N$, $\varphi \in H$, and $\psi \in H^1_{\Gamma_2}(\Omega)$,

$$\begin{cases}
\frac{1}{\tau}(e_k - e_{k-1}, \varphi) + a_0(e^*_k, \varphi) - (q_k, \operatorname{div}\varphi) = \langle R_k + R_k^{\text{n.l.}}, \varphi\rangle_H \\
\tau(\nabla q_k, \nabla\psi) + (\operatorname{div} e^*_k, \psi) = -\tau(\nabla P(t_k)\nabla\psi), \\
e_k = e^*_k - \tau\nabla(q_k + P(t_k)) \text{ in } L^2(\Omega)^d,
\end{cases} \tag{5.4.19}$$

where we have used $(\nabla q_k, \varphi) = -(q_k, \operatorname{div}\varphi)$.

In order to prove Theorems 5.3.3 and 5.3.8, we prepare Lemmas 5.4.2 and 5.4.3.

Lemma 5.4.2. *(i) Under Hypothesis 5.3.2, we have*

$$\|\bar{R}_\tau\|^2_{L^2(H^*)} \leq \frac{\tau^2}{3}\left\|\frac{\partial^2 u}{\partial t^2}\right\|^2_{L^2(H^*)}.$$

(ii) Furthermore, if Hypothesis 5.3.7 holds, then we have

$$\sum_{k=2}^{N} \tau\|D_\tau R_k\|^2_{H^*} \leq \frac{2}{3}\tau^2\left\|\frac{\partial^3 u}{\partial t^3}\right\|^2_{L^2(H^*)}.$$

Proof. It holds that for all $\varphi \in H$ and $k = 1, 2, \ldots, N$,

$$\begin{aligned}
\langle R_k, \varphi\rangle_H &= \left\langle \frac{u(t_k) - u(t_{k-1})}{\tau} - \frac{\partial u}{\partial t}(t_k), \varphi \right\rangle \\
&= \tau\int_0^1 \left\langle s\frac{\partial^2 u}{\partial t^2}(t_{k-1} + s\tau), \varphi\right\rangle_H ds \\
&\leq \tau\int_0^1 s\left\|\frac{\partial^2 u}{\partial t^2}(t_{k-1} + s\tau)\right\|_{H^*}\|\varphi\|_1 ds \\
&\leq \tau\|\varphi\|_1\sqrt{\int_0^1 s^2 dt}\sqrt{\int_0^1 \left\|\frac{\partial^2 u}{\partial t^2}(t_{k-1} + s\tau)\right\|^2_{H^*} ds} \\
&= \sqrt{\frac{\tau}{3}}\|\varphi\|_1\left\|\frac{\partial^2 u}{\partial t^2}\right\|_{L^2(t_{k-1}, t_k; H^*)},
\end{aligned}$$

which implies that

$$\|\bar{R}_\tau\|^2_{L^2(H^*)} = \sum_{k=1}^N \tau \left(\sup_{0\neq\varphi\in H} \frac{\langle R_k, \varphi\rangle_H}{\|\varphi\|_1} \right)^2 \leq \sum_{k=1}^N \tau \frac{\tau}{3} \left\| \frac{\partial^2 u}{\partial t^2} \right\|^2_{L^2(t_{k-1},t_k;H^*)} \leq \frac{1}{3}\tau^2 \left\| \frac{\partial^2 u}{\partial t^2} \right\|^2_{L^2(H^*)}.$$

Next, we show the second inequality of the conclusion. For all $\varphi \in H$ and $k = 2, 3, \dots, N$, we have

$$\begin{aligned}
\langle D_\tau R_k, \varphi\rangle_H &= \left\langle \frac{R_k - R_{k-1}}{\tau}, \varphi \right\rangle_H \\
&= \int_0^1 \left\langle s_1 \frac{\partial^2 u}{\partial t^2}(t_{k-1} + s_1\tau) - s_1 \frac{\partial^2 u}{\partial t^2}(t_{k-2} + s_1\tau), \varphi \right\rangle_H ds_1 \\
&= \tau \int_0^1 \int_0^1 \left\langle s_1 \frac{\partial^3 u}{\partial t^3}(t_{k-2} + s_1\tau + s_2\tau), \varphi \right\rangle_H ds_1 ds_2 \\
&\leq \tau \int_0^1 \int_0^1 s_1 \left\| \frac{\partial^3 u}{\partial t^3}(t_{k-2} + s_1\tau + s_2\tau) \right\|_{H^*} \|\varphi\|_1 ds_1 ds_2 \\
&\leq \tau \|\varphi\|_1 \sqrt{\int_0^1 \int_0^1 s_1^2 ds_1 ds_2} \sqrt{\int_0^1 \int_0^1 \left\| \frac{\partial^3 u}{\partial t^3}(t_{k-2} + s_1\tau + s_2\tau) \right\|^2_{H^*} ds_1 ds_2} \\
&\leq \tau \|\varphi\|_1 \sqrt{\frac{1}{3}} \sqrt{\int_{-1}^1 \int_0^2 \left\| \frac{\partial^3 u}{\partial t^3}(t_{k-2} + \tilde{s}_1\tau) \right\|^2_{H^*} \frac{1}{2} d\tilde{s}_1 d\tilde{s}_2} \\
&= \sqrt{\frac{\tau}{3}} \|\varphi\|_1 \left\| \frac{\partial^3 u}{\partial t^3} \right\|_{L^2(t_{k-2},t_k;H^*)},
\end{aligned}$$

where we have used the coordinate transformation $(s_1, s_2) \mapsto (\tilde{s}_1, \tilde{s}_2) := (s_1+s_2, -s_1+s_2)$. Therefore, we obtain

$$\begin{aligned}
\sum_{k=2}^N \tau \|D_\tau R_k\|^2_{H^*} &= \sum_{k=2}^N \tau \left(\sup_{0\neq\varphi\in H} \frac{\langle D_\tau R_k, \varphi\rangle}{\|\varphi\|_1} \right)^2 \\
&\leq \sum_{k=2}^N \tau \frac{\tau}{3} \left\| \frac{\partial^3 u}{\partial t^3} \right\|^2_{L^2(t_{k-2},t_k;H^*)} \\
&\leq \frac{2}{3}\tau^2 \left\| \frac{\partial^3 u}{\partial t^3} \right\|^2_{L^2(H^*)}.
\end{aligned}$$

□

Lemma 5.4.3. *Let $(E, (\cdot,\cdot)_E)$ be a Hilbert space and let $x \in C([0,T]; E)$ satisfy that $\frac{\partial x}{\partial t} \in L^2(0,T;E)$.*
(i) It holds that for all $k = 1, 2, \dots, N$,

$$\|D_\tau x(t_k)\|_E \leq \frac{1}{\sqrt{\tau}} \left\| \frac{\partial x}{\partial t} \right\|_{L^2(t_{k-1},t_k;E)}.$$

(ii) It holds that

$$\|x - x_\tau\|_{L^\infty(E)} \leq \sqrt{\tau} \left\| \frac{\partial x}{\partial t} \right\|_{L^2(E)}, \qquad \|x - x_\tau\|_{L^2(E)} \leq \frac{\tau}{\sqrt{2}} \left\| \frac{\partial x}{\partial t} \right\|_{L^2(E)}.$$

Proof. It holds that for all $k = 1, 2, \dots, N$ and $t \in [t_{k-1}, t_k]$,

$$\|x(t_k) - x(t)\|_E \le \int_t^{t_k} \left\| \frac{\partial x}{\partial t}(s) \right\|_E ds \le \sqrt{t_k - t} \left\| \frac{\partial x}{\partial t} \right\|_{L^2(t_{k-1}, t_k; E)},$$

which implies that $\|x - x_\tau\|_{L^\infty(E)} \le \sqrt{\tau} \left\| \frac{\partial x}{\partial t} \right\|_{L^2(E)}$ and

$$\|D_\tau x(t_k)\|_E = \frac{1}{\tau} \|x(t_k) - x(t_{k-1})\|_E \le \frac{1}{\sqrt{\tau}} \left\| \frac{\partial x}{\partial t} \right\|_{L^2(t_{k-1}, t_k; E)}.$$

On the other hand, we have

$$\begin{aligned} \|x - x_\tau\|^2_{L^2(E)} &= \sum_{k=1}^N \int_{t_{k-1}}^{t_k} \|x(t) - x(t_k)\|_E^2 dt \\ &\le \sum_{k=1}^N \int_{t_{k-1}}^{t_k} (t_k - t) dt \left\| \frac{\partial x}{\partial t} \right\|^2_{L^2(t_{k-1}, t_k; E)} \\ &= \sum_{k=1}^N \int_{t_{k-1}}^{t_k} \left\| \frac{\partial x}{\partial t}(s) \right\|_E^2 ds \int_{t_{k-1}}^{t_k} (t_k - t) dt \\ &= \frac{1}{2} \tau^2 \left\| \frac{\partial x}{\partial t} \right\|^2_{L^2(E)}. \end{aligned}$$

□

By using Lemmas 5.4.1, 5.4.2, and 5.4.3, we prove Theorem 5.3.3.

Proof of Theorem 5.3.3. For all $\delta > 0$ and $k = 1, 2, \dots, N$, by Lemmas 5.2.3, 5.2.4, and 5.4.3, we have

$$\begin{aligned} \langle R_k^{\text{n.l.}}, e_k^* \rangle_H &= -a_1(u_{k-1}^*, e_k^*, e_k^*) - a_1(e_{k-1}^*, u(t_k), e_k^*) + \tau a_1(D_\tau u(t_k), u(t_k), e_k^*) \\ &\le c_d \|e_{k-1}^*\|_0 \|u(t_k)\|_2 \|e_k^*\|_1 + c_d \tau \|D_\tau u(t_k)\|_0 \|u(t_k)\|_2 \|e_k^*\|_1 \\ &\le \frac{\delta}{2} \|e_k^*\|_1^2 + \frac{c_d^2 \|u(t_k)\|_2^2}{2\delta} \|e_{k-1}^*\|_0^2 + \frac{\delta}{2} \|e_k^*\|_1^2 + \frac{c_d^2 \|u(t_k)\|_2^2}{2\delta} \tau^2 \|D_\tau u(t_k)\|_0^2 \qquad (5.4.20) \\ &\le \delta \|e_k^*\|_1^2 + \frac{c_d^2 c_{\max}^2}{2\delta} \|e_{k-1}^*\|_0^2 + \frac{c_d^2 c_{\max}^2}{2\delta} \tau \left\| \frac{\partial u}{\partial t} \right\|^2_{L^2(t_{k-1}, t_k; L^2(\Omega)^d)} \end{aligned}$$

where $c_{\max} := \|u\|_{C([0,T], H^2(\Omega)^d)}$. By (5.4.19) and Lemmas 5.4.1, 5.4.2, and 5.4.3, there exist two constants $\tau_0, c_1 > 0$ such that for all $0 < \tau < \tau_0$,

$$\begin{aligned} &\|\bar{e}_\tau\|^2_{L^\infty(L^2)} + \|\bar{e}_\tau^*\|^2_{L^\infty(L^2)} + \|\bar{e}_\tau^*\|^2_{L^2(H^1)} + \frac{1}{\tau} \|\bar{e}_\tau - \bar{e}_\tau^*\|^2_{L^2(L^2)} \\ \le\, &c_1 \left(\|\bar{R}_\tau\|^2_{L^2(H^*)} + \tau \|P_\tau\|^2_{L^2(H^1)} + \tau^2 \left\| \frac{\partial u}{\partial t} \right\|^2_{L^2(L^2)} \right) \\ \le\, &c_1 \left(\frac{\tau^2}{3} \left\| \frac{\partial^2 u}{\partial t^2} \right\|^2_{L^2(H^*)} + 2\tau \|P\|^2_{L^2(H^1)} + \tau^3 \left\| \frac{\partial P}{\partial t} \right\|^2_{L^2(H^1)} + \tau^2 \left\| \frac{\partial u}{\partial t} \right\|^2_{L^2(L^2)} \right), \end{aligned}$$

which implies that

$$\|\bar{u}_\tau - u_\tau\|_{L^\infty(L^2)} + \|\bar{u}_\tau^* - u_\tau\|_{L^\infty(L^2)} + \|\bar{u}_\tau^* - u_\tau\|_{L^2(H^1)} \le c_2 \sqrt{\tau},$$
$$\|\bar{u}_\tau - \bar{u}_\tau^*\|_{L^2(L^2)} \le c_2 \tau$$

for a constant $c_2 > 0$, where we have used $\bar{e}_\tau = \bar{u}_\tau - u_\tau$ and $\bar{e}^*_\tau = \bar{u}^*_\tau - u_\tau$. By the triangle inequality and Lemma 5.4.3, it holds that $\|u - \bar{u}_\tau\|_{L^\infty(L^2)} + \|u - \bar{u}^*_\tau\|_{L^\infty(L^2)} \le c_2\sqrt{\tau} + 2\sqrt{\tau}\|\frac{\partial u}{\partial t}\|_{L^2(L^2)}$. To complete the first inequality of Theorem 5.3.3, it is sufficient to prove that $\|u - u_\tau\|_{L^2(H^1)} \le c_3\sqrt{\tau}$ for a constant $c_3 > 0$. Since $u(t) \in H \cap H^2(\Omega)^d$ and $\operatorname{div} u(t) = 0 \in H^1(\Omega)$ for all $t \in [0,T]$, by Proposition 5.2.6, Theorem 2.2.19, and Lemma 5.4.3, we find that

$$
\begin{aligned}
\|u - u_\tau\|^2_{L^2(H^1)} =& \sum_{k=1}^N \int_{t_{k-1}}^{t_k} \|u(t) - u(t_k)\|_1^2 dt \\
\le& c_a \sum_{k=1}^N \int_{t_{k-1}}^{t_k} a_0(u(t) - u(t_k), u(t) - u(t_k)) dt \\
=& c_a \sum_{k=1}^N \int_{t_{k-1}}^{t_k} (-\Delta(u(t) - u(t_k)), u(t) - u(t_k))\, dt \\
\le& c_a \sum_{k=1}^N \int_{t_{k-1}}^{t_k} \|\Delta(u(t) - u(t_k))\|_0 \|u(t) - u(t_k)\|_0 dt \\
\le& \sqrt{d} c_a \sum_{k=1}^N \int_{t_{k-1}}^{t_k} \|u(t) - u(t_k)\|_2 \|u(t) - u(t_k)\|_0 dt \\
\le& 2\sqrt{d} c_a c_{\max} \int_0^T \|u(t) - u(t_k)\|_0 dt \\
\le& 2\sqrt{dT} c_a c_{\max} \|u - u_\tau\|_{L^2(L^2)} \\
\le& \sqrt{2dT} c_a c_{\max} \tau \left\| \frac{\partial u}{\partial t} \right\|_{L^2(L^2)}.
\end{aligned}
$$

□

We improve the error estimates for the velocity and pressure in the $L^2(L^2)$-norm. In order to prove Theorem 5.3.8, we prepare Proposition 5.4.4 and Lemma 5.4.5.

Proposition 5.4.4. *Under Hypothesis 5.3.6, for all $e \in L^2(\Omega)$, the pair of functions $(w, r) = T(e)$ belongs to $H^2(\Omega) \times H^1_{\Gamma_2}(\Omega)$.*

Proof. By Hypothesis 5.3.6, $(w, r) \in H^2(\Omega)^d \times H^1(\Omega)$. Since it holds that for all $\varphi \in H$,

$$
0 = a_0(w, \varphi) - (r, \operatorname{div} \varphi) - (e, \varphi) = \int_\Omega (\nabla \times (\nabla \times w) + \nabla r - e) \cdot \varphi dx - \int_{\Gamma_2} r\varphi \cdot n ds,
$$

we obtain $r \in H^1_{\Gamma_2}(\Omega)$. □

Lemma 5.4.5. *Under the assumption of Lemma 5.4.1 and Hypothesis 5.3.6, if we assume the following conditions: if $(w_k, r_k) := T(v_k)$ for all $k = 0, 1, \ldots, N$, then for all $\delta > 0$ there exist a constant $A_\delta > 0$ independent of k and τ, and a sequence $(\gamma_k)_{k=1}^N \in \mathbb{R}$ such that for all $k = 1, 2, \ldots, N$,*

$$
\langle G_k, w_k \rangle_H \le \delta(\|v^*_{k-1}\|_0^2 + \|v^*_k\|_0^2) + A_\delta(\|w_k\|_1^2 + \gamma_k^2), \tag{5.4.21}
$$

then there exist two constants $\tau_0, c > 0$ independent of τ such that for all $0 < \tau < \tau_0$,

$$
\|\bar{v}_\tau\|^2_{L^2(L^2)} \le c(\|v_0\|^2_{V^*} + \tau\|v^*_0\|_0^2 + \|\bar{v}^*_\tau - \bar{v}_\tau\|^2_{L^2(L^2)} + \|\bar{F}_\tau\|^2_{L^2(H^*)} + \|\bar{\gamma}_\tau\|^2_{L^2(0,T)}).
$$

Proof. Let $(w_k, r_k) := T(v_k)$ for all $k = 0, 1, \dots, N$. It follows from Proposition 5.4.4 that $r_k \in H^1_{\Gamma_2}(\Omega)$. The first equation of (5.4.13) implies that for all $k = 1, 2, \dots, N$,

$$\frac{1}{\tau}(v_k - v_{k-1}, w_k) + a_0(v_k^*, w_k) = \langle F_k, w_k\rangle_H + \langle G_k, w_k\rangle_H, \tag{5.4.22}$$

where we have used $\operatorname{div} w_k = 0$ in $L^2(\Omega)$. By Lemma 5.2.2, we obtain

$$\begin{aligned}(v_k - v_{k-1}, w_k) &= (v_k, w_k) - (v_{k-1}, w_k)\\ &= a_0(w_k, w_k) - (r_k, \operatorname{div} w_k) - a_0(w_{k-1}, w_k) + (r_{k-1}, \operatorname{div} w_k)\\ &= a_0(w_k - w_{k-1}, w_k)\\ &= \frac{1}{2}\left(\|w_k\|_{a_0}^2 - \|w_{k-1}\|_{a_0}^2 + \|w_k - w_{k-1}\|_{a_0}^2\right)\\ &\geq \frac{c_1}{2}\left(\|w_k\|_1^2 - \|w_{k-1}\|_1^2\right)\end{aligned}$$

where $c_1 := \min\{c_a, c_a^{-1}\}$. For the second term of the left hand side of (5.4.22), by the definition of the operator T, we have

$$\begin{aligned}a_0(v_k^*, w_k) &= (v_k, v_k^*) + (r_k, \operatorname{div} v_k^*)\\ &= \|v_k\|_0^2 + (v_k, v_k^* - v_k) - (\nabla r_k, v_k^*)\\ &= \|v_k\|_0^2 + (v_k, v_k^* - v_k) - (\nabla r_k, v_k^* - v_k),\end{aligned}$$

where we have used the third equation of (5.4.13) and

$$(\nabla r_k, v_k) = (\nabla r_k, v_k^*) - \tau(\nabla r_k, \nabla(q_k + Q_k)) = 0.$$

By Hypothesis 5.3.6, it holds that

$$\begin{aligned}|(v_k, v_k^* - v_k) - (\nabla r_k, v_k^* - v_k)| &\leq (\|v_k\|_0 + \|\nabla r_k\|_0)\|v_k^* - v_k\|_0\\ &\leq c_2\|v_k\|_0\|v_k^* - v_k\|_0\\ &\leq \frac{1}{4}\|v_k\|_0^2 + c_2^2\|v_k^* - v_k\|_0^2\end{aligned}$$

for a constant $c_2 > 0$. Hence, we have

$$a_0(v_k^*, w_k) \geq \frac{3}{4}\|v_k\|_0^2 - c_2^2\|v_k^* - v_k\|_0^2.$$

For the first term of the right hand side of (5.4.22), by Lemma 5.2.2, we have

$$\langle F_k, w_k\rangle_H \leq \|F_k\|_{H^*}\|w_k\|_1 \leq c_3\|F_k\|_{H^*}\|v_k\|_0 \leq \frac{1}{4}\|v_k\|_0^2 + c_3^2\|F_k\|_{H^*}^2$$

for a constant $c_3 > 0$. Hence, we have that for all $k = 1, 2, \dots, N$,

$$\|w_k\|_1^2 - \|w_{k-1}\|_1^2 + \frac{\tau}{c_1}\|v_k\|_0^2 \leq \frac{2\tau}{c_1}(c_2^2\|v_k^* - v_k\|_0^2 + c_3^2\|F_k\|_{H^*}^2 + \langle G_k, w_k\rangle_H).$$

By summing up for $k = 1, 2, \dots, m$ with an arbitrary natural number $m \leq N$, it holds that

$$\|w_m\|_1^2 - \|w_0\|_1^2 + \frac{\tau}{c_1}\sum_{k=1}^m \|v_k\|_0^2 \leq \frac{2\tau}{c_1}\sum_{k=1}^m (c_2^2\|v_k^* - v_k\|_0^2 + c_3^2\|F_k\|_{H^*}^2 + \langle G_k, w_k\rangle_H).$$

From the assumption (5.4.21) with $\delta := \frac{1}{16}$, we obtain for all $m = 1, 2, \ldots, N$,

$$\begin{aligned}\sum_{k=1}^{m} \langle G_k, w_k \rangle_H &\le \sum_{k=1}^{m} \left\{ \frac{1}{16} (\|v_{k-1}^*\|_0^2 + \|v_k^*\|_0^2) + A_{\frac{1}{16}} (\|w_k\|_1^2 + \gamma_k^2) \right\} \\ &\le \frac{1}{16} \|v_0^*\|_0^2 + \frac{1}{8} \sum_{k=1}^{m} \|v_k^*\|_0^2 + A_{\frac{1}{16}} \sum_{k=1}^{m} (\|w_k\|_1^2 + \gamma_k^2) \\ &\le \frac{1}{16} \|v_0^*\|_0^2 + \frac{1}{4} \sum_{k=1}^{m} (\|v_k\|_0^2 + \|v_k^* - v_k\|_0^2) + A_{\frac{1}{16}} \sum_{k=1}^{m} (\|w_k\|_1^2 + \gamma_k^2)\end{aligned}$$

and hence,

$$\begin{aligned}&\|w_m\|_1^2 - \|w_0\|_1^2 + \frac{\tau}{2c_1} \sum_{k=1}^{m} \|v_k\|_0^2 \\ &\le \tau \sum_{k=1}^{m} \frac{2A_{\frac{1}{16}}}{c_1} \|w_k\|_1^2 + \frac{\tau}{8c_1} \|v_0^*\|_0^2 + \tau \sum_{k=1}^{m} c_4 (\|v_k^* - v_k\|_0^2 + \|F_k\|_{H^*}^2 + \gamma_k^2),\end{aligned}$$

where $c_4 := 2c_1^{-1} \max\{c_2^2 + 1/4, c_3^2, A_{\frac{1}{16}}\}$. By the discrete Gronwall inequality, if $\tau \le \tau_0 := c_1/A_{\frac{1}{16}}$, then we have

$$\begin{aligned}&\|w_N\|_1^2 + \frac{\tau}{2c_1} \sum_{k=1}^{N} \|v_k\|^2 \\ &\le \exp\left(\frac{4A_{\frac{1}{16}}}{c_1} \right) \left\{ \|w_0\|_1^2 + \frac{\tau}{8c_1} \|v_0^*\|_0^2 + \tau \sum_{k=1}^{N} c_4 \left(\|v_k^* - v_k\|_0^2 + \|F_k\|_{H^*}^2 + \gamma_k^2 \right) \right\}.\end{aligned}$$

Therefore, by Lemma 5.2.2, we obtain

$$\|\bar{v}_\tau\|_{L^2(L^2)}^2 \le c_5 \Big(\|v_0\|_{V^*}^2 + \tau \|v_0^*\|_0^2 + \|\bar{v}_\tau^* - \bar{v}_\tau\|_{L^2(L^2)}^2 + \|F_\tau\|_{L^2(H^*)}^2 + \|\bar{\gamma}_\tau\|_{L^2}^2 \Big)$$

for a constant $c_5 > 0$. □

We prove the first inequality of Theorem 5.3.8

Proof of the first inequality of Theorem 5.3.8. We apply Lemmas 5.4.5 for (5.4.19). Let $(w_k, r_k) := T(e_k)$ for all $k = 0, 1, \ldots, N$. It holds that for all $k = 1, 2, \ldots, N$,

$$\langle R_k^{\text{n.l.}}, w_k \rangle_H = -a_1(e_{k-1}^*, u_k^*, w_k) - a_1(u(t_{k-1}), e_k^*, w_k) + \tau a_1(D_\tau u(t_k), u(t_k), w_k).$$

Hypothesis 5.3.6 and Theorem 5.3.3 implies that there exists a constant $c_1 > 0$ such that

$\|w_k\|_2 \leq c_1\sqrt{\tau}$ for all $k = 1, 2, \ldots, N$. It holds that for all $\delta > 0$,

$$\begin{aligned}
-a_1(e_{k-1}^*, u_k^*, w_k) &= -a_1(e_{k-1}^*, u(t_k), w_k) - a_1(e_{k-1}^*, e_k^*, w_k) \\
&\leq c_d\|e_{k-1}^*\|_0\|u(t_k)\|_2\|w_k\|_1 + c_d\|e_{k-1}^*\|_0\|e_k^*\|_1\|w_k\|_2 \\
&\leq c_d c_{\max}\|e_{k-1}^*\|_0\|w_k\|_1 + c_d c_1\sqrt{\tau}\|e_{k-1}^*\|_0\|e_k^*\|_1 \\
&\leq \frac{\delta}{2}\|e_{k-1}^*\|_0^2 + \frac{c_d^2 c_{\max}^2}{2\delta}\|w_k\|_1^2 + \frac{\delta}{2}\|e_{k-1}^*\|_0^2 + \frac{c_d^2 c_1^2}{2\delta}\tau\|e_k^*\|_1^2 \\
&\leq \delta\|e_{k-1}^*\|_0^2 + \frac{c_d^2 c_{\max}^2}{2\delta}\|w_k\|_1^2 + \frac{c_d^2 c_1^2}{2\delta}\tau\|e_k^*\|_1^2, \\
-a_1(u(t_{k-1}), e_k^*, w_k) &= a_1(u(t_{k-1}), w_k, e_k^*) \\
&\leq c_d\|u(t_{k-1})\|_2\|w_k\|_1\|e_k^*\|_0 \\
&\leq \delta\|e_k^*\|_0^2 + \frac{c_d^2 c_{\max}^2}{4\delta}\|w_k\|_1^2 \\
\tau a_1(D_\tau u(t_k), u(t_k), w_k) &\leq c_d\tau\|D_\tau u(t_k)\|_0\|u(t_k)\|_2\|w_k\|_1 \\
&\leq c_d c_{\max}\tau\|D_\tau u(t_k)\|_0\|w_k\|_1 \\
&\leq \frac{c_d^2 c_{\max}^2}{4\delta}\|w_k\|_1^2 + \delta\tau^2\|D_\tau u(t_k)\|_0^2,
\end{aligned}$$

where $c_{\max} := \|u\|_{C([0,T];H^2(\Omega)^d)}$. Hence, by Lemma 5.4.3, it holds that for all $k = 1, 2, \ldots, N$,

$$\begin{aligned}
\langle R_k^{\text{n.l.}}, w_k\rangle_H &\leq \delta\|e_{k-1}^*\|_0^2 + \delta\|e_k^*\|_0^2 + c_\delta(\|w_k\|_1^2 + \tau\|e_k^*\|_1^2 + \tau^2\|D_\tau u(t_k)\|_0^2) \\
&\leq \delta\|e_{k-1}^*\|_0^2 + \delta\|e_k^*\|_0^2 + c_\delta\left(\|w_k\|_1^2 + \tau\|e_k^*\|_1^2 + \tau\left\|\frac{\partial u}{\partial t}\right\|_{L^2(t_{k-1},t_k;L^2(\Omega)^d)}^2\right),
\end{aligned}$$

where $c_\delta := \max\{c_d^2 c_{\max}^2/\delta, c_d^2 c_1^2/(2\delta), \delta\}$. By Lemma 5.4.5, there exist two constants $c_2, \tau_0 > 0$ such that for all $0 < \tau < \tau_0$,

$$\|\bar{e}_\tau\|_{L^2(L^2)}^2 \leq c_2\left(\|\bar{e}_\tau^* - \bar{e}_\tau\|_{L^2(L^2)}^2 + \|\bar{R}_\tau\|_{L^2(H^*)}^2 + \tau\|\bar{e}_\tau^*\|_{L^2(H^1)}^2 + \tau^2\left\|\frac{\partial u}{\partial t}\right\|_{L^2(L^2)}^2\right).$$

By Lemma 5.4.2 and Theorem 5.3.3, there exists a constant $c_3 > 0$ such that for all $0 < \tau < \tau_0$,

$$\|u_\tau - \bar{u}_\tau\|_{L^2(L^2)} \leq c_3\tau.$$

By Lemma 5.4.3 and Theorem 5.3.3, we obtain the first inequality of Theorem 5.3.8;

$$\begin{aligned}
&\|u - \bar{u}_\tau\|_{L^2(L^2)} + \|u - \bar{u}_\tau^*\|_{L^2(L^2)} \\
&= \|u - u_\tau + u_\tau - \bar{u}_\tau\|_{L^2(L^2)} + \|u - u_\tau + u_\tau - \bar{u}_\tau + \bar{u}_\tau - \bar{u}_\tau^*\|_{L^2(L^2)} \\
&\leq 2\|u - u_\tau\|_{L^2(L^2)} + 2\|u_\tau - \bar{u}_\tau\|_{L^2(L^2)} + \|\bar{u}_\tau - \bar{u}_\tau^*\|_{L^2(L^2)} \\
&\leq c_4\tau
\end{aligned}$$

for a constant $c_4 > 0$. □

To prove the second inequality of Theorem 5.3.8, we prepare the following two lemmas:

Lemma 5.4.6. *Under Hypothesis 5.3.2, there exists a constant $c > 0$ independent of τ such that*

$$\|D_\tau e_1\|_{V^*} \leq c\sqrt{\tau}, \qquad \|D_\tau e_1\|_0 + \|D_\tau e_1^*\|_0 \leq c, \qquad \|D_\tau e_1^*\|_1 \leq \frac{c}{\sqrt{\tau}}.$$

Proof. By (5.4.19) and (5.4.16) with $k := 1$ in the proof of Lemma 5.4.1, we obtain

$$\|e_1\|_0^2+\frac{1}{2}\|e_1-e_1^*\|_0^2+\frac{\tau}{c_a}\|e_1^*\|_1^2\le c_a\tau\|R_1\|_{H^*}^2+6\tau^2\|\nabla P(t_1)\|_0^2+2\tau\langle R_1^{\text{n.l.}},e_1^*\rangle_H.$$

Putting $k := 1$ and $\delta := \frac{1}{4c_a}$ in (5.4.20), it holds that

$$\langle R_1^{\text{n.l.}},e_1^*\rangle_H \le \frac{1}{4c_a}\|e_1^*\|_1^2 + 2c_a c_d^2 c_{\max}^2 \tau \left\|\frac{\partial u}{\partial t}\right\|_{L^2(0,t_1;L^2(\Omega)^d)}^2$$

where $c_{\max} := \|u\|_{C([0,T];H^2(\Omega)^d)}$. Hence, by Lemma 5.4.2, we have

$$\begin{aligned}
&\|e_1\|_0^2 + \frac{1}{2}\|e_1-e_1^*\|_0^2 + \frac{1}{2c_a}\tau\|e_1^*\|_1^2 \\
&\le c_a\tau\|R_1\|_{H^*}^2 + 6\tau^2\|\nabla P(t_1)\|_0^2 + 4c_a c_d^2 c_{\max}^2\tau^2 \left\|\frac{\partial u}{\partial t}\right\|_{L^2(0,t_1;L^2(\Omega)^d)}^2 \\
&\le \frac{c_a\tau^2}{3}\left\|\frac{\partial u}{\partial t}\right\|_{L^2(H^*)}^2 + 6\tau^2\|P\|_{C([0,T];H^1)}^2 + 4c_a c_d^2 c_{\max}^2\tau^2\left\|\frac{\partial u}{\partial t}\right\|_{L^2(L^2)}^2 \le c_2\tau^2
\end{aligned}$$

where $c_2 := c_a(\frac{1}{3}+4c_d^2c_{\max}^2)\left\|\frac{\partial u}{\partial t}\right\|_{L^2(L^2)}^2 + 6\|P\|_{C([0,T];H^1)}^2$, which implies that $\|D_\tau e_1\|_0 = \tau^{-1}\|e_1\|_0 \le \sqrt{c_2}$, $\|D_\tau e_1^*\|_1 \le \sqrt{2c_a c_2}\tau^{-1/2}$ and

$$\|D_\tau e_1^*\|_0 = \frac{1}{\tau}\|e_1^*\|_0 \le \frac{1}{\tau}(\|e_1\|_0 + \|e_1-e_1^*\|_0) \le (1+\sqrt{2})\sqrt{c_2}.$$

On the other hand, by (5.4.19), Theorem 2.2.19, and Lemma 5.4.2,

$$\begin{aligned}
&\|D_\tau e_1\|_{V^*} \\
&= \sup_{0\neq\varphi\in V}\frac{|(e_1-e_0,\varphi)|}{\tau\|\varphi\|_1} \\
&= \sup_{0\neq\varphi\in V}\frac{|-a_0(e_1^*,\varphi)+(q_1,\operatorname{div}\varphi)+\langle R_1,\varphi\rangle_H - a_1(u_0,e_1^*,\varphi)+\tau a_1(D_\tau u(t_1),u(t_1),\varphi)|}{\|\varphi\|_1} \\
&\le c_a\|e_1^*\|_1 + \|R_1\|_{H^*} + c_d(\|u_0\|_1\|e_1^*\|_1 + \tau\|D_\tau u(t_1)\|_0\|u(t_1)\|_2) \\
&\le c_a\|e_1^*\|_1 + \sqrt{\frac{\tau}{3}}\left\|\frac{\partial u}{\partial t}\right\|_{L^2(0,t_1;H^*)} + c_d c_{\max}\left(\|e_1^*\|_1 + \sqrt{\tau}\left\|\frac{\partial u}{\partial t}\right\|_{L^2(0,t_1,L^2(\Omega)^d)}\right) \\
&\le \sqrt{\tau}\left\{(c_a+c_d c_{\max})\sqrt{2c_a c_2} + \left(\frac{1}{\sqrt{3}}+c_d c_{\max}\right)\left\|\frac{\partial u}{\partial t}\right\|_{L^2(L^2)}\right\},
\end{aligned}$$

where $c_{\max} := \|u\|_{C([0,T];H^2(\Omega)^d)}$. □

Lemma 5.4.7. *Under Hypotheses 5.3.2, 5.3.6, and 5.3.7, there exist two constants $c, \tau_0 > 0$ independent of τ such that for all $0 < \tau < \tau_0$,*

$$\left\|\frac{\partial \hat{e}_\tau}{\partial t}\right\|_{L^2(L^2)} \le c\sqrt{\tau}.$$

Proof. By (5.4.19), it holds that $(D_\tau e_k^*, D_\tau q_k, D_\tau e_k)_{k=2}^N \subset H \times H^1_{\Gamma_2}(\Omega) \times L^2(\Omega)^d$ and for all $k = 2, 3, \ldots, N$, $\varphi \in H$, and $\psi \in H^1_{\Gamma_2}(\Omega)$,

$$\begin{cases} \left(\dfrac{D_\tau e_k - D_\tau e_{k-1}}{\tau}, \varphi\right) + a_0(D_\tau e_k^*, \varphi) - (D_\tau q_k, \operatorname{div}\varphi) = \langle D_\tau R_k + D_\tau R_k^{\text{n.l.}}, \varphi\rangle_H, \\ \tau(\nabla D_\tau q_k, \nabla\psi) + (\operatorname{div} D_\tau e_k^*, \psi) = -(\nabla D_\tau P(t_k), \nabla\psi), \\ D_\tau e_k = D_\tau e_k^* - \tau\nabla D_\tau(q_k + P(t_k)) \text{ in } L^2(\Omega)^d \end{cases} \tag{5.4.23}$$

with $D_\tau e_1 = \tau^{-1}(e_1 - e_0) = \tau^{-1} e_1$. It holds for all $k = 2, 3, \ldots, N$ and $\varphi \in H$,

$$\begin{aligned} \tau\langle D_\tau R_k^{\text{n.l.}}, \varphi\rangle_H = &-\tau a_1(u_{k-2}^*, D_\tau e_k^*, \varphi) - \tau a_1(D_\tau u_{k-1}^*, e_k^*, \varphi) + a_1(e_{k-2}^*, u(t_{k-1}), \varphi) \\ &- a_1(e_{k-1}^*, u(t_k), \varphi) - \tau a_1(D_\tau u(t_{k-1}), u(t_{k-1}), \varphi) + \tau a_1(D_\tau u(t_k), u(t_k), \varphi). \end{aligned} \tag{5.4.24}$$

Here, by Lemma 5.2.4, the right hand side except for the first and second terms are evaluated from above for all $k = 2, 3, \ldots, N$, $\varphi \in H$ and $\delta > 0$,

$$\begin{aligned} &a_1(e_{k-2}^*, u(t_{k-1}), \varphi) - a_1(e_{k-1}^*, u(t_k), \varphi) \\ &\quad - \tau a_1(D_\tau u(t_{k-1}), u(t_{k-1}), \varphi) + \tau a_1(D_\tau u(t_k), u(t_k), \varphi) \\ &\le c_d(\|e_{k-2}^*\|_0 \|u(t_{k-1})\|_2 + \|e_{k-1}^*\|_0 \|u(t_k)\|_2 \\ &\quad + \tau\|D_\tau u(t_{k-1})\|_0 \|u(t_{k-1})\|_2 + \tau\|D_\tau u(t_k)\|_0 \|u(t_k)\|_2)\|\varphi\|_1 \\ &\le c_d c_{\max}\left(\|e_{k-2}^*\|_0 + \|e_{k-1}^*\|_0 + \tau\|D_\tau u(t_{k-1})\|_0 + \tau\|D_\tau u(t_k)\|_0\right)\|\varphi\|_1 \\ &\le \frac{\delta}{2}\|\varphi\|_1^2 + \frac{2c_d^2 c_{\max}^2}{\delta}\sum_{i=0}^{1}\left(\|e_{k-i-1}^*\|_0^2 + \tau^2\|D_\tau u(t_{k-i})\|_0^2\right), \end{aligned} \tag{5.4.25}$$

where $c_{\max} := \|u\|_{C([0,T];H^2(\Omega)^d)}$. By Lemma 5.2.3, it holds that

$$-\tau a_1(u_{k-2}^*, D_\tau e_k^*, D_\tau e_k^*) = 0.$$

By Theorem 5.3.3, there exist two constants $\tau_1, c_1 > 0$ such that $\|\bar{e}_\tau^*\|_{L^2(H^1)} \le c_1$ for all $0 < \tau < \tau_1$, and hence for all $k = 1, 2, \ldots, N$, $\|e_k^*\|_1 \le c_1$ and

$$\begin{aligned} &-\tau a_1(D_\tau u_{k-1}^*, e_k^*, D_\tau e_k^*) \\ &= -\tau a_1(D_\tau u(t_{k-1}), e_k^*, D_\tau e_k^*) - a_1(D_\tau e_{k-1}^*, e_k^*, e_k^*) + a_1(D_\tau e_{k-1}^*, e_k^*, e_{k-1}^*) \\ &\le c_d\tau\|D_\tau u(t_{k-1})\|_1 \|e_k^*\|_1 \|D_\tau e_k^*\|_1 + c_d\|D_\tau e_{k-1}^*\|_1 \|e_k^*\|_1 \|e_{k-1}^*\|_1 \\ &\le c_d c_1 \tau\|D_\tau u(t_{k-1})\|_1 \|D_\tau e_k^*\|_1 + c_d c_1\|D_\tau e_{k-1}^*\|_1 \|e_{k-1}^*\|_1 \\ &\le \frac{\delta}{2}\|D_\tau e_k^*\|_1^2 + \frac{c_d^2 c_1^2}{2\delta}\tau^2\|D_\tau u(t_{k-1})\|_1^2 + \delta\|D_\tau e_{k-1}^*\|_1^2 + \frac{c_d^2 c_1^2}{4\delta}\|e_{k-1}^*\|_1^2. \end{aligned}$$

Hence, by (5.4.24) with $\varphi := D_\tau e_k^*$ and Lemma 5.4.3, for all $0 < \tau < \tau_1$, $k = 2, 3, \ldots, N$ and $\delta > 0$,

$$\begin{aligned} \tau\langle D_\tau R_k^{\text{n.l.}}, D_\tau e_k^*\rangle_H &\le \delta(\|D_\tau e_k^*\|_1^2 + \|D_\tau e_{k-1}^*\|_1^2) + c_\delta\sum_{i=0}^{1}\left(\|e_{k-i-1}^*\|_1^2 + \tau^2\|D_\tau u(t_{k-i})\|_1^2\right) \\ &\le \delta(\|D_\tau e_k^*\|_1^2 + \|D_\tau e_{k-1}^*\|_1^2) + c_\delta\sum_{i=0}^{1}\left(\|e_{k-i-1}^*\|_1^2 + \tau\left\|\frac{\partial u}{\partial t}\right\|_{L^2(t_{k-i-1}, t_{k-i}; H^1(\Omega)^d)}^2\right), \end{aligned}$$

where $c_\delta := \delta^{-1}(2c_d^2c_{\max}^2 + 2^{-1}c_d^2c_1^2)$. Putting $\delta := 1/(4c_a)$, by (5.4.23) and (5.4.16) in the proof of Lemma 5.4.1, we have for all $0 < \tau < \tau_1$ and $k = 2, 3, \ldots, N$,

$$
\begin{aligned}
&\|D_\tau e_k\|_0^2 - \|D_\tau e_{k-1}\|_0^2 + \frac{1}{2}\|D_\tau e_k^* - D_\tau e_k\|_0^2 + \frac{\tau}{c_a}\|D_\tau e_k^*\|_1^2 \\
&\le c_a\tau\|D_\tau R_k\|_{H^*}^2 + 6\tau^2\|\nabla D_\tau P(t_k)\|_0^2 + 2\tau\langle D_\tau R_k^{\text{n.l.}}, D_\tau e_k^*\rangle_H \\
&\le c_a\tau\|D_\tau R_k\|_{H^*}^2 + 6\tau^2\|\nabla D_\tau P(t_k)\|_0^2 + \frac{\tau}{2c_a}(\|D_\tau e_k^*\|_1^2 + \|D_\tau e_{k-1}^*\|_1^2) \\
&+ 2c_{\frac{1}{4c_a}}\tau\sum_{i=0}^{1}\left(\|e_{k-i-1}^*\|_1^2 + \tau\left\|\frac{\partial u}{\partial t}\right\|_{L^2(t_{k-i-1},t_{k-i};H^1(\Omega)^d)}^2\right).
\end{aligned}
$$

Summing up for $k = 2, 3, \ldots, m$ with an arbitrary natural number $m \le N$, by Lemmas 5.4.2 and 5.4.3, it holds that

$$
\begin{aligned}
&\|D_\tau e_m\|_0^2 + \frac{\tau}{2c_a}\|D_\tau e_m^*\|_1^2 + \tau\sum_{k=2}^{m}\frac{1}{2\tau}\|D_\tau e_k^* - D_\tau e_k\|_0^2 \\
&\le \|D_\tau e_1\|_0^2 + \frac{\tau}{2c_a}\|D_\tau e_1^*\|_1^2 + \tau\sum_{k=2}^{m}(c_a\|D_\tau R_k\|_{H^*}^2 + 6\tau\|D_\tau P(t_k)\|_1^2) \\
&+ 4c_{\frac{1}{4c_a}}\tau\sum_{k=1}^{m}\left(\|e_k^*\|_1^2 + \tau\left\|\frac{\partial u}{\partial t}\right\|_{L^2(t_{k-1},t_k;H^1(\Omega)^d)}^2\right) \\
&\le c_2\left\{\|D_\tau e_1\|_0^2 + \tau\|D_\tau e_1^*\|_1^2 + \tau^2\left\|\frac{\partial^3 u}{\partial t^3}\right\|_{L^2(H^*)}^2 + \tau\left\|\frac{\partial P}{\partial t}\right\|_{L^2(H^1)}^2 + \|\bar{e}_\tau^*\|_{L^2(H^1)}^2 + \tau^2\left\|\frac{\partial u}{\partial t}\right\|_{L^2(H^1)}^2\right\},
\end{aligned}
$$

where $c_2 := \max\{2^{-1}c_a^{-1}, c_a, 6, 4c_{\frac{1}{4c_a}}\}$. Hence, by Lemma 5.4.6, there exist two constants $c_3 > 0$ such that for all $0 < \tau < \tau_1$,

$$
\max_{k=1,\ldots,N}\|D_\tau e_k\|_0^2 + \tau\sum_{k=2}^{N}\frac{1}{\tau}\|D_\tau e_k - D_\tau e_k^*\|_0^2 \le c_3. \tag{5.4.26}
$$

To use Lemma 5.4.5 for (5.4.23), we set $(w_k, r_k) = T(D_\tau e_k)$ for all $k = 1, 2, \ldots, N$. By Hypothesis 5.3.6 and (5.4.26), there exists a constant $c_4 > 0$ such that for all $0 < \tau < \tau_1$ and $k = 1, 2, \ldots, N$, $\|w_k\|_2 \le c_4$, and hence, for all $0 < \tau < \tau_1$, $\delta > 0$ and $k = 2, 3, \ldots, N$,

$$
\begin{aligned}
-\tau a_1(u_{k-2}^*, D_\tau e_k^*, w_k) &= \tau a_1(u_{k-2}^*, w_k, D_\tau e_k^*) \\
&\le c_d\tau\|u_{k-2}^*\|_1\|w_k\|_2\|D_\tau e_k^*\|_0 \\
&\le c_dc_4\tau\|u_{k-2}^*\|_1\|D_\tau e_k^*\|_0 \\
&= \delta\|D_\tau e_k^*\|_0^2 + \frac{c_d^2c_4^2}{4\delta}\tau^2\|u_{k-2}^*\|_1^2, \\
-\tau a_1(D_\tau u_{k-1}^*, e_k^*, w_k) &= -\tau a_1(D_\tau u(t_{k-1}), e_k^*, w_k) - \tau a_1(D_\tau e_{k-1}^*, e_k^*, w_k) \\
&\le c_d\tau\|D_\tau u(t_{k-1})\|_1\|e_k^*\|_1\|w_k\|_1 + c_d\|D_\tau e_{k-1}^*\|_0\|e_k^*\|_1\|w_k\|_2 \\
&\le c_dc_1\tau\|D_\tau u(t_{k-1})\|_1\|w_k\|_1 + c_dc_4\|D_\tau e_{k-1}^*\|_0\|e_k^*\|_1 \\
&\le \frac{\delta}{2}\|w_k\|_1^2 + \frac{c_d^2c_1^2}{2\delta}\tau^2\|D_\tau u(t_{k-1})\|_1^2 + \delta\|D_\tau e_{k-1}^*\|_0^2 + \frac{c_d^2c_4^2}{4\delta}\|e_{k-1}^*\|_1^2.
\end{aligned}
$$

By (5.4.24) and (5.4.25) with $\varphi := w_k$, we have

$$\begin{aligned}\langle D_\tau R_k^{\text{n.l.}}, w_k\rangle_H &\le \delta(\|D_\tau e_{k-1}^*\|_0^2 + \|D_\tau e_k^*\|_0^2)\\ &\quad + \tilde{c}_\delta \left\{ \|w_k\|_1^2 + \tau^2\|u_{k-2}^*\|_1^2 + \sum_{i=0}^{1}\left(\|e_{k-i-1}^*\|_1^2 + \tau^2\|D_\tau u(t_{k-i})\|_1^2\right)\right\},\end{aligned}$$

where $\tilde{c}_\delta := \max\{\delta, \delta^{-1}c_d^2(2c_{\max}^2 + 2^{-1}c_1^2 + 4^{-1}c_4^2)\}$. By Lemmas 5.4.5 and 5.4.3, there exist two constants $0 < \tau_2 \le \tau_1$ and $c_5 > 0$ such that for all $0 < \tau < \tau_2$,

$$\begin{aligned}\tau\sum_{k=2}^{N}\|D_\tau e_k\|_0^2 &\le c_5\Bigg(\|D_\tau e_1\|_{V^*}^2 + \tau\|D_\tau e_1^*\|_0^2 + \tau^2\|\bar{u}_\tau^*\|_{L^2(H^1)}^2 + \|\bar{e}_\tau^*\|_{L^2(H^1)}^2\\ &\qquad + \tau^2\left\|\frac{\partial u}{\partial t}\right\|_{L^2(H^1)}^2 + \tau\sum_{k=2}^{N}(\|D_\tau e_k - D_\tau e_k^*\|_0^2 + \|D_\tau R_k\|_{H^*}^2)\Bigg).\end{aligned}$$

Hence, by Theorems 5.3.1, 5.3.3, Lemmas 5.4.2, 5.4.6, and (5.4.26), it holds that for all $0 < \tau < \tau_2$,

$$\left\|\frac{\partial \hat{e}_\tau}{\partial t}\right\|_{L^2(L^2)}^2 = \tau\|D_\tau e_1\|_0^2 + \tau\sum_{k=2}^{N}\|D_\tau e_k\|_0^2 \le c_7\tau$$

for a constant $c_7 > 0$, where we have used $\frac{\partial \hat{e}_\tau}{\partial t} = (\overline{D_\tau e})_\tau$ on (t_{k-1}, t_k) for all $k = 1, 2, \ldots, N$ □

Finally, we prove the second inequality of Theorem 5.3.8.

Proof of the second inequality of Theorem 5.3.8. By (5.4.19), Lemma 5.2.1, and Theorem 2.2.19, there exists a constant $c_1 > 0$ such that for all $k = 1, 2, \ldots, N$,

$$\begin{aligned}\|q_k\|_0 &\le c_1 \sup_{0\ne\varphi\in H}\frac{|(q_k, \operatorname{div}\varphi)|}{\|\varphi\|_1} = c_1 \sup_{0\ne\varphi\in H}\frac{\left|(D_\tau e_k, \varphi) + a_0(e_k^*, \varphi) - \langle R_k + R_k^{\text{n.l.}}, \varphi\rangle_H\right|}{\|\varphi\|_1}\\ &\le c_1\left(\|D_\tau e_k\|_0 + c_a\|e_k^*\|_1 + \|R_k\|_{H^*} + \|R_k^{\text{n.l.}}\|_{H^*}\right).\end{aligned}$$

By Hypothesis 5.3.2 and Theorem 5.3.3, there exist two constants $\tau_1, c_2 > 0$ such that $\|u(t_k)\|_2$, $\tau^{-1/2}\|\bar{e}_\tau^*\|_{L^2(H^1)} \le c_2$ for all $0 < \tau < \tau_1$ and $k = 0, 1, \ldots, N$. By Lemma 5.4.3, it holds that for all $0 < \tau < \tau_1$, $k = 1, 2, \ldots, N$ and $\varphi \in H$,

$$\begin{aligned}&|\langle R_k^{\text{n.l.}}, \varphi\rangle_H|\\ &= |-a_1(e_{k-1}^*, u(t_k), \varphi) - a_1(e_{k-1}^*, e_k^*, \varphi) - a_1(u(t_{k-1}), e_k^*, \varphi) + \tau a_1(D_\tau u(t_k), u(t_k), \varphi)|\\ &\le c_d\big(\|e_{k-1}^*\|_1\|u(t_k)\|_1 + \|e_{k-1}^*\|_1\|e_k^*\|_1 + \|u(t_{k-1})\|_1\|e_k^*\|_1 + \tau\|D_\tau u(t_k)\|_0\|u(t_k)\|_2\big)\|\varphi\|_1\\ &\le c_d c_2\left(\|e_{k-1}^*\|_1 + 2\|e_k^*\|_1 + \tau\|D_\tau u(t_k)\|_0\right)\|\varphi\|_1\\ &\le c_d c_2\left(\|e_{k-1}^*\|_1 + 2\|e_k^*\|_1 + \sqrt{\tau}\left\|\frac{\partial u}{\partial t}\right\|_{L^2(t_{k-1}, t_k, L^2(\Omega)^d)}\right)\|\varphi\|_1,\end{aligned}$$

where we have used $\|e_k\|_1 \le c_2$ for all $k = 0, 1, \ldots, N$. Hence, we have for all $0 < \tau < \tau_1$ and $k = 1, 2, \ldots, N$,

$$\|q_k\|_0 \le c_3\left(\|D_\tau e_k\|_0 + \|e_{k-1}^*\|_1 + \|e_k^*\|_1 + \sqrt{\tau}\left\|\frac{\partial u}{\partial t}\right\|_{L^2(t_{k-1}, t_k, L^2(\Omega)^d)} + \|R_k\|_{H^*}\right)$$

for a constant $c_3 > 0$. By Lemmas 5.4.2 and 5.4.7, there exist three constants $\tau_2, c_4, c_5 > 0$ such that for all $0 < \tau < \tau_2 \leq \tau_1$,

$$\begin{aligned}\|\bar{P}_\tau - P_\tau\|^2_{L^2(L^2)} &\leq c_4 \left(\left\| \frac{\partial \hat{e}_\tau}{\partial t} \right\|^2_{L^2(L^2)} + \|\bar{e}^*_\tau\|^2_{L^2(H^1)} + \tau^2 \left\| \frac{\partial u}{\partial t} \right\|^2_{L^2(L^2)} + \|\bar{R}_\tau\|^2_{L^2(H^*)} \right) \\ &\leq c_4 \left(\left\| \frac{\partial \hat{e}_\tau}{\partial t} \right\|^2_{L^2(L^2)} + c_2^2 \tau + \tau^2 \left\| \frac{\partial u}{\partial t} \right\|^2_{L^2(L^2)} + \frac{\tau^2}{3} \left\| \frac{\partial^2 u}{\partial t^2} \right\|^2_{L^2(H^*)} \right) \leq c_5 \tau.\end{aligned}$$

Therefore, by Lemma 5.4.3, we conclude the proof:

$$\|P - \bar{P}_\tau\|_{L^2(L^2)} \leq \|P - P_\tau\|_{L^2(L^2)} + \|P_\tau - \bar{P}_\tau\|_{L^2(L^2)} \leq \sqrt{\tau} \left(\left\| \frac{\partial P}{\partial t} \right\|_{L^2(L^2)} + \sqrt{c_5} \right).$$

□

5.4.3 Proof of Corollary 5.3.10

We prove Corollary 5.3.10 by using the boundedness from Theorem 5.3.1 and the Aubin–Lions compactness lemma.

Proof of Corollary 5.3.10. By the first and third equations of (PM), it holds that for all $v \in V$ and $k = 1, 2, \ldots, N$,

$$\begin{aligned}(D_\tau u_k, v) + a_0(u^*_k, v) + (g_k, v) + (h_k, \nabla v) &= \langle f_k, v \rangle_H - (\nabla P_k, v) \\ &= \langle f_k, v \rangle_H - \int_{\Gamma_2} p^b_k v \cdot n ds,\end{aligned}$$

where g_k and h_k are defined[2] by

$$g_k := (\nabla u^*_k)^T u^*_{k-1} - u^*_{k-1} \operatorname{div} u^*_k, \qquad h_k := -u^*_k (u^*_{k-1})^T,$$

which implies that for all $v \in V$ and $\theta \in C^\infty_0(0, T)$,

$$\int_0^T \left(\left(\frac{\partial \hat{u}_\tau}{\partial t}, v \right) + a_0(\bar{u}^*_\tau, v) + (\bar{g}_\tau, v) + (\bar{h}_\tau, \nabla v) \right) \theta dt = \int_0^T \left(\langle \bar{f}_\tau, v \rangle_H - \int_{\Gamma_2} \bar{p}^b_\tau v \cdot n ds \right) \theta dt. \tag{5.4.27}$$

Here, $\bar{f}_\tau \to f$ strongly in $L^2(H^*)$ and $\bar{p}^b_\tau \to p^b$ strongly in $L^2(H^1(\Omega))$ as $\tau \to 0$. By Theorem 5.3.1 and Lemma 5.4.1, there exists a constant $c_1 > 0$ such that

$$\|\bar{u}_\tau\|^2_{L^\infty(L^2)} + \|\bar{u}^*_\tau\|^2_{L^\infty(L^2)} + \|\bar{u}^*_k\|^2_{L^2(H^1)} + \tau \left\| \frac{\partial \hat{u}_\tau}{\partial t} \right\|^2_{L^2(L^2)} + \frac{1}{\tau} \|\bar{u}_\tau - \bar{u}^*_\tau\|^2_{L^2(L^2)} \leq c_1. \tag{5.4.28}$$

[2]Here, it holds that for all $i, j = 1, \ldots, d$ and $k = 1, 2, \ldots, N$,

$$(g_k)_i := \sum_{l=1}^d \frac{\partial (u^*_k)_l}{\partial x_i} (u^*_{k-1})_l - (u^*_{k-1})_i \operatorname{div} u^*_k, \qquad (h_k)_{ij} := -(u^*_k)_i (u^*_{k-1})_j.$$

In particular, it holds that

$$\|u_1^*\|_0^2 + \tau\|u_1^*\|_1^2 + \|u_1 - u_0\|_0^2 + \|u_1 - u_1^*\|_0^2 \le c_1, \tag{5.4.29}$$

which implies that $\|u_1^* - u_0\|_0 \le \|u_1^* - u_1\|_0 + \|u_1 - u_0\|_0 \le 2\sqrt{c_1}$. Furthermore, by the first equation of (PM), Theorem 2.2.19, and Lemma 5.2.4, we have

$$\begin{aligned}\|u_1^* - u_0\|_{V^*} &= \sup_{0\neq v\in H} \frac{\tau}{\|v\|_1} |-a_0(u_1^*, v) - a_1(u_0, u_1^*, v) + \langle f_1, v\rangle_H| \\ &\le c_a\tau\|u_1^*\|_1 + c_d\tau\|u_0\|_{L^{p_d}}\|u^*\|_1 + \tau\|f_1\|_{H^*} \\ &\le c_2\sqrt{\tau}.\end{aligned} \tag{5.4.30}$$

where $c_2 := \sqrt{c_1}(c_a + c_d\|u_0\|_{L^{p_d}}) + \|f\|_{L^2(H^*)}$. Let $u_0^\circ := u_1^*$, $u_k^\circ := u_k^*$ for all $k = 1, 2, \dots, N$ and let $\hat{u}_\tau^\circ$ be the piecewise linear interpolant of $(u_k^\circ)_{k=0}^N \subset H$.

From the uniform estimates (5.4.28), one can show that there exist a sequence $(\tau_k)_{k\in\mathbb{N}}$ and three functions $u \in L^2(H)\cap L^\infty(L^2(\Omega)^d)\cap W^{1,4/p_d}(V^*)$ (in particular, $u \in C([0,T];V^*)$), $g \in L^{4/p_d}(L^{\tilde{p}_d}(\Omega)^d)$ and $h \in L^{4/p_d}(L^2(\Omega)^{d\times d})$ such that $\tau_k \to 0$ and

$$\bar{u}_{\tau_k}^* \to u \quad \text{weakly in } L^2(H), \tag{5.4.31}$$

$$\text{strongly in } L^2(L^2(\Omega)^d), \tag{5.4.32}$$

$$\hat{u}_{\tau_k}^\circ \to u \quad \text{strongly in } L^2(L^2(\Omega)^d), \tag{5.4.33}$$

$$\text{strongly in } C([0,T];V^*), \tag{5.4.34}$$

$$\hat{u}_{\tau_k} \to u \quad \text{strongly in } L^2(L^2(\Omega)^d), \tag{5.4.35}$$

$$\text{weakly in } W^{1,4/p_d}(V^*), \tag{5.4.36}$$

$$\bar{g}_{\tau_k} \rightharpoonup g \quad \text{weakly in } L^{4/p_d}(L^{\tilde{p}_d}(\Omega)^d), \tag{5.4.37}$$

$$\bar{h}_{\tau_k} \rightharpoonup h \quad \text{weakly in } L^{4/p_d}(L^2(\Omega)^{d\times d}), \tag{5.4.38}$$

as $k \to \infty$. Here, we note that $\bar{u}_{\tau_k}^*$, $\hat{u}_{\tau_k}^\circ$ and $\hat{u}_{\tau_k}$ possess a common limit function. Indeed, the weak convergence (5.4.31) of $\bar{u}_\tau^*$ immediately follows from the uniform estimates (5.4.28). Since we have $1/\tilde{p}_d = 1/2 + 1/p_d$, $p_d/4 = 1/2 + p_d/(2\tilde{q}_d)$, and

$$\|\bar{u}_\tau^*\|_{L^{2\tilde{q}_d/p_d}(L^{p_d})} \le \|\bar{u}_\tau^*\|_{L^2(L^{\tilde{q}_d})}^{p_d/\tilde{q}_d}\|\bar{u}_\tau^*\|_{L^\infty(L^2)}^{1-p_d/\tilde{q}_d} \le c_3\|\bar{u}_\tau^*\|_{L^2(H^1)}^{p_d/\tilde{q}_d}\|\bar{u}_\tau^*\|_{L^\infty(L^2)}^{1-p_d/\tilde{q}_d}$$

for a constant $c_3 > 0$ (cf. [14, Theorem II.5.5])[3], it holds that

$$\begin{aligned}\|\bar{g}_\tau\|_{L^{4/p_d}(L^{\tilde{p}_d})} &\le \left\{\tau\sum_{k=1}^N \left(\|\nabla u_k^*\|_{L^2}\|u_{k-1}^*\|_{L^{p_d}} + \|u_{k-1}^*\|_{L^{p_d}}\|\operatorname{div} u_k^*\|_{L^2}\right)^{4/p_d}\right\}^{p_d/4} \\ &\le c_4\left(\tau\sum_{k=1}^N \|u_k^*\|_1^{4/p_d}\|u_{k-1}^*\|_{L^{p_d}}^{4/p_d}\right)^{p_d/4} \\ &\le c_4\left(\tau\sum_{k=1}^N \|u_k^*\|_1^2\right)^{1/2}\left(\tau\sum_{k=1}^N \|u_{k-1}^*\|_{L^{p_d}}^{2\tilde{q}_d/p_d}\right)^{p_d/(2\tilde{q}_d)} \\ &\le c_4\left(\|\bar{u}_\tau^*\|_{L^2(H^1)}^2 + \|\bar{u}_\tau^*\|_{L^{2\tilde{q}_d/p_d}(L^{p_d})}^2 + \tau^{p_d/\tilde{q}_d}\|u_0\|_{L^{p_d}}^2\right) \\ &\le c_4\left(\|\bar{u}_\tau^*\|_{L^2(H^1)}^2 + c_3\|\bar{u}_\tau^*\|_{L^2(H^1)}^{2p_d/\tilde{q}_d}\|\bar{u}_\tau^*\|_{L^\infty(L^2)}^{2-2p_d/\tilde{q}_d} + \tau^{p_d/\tilde{q}_d}\|u_0\|_{L^{p_d}}^2\right),\end{aligned}$$

[3] Since $p_2 = 2 + \varepsilon$ and $p_3 = 3$, we have $p_2/\tilde{q}_2 = \varepsilon/2$ and $p_3/\tilde{q}_3 = 1/2$.

$$
\begin{aligned}
\|\bar{h}_\tau\|_{L^{4/p_d}(L^2)} &\le \left(\tau\sum_{k=1}^{N}\|u_k^*\|_{L^{\tilde{q}_d}}^{4/p_d}\|u_{k-1}^*\|_{L^{p_d}}^{4/p_d}\right)^{p_d/4}\\
&\le c_5\left(\tau\sum_{k=1}^{N}\|u_k^*\|_1^{4/p_d}\|u_{k-1}^*\|_{L^{p_d}}^{4/p_d}\right)^{p_d/4}\\
&\le c_5\left(\|\bar{u}_\tau^*\|_{L^2(H^1)}^2 + c_3\|\bar{u}_\tau^*\|_{L^2(H^1)}^{2p_d/\tilde{q}_d}\|\bar{u}_\tau^*\|_{L^\infty(L^2)}^{2-2p_d/\tilde{q}_d} + \tau^{p_d/\tilde{q}_d}\|u_0\|_{L^{p_d}}^2\right)
\end{aligned}
$$

for constants c_4 and c_5. Hence, by (5.4.28), the weak convergences (5.4.37) and (5.4.38) hold. Moreover, since there exists a constant $c_6 > 0$ such that

$$
|(g_k, v)| \le \|g_k\|_{L^{\tilde{p}_d}}\|v\|_{L^{\tilde{q}_d}} \le c_6\|g_k\|_{L^{\tilde{p}_d}}\|v\|_1
$$

for all $k = 1, 2, \ldots, N$ and $v \in H^1(\Omega)^d$, we have

$$
\begin{aligned}
&\left\|\frac{\partial \hat{u}_\tau}{\partial t}\right\|_{L^{4/p_d}(V^*)}\\
&= \left\{\int_0^T\left(\sup_{0\ne v\in V}\frac{1}{\|v\|_1}\left|-a_0(\bar{u}_\tau^*(t), v) - (\bar{g}_\tau(t), v) - (\bar{h}_\tau(t), \nabla v)\right.\right.\right.\\
&\qquad\qquad \left.\left.\left. + \langle f_\tau(t), v\rangle_H - \int_{\Gamma_2} p_\tau^b(t) v\cdot n ds\right|\right)^{4/p_d} dt\right\}^{p_d/4}\\
&= \left\{\int_0^T\left(c_a\|\bar{u}_\tau^*(t)\|_1 + c_6\|\bar{g}_\tau(t)\|_{L^{\tilde{p}_d}} + \|\bar{h}_\tau(t)\|_0 + \|f_\tau(t)\|_{H^*} + \|p_\tau^b(t)\|_1\right)^{4/p_d} dt\right\}^{p_d/4}\\
&\le c_a\|\bar{u}_\tau^*\|_{L^{4/p_d}(H^1)} + c_6\|\bar{g}_\tau\|_{L^{4/p_d}(L^{\tilde{p}_d})} + \|\bar{h}_\tau\|_{L^{4/p_d}(L^2)} + \|f_\tau\|_{L^{4/p_d}(H^*)} + \|p_\tau^b\|_{L^{4/p_d}(H^1)}\\
&\le T^{p_d/(2\tilde{q}_d)}(c_a\sqrt{c_1} + \|f\|_{L^2(H^*)} + \|p^b\|_{L^2(H^1)}) + c_6\|\bar{g}_\tau\|_{L^{4/p_d}(L^{\tilde{p}_d})} + \|\bar{h}_\tau\|_{L^{4/p_d}(L^2)},
\end{aligned}
$$

$$
\begin{aligned}
&\left\|\frac{\partial \hat{u}_\tau^\circ}{\partial t} - \frac{\partial \hat{u}_\tau}{\partial t}\right\|_{L^{4/p_d}(V^*)}\\
&= \left\{\tau\sum_{k=1}^{N}\left(\sup_{0\ne v\in V}\frac{|(u_k^\circ - u_{k-1}^\circ - u_k + u_{k-1}, v)|}{\tau\|v\|_1}\right)^{4/p_d}\right\}^{p_d/4}\\
&\le 2\left(\tau\sum_{k=1}^{N}\sup_{0\ne v\in V}\frac{|(\nabla P_k, v)|^{4/p_d}}{\|v\|_1^{4/p_d}} + \tau\|u_1^* - u_0\|_{V^*}^{4/p_d}\right)^{p_d/4}\\
&\le 2\left(\tau\sum_{k=1}^{N}\sup_{0\ne v\in V}\frac{1}{\|v\|_1^{4/p_d}}\left|\int_{\Gamma_2} p^b(t_k) v\cdot n ds\right|^{4/p_d}\right)^{p_d/4} + 2\tau^{p_d/4}\|u_1^* - u_0\|_{V^*}\\
&\le 2\left(\tau\sum_{k=1}^{N}\|p^b(t_k)\|_1^{4/p_d}\right)^{p_d/4} + 2c_2\tau^{p_d/4+1/2}\\
&\le 2\|\bar{p}_\tau^b\|_{L^{4/p_d}(H^1)} + 2c_2\tau^{1+p_d/(2\tilde{q}_d)}\\
&\le 2T^{p_d/(2\tilde{q}_d)}(\|p^b\|_{L^2(H^1)} + c_2 T),
\end{aligned}
$$

and $\|\frac{\partial \hat{u}_\tau}{\partial t}\|_{L^{4/p_d}(V^*)}$ and $\|\frac{\partial \hat{u}_\tau^\circ}{\partial t}\|_{L^{4/p_d}(V^*)}$ are also bounded. Hence, (5.4.36) holds. Furthermore, $\|\hat{u}^\circ_\tau\|_{L^2(H^1)}$ is bounded: by (5.4.28) and (5.4.29),

$$\begin{aligned}\|\hat{u}^\circ_\tau\|^2_{L^2(H^1)} &= \sum_{k=1}^N \int_0^1 \|(1-s)u^\circ_{k-1} + su^\circ_k\|_1^2 \tau ds\\ &\le \sum_{k=1}^N \tau(\|u^\circ_{k-1}\|_1^2 + \|u^\circ_k\|_1^2)\int_0^1 \{(1-s)^2+s^2\}ds\\ &= \frac{2}{3}\sum_{k=1}^N \tau(\|u^*_{k-1}\|_1^2 + \|u^*_k\|_1^2)\\ &\le \frac{4}{3}\|\bar{u}^*_\tau\|_{L^2(H^1)} + \frac{2}{3}\tau\|u^*_1\|_1^2\\ &\le \frac{4}{3}\|\bar{u}^*_\tau\|^2_{L^2(H^1)} + \frac{2c_1}{3}\\ &\le 2c_1,\end{aligned}$$

which implies the strong convergence (5.4.33) of $\hat{u}^\circ_\tau$ in $L^2(L^2(\Omega)^d)$ from the Aubin–Lions lemma [14, Theorem II.5.16 (i)]. Since we have for all $t \in (t_{k-1}, t_k), k = 1, 2, \ldots, N$,

$$\|\bar{u}^*_\tau(t) - \hat{u}^\circ_\tau(t)\|_0 = \left|\frac{t_k - t}{\tau}\right| \|u^\circ_k - u^\circ_{k-1}\|_0 \le \|u^\circ_k - u_k\|_0 + \tau\|D_\tau u_k\|_0 + \|u_{k-1} - u^\circ_{k-1}\|_0,$$

$$\|\bar{u}^*_\tau(t) - \hat{u}_\tau(t)\|_0 \le \|\bar{u}^*_\tau(t) - \bar{u}_\tau(t)\|_0 + \|\bar{u}_\tau(t) - \hat{u}_\tau(t)\|_0 \le \|u^*_k - u_k\|_0 + \tau\|D_\tau u_k\|_0,$$

the functions $\bar{u}^*_{\tau_k}$, $\hat{u}^\circ_{\tau_k}$ and $\hat{u}_{\tau_k}$ possess a common limit function u, and the strong convergences (5.4.32) and (5.4.35) hold: by (5.4.28) and (5.4.29),

$$\begin{aligned}\|\bar{u}^*_\tau - \hat{u}^\circ_\tau\|_{L^2(L^2)} &\le \left(\tau\sum_{k=1}^N (\|u^\circ_k - u_k\|_0 + \tau\|D_\tau u_k\|_0 + \|u_{k-1} - u^\circ_{k-1}\|_0)^2\right)^{1/2}\\ &\le 2\sqrt{3}\|\bar{u}^*_\tau - \bar{u}_\tau\|_{L^2(L^2)} + \sqrt{3}\tau\left\|\frac{\partial \hat{u}_\tau}{\partial t}\right\|_{L^2(L^2)} + \sqrt{3\tau}\|u_0 - u^*_1\|_0\\ &\le 5\sqrt{3c_1\tau},\end{aligned}$$

$$\|\bar{u}^*_\tau - \hat{u}_\tau\|_{L^2(L^2)} \le \sqrt{2}\|\bar{u}^*_\tau - \bar{u}_\tau\|_{L^2(L^2)} + \sqrt{2}\tau\left\|\frac{\partial \hat{u}_\tau}{\partial t}\right\|_{L^2(L^2)} \le 2\sqrt{2c_1\tau}.$$

It also holds that

$$\|\bar{u}^*_\tau - \hat{u}^\circ_\tau\|_{L^\infty(L^2)} \le \max_{k=1,2,\ldots,N}(\|u^\circ_k\|_0 + \|u^\circ_{k-1}\|_0) \le 2\sqrt{c_1}.$$

Since $\|\hat{u}^\circ_\tau\|_{L^\infty(L^2)}$ and $\|\frac{\partial \hat{u}^\circ_\tau}{\partial t}\|_{L^{4/p_d}(V^*)}$ are bounded, we obtain the strong convergence (5.4.34) of $\hat{u}^\circ_\tau$ in $C([0,T]; V^*)$ [14, Theorem II.5.16 (ii)]. In particular, $\hat{u}^\circ_\tau(0)$ converges to $u(0)$ in V^*. On the other hand, by (5.4.30), $\hat{u}^\circ_\tau(0) = u^*_1$ converges to u_0 in V^*. Through the uniqueness of the limit in V^*, we have indeed obtained that $u(0) = u_0$.

From (5.4.27) with $\varepsilon := \varepsilon_k$, taking $k \to \infty$, it holds that for all $v \in V$ and $\theta \in C_0^\infty(0,T)$,

$$\begin{aligned}&\int_0^T \left(\left\langle \frac{\partial u}{\partial t}, \theta v\right\rangle_V + a_0(u, \theta v) + (g, \theta v) + (h, \nabla(\theta v))\right) dt\\ &= \int_0^T \left(\langle f, \theta v\rangle_H - \int_{\Gamma_2} p^b \theta v \cdot n ds\right) dt.\end{aligned}$$

Next, we show that

$$g = (\nabla u)^T u - u \operatorname{div} u, \qquad h = -u(u)^T. \tag{5.4.39}$$

We set $\bar{v}_\tau(t) := u^*_{k-1}$ for $t \in (t_{k-1}, t_k], k = 1, 2, \ldots, N$. Then, it holds that

$$\begin{aligned}
\|\bar{v}_\tau - \bar{u}^*_\tau\|_{L^2(L^2)} &\le \left(\tau \sum_{k=1}^{N} (\|u^*_k - u_k\|_0 + \tau \|D_\tau u_k\|_0 + \|u_{k-1} - u^*_{k-1}\|_0)^2 \right)^{1/2} \\
&\le 2\sqrt{3} \|\bar{u}^*_\tau - \bar{u}_\tau\|_{L^2(L^2)} + \sqrt{3}\tau \left\| \frac{\partial \hat{u}_\tau}{\partial t} \right\|_{L^2(L^2)} \\
&\le 3\sqrt{3 c_1 \tau},
\end{aligned}$$

and hence it follows from (5.4.32) that $\bar{v}_{\tau_k} \to u$ strongly in $L^2(L^2(\Omega)^d)$ as $k \to \infty$. Since $\nabla \bar{u}^*_\tau \rightharpoonup \nabla u$ weakly in $L^2(L^2(\Omega)^{d\times d})$ and $\operatorname{div} \bar{u}^*_\tau \rightharpoonup \operatorname{div} u$ weakly in $L^2(L^2(\Omega))$ as $k \to \infty$, we have

$$\begin{aligned}
\bar{g}_\tau = (\nabla \bar{u}^*_\tau)^T \bar{v}_\tau - \bar{v}_\tau \operatorname{div} \bar{u}^*_\tau \quad &\rightharpoonup \quad (\nabla u)^T u - u \operatorname{div} u \quad \text{weakly in } L^1(L^1(\Omega)^d), \\
\bar{h}_\tau = -\bar{u}^*_\tau (\bar{v}_\tau)^T \quad &\to \quad -u(u)^T \qquad\qquad\quad \text{strongly in } L^1(L^1(\Omega)^{d\times d})
\end{aligned}$$

as $k \to \infty$ (cf. [14, Proposition II.2.12]). On the other hand, we also know (5.4.37) and (5.4.38). The convergence in these spaces imply the convergence in the distributions sense, therefore (5.4.39) holds by the uniqueness of the limit in $\mathcal{D}'((0,T) \times \Omega)$. Hence, it holds that for all $v \in V$ and $\theta \in C_0^\infty(0,T)$,

$$\begin{aligned}
&\int_0^T \left(\left\langle \frac{\partial u}{\partial t}, v \right\rangle_V + a_0(u,v) + ((\nabla u)u - u \operatorname{div} u, v) - (u(u)^T, \nabla v) \right) \theta dt \\
&= \int_0^T \left(\langle f, v \rangle_H - \int_{\Gamma_2} p^b v \cdot n ds \right) \theta dt,
\end{aligned}$$

which is equivalent to the following

$$\int_0^T \left(\left\langle \frac{\partial u}{\partial t}, v \right\rangle_V + a_0(u,v) + a_1(u,u,v) \right) \theta dt = \int_0^T \left(\langle f, v \rangle_H - \int_{\Gamma_2} p^b v \cdot n ds \right) \theta dt.$$

□

5.5 Numerical examples

For our simulation, we set $T = 1$ and

$$\begin{aligned}
\Omega &= \left\{ (r\cos\theta, r\sin\theta) \in \mathbb{R}^2 \;\middle|\; r_1 < r < r_2, \theta_1 < \theta < \theta_2 \right\}, \\
\Gamma_1 &= \left\{ (r\cos\theta, r\sin\theta) \in \mathbb{R}^2 \;\middle|\; r \in \{r_1, r_2\}, \theta_1 < \theta < \theta_2 \right\}, \\
\Gamma_2 &= \left\{ (r\cos\theta, r\sin\theta) \in \mathbb{R}^2 \;\middle|\; r_1 < r < r_2, \theta \in \{\theta_1, \theta_2\} \right\},
\end{aligned}$$

where $r_1 := 2, r_2 = 3, \theta_1 = 0, \theta_2 := \pi/2$ (Fig. 5.1), and define the following constants:

$$\begin{aligned}
p_{\text{in}} &:= 1, \qquad p_{\text{out}} := -1, \qquad \alpha := \frac{p_{\text{in}} - p_{\text{out}}}{\theta_2 - \theta_1}, \\
C &:= \frac{1}{2} r_1^2 r_2^2 \frac{\log\theta_2 - \log\theta_1}{r_2^2 - r_1^2}, \quad D := -\frac{1}{2} \frac{r_2^2 \log r_2 - r_1^2 \log r_1}{r_2^2 - r_1^2}.
\end{aligned}$$

The following functions

$$u(x,y,t) := \begin{pmatrix} U(r)e^{-t}\sin\theta \\ -U(r)e^{-t}\cos\theta \end{pmatrix}, \qquad p(x,y,t) := p_0(\theta)e^{-t},$$

where $(r,\theta) = (r(x,y),\theta(x,y))$ are the polar coordinates and

$$U(r) = \alpha\left(\frac{1}{2}r\log r + \frac{C}{r} + Dr\right), \qquad p_0(\theta) = \frac{p_{\text{in}}(\theta-\theta_1) + p_{\text{out}}(\theta_2-\theta)}{\theta_2-\theta_1},$$

satisfy (5.1.1) with $\nu = \rho = 1$ and

$$f(x,y,t) := \begin{pmatrix} -\dfrac{U^2(r)}{r}e^{-2t}\cos\theta - U(r)e^{-t}\sin\theta \\ -\dfrac{U^2(r)}{r}e^{-2t}\sin\theta + U(r)e^{-t}\cos\theta \end{pmatrix} = \left\{\frac{\partial u}{\partial t} + (u\cdot\nabla)u\right\}(x,y,t),$$

$$p^b(x,y,t) := p_0(\theta)e^{-t} + \frac{U^2(r)}{2}e^{-2t}, \qquad u_0(x,y) := \begin{pmatrix} U(r)\sin\theta \\ -U(r)\cos\theta \end{pmatrix}.$$

Fig. 5.2 shows the initial value u_0 of the velocity and the pressure p at $t = 0$.

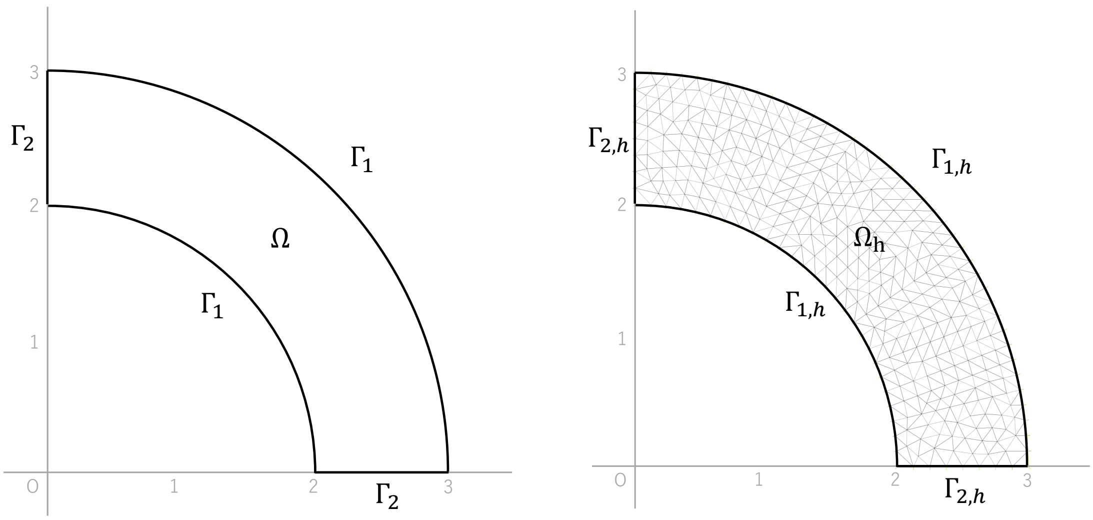

Figure 5.1: The domain Ω with boundary Γ_1, Γ_2 (left), and $\Omega_h, \Gamma_{1,h}, \Gamma_{2,h}$ with mesh (right).

We introduce a domain Ω_h to approximate the domain Ω, with boundary $\partial\Omega_h = \Gamma_{1,h} \cup \Gamma_{2,h}$ (Fig. 5.1). We also introduce a regular triangulation $\mathcal{T}_h$ to Ω_h, with $h = \max_{K\in\mathcal{T}_h}\operatorname{diam}(K)$ and $\overline{\Omega_h} = \cup_{K\in\mathcal{T}_h}\overline{K}$. To consider the P2 and P1 element approximation for velocity and pressure, respectively, we define the function spaces: for $i = 1, 2$,

$$\begin{aligned} X_h^i &:= \left\{\psi_h \in C(\overline{\Omega_h}) \;\middle|\; \varphi_h|_K \in P_i(K), \forall K \in \mathcal{T}_h\right\}, \\ H_h &:= \left\{\varphi_h \in (X_h^2)^2 \;\middle|\; \varphi_h = 0 \text{ on } \Gamma_{1,h},\ \varphi_h \times n_h = 0 \text{ on } \Gamma_{2,h}\right\}, \\ Q_h &:= \left\{\psi_h \in X_h^1 \;\middle|\; \psi_h = 0 \text{ on } \Gamma_{2,h}\right\}, \end{aligned}$$

where $P_i(K)$ is the set of polynomials of degree i or less on K and n_h is the unit outward normal vector for $\Gamma_{2,h}$. Here, since $\Gamma_{2,h}$ is flat, the normal component of $\varphi_h \in H_h$ is not

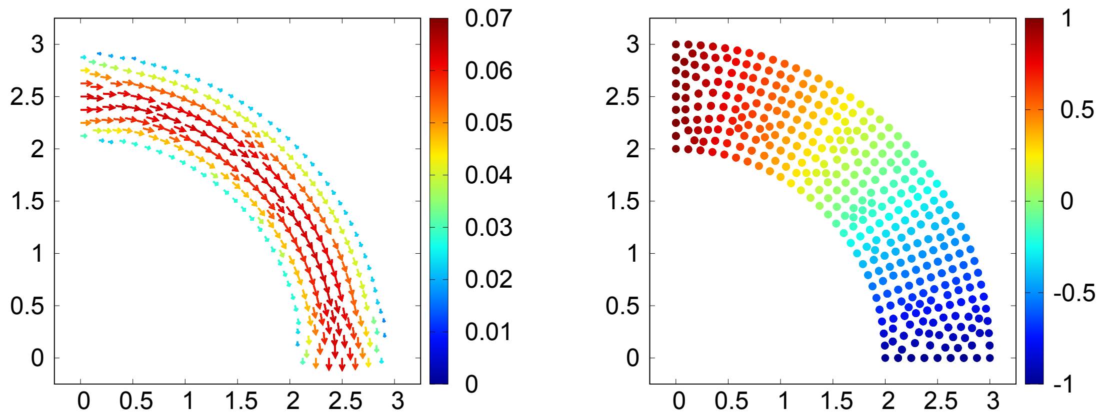

Figure 5.2: The initial value u_0 of the velocity (left) and the pressure p at $t = 0$ (right). In the left figure, the color scale indicates the length of $|u_0(\xi)|$ at each node ξ.

determined. If $\Gamma_{2,h}$ is not flat, then n_h is discontinuous on $\Gamma_{2,h}$ and $\varphi_h = 0$ on $\Gamma_{2,h}$ (cf. [13]). Let $\Pi_h^i : C(\overline{\Omega_h}) \to X_h^i$ $(i = 1, 2)$ be the Lagrange interpolation operator (on each triangle). By replacing u_{k-1} in the first equation of (PM) with the third equation of (PM) at the previous step (Remark 5.2.11), we consider the following discrete problem:

Problem 5.5.1. *For all $k = 1, 2, \ldots, N$, find $(u_k^*, P_k) \in H_h \times X_h^1$ such that $P_k - \Pi_h^1 p^b(t_k) \in Q_h$ and for all $\varphi \in H_h$ and $\psi \in Q_h$,*

$$\begin{cases} \dfrac{1}{\tau}(u_k^* - u_{k-1}^*, \varphi) + a_0(u_k^*, \varphi) + a_1(u_{k-1}^*, u_k^*, \varphi) + (\nabla P_{k-1}, \varphi) = (f(t_k), \varphi), \\ \tau(\nabla P_k, \nabla \psi) = -(\operatorname{div} u_k^*, \psi), \end{cases} \tag{5.5.40}$$

where $P_0 := 0$.

For all $k = 1, 2, \ldots, N$, we set $u_k := u_k^* - \tau \nabla P_k$. See [39, 40] for details on u_k and its divergence.

On a mesh with $h = 2^{-6}$, we solve the problems (5.5.40) numerically by using the FreeFEM software [43]. We compute the error estimates between the numerical solutions of (5.5.40) and the interpolation $(\Pi_h^2 u, \Pi_h^1 P)$ of the exact solution (u, P), where $P := p + |u|^2/2$. In Fig. 5.3, the numerical errors $\|\bar{u}_\tau - \Pi_h^2 u_\tau\|_{L^2(L^2(\Omega_h)^d)}$, $\|\bar{u}_\tau^* - \Pi_h^2 u_\tau\|_{L^2(L^2(\Omega_h)^d)}$, $\|\bar{P}_\tau - \Pi_h^1 P_\tau\|_{L^2(L^2(\Omega_h))}$, and $\|\bar{u}_\tau^* - \Pi_h^2 u_\tau\|_{L^2(H^1(\Omega_h)^d)}$ are presented. It can be observed that $\|\bar{u}_\tau - \Pi_h^2 u_\tau\|_{L^2(L^2(\Omega_h)^d)}$ and $\|\bar{u}_\tau^* - \Pi_h^2 u_\tau\|_{L^2(L^2(\Omega_h)^d)}$ are almost of first order in τ and that $\|\bar{P}_\tau - \Pi_h^1 P_\tau\|_{L^2(L^2(\Omega_h))}$ is of 0.5th order in τ, as expected from Theorem 5.3.8. Furthermore, the error $\|\bar{u}_\tau^* - \Pi_h^2 u_\tau\|_{L^2(H^1(\Omega_h)^d)}$ is almost of first order in τ, which is better than the theoretically predicted rate (Theorem 5.3.3).

5.6 Conclusion

We have proposed the projection method (5.1.3), (5.1.4), and (5.1.5) for Navier–Stokes equations (5.1.1) with a total pressure boundary condition. We have shown the stability of the projection method in Theorem 5.3.1 and established error estimates for the velocity and the pressure in suitable norms between the solution to (NS) and (PM) in Theorems 5.3.3 and 5.3.8. The convergence rates are the same as the case of the usual full-Dirichlet

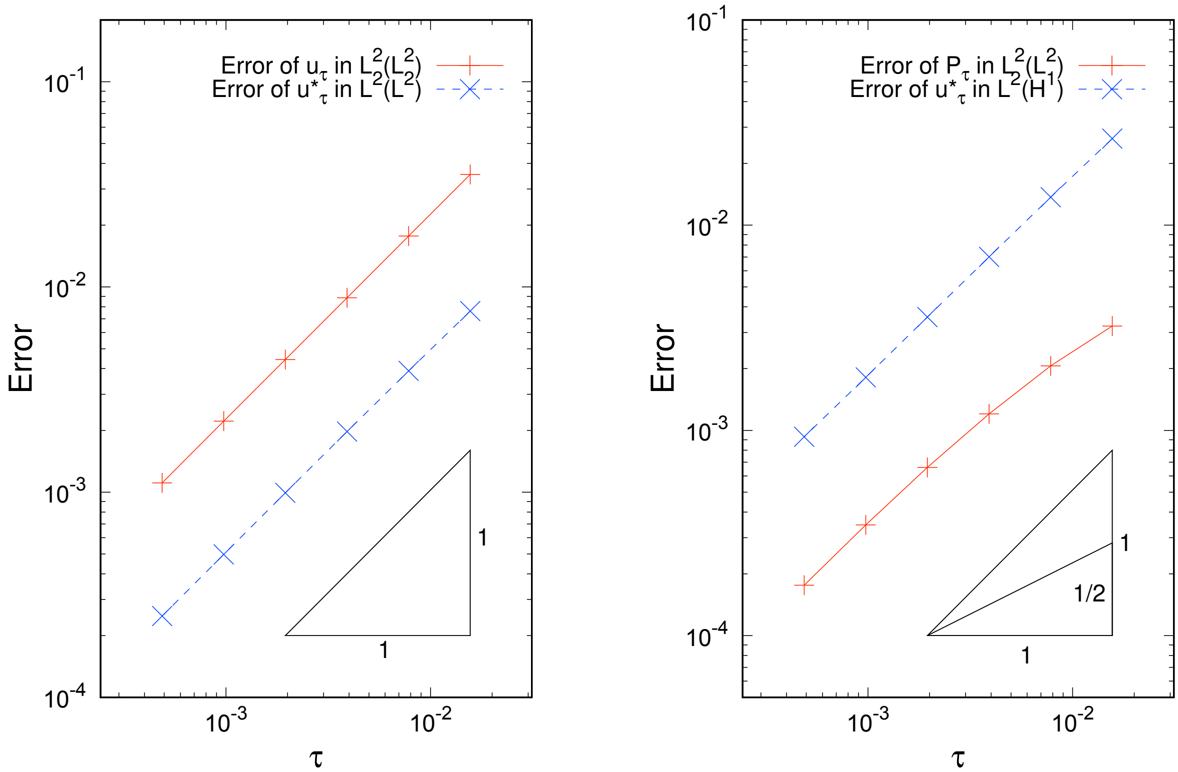

Figure 5.3: The errors of the log scale: $\|\bar{u}_\tau - \Pi_h^2 u_\tau\|_{L^2(L^2(\Omega_h)^d)}$, $\|\bar{u}^*_\tau - \Pi_h^2 u_\tau\|_{L^2(L^2(\Omega_h)^d)}$ (left), $\|\bar{P}_\tau - \Pi_h^1 P_\tau\|_{L^2(L^2(\Omega_h))}$, and $\|\bar{u}^*_\tau - \Pi_h^2 u_\tau\|_{L^2(H^1(\Omega_h)^d)}$ (right). The triangles show the slope of $O(\tau)$ and $O(\sqrt{\tau})$.

boundary condition for velocity [76]. The traction boundary condition is often used to apply Dirichlet boundary conditions for pressure; however, the convergence rates are worse than our case (Compare [37] and [41]).

The projection method is still evolving, and many high-convergence methods have been proposed (e.g., [41]). The application of the boundary conditions proposed in this paper to these methods will be a focus of our future works. As another future direction, the case in which Γ_2 is not flat in numerical calculations is an important problem (cf. [13]). In addition, since the nonlinear term $(\nabla \times u) \times u$ is different from the standard advection term $(u \cdot \nabla)u$, it cannot be applied to methods using the Lagrangian coordinates, such as the characteristic curve method and particle methods; this problem remains open for further study.

List of publications

Preprints:

1. K. Matsui: A projection method for Navier–Stokes equations with a boundary condition including the total pressure. arXiv:2105.13014 (submitted), 2021.

Refereed journal paper:

2. Matsui, K.: Sharp consistency estimates for a pressure-Poisson problem with Stokes boundary value problems. Discrete & Continuous Dynamical Systems - S **14** (3), 1001–1015 (2021). DOI 10.3934/dcdss.2020380

3. M. Kimura, K. Matsui, A. Muntean, and H. Notsu: Analysis of a projection method for the Stokes problem using an ε-Stokes approach. Japan Journal of Industrial and Applied Mathematics **36**, 959–985 (2019). DOI 10.1007/s13160-019-00373-3

4. K. Matsui and A. Muntean: Asymptotic analysis of an ε-Stokes problem connecting Stokes and pressure-Poisson problems. Advances in Mathematical Sciences and Applications **27**, 181–191 (2018).

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Bibliography

[1] Acosta, G., Durán, R.G., Muschietti, M.A.: Solutions of the divergence operator on john domains. Advances in Mathematics **206**(2), 373–401 (2006). DOI 10.1016/j.aim.2005.09.004

[2] Amrouche, C., Ciarlet, P.G., Mardare, C.: On a lemma of Jacques-Louis lions and its relation to other fundamental results. Journal de Mathématiques Pures et Appliquées **104**(2), 207–226 (2015). DOI 10.1016/j.matpur.2014.11.007

[3] Amrouche, C., Seloula, N.E.H.: L^p-theory for the Navier–Stokes equations with pressure boundary conditions. Discrete & Continuous Dynamical Systems - S **6**(5), 1113–1137 (2013). DOI 10.3934/dcdss.2013.6.1113

[4] Amsden, A.A., Harlow, F.H.: A simplified MAC technique for incompressible fluid flow calculations. Journal of Computational Physics **6**(2), 322–325 (1970). DOI 10.1016/0021-9991(70)90029-X

[5] Amsden, A.A., Harlow, F.H.: The SMAC method: A numerical technique for calculating incompressible fluid flows. Tech. Rep. LA-4370, Los Alamos National Laboratory (1970)

[6] Badia, S., Codina, R.: Convergence analysis of the FEM approximation of the first order projection method for incompressible flows with and without the inf-sup condition. Numerische Mathematik **107**, 533–557 (2007). DOI 10.1007/s00211-007-0099-5

[7] Begue, C., Conca, C., Murat, F., Pironneau, O.: A nouveau sur les equations de Stokes et de Navier-Stokes avec des conditions aux limites sur la pression. C. R. Acad. Sc. Paris, Serie I **304**, 23–28 (1987)

[8] Begue, C., Conca, C., Murat, F., Pironneau, O.: Les équations de Stokes et de Navier–Stokes avec des conditions aux limites sur la pression. In: H. Brezis, J.L. Lions (eds.) Nonlinear Partial Differential Equations and Their Applications, College de France, Seminar, vol. IX, pp. 179–264. Longman Scientific and Technical (1988)

[9] Bernard, J.M.: Non-standard Stokes and Navier–Stokes problems: existence and regularity in stationary case. Mathematical Methods in the Applied Sciences **25**(8), 627–661 (2002). DOI 10.1002/mma.260

[10] Bernard, J.M.: Time-dependent Stokes and Navier–Stokes problems with boundary conditions involving pressure, existence and regularity. Nonlinear Analysis: Real World Applications **4**(5), 805–839 (2003). DOI 10.1016/S1468-1218(03)00016-6

[11] Bernardi, C., Canuto, C., Maday, Y.: Spectral approximations of the Stokes equations with boundary conditions on the pressure. SIAM Journal on Numerical Analysis **28**(2), 333–362 (1991). DOI 10.1137/0728019

[12] Bernardi, C., Rebollo, T.C., Yakoubi, D.: Finite element discretization of the Stokes and Navier–Stokes equations with boundary conditions on the pressure. SIAM Journal on Numerical Analysis **53**(3), 1256–1279 (2015). DOI 10.1137/140972299

[13] Bertoluzza, S., Chabannes, V., Prud'homme, C., Szopos, M.: Boundary conditions involving pressure for the Stokes problem and applications in computational hemodynamics. Computer Methods in Applied Mechanics and Engineering **322**, 58–80 (2017). DOI 10.1016/j.cma.2017.04.024

[14] Boyer, F., Fabrie, P.: Mathematical tools for the study of the incompressible Navier–Stokes equations and related models, *Applied Mathematical Sciences*, vol. 183. Springer, New York (2013). DOI 10.1007/978-1-4614-5975-0

[15] Bramble, J.H.: A proof of the inf-sup condition for the Stokes equations on Lipschitz domains. Mathematical Models and Methods in Applied Sciences **13**(03), 361–371 (2003). DOI 10.1142/S0218202503002544

[16] Brezis, H.: Functional Analysis, Sobolev Spaces and Partial Differential Equations. Universitext. Springer, New York (2011). DOI 10.1007/978-0-387-70914-7

[17] Brezzi, F., Fortin, M.: Mixed and Hybrid Finite Element Methods, *Springer Series in Computational Mathematics*, vol. 15. Springer (1991). DOI 10.1007/978-1-4612-3172-1

[18] Chan, R.K.C., Street, R.L.: A computer study of finite-amplitude water waves. Journal of Computational Physics **6**(1), 68–94 (1970). DOI 10.1016/0021-9991(70)90005-7

[19] Cho, J.R.: Projection of the rotation form Navier–Stokes equation onto the half-staggered grid. Journal of Mechanical Science and Technology **30**(7), 3159–3164 (2016). DOI 10.1007/s12206-016-0626-3

[20] Chorin, A.J.: A numerical method for solving incompressible viscous flow problems. Journal of Computational Physics **2**(1), 12–26 (1967). DOI 10.1016/0021-9991(67)90037-X

[21] Chorin, A.J.: Numerical solution of the Navier–Stokes equations. Mathematics of Computation **22**(104), 745–762 (1968). DOI 10.1090/S0025-5718-1968-0242392-2

[22] Ciarlet, P.G.: Mathematical Elasticity Volume I: Three-Dimensional Elasticity. North Holland (1988)

[23] Conca, C., Murat, F., Pironneau, O.: The Stokes and Navier–Stokes equations with boundary conditions involving the pressure. Japanese Journal of Mathematics. New Series **20**(2), 279–318 (1994). DOI 10.4099/math1924.20.279

[24] Conca, C., Pares, C., Pironneau, O., Thiriet, M.: Navier–Stokes equations with imposed pressure and velocity fluxes. International Journal for Numerical Methods in Fluids **20**(4), 267–287 (1995). DOI 10.1002/fld.1650200402

[25] Cummins, S.J., Rudman, M.: An SPH projection method. Journal of Computational Physics **152**(2), 584–607 (1999). DOI 10.1006/jcph.1999.6246

[26] Douglas, J., Wang, J.: An absolutely stabilized finite element method for the Stokes problem. Mathematics of Computation **52**(186), 495–508 (1989). DOI 10.1090/S0025-5718-1989-0958871-X

[27] Duvaut, G., Lions, J.L.: Inequalities in Mechanics and Physics, *Grundlehren der mathematischen Wissenschaften*, vol. 219. Springer, Berlin, Heidelberg (1976). DOI 10.1007/978-3-642-66165-5

[28] Evans, L.: Partial Differential Equations, *Graduate Studies in Mathematics*, vol. 19, 2 edn. American Mathematical Society, Providence, Rhode Island (2010)

[29] Eymard, R., Gallouët, T., Herbin, R.: Finite volume methods. In: Solution of Equation in $\mathbb{R}^n$ (Part 3), Techniques of Scientific Computing (Part 3), *Handbook of Numerical Analysis*, vol. 7, pp. 713–1018. Elsevier (2000). DOI 10.1016/S1570-8659(00)07005-8

[30] Forsythe, G.E., Wasow, W.: Finite-Difference Methods for Partial Differential Equations, 4 edn. John Wiley and Sons (1967)

[31] de Frutos, J., García-Archilla, B., Novo, J.: Error analysis of projection methods for non inf-sup stable mixed finite elements: The Navier–Stokes equations. Journal of Scientific Computing **74**, 426–455 (2018). DOI 10.1007/s10915-017-0446-3

[32] Girault, V., Raviart, P.A.: Finite Element Methods for Navier–Stokes Equations: Theory and Algorithms, *Springer Series in Computational Mathematics*, vol. 5. Springer-Verlag, Berlin, Heidelberg (1986). DOI 10.1007/978-3-642-61623-5

[33] Glowinski, R.: Finite element methods for incompressible viscous flow. In: Numerical Methods for Fluids (Part 3), *Handbook of Numerical Analysis*, vol. 9, pp. 3–1176. Elsevier (2003). DOI 10.1016/S1570-8659(03)09003-3

[34] Gresho, P.M.: Incompressible fluid dynamics: Some fundamental formulation issues. Annual Review of Fluid Mechanics **23**(1), 413–453 (1991). DOI 10.1146/annurev.fl.23.010191.002213

[35] Gresho, P.M., Sani, R.L.: On pressure boundary conditions for the incompressible Navier–Stokes equations. International Journal for Numerical Methods in Fluids **7**(10), 1111–1145 (1987). DOI 10.1002/fld.1650071008

[36] Grisvard, P.: Elliptic Problems in Nonsmooth Domains. Pitman (1985)

[37] Guermond, J.L., Minev, P., Shen, J.: Error analysis of pressure-correction schemes for the time-dependent Stokes equations with open boundary conditions. SIAM Journal on Numerical Analysis **43**(1), 239–258 (2005). DOI 10.1137/040604418

[38] Guermond, J.L., Minev, P., Shen, J.: An overview of projection methods for incompressible flows. Computer Methods in Applied Mechanics and Engineering **195**(44), 6011–6045 (2006). DOI 10.1016/j.cma.2005.10.010

[39] Guermond, J.L., Quartapelle, L.: Calculation of incompressible viscous flows by an unconditionally stable projection FEM. Journal of Computational Physics **132**(1), 12–33 (1997). DOI 10.1006/jcph.1996.5587

[40] Guermond, J.L., Quartapelle, L.: On the approximation of the unsteady Navier–Stokes equations by finite element projection methods. Numerische Mathematik **80**(2), 207–238 (1998). DOI 10.1007/s002110050366

[41] Guermond, J.L., Shen, J.: On the error estimates for the rotational pressure-correction projection methods. Mathematics of Computation **73**(248), 1719–1737 (2004). DOI 10.1090/S0025-5718-03-01621-1

[42] Harlow, F.H., Welch, J.E.: Numerical calculation of time-dependent viscous incompressible flow of fluid with a free surface. The Physics of Fluids **8**, 2182–2189 (1965)

[43] Hecht, F.: New development in FreeFem++. Journal of Numerical Mathematics **20**(3-4), 251–265 (2012)

[44] Heywood, J.G., Rannacher, R.: Finite-element approximation of the nonstationary Navier–Stokes problem part IV: Error analysis for second-order time discretization. SIAM Journal on Numerical Analysis **27**(2), 353–384 (1990). URL `http://www.jstor.org/stable/2157967`

[45] Holman, J.P.: Experimental Methods for Engineers, 7 edn. McGraw-Hill (2001)

[46] Hoover, W.G.: Smooth Particle Applied Mechanics: The State of the Art, *Advanced Series in Nonlinear Dynamics*, vol. 25. World Scientific (2006). DOI 10.1142/6218

[47] Hosseini, S.M., Feng, J.J.: Pressure boundary conditions for computing incompressible flows with SPH. Journal of Computational Physics **230**(19), 7473–7487 (2011). DOI 10.1016/j.jcp.2011.06.013

[48] Hughes, T., Franca, L., Balestra, M.: A new finite element formulation for computational fluid dynamics: V. circumventing the Babuška–Brezzi condition: A stable Petrov–Galerkin formulation of the Stokes problem accommodating equal-order interpolations. Computer Methods in Applied Mechanics and Engineering **59**(1), 85–99 (1986). DOI 10.1016/0045-7825(86)90025-3

[49] Imoto, Y.: Unique solvability and stability analysis for incompressible smoothed particle hydrodynamics method. Computational Particle Mechanics **6**(2), 297–309 (2019). DOI 10.1007/s40571-018-0214-7

[50] Johnston, H., Liu, J.G.: Accurate, stable and efficient Navier–Stokes solvers based on explicit treatment of the pressure term. Journal of Computational Physics **199**(1), 221–259 (2004). DOI 10.1016/j.jcp.2004.02.009

[51] Kangro, U., Nicolaides, R.: Divergence boundary conditions for vector Helmholtz equations with divergence constraints. ESAIM: Mathematical Modelling and Numerical Analysis **33**(3), 479–492 (1999). DOI 10.1051/m2an:1999148

[52] Karniadakis, G., Sherwin, S.J.: Spectral/hp Element Methods for Computational Fluid Dynamics, 2 edn. Numerical mathematics and scientific computation. Oxford University Press (2005). DOI 10.1093/acprof:oso/9780198528692.001.0001

[53] Kim, J., Moin, P.: Application of a fractional-step method to incompressible Navier–Stokes equations. Journal of Computational Physics **59**(2), 308–323 (1985). DOI 10.1016/0021-9991(85)90148-2

[54] Kim, T.: Regularity of solutions to the Navier–Stokes equations with a nonstandard boundary condition. Acta Mathematicae Applicatae Sinica, English Series **31**(3), 707–718 (2015). DOI 10.1007/s10255-015-0497-x

[55] Kim, T., Cao, D.: Some properties on the surfaces of vector fields and its application to the Stokes and Navier–Stokes problems with mixed boundary conditions. Nonlinear Analysis: Theory, Methods and Applications **113**, 94–114 (2015). DOI 10.1016/j.na.2014.09.017. Erratum, ibid **135**, 249–250 (2016). DOI 10.1016/j.na.2016.01.003

[56] Kim, T., Cao, D.: Equations of Motion for Incompressible Viscous Fluids: With Mixed Boundary Conditions. Advances in Mathematical Fluid Mechanics. Birkhäuser, Cham (2021). DOI 10.1007/978-3-030-78659-5

[57] Koshizuka, S., Oka, Y.: Moving-particle semi-implicit method for fragmentation of incompressible fluid. Nuclear Science and Engineering **123**(3), 421–434 (1996)

[58] Koshizuka, S., Shibata, K., Kondo, M., Matsunaga, T.: Moving Particle Semi-implicit Method: A Meshfree Particle Method for Fluid Dynamics. Academic Press (2018). DOI 10.1016/C2016-0-03952-9

[59] Li, L.: A split-step finite-element method for incompressible Navier–Stokes equations with high-order accuracy up-to the boundary. Journal of Computational Physics **408**, 109,274 (2020). DOI 10.1016/j.jcp.2020.109274

[60] Liu, J.: Open and traction boundary conditions for the incompressible Navier–Stokes equations. Journal of Computational Physics **228**(19), 7250–7267 (2009). DOI 10.1016/j.jcp.2009.06.021

[61] Liu, M.B., Liu, G.R.: Smoothed particle hydrodynamics (SPH): an overview and recent developments. Archives of Computational Methods in Engineering **17**, 25–76 (2010). DOI 10.1007/s11831-010-9040-7

[62] Liu, M.B., Liu, G.R.: Particle Methods for Multi-Scale and Multi-Physics. World Scientific (2015). DOI 10.1142/9017

[63] Marušić, S.: On the Navier–Stokes system with pressure boundary condition. Annali dell'Universita' di Ferrara **53**(2), 319–331 (2007). DOI 10.1007/s11565-007-0024-y

[64] McKee, S., Tomé, M.F., Cuminato, J.A., Castelo, A., Ferreira, V.G.: Recent advances in the marker and cell method. Archives of Computational Methods in Engineering **11**(2), 107–142 (2004). DOI 10.1007/BF02905936

[65] Moukalled, F., Mangani, L., Darwish, M.: The Finite Volume Method in Computational Fluid Dynamics: An Advanced Introduction with OpenFOAM® and Matlab. Fluid Mechanics and Its Applications. Springer, Cham (2016). DOI 10.1007/978-3-319-16874-6

[66] Nečas, J.: Direct Methods in the Theory of Elliptic Equations. Springer Monographs in Mathematics. Springer (1967). DOI 10.1007/978-3-642-10455-8. Translated in 2012 by Kufner, A. and Tronel, G.

[67] Olshanskii, M.A., Reusken, A.: Navier–Stokes equations in rotation form: A robust multigrid solver for the velocity problem. SIAM Journal on Scientific Computing **23**(5), 1683–706 (2002). DOI 10.1137/S1064827500374881

[68] Perot, J.B.: An analysis of the fractional step method. Journal of Computational Physics **108**(1), 51–58 (1993). DOI 10.1006/jcph.1993.1162

[69] Pompe, W.: Korn's first inequality with variable coefficients and its generalization. Commentationes Mathematicae Universitatis Carolinae **44**(1), 57–70 (2003). URL http://eudml.org/doc/249158

[70] Prohl, A.: Projection and Quasi-Compressibility Methods for Solving the Incompressible Navier–Stokes Equations. Advances in Numerical Mathematics. Vieweg+Teubner Verlag, Wiesbaden (1997). DOI 10.1007/978-3-663-11171-9

[71] Rannacher, R.: On Chorin's projection method for the incompressible Navier–Stokes equations. In: J.G. Heywood, K. Masuda, R. Rautmann, V.A. Solonnikov (eds.) The Navier–Stokes Equations II — Theory and Numerical Methods, *Lecture Notes in Mathematics*, vol. 1530, pp. 167–183. Springer, Berlin, Heidelberg (1992). DOI 10.1007/BFb0090341

[72] Saito, N., Fujita, H.: Remarks on traces of H^1-functions defined in a domain with corners. Journal of Mathematical Sciences, the University of Tokyo **7**(2), 325–345 (2000)

[73] Sani, R.L., Shen, J., Pironneau, O., Gresho, P.M.: Pressure boundary condition for the time-dependent incompressible Navier–Stokes equations. International Journal for Numerical Methods in Fluids **50**(6), 673–682 (2006). DOI 10.1002/fld.1062

[74] Shakibaeinia, A., Jin, Y.C.: MPS mesh-free particle method for multiphase flows. Computer Methods in Applied Mechanics and Engineering **229–232**, 13–26 (2012). DOI 10.1016/j.cma.2012.03.013

[75] Shao, S., Lo, E.Y.M.: Incompressible sph method for simulating newtonian and non-newtonian flows with a free surface. Advances in Water Resources **26**(7), 787–800 (2003). DOI 10.1016/S0309-1708(03)00030-7

[76] Shen, J.: On error estimates of projection methods for Navier–Stokes equations: First-order schemes. SIAM Journal on Numerical Analysis **29**(1), 57–77 (1992). DOI 10.1137/0729004

[77] Shen, J.: Remarks on the pressure error estimates for the projection methods. Numerische Mathematik **67**(4), 513–520 (1994). DOI 10.1007/s002110050042

[78] Smith, G.D.: Numerical Solution of Partial Differential Equations: Finite Difference Methods, 3 edn. Oxford University Press (1986)

[79] Temam, R.: Sur l'approximation de la solution des équations de Navier–Stokes par la méthode des pas fractionnaires (II). Archive for Rational Mechanics and Analysis **33**(5), 377–385 (1969). DOI 10.1007/BF00247696

[80] Temam, R.: Navier–Stokes Equations. North Holland (1979)

[81] Thomas, J.W.: Numerical Partial Differential Equations: Finite Difference Methods, *Texts in Applied Mathematics*, vol. 22. Springer (1995). DOI 10.1007/978-1-4899-7278-1

[82] Tomiyama, A., Hirano, M.: An improvement of the computational efficiency of the sola method. Transactions of the Japan Society of Mechanical Engineers Series B **58**(551), 2171–2176 (1992). DOI 10.1299/kikaib.58.2171

[83] Versteeg, H.K., Malalasekera, W.: An Introduction to Computational Fluid Dynamics: The Finite Volume Method, 2 edn. Prentice Hall (2007)

[84] Viecelli, J.A.: A computing method for incompressible flows bounded by moving walls. Journal of Computational Physics **8**(1), 119–143 (1971). DOI 10.1016/0021-9991(71)90039-8

[85] Welch, J.E., Harlow, F.H., Shannon, J.P., Daly, B.J.: The MAC method: A computing technique for solving viscous, incompressible, transient fluid-flow problems involving free surfaces. Tech. Rep. LA-3425, Los Alamos National Laboratory (1966). DOI 10.2172/4563173. Errata (1968)

[86] Yang, L., Badia, S., Codina, R.: A pseudo-compressible variational multiscale solver for turbulent incompressible flows. Computational Mechanics **58**(6), 1051–1069 (2016). DOI 10.1007/s00466-016-1332-9

[87] Zienkiewicz, O.C., Taylor, R.L., Zhu, J.Z.: The Finite Element Method: Its Basis and Fundamentals, 7 edn. Butterworth-Heinemann (2013). DOI 10.1016/C2009-0-24909-9

Chapter A

Proofs

After defining the Lipschitz domain in Section A.1, we prove the Nečas inequality (Theorem 2.2.13) in Section A.2 and its corollaries (Theorems 2.2.14 and 2.2.15) in Section A.3.

A.1 Bounded Lipschitz domain

We introduce the notation used in this appendix.

For $\alpha, \beta > 0$ and an open set $U \subset \mathbb{R}^{n-1}$, we set

$$\begin{aligned} B(\alpha) &:= \left\{ x' = (x_1, \cdots, x_{n-1}) \in \mathbb{R}^{n-1} \;\middle|\; |x'| < \alpha \right\}, \\ K(\alpha, \beta) &:= B(\alpha) \times (0, \beta), \\ M(\alpha, \beta) &:= B(\alpha) \times (-\beta, \beta), \end{aligned}$$

where $|x'| := \sqrt{x_1^2 + \cdots + x_{n-1}^2}$. A function $g : \overline{U} \to \mathbb{R}$ is called Lipschitz continuous if there exists a constant $c > 0$ such that

$$|g(x') - g(y')| \le c|x' - y'| \quad \text{for all } x', y' \in \overline{U}.$$

The constant c is called a Lipschitz constant c_g for g.

We use the following theorems and lemmas.

Theorem A.1.1 (partition of unity)**.** [16, Lemma 9.3] *Let $\Omega \subset \mathbb{R}^n$ be a bounded open subset and let open subsets $U_0, U_1, \cdots, U_m \subset \mathbb{R}^n$ satisfy $\overline{\Omega} \subset \bigcup_{r=0}^m U_r$. Then, there exists functions $\eta_0, \eta_1, \cdots, \eta_m \in C^\infty(\mathbb{R}^n)$ such that*

$$\begin{aligned} &\eta_r \in C_0^\infty(U_r) && \text{for all } r = 0, 1, \cdots, m, \\ &0 \le \eta_r(x) \le 1 && \text{for all } r = 0, 1, \cdots, m, x \in U_r, \\ &\sum_{r=0}^m \eta_r(x) = 1 && \text{for all } x \in \overline{\Omega}. \end{aligned}$$

Definition A.1.1. *A bounded open set Ω is called a Lipschitz domain if there exist two real numbers $\alpha, \beta > 0$, an integer $m \in \mathbb{N}$, systems of local charts $(x_{r1}, \cdots, x_{rn})$ $(r = 1, 2, \cdots, m)$ and Lipschitz continuous functions $g_r : \overline{B_r(\alpha)} := \{x'_r \in \mathbb{R}^{n-1} \mid |x_{ri}| \le \alpha, i = 1, 2, \cdots, n-1\} \to \mathbb{R}$ such that*

$$\Gamma = \bigcup_{r=1}^m \{x_r \in \mathbb{R}^n \mid x_r = (x'_r, x_{rn}), x'_r \in B_r(\alpha), x_{rn} = g_r(x'_r)\}$$

and it follows that for all $y'_r \in \overline{B_r(\alpha)}$

$$\begin{aligned} g_r(y'_r) < y_{rn} < g_r(y'_r) + \beta \quad &\Rightarrow (y'_r, y_{rn}) \in \Omega, \\ g_r(y'_r) - \beta < y_{rn} < g_r(y'_r) \quad &\Rightarrow (y'_r, y_{rn}) \in \mathbb{R}^n \backslash \Omega. \end{aligned}$$

By using systems of local charts in Definition A.1.1, we define subsets;

$$\begin{aligned} B_r(\alpha') &:= \{x'_r \in \mathbb{R}^{n-1} \mid |x'_r| < \alpha'\}, \\ U_r(\alpha', \beta') &:= \{x_r = (x'_r, x_{rn}) \in \mathbb{R}^n \mid x'_r \in B_r(\alpha'), g_r(x'_r) - \beta' < x_{rn} < g_r(x'_r) + \beta'\}, \\ U_r^+(\alpha', \beta') &:= \{x_r = (x'_r, x_{rn}) \in \mathbb{R}^n \mid x'_r \in B_r(\alpha'), g_r(x'_r) < x_{rn} < g_r(x'_r) + \beta'\} \end{aligned}$$

for $0 < \alpha' \leq \alpha, 0 < \beta' \leq \beta$ and $r = 1, 2, \cdots, m$.

A.2 Nečas inequality on a bounded Lipschitz domain

The Nečas inequality (Lemma 2.2.13) is important for the proof of the existence of the solution to the Stokes problem and the Korn inequality, cf. [27, 32, 80]. The Nečas inequality on a bounded Lipschitz domain was proven by Nečas [66]. The Nečas inequality also holds on a John domain that is a weaker condition than Lipschitz domain [1].

Nečas proceeds with the proof in two steps:

1 Interior of Ω. Here, the proof follows the case $\Omega = \mathbb{R}^n$.

2 Neighborhood V near the boundary Γ. Here, the proof follows the case Ω as the half space $\{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid x_n > 0\}$.

There are other methods for the proof of the Nečas inequality [2]. In [27], the authors prove that

$$\left\{ p \in H^{-1}(\Omega) \;\middle|\; \frac{\partial p}{\partial x_i} \in H^{-1}(\Omega) \text{ for all } i = 1, \cdots, n \right\} = L^2(\Omega)$$

holds with a $C^{1,1}$-class boundary. The equation is equivalent to the Nečas inequality. See also [15].

The purpose of this appendix is to provide the Nečas style proof. In A.2.1, we introduce the notations and symbols used in this appendix. We prove the case $\Omega = \mathbb{R}^n$ in A.2.2 and the case Ω is a subset K of the half space in A.2.3. In A.2.4 we define mollifiers and show several properties. We also make the mapping $T : K \to V$ using the mollifiers. In A.2.5, we prove the Nečas inequality.

A.2.1 Preliminaries

We use the following theorems and lemmas.

Lemma A.2.1. *Let* $U \subset \mathbb{R}^n$ *be a open set. We have*

$$\|\nabla p\|_{H^{-1}(U)^n} \leq \sum_{i=1}^{n} \left\| \frac{\partial p}{\partial x_i} \right\|_{H^{-1}(U)} \leq \sqrt{n} \|\nabla p\|_{H^{-1}(U)^n}$$

for all $p \in L^2(U)$.

Proof.

$$\begin{aligned}
\sum_{i=1}^{n}\left\|\frac{\partial p}{\partial x_i}\right\|_{H^{-1}(U)} &= \sum_{i=1}^{n}\sup_{\varphi_i\in H^1(U),\ \|\varphi_i\|_{H^1(U)}=1}\left\langle\frac{\partial p}{\partial x_i},\varphi_i\right\rangle\\
&= \sup_{\varphi\in H^1(U)^n,\ \|\varphi_1\|_{H^1(U)}=\cdots=\|\varphi_n\|_{H^1(U)}=1}\langle\nabla p,\varphi\rangle\\
&\le \sup_{\varphi\in H^1(U)^n,\ \|\varphi\|_{H^1(U)^n}=\sqrt{n}}\langle\nabla p,\varphi\rangle\\
&= \sqrt{n}\sup_{\varphi\in H^1(U)^n,\ \|\varphi\|_{H^1(U)^n}=\sqrt{n}}\langle\nabla p,\frac{1}{\sqrt{n}}\varphi\rangle\\
&= \sqrt{n}\sup_{\varphi\in H^1(U)^n,\ \|\varphi\|_{H^1(U)^n}=1}\langle\nabla p,\varphi\rangle\\
&= \sqrt{n}\|\nabla p\|_{H^{-1}(U)^n}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\|\nabla p\|_{H^{-1}(U)^n} &= \sup_{0\neq\varphi\in H^1(U)^n,\ \|\varphi\|_{H^1(U)^n}\le 1}\langle\nabla p,\varphi\rangle\\
&= \sup_{0\neq\varphi\in H^1(U)^n,\ \|\varphi_1\|^2_{H^1(U)}+\cdots+\|\varphi_n\|^2_{H^1(U)}\le 1}\langle\nabla p,\varphi\rangle\\
&\le \sum_{i=1}^{n}\sup_{0\neq\varphi_i\in H^1(U),\ \|\varphi_i\|_{H^1(U)}\le 1}\left\langle\frac{\partial p}{\partial x_i},\varphi_i\right\rangle\\
&= \sum_{i=1}^{n}\left\|\frac{\partial p}{\partial x_i}\right\|_{H^{-1}(U)}.
\end{aligned}$$

□

Lemma A.2.2. *Let Ω be an open set in $\mathbb{R}^n$. If there exists a constant $c>0$ such that*

$$\|p\|_{L^2(\Omega)}\le c(\|p\|_{H^{-1}(\Omega)}+\|\nabla p\|_{H^{-1}(\Omega)^n})\qquad \textit{for all } p\in C_0^\infty(\Omega),$$

then it holds that

$$\|p\|_{L^2(\Omega)}\le c(\|p\|_{H^{-1}(\Omega)}+\|\nabla p\|_{H^{-1}(\Omega)^n})\qquad \textit{for all } p\in L^2(\Omega).$$

Proof. For $p\in L^2(\Omega)$, we have

$$\begin{aligned}
\|p\|_{H^{-1}(\Omega)} &= \sup_{\psi\in H_0^1(\Omega),\ \|\psi\|_{H^1(\Omega)}=1}\int_\Omega p\psi\,dx\\
&\le \sup_{\psi\in H_0^1(\Omega),\ \|\psi\|_{H^1(\Omega)}=1}\|p\|_{L^2(\Omega)}\|\psi\|_{L^2(\Omega)}\\
&\le \sup_{\psi\in H_0^1(\Omega),\ \|\psi\|_{H^1(\Omega)}=1}\|p\|_{L^2(\Omega)}\|\psi\|_{H^1(\Omega)}\\
&= \|p\|_{L^2(\Omega)},\\
\|\nabla p\|_{H^{-1}(\Omega)^n} &= \sup_{\varphi\in H_0^1(\Omega)^n,\ \|\varphi\|_{H^1(\Omega)^n}=1}\int_\Omega p\operatorname{div}\varphi\,dx\\
&\le \sup_{\varphi\in H_0^1(\Omega)^n,\ \|\varphi\|_{H^1(\Omega)^n}=1}\|p\|_{L^2(\Omega)}\|\operatorname{div}\varphi\|_{L^2(\Omega)}\\
&\le \sqrt{n}\sup_{\varphi\in H_0^1(\Omega)^n,\ \|\varphi\|_{H^1(\Omega)^n}=1}\|p\|_{L^2(\Omega)}\|\nabla\varphi\|_{L^2(\Omega)^{n\times n}}\\
&\le \sqrt{n}\|p\|_{L^2(\Omega)}.
\end{aligned}$$

Since $C_0^\infty(\Omega)$ is dense in $L^2(\Omega)$, we obtain the result. □

A.2.2 Total space

If $p, q : \mathbb{R}^n \to \mathbb{R}$ is a continuous function with compact support, its Fourier transform $\hat{p}$ and inverse Fourier transform $\check{q}$ is defined by

$$\begin{aligned}\hat{p}(\xi) &:= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} p(x)dx \quad \text{for all } \xi \in \mathbb{R}^n,\\ \check{q}(x) &:= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} q(\xi)d\xi \quad \text{for all } x \in \mathbb{R}^n,\end{aligned}$$

where $x \cdot \xi = \sum_{k=1}^n x_k \xi_k$. It is easy to see that $\hat{q}(x) = \check{q}(-x)$ for all $x \in \mathbb{R}^n$. It is well known property of the Fourier transform that

$$\widehat{\frac{\partial p}{\partial x_k}}(\xi) = i\xi_k \hat{p}(\xi) \quad \text{for all } k = 1, \cdots, n.$$

One proves (Plancherel theorem [28, Theorem 4.3.2 (ii)]) that if $p \in L^2(\mathbb{R}^n)$ then $\hat{p} \in L^2(\mathbb{R}^n)$ and $\|\hat{p}\|_{L^2(\mathbb{R}^n)} = \|p\|_{L^2(\mathbb{R}^n)}$. By continuous extension, one can therefore define $\mathcal{F} : L^2(\mathbb{R}^n) \ni p \to \hat{p} \in L^2(\mathbb{R}^n)$ and $\mathcal{F}^* : L^2(\mathbb{R}^n) \ni q \to \check{q} \in L^2(\mathbb{R}^n)$. The linear isometric mapping $\mathcal{F}$ is an unitary map and has the inverse map:

Theorem A.2.3. [28, Theorem 4.3.2 (i)]

$$\int_{\mathbb{R}^n} p(x)\overline{q(x)}dx = \int_{\mathbb{R}^n} \hat{p}(\xi)\overline{\hat{q}(\xi)}d\xi \quad \textit{for all } p, q \in L^2(\mathbb{R}^n),$$

Theorem A.2.4. [28, Theorem 4.3.2 (iv)]

$$\mathcal{F}\mathcal{F}^* p = \mathcal{F}^*\mathcal{F}p = p \quad \textit{for all } p \in L^2(\mathbb{R}^n).$$

For Sobolev space $H^1(\mathbb{R}^n)$, we have a lemma:

Theorem A.2.5. [28, Theorem 5.8.8]

$$p \in H^1(\mathbb{R}^n) \Leftrightarrow \int_{\mathbb{R}^n} |\hat{p}(\xi)|(1 + |\xi|^2)d\xi < +\infty$$

and we have

$$\|p\|_{H^1(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |\hat{p}(\xi)|(1 + |\xi|^2)d\xi\right)^{1/2} \quad \textit{for all } p \in H^1(\mathbb{R}^n).$$

By using Fourier transform, we prove Theorem 2.2.13 when $\Omega = \mathbb{R}^n$.

Lemma A.2.6. *There exists a constant $c > 0$ such that*

$$\|p\|_{L^2(\mathbb{R}^n)} \leq \sqrt{n}(\|p\|_{H^{-1}(\mathbb{R}^n)} + \|\nabla p\|_{H^{-1}(\mathbb{R}^n)^n})$$

for all $p \in L^2(\mathbb{R}^n)$.

Proof. Using Fourier transform, we get

$$\begin{aligned} \|p\|^2_{L^2(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} |\hat{p}(\xi)|^2 d\xi \\ &= \int_{\mathbb{R}^n} |\hat{p}(\xi)|^2 (1+|\xi|^2)(1+|\xi|^2)^{-1} d\xi \\ &= \int_{\mathbb{R}^n} |\hat{p}(\xi)|^2 (1+|\xi|^2)^{-1} d\xi + \sum_{j=1}^{n} \int_{\mathbb{R}^n} |\xi_j \hat{p}(\xi)|^2 (1+|\xi|^2)^{-1} d\xi. \end{aligned} \tag{A.2.1}$$

Here it follows that

$$\|p\|^2_{H^{-1}(\mathbb{R}^n)} = \sup_{0\neq\varphi\in H^1(\mathbb{R}^n)} \frac{\left(\int_{\mathbb{R}^n} p\varphi\, dx\right)^2}{\|\varphi\|^2_{H^1(\mathbb{R}^n)}} = \sup_{0\neq\varphi\in H^1(\mathbb{R}^n)} \frac{\left(\int_{\mathbb{R}^n} \hat{p}\ \overline{\hat{\varphi}}\, d\xi\right)^2}{\int_{\mathbb{R}^n} |\hat{\varphi}(\xi)|^2 (1+|\xi|^2) d\xi}. \tag{A.2.2}$$

Putting $\varphi := \mathcal{F}^{-1}[(1+|\xi|^2)^{-1}\hat{p}(\xi)]$, we obtain

$$\int_{\mathbb{R}^n} |\hat{\varphi}(\xi)|^2 (1+|\xi|^2) d\xi = \int_{\mathbb{R}^n} |\hat{p}(\xi)|^2 (1+|\xi|^2)^{-1} d\xi \le \int_{\mathbb{R}^n} |\hat{p}(\xi)|^2 d\xi < +\infty.$$

By Lemma A.2.5, $\varphi \in H^1(\mathbb{R}^n)$, hence, it follows from (A.2.2) that

$$\|p\|^2_{H^{-1}(\mathbb{R}^n)} \ge \frac{\left(\int_{\mathbb{R}^n} \hat{p}(\xi)\overline{(1+|\xi|^2)^{-1}\hat{p}(\xi)} d\xi\right)^2}{\int_{\mathbb{R}^n} |(1+|\xi|^2)^{-1}\hat{p}(\xi)|^2 (1+|\xi|^2) d\xi} = \int_{\mathbb{R}^n} |\hat{p}(\xi)|^2 (1+|\xi|^2)^{-1} d\xi. \tag{A.2.3}$$

On the other hand, for $j = 1, \cdots, n$, it holds that

$$\left\|\frac{\partial p}{\partial x_j}\right\|^2_{H^{-1}(\mathbb{R}^n)} = \sup_{0\neq\varphi\in H^1(\mathbb{R}^n)} \frac{\left(\int_{\mathbb{R}^n} \frac{\partial p}{\partial x_j}\varphi\, dx\right)^2}{\|\varphi\|^2_{H^1(\mathbb{R}^n)}} = \sup_{0\neq\varphi\in H^1(\mathbb{R}^n)} \frac{\left(\int_{\mathbb{R}^n} i\xi_j \hat{p}(\xi)\overline{\hat{\varphi}(\xi)} d\xi\right)^2}{\int_{\mathbb{R}^n} |\hat{\varphi}(\xi)|^2 (1+|\xi|^2) d\xi}$$

Putting $\varphi := \mathcal{F}^{-1}[i\xi_j(1+|\xi|^2)^{-1}\hat{p}(\xi)]$, we have

$$\|\varphi\|^2_{H^1(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |\hat{\varphi}(\xi)|^2 (1+|\xi|^2) d\xi = \int_{\mathbb{R}^n} |\hat{p}(\xi)|^2 \frac{\xi_j^2}{1+|\xi|^2} d\xi \le \int_{\mathbb{R}^n} |\hat{p}(\xi)|^2 d\xi < +\infty.$$

By Lemma A.2.5, we obtain $\varphi \in H^1(\mathbb{R}^n)$, hence,

$$\left\|\frac{\partial p}{\partial x_j}\right\|^2_{H^{-1}(\mathbb{R}^n)} \ge \frac{\left(\int_{\mathbb{R}^n} i\xi_j \hat{p}(\xi)\overline{i\xi_j(1+|\xi|^2)^{-1}\hat{p}(\xi)} d\xi\right)^2}{\int_{\mathbb{R}^n} |i\xi_j(1+|\xi|^2)^{-1}\hat{p}(\xi)|^2 (1+|\xi|^2) d\xi} = \int_{\mathbb{R}^n} |\xi_j \hat{p}(\xi)|^2 (1+|\xi|^2)^{-1} d\xi. \tag{A.2.4}$$

By (A.2.1), (A.2.3) and (A.2.4),

$$\|p\|^2_{L^2(\mathbb{R}^n)} \leq \|p\|^2_{H^{-1}(\mathbb{R}^n)} + \sum_{j=1}^{n} \left\| \frac{\partial p}{\partial x_j} \right\|^2_{H^{-1}(\mathbb{R}^n)}.$$

By Lemma A.2.1, we obtain the result;

$$\begin{aligned} \|p\|_{L^2(\mathbb{R}^n)} &\leq \sqrt{\|p\|^2_{H^{-1}(\mathbb{R}^n)} + \sum_{j=1}^{n} \left\| \frac{\partial p}{\partial x_j} \right\|^2_{H^{-1}(\mathbb{R}^n)}} \\ &\leq \|p\|_{H^{-1}(\mathbb{R}^n)} + \sum_{j=1}^{n} \left\| \frac{\partial p}{\partial x_j} \right\|_{H^{-1}(\mathbb{R}^n)} \\ &\leq \sqrt{n}\left(\|p\|_{H^{-1}(\mathbb{R}^n)} + \|\nabla p\|_{H^{-1}(\mathbb{R}^n)^n}\right). \end{aligned}$$

□

Using Lemma A.2.6, we prove the following lemma.

Lemma A.2.7. *Let $\Omega \subset \mathbb{R}^n$ be an open set and let a bounded open set $U \subset \Omega$ satisfy that $\overline{U} \subset \Omega$. There exists a constant $c = c(\Omega, U) > 0$ depending only on U such that*

$$\|p\|_{L^2(\Omega)} \leq c(\|p\|_{H^{-1}(\Omega)} + \|\nabla p\|_{H^{-1}(\Omega)^n}) \qquad \text{for all } p \in C_0^\infty(U)\ (\subset C_0^\infty(\Omega)). \tag{A.2.5}$$

Proof. For $p \in C_0^\infty(\Omega)$, we set

$$\tilde{p}(x) = \begin{cases} p(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \mathbb{R}^n \setminus \Omega. \end{cases}$$

It is easy to see that $\tilde{p} \in L^2(\mathbb{R}^n)$ and $\|\tilde{p}\|_{L^2(\mathbb{R}^n)} = \|p\|_{L^2(\Omega)}$. One can make a function $\rho \in C_0^\infty(\mathbb{R}^n)$ such that

$$\rho(x) = 0\ (x \notin \Omega), \qquad 0 \leq \rho(x) \leq 1\ (x \in \Omega), \qquad \rho(x) = 1\ (x \in U).$$

Since $\rho \in H^1(\Omega) \subset H^1(\mathbb{R}^n)$, there exists a constant $c > 0$ such that

$$\|\rho\psi\|_{H^1(\Omega)} \leq c\|\rho\|_{H^1(\Omega)}\|\psi\|_{H^1(\mathbb{R}^n)} \qquad \text{for all } \psi \in H^1(\mathbb{R}^n).$$

Thus it follows that for $p \in C_0^\infty(U)$ $(\subset C_0^\infty(\Omega))$

$$\begin{aligned} \|\tilde{p}\|_{H^{-1}(\mathbb{R}^n)} &= \sup_{0 \neq \psi \in H^1(\mathbb{R}^n)} \frac{\displaystyle\int_{\mathbb{R}^n} \tilde{p}\psi\, dx}{\|\psi\|_{H^1(\mathbb{R}^n)}} \\ &= \sup_{0 \neq \psi \in H^1(\mathbb{R}^n)} \frac{\displaystyle\int_{\Omega} p\rho\psi\, dx}{\|\psi\|_{H^1(\mathbb{R}^n)}} \qquad (\text{by } \operatorname{supp}(p) \subset U) \\ &\leq c\|\rho\|_{H^1(\Omega)} \sup_{0 \neq \psi \in H^1(\mathbb{R}^n)} \frac{\displaystyle\int_{\Omega} p\rho\psi\, dx}{\|\rho\psi\|_{H^1(\Omega)}} \\ &\leq c\|\rho\|_{H^1(\Omega)} \sup_{0 \neq \tilde{\psi} \in H_0^1(\Omega)} \frac{\displaystyle\int_{\Omega} p\tilde{\psi}\, dx}{\|\tilde{\psi}\|_{H^1(\Omega)}} \\ &\leq c\|\rho\|_{H^1(\Omega)}\|p\|_{H^{-1}(\Omega)}, \end{aligned}$$

and

$$\|\nabla\tilde{p}\|_{H^{-1}(\mathbb{R}^n)^n} \le c\|\rho\|_{H^1(\Omega)}\|\nabla\tilde{p}\|_{H^{-1}(\Omega)^n}.$$

By Lemma A.2.6, we obtain the result;

$$\begin{aligned}\|p\|_{L^2(\Omega)} &= \|\tilde{p}\|_{L^2(\mathbb{R}^n)}\\ &\le \sqrt{n}(\|\tilde{p}\|_{H^{-1}(\mathbb{R}^n)} + \|\nabla\tilde{p}\|_{H^{-1}(\mathbb{R}^n)^n})\\ &\le c\sqrt{n}\|\rho\|_{H^1(\Omega)}(\|p\|_{H^{-1}(\Omega)} + \|\nabla p\|_{H^{-1}(\Omega)^n}).\end{aligned}$$

□

A.2.3 Extension of $K(\alpha,\beta)$ to $M(\alpha,\beta)$

We shall consider Lemma A.2.7 on the subset $K(\alpha,\beta)$ of the half space $\{(x_1,\cdots,x_n)\in\mathbb{R}^n \mid x_n>0\}$. We prepare the following lemma.

Lemma A.2.8. *Let $\alpha,\beta>0$, $K=K(\alpha,\beta)$ and $M=M(\alpha,\beta)$. There exist an extension operator $C_0^\infty(K)\ni p\mapsto\tilde{p}\in C_0^\infty(M)$ and a constant $c>0$ independent of α and β such that*

$$\begin{aligned}\|p\|_{L^2(K)} &\le \|\tilde{p}\|_{L^2(M)},\\ \|\tilde{p}\|_{H^{-1}(M)} &\le c\|p\|_{H^{-1}(K)},\\ \|\nabla\tilde{p}\|_{H^{-1}(M)^n} &\le c\|\nabla p\|_{H^{-1}(K)^n},\end{aligned}$$

for all $p\in C_0^\infty(K)$, and if $p\in C_0^\infty(K(\alpha',\beta'))$ $(0<\alpha'<\alpha, 0<\beta'<\beta)$ then $\tilde{p}\in C_0^\infty(M(\alpha',\beta'))$.

Proof. Let λ_1,λ_2 are solutions of the linear system:

$$\lambda_1+\lambda_2=1,\qquad \lambda_1+2\lambda_2=-1, \tag{A.2.6}$$

(i.e., $\lambda_1=3,\lambda_2=-2$). For $p\in C_0^\infty(K)$, we define $\tilde{p}\in C_0^\infty(M)$ as follows:

$$\tilde{p}(x',x_n)=\begin{cases} p(x',x_n) & \text{if } 0<x_n<\beta,\\ 0 & \text{if } x_n=0,\\ \lambda_1 p(x',-x_n)+\lambda_2 p\left(x',-\dfrac{x_n}{2}\right) & \text{if } -\beta<x_n<0,\end{cases}$$

for $x'\in B(\alpha)$ and $x_n\in(-\beta,\beta)$. It is easy to see that

$$\|p\|_{L^2(K)}\le\|\tilde{p}\|_{L^2(M)}$$

and if $p\in C_0^\infty(K(\alpha',\beta'))$ then $\tilde{p}\in C_0^\infty(M(\alpha',\beta'))$. Moreover, if $p\in C_0^\infty(K(\alpha',\beta'))$ then $\mathrm{supp}(\tilde{p})\subset M(\alpha',\beta')$.

For $p\in C_0^\infty(K), v\in C_0^\infty(M)$ and $i=1,2,\cdots,n-1$, we have

$$\begin{aligned}&\int_M \frac{\partial\tilde{p}}{\partial x_i}(x',x_n)v(x',x_n)dx\\ =&\int_K \frac{\partial p}{\partial x_i}(x',x_n)v(x',x_n)dx+\int_{K^-}\left(\lambda_1\frac{\partial p}{\partial x_i}(x',-x_n)+\lambda_2\frac{\partial p}{\partial x_i}\left(x',-\frac{x_n}{2}\right)\right)v(x',x_n)dx\\ =&\int_K \frac{\partial p}{\partial x_i}(x',x_n)\left(v(x',x_n)+\lambda_1 v(x',-x_n)+2\lambda_2 v(x',-2x_n)\right)dx\\ =&\int_K \frac{\partial p}{\partial x_i}(x',x_n)P_1v(x',x_n)dx\end{aligned}$$

where $K^- := B(\alpha)\times(-\beta,0)$ and $P_1v(x',x_n) := v(x',x_n)+\lambda_1 v(x',-x_n)+2\lambda_2 v(x',-2x_n)$. By (A.2.6), we obtain that for $x'\in B(\alpha)$

$$P_1v(x',0) = v(x',0)+\lambda_1 v(x',0)+2\lambda_2 v(x',0) = (1+\lambda_1+2\lambda_2)v(x',0) = 0.$$

It implies that $P_1 v \in H_0^1(K)$. Moreover, it follows that

$$\|P_1 v\|_{H^1(K)} \le c_1 \|v\|_{H^1(M)} \qquad \text{for all } v\in C_0^\infty(M).$$

Thus we can extend P_1 as a bounded operator on $H_0^1(M)$, which satisfies that

$$\int_M \frac{\partial \tilde{p}}{\partial x_i}(x)v(x)dx = \int_K \frac{\partial p}{\partial x_i}(x)P_1v(x)dx$$

for all $v\in H_0^1(M)$ and $i=1,2,\cdots,n-1$. The same argument works for $\int_M \tilde{p}vdx$. It implies that

$$\|\tilde{p}\|_{H^{-1}(M)} \le c_1\|p\|_{H^{-1}(K)}, \qquad \left\|\frac{\partial \tilde{p}}{\partial x_i}\right\|_{H^{-1}(M)} \le c_1 \left\|\frac{\partial p}{\partial x_i}\right\|_{H^{-1}(K)}$$

for $i=1,2,\cdots,n-1$. For $v\in C_0^\infty(M)$, we have

$$\begin{aligned}
&\int_M \frac{\partial \tilde{p}}{\partial x_n}(x',x_n)v(x',x_n)dx\\
=\ &\int_K \frac{\partial p}{\partial x_n}(x',x_n)v(x',x_n)dx + \int_{K^-}\left(-\lambda_1\frac{\partial p}{\partial x_n}(x',-x_n) - \frac{\lambda_2}{2}\frac{\partial p}{\partial x_n}\left(x',-\frac{x_n}{2}\right)\right)v(x',x_n)dx\\
=\ &\int_K \frac{\partial p}{\partial x_n}(x',x_n)\left(v(x',x_n)-\lambda_1 v(x',-x_n)-\lambda_2 v(x',-2x_n)\right)dx\\
=\ &\int_K \frac{\partial p}{\partial x_n}(x',x_n)P_2v(x',x_n)dx
\end{aligned}$$

where $P_2v(x',x_n) := v(x',x_n)-\lambda_1 v(x',-x_n)-\lambda_2 v(x',-2x_n)$. By (A.2.6), we obtain that for $x'\in B(\alpha)$

$$P_2v(x',0) = v(x',0)-\lambda_1 v(x',0)-\lambda_2 v(x',0) = (1-\lambda_1-\lambda_2)v(x',0) = 0.$$

It implies that $P_2 v\in H_0^1(K)$. Moreover, it follows that

$$\|P_2 v\|_{H^1(K)} \le c_2 \|v\|_{H^1(M)} \qquad \text{for all } v\in C_0^\infty(M).$$

Thus we can extend P_2 as a bounded operator on $H_0^1(M)$, which satisfies that

$$\int_M \frac{\partial \tilde{p}}{\partial x_n}(x)v(x)dx = \int_K \frac{\partial p}{\partial x_n}(x)P_2v(x)dx$$

for all $v\in H_0^1(M)$. It implies

$$\left\|\frac{\partial \tilde{p}}{\partial x_n}\right\|_{H^{-1}(M)} \le c_2 \left\|\frac{\partial p}{\partial x_n}\right\|_{H^{-1}(K)}.$$

□

By Lemmas A.2.7 and A.2.8, the following lemma holds.

Lemma A.2.9. *Let* $0 < \alpha' < \alpha$, $0 < \beta' < \beta$ *and* $K = K(\alpha,\beta)$. *There exists a constant* $c = c(\alpha,\beta,\alpha',\beta') > 0$ *such that*

$$\|p\|_{L^2(K)} \le c(\|p\|_{H^{-1}(K)} + \|\nabla p\|_{H^{-1}(K)^n}) \qquad \text{for all } p \in C_0^\infty(K(\alpha',\beta')).$$

Proof. By Lemma A.2.8, we have $\tilde{p} \in C_0^\infty(M(\alpha',\beta'))$. Here, it is immediate to check that $\overline{M(\alpha',\beta')} \subset M = M(\alpha,\beta)$. By Lemmas A.2.7 and A.2.8, it follows that

$$\|p\|_{L^2(K)} \le \|\tilde{p}\|_{L^2(M)} \le c_1(\|\tilde{p}\|_{H^{-1}(M)} + \|\nabla\tilde{p}\|_{H^{-1}(M)^n}) \le c_2(\|p\|_{H^{-1}(K)} + \|\nabla p\|_{H^{-1}(K)^n}).$$

□

A.2.4 Local Lipschitz boundary

We shall consider a neighborhood of the boundary Γ. For $0 < \alpha' \le \alpha$ and $0 < \beta' \le \beta$, let a function $g : \overline{\Delta(\alpha)} \to \mathbb{R}$ be Lipschitz continuous and let

$$U_g^+(\alpha',\beta') := \{x = (x', x_n) \in \mathbb{R}^n \mid x' \in B(\alpha'), g(x') < x_n < g(x') + \beta'\}.$$

In this subsection, we make a mapping $K(\alpha',\beta') \to U_g^+(\alpha',\beta)$ and extend Lemma A.2.9 to $U_g^+(\alpha,\beta)$. The simple mapping $K(\alpha,\beta) \ni (y', y_n) \mapsto (y', g(y') + y_n) \in U_g^+(\alpha,\beta)$ is not smooth enough to prove the lemma if g is not sufficient smooth, thus we define a mapping $K(\alpha',\beta') \to U_g^+(\alpha',\beta)$ using mollifiers according to the Nečas's proof.

For $f \in L^1(B(h))$ and $g \in L^\infty(B(\alpha))$ with $0 < \alpha' < \alpha$ and $h = \alpha - \alpha'$, one can define the convolution product of f and g;

$$(f * g)(x') := \int_{B(h)} f(y')g(x' - y')dy' \quad \text{for a.e. } x' \in B(\alpha').$$

It is easy to see that

$$\|f * g\|_{L^\infty(B(\alpha'))} \le \|f\|_{L^1(B(h))}\|g\|_{L^\infty(B(\alpha))}. \tag{A.2.7}$$

Let $\rho_1 \in C_0^\infty(\mathbb{R}^{n-1})$ satisfy that $\text{supp}(\rho_1) \subset \overline{B(1)}$, $\rho_1 \ge 0$ on $\mathbb{R}^{n-1}$ and $\int_{B(1)} \rho_1 = 1$; for example the function

$$\rho_1(x') := \begin{cases} P_0 \exp\left(\dfrac{1}{|x'|^2 - 1}\right) & \text{if } |x'| < 1, \\ 0 & \text{if } |x'| \ge 1, \end{cases}$$

where $P_0 = 1/\int_{\mathbb{R}^{n-1}} \exp(\frac{1}{|x'|^2-1})dx'$. For $h > 0$, we set

$$\rho_h(x') := \frac{1}{h^{n-1}}\rho_1\left(\frac{x'}{h}\right)$$

for $x' \in \mathbb{R}^{n-1}$.

We show some properties of the mollifiers.

Proposition A.2.10. *For $h > 0$, we have*

$$\int_{B(h)} \rho_h \, dx' = 1. \tag{A.2.8}$$

Furthermore, there exists a constant $c > 0$ independent of h such that

$$\left|\frac{\partial \rho_h}{\partial x_i}(x')\right| < \frac{c}{h^n}, \quad \left|\frac{\partial \rho_h}{\partial h}(x')\right| < \frac{c}{h^n} \qquad \text{for all } h > 0, x' \in \mathbb{R}^{n-1}, i = 1, \cdots, n-1.$$

Proof. We compute

$$\int_{B(h)} \rho_h \, dx' = \frac{1}{h^{n-1}} \int_{B(h)} \rho_1 \left(\frac{x'}{h}\right) dx' = \frac{1}{h^{n-1}} \int_{B(1)} \rho_1(x') h^{n-1} dx' = \int_{B(1)} \rho_1(x') dx' = 1.$$

For $x' = (x_1, \cdots, x_{n-1}) \in B(h)$ and $i = 1, \cdots, n-1$, we have

$$\begin{aligned}
\frac{\partial \rho_h}{\partial x_i}(x') &= \frac{\partial}{\partial x_i}\left(\frac{1}{h^{n-1}}\rho_1\left(\frac{x'}{h}\right)\right) = \frac{1}{h^n}\frac{\partial \rho_1}{\partial x_i}\left(\frac{x'}{h}\right), \\
\frac{\partial \rho_h}{\partial h}(x') &= -\frac{n-1}{h^n}\rho_1\left(\frac{x'}{h}\right) + \frac{1}{h^{n-1}}\sum_{j=1}^{n-1}\left(-\frac{x_j}{h^2}\right)\frac{\partial \rho_1}{\partial x_j}\left(\frac{x'}{h}\right) \\
&= \frac{1}{h^n}\left\{(1-n)\rho_1\left(\frac{x'}{h}\right) - \sum_{j=1}^{n-1}\frac{x_j}{h}\frac{\partial \rho_1}{\partial x_j}\left(\frac{x'}{h}\right)\right\}.
\end{aligned}$$

Since $\rho_1 \in C^\infty(\mathbb{R}^{n-1})$ and $\mathrm{supp}(\rho_1) \subset \overline{B(1)}$, functions $\frac{\partial \rho_1}{\partial x_i}\left(\frac{x'}{h}\right)$ and $\rho_1\left(\frac{x'}{h}\right)$ are bounded on $B(h)$. Therefore, there exists a constant $c > 0$ such that

$$\left|\frac{\partial \rho_h}{\partial x_i}(x')\right| < \frac{c}{h^n}, \quad \left|\frac{\partial \rho_h}{\partial h}(x')\right| < \frac{c}{h^n} \qquad \text{for all } h > 0, x' \in \mathbb{R}^{n-1}, i = 1, \cdots, n-1.$$

□

Lemma A.2.11. *Let $0 < \alpha' < \alpha$. For all Lipschitz continuous function $g : \overline{B(\alpha)} \to \mathbb{R}$, there exists a constant $M = M(\alpha, g) > 0$ such that*

$$\left|\frac{\partial}{\partial x_i}(\rho_h * g)(x')\right| < M, \quad \left|\frac{\partial}{\partial h}(\rho_h * g)(x')\right| < M$$

for all $0 < h < \alpha - \alpha'$ and $x' \in B(\alpha')$.

Proof. For a Lipschitz constant c_g for g, it follows that

$$\left|\frac{\partial g}{\partial x_i}(x')\right| \le c_g \qquad \text{a.e. for } x' \in B(\alpha)$$

with $i = 1, \cdots, n-1$. For $h > 0, x' \in B(\alpha')$ and $i = 1, \cdots, n-1$,

$$\begin{aligned}
\left|\frac{\partial}{\partial x_i}(\rho_h * g)(x')\right| &= \left|\frac{\partial}{\partial x_i}\int_{B(h)} \rho_h(y') g(x' - y') dy'\right| \\
&= \left|\int_{B(h)} \rho_h(y') \frac{\partial g}{\partial x_i}(x' - y') dy'\right| \\
&\le \int_{B(h)} \left|\rho_h(y') \frac{\partial g}{\partial x_i}(x' - y')\right| dy' \\
&\le c_g \int_{B(h)} |\rho_h(y')| dy' = c_g,
\end{aligned}$$

$$\begin{aligned}
\left|\frac{\partial}{\partial h}(\rho_h * g)(x')\right| &= \left|\frac{\partial}{\partial h}\int_{B(h)} \rho_h(y')g(x'-y')dy'\right| \\
&= \left|\frac{\partial}{\partial h}\int_{B(h)} \frac{1}{h^{n-1}}\rho_1\left(\frac{y'}{h}\right) g(x'-y')dy'\right| \\
&= \left|\frac{\partial}{\partial h}\int_{B(1)} \frac{1}{h^{n-1}}\rho_1(z')g(x'-hz')h^{n-1}dz'\right| \\
&= \left|\int_{B(1)} \frac{\partial}{\partial h}\{\rho_1(z')g(x'-hz')\}dz'\right| \\
&= \left|\int_{B(1)} \rho_1(z')\sum_{j=1}^{n-1} z_j\frac{\partial g}{\partial z_j}(x'-hz')dz'\right| \\
&\le \int_{B(1)} |\rho_1(z')|\sum_{j=1}^{n-1}|z_j|\left|\frac{\partial g}{\partial z_j}(x'-hz')\right|dz' \\
&\le (n-1)c_g\int_{B(1)}|\rho_1(z')|dz' = (n-1)c_g.
\end{aligned}$$

□

Lemma A.2.12. *Let $0 < \alpha' < \alpha$. For all Lipschitz continuous function $g : \overline{B(\alpha)} \to \mathbb{R}$, there exists a constant $c = c(\alpha, g) > 0$ such that*

$$\begin{aligned}
\left\|\frac{\partial^2}{\partial x_i\partial x_j}(\rho_h * g)\right\|_{L^\infty(B(\alpha'))} &\le \frac{c}{h}, \\
\left\|\frac{\partial^2}{\partial x_i\partial h}(\rho_h * g)\right\|_{L^\infty(B(\alpha'))} &\le \frac{c}{h}, \\
\left\|\frac{\partial^2}{\partial h^2}(\rho_h * g)\right\|_{L^\infty(B(\alpha'))} &\le \frac{c}{h}
\end{aligned}$$

for all $i, j = 1, \cdots, n-1$ and $0 < h < \alpha - \alpha'$.

Proof. For $i, j = 1, \cdots, n-1$ and $0 < h < \alpha - \alpha'$, by (A.2.7), we obtain

$$\begin{aligned}
\left\|\frac{\partial^2}{\partial x_i\partial x_j}(\rho_h * g)\right\|_{L^\infty(B(\alpha'))} &= \frac{1}{h}\left\|\left(h\frac{\partial\rho_h}{\partial x_j}\right) * \frac{\partial g}{\partial x_i}\right\|_{L^\infty(B(\alpha'))} \\
&\le \frac{1}{h}\left\|h\frac{\partial\rho_h}{\partial x_j}\right\|_{L^1(B(h))}\left\|\frac{\partial g}{\partial x_i}\right\|_{L^\infty(B(\alpha))} \\
&\le \frac{c_g}{h}\left\|h\frac{\partial\rho_h}{\partial x_j}\right\|_{L^1(B(h))}, \\
\left\|\frac{\partial^2}{\partial x_i\partial h}(\rho_h * f)\right\|_{L^\infty(B(\alpha'))} &\le \frac{1}{h}\left\|h\frac{\partial\rho_h}{\partial h}\right\|_{L^1(B(h))}\left\|\frac{\partial g}{\partial x_i}\right\|_{L^\infty(B(\alpha))} \\
&\le \frac{c_g}{h}\left\|h\frac{\partial\rho_h}{\partial x_j}\right\|_{L^1(B(h))}.
\end{aligned}$$

By Proposition A.2.10, it follows that

$$\left\|h\frac{\partial\rho_h}{\partial x_j}\right\|_{L^1(B(h))} = h\int_{B(h)}\left|\frac{\partial\rho_h}{\partial x_j}(x')\right|dx \le \frac{c}{h^{n-1}}\int_{B(h)}dx = c|B(1)|,$$

where $|B(1)|$ is the volume of an $(n-1)$-dimensional unit ball. Hence, we get

$$\begin{aligned}\left\|\frac{\partial^2}{\partial x_i \partial x_j}(\rho_h * g)\right\|_{L^\infty(B(\alpha'))} &\leq \frac{cc_g|B(1)|}{h},\\ \left\|\frac{\partial^2}{\partial x_i \partial h}(\rho_h * g)\right\|_{L^\infty(B(\alpha'))} &\leq \frac{cc_g|B(1)|}{h}.\end{aligned}$$

For all $x' \in B(\alpha')$, we have

$$\frac{\partial}{\partial h}(\rho_h * g)(x') = \int_{B(1)} \rho_1(z') \sum_{k=1}^{n-1} z_i \frac{\partial g}{\partial z_k}(x' - hz')dz' = \sum_{k=1}^{n-1} \int_{B(h)} \frac{y_k}{h} \rho_h(y') \frac{\partial g}{\partial y_k}(x' - y')dy'.$$

Thus, it holds that

$$\frac{\partial^2}{\partial h^2}(\rho_h * g)(x') = \frac{1}{h}\sum_{k=1}^{n-1} \int_{B(h)} \left(-\frac{y_k}{h}\rho_h(y') + \frac{y_k}{h} h \frac{\partial \rho_h}{\partial h}(y')\right) \frac{\partial g}{\partial y_k}(x' - y')dy'.$$

By (A.2.7) and Proposition A.2.10, it follows that

$$\begin{aligned}&\left\|\frac{\partial^2}{\partial h^2}(\rho_h * g)\right\|_{L^\infty(B(\alpha'))}\\ \leq\ & \frac{1}{h}\sum_{k=1}^{n-1} \left\|\frac{\partial g}{\partial y_k}\right\|_{L^\infty(B(\alpha))} \left\{\int_{B(h)} \left|\frac{y_k}{h}\rho_h(y')\right| dy' + \int_{B(h)} \left|\frac{y_k}{h} h \frac{\partial \rho_h}{\partial h}(y')\right| dy'\right\}\\ \leq\ & \frac{c_g}{h}\sum_{k=1}^{n-1} \left\{\int_{B(h)} |\rho_h(y')|dy' + \int_{B(h)} \frac{c}{h^{n-1}}dy'\right\}\\ =\ & \frac{(n-1)c_g(1 + c|B(1)|)}{h}.\end{aligned}$$

□

For $0 < \alpha' < \alpha$ and $0 < \beta' < \beta$, we make a mapping $K(\alpha', \beta') \to U_g^+(\alpha', \beta)$ using the mollifiers according to the Nečas's proof.

Lemma A.2.13. *Let $0 < \alpha' < \alpha$ and $0 < \beta' < \beta$. For all Lipschitz continuous function $g : \overline{B(\alpha)} \to \mathbb{R}$, there exist two constants $\delta = \delta(\alpha, \beta, \alpha', \beta', g), M = M(\alpha, g) > 0$ such that the mapping $T : K(\alpha', \beta') \to U_g^+(\alpha', \beta)$;*

$$T(y) := (y', G(y', \delta y_n) + (1 + \delta M)y_n) \quad \textit{for } y = (y', y_n) \in K(\alpha', \beta'),$$

where

$$G(y', h) := (\rho_h * g)(y') = \int_{B(h)} \rho_h(\xi')g(y' - \xi')d\xi',$$

satisfies the following statements:

- *The mapping $T : K(\alpha', \beta') \to V(\alpha', \beta') := T(K(\alpha', \beta'))$ is C^∞-diffeomorphism.*
- *$U_g^+(\alpha', \beta') \subset V(\alpha', \beta') \subset U_g^+(\alpha', \beta)$ (Fig. A.1).*

- *Let* $(x_1, \cdots, x_n) = T(y_1, \cdots, y_n)$ *for* $(y_1, \cdots, y_n) \in K(\alpha', \beta')$*. The Jacobian* $\mathrm{Jac}(T) := \det(\frac{\partial x_i}{\partial y_j})_{1\le i,j\le n}$ *satisfies*

$$1 \le \mathrm{Jac}(T) \le 1 + 2\delta M.$$

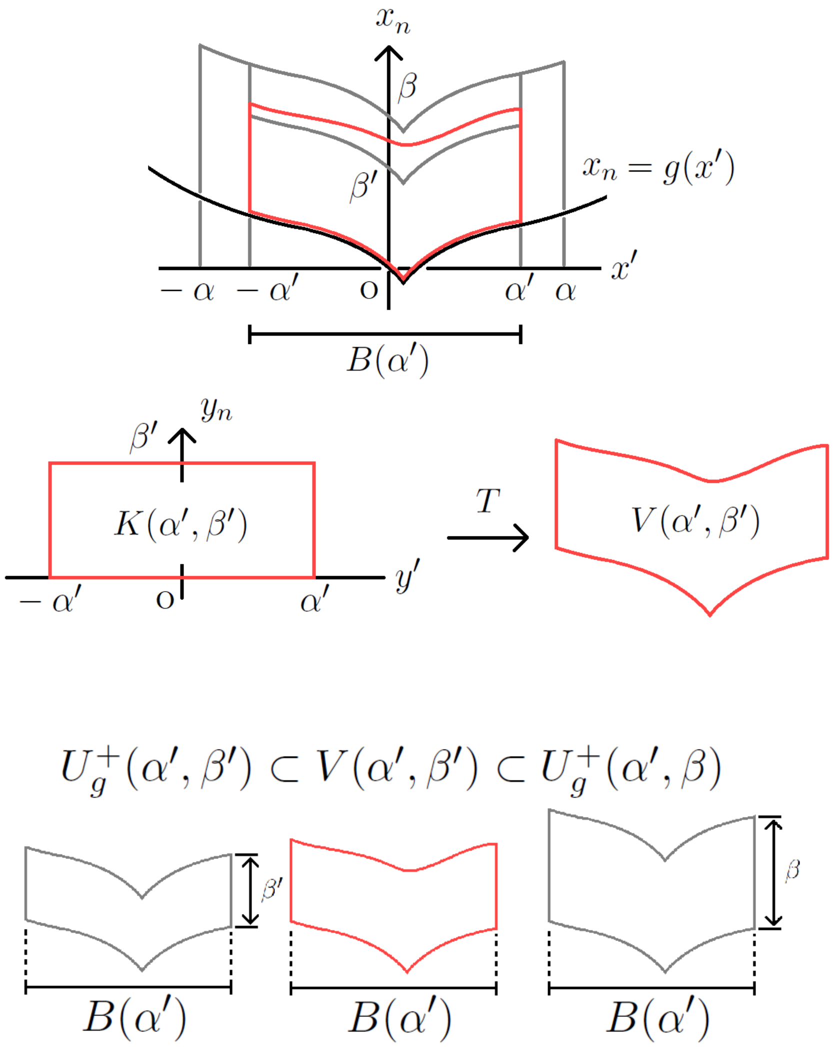

Figure A.1: Sketch of the mapping T

Proof. Since $\rho_h \in C_0^\infty(\mathbb{R}^{n-1})$ and g is a Lipschitz continuous function, the function G is infinitely differentiable on $B(\alpha') \times (0, \alpha - \alpha')$. By Lemma A.2.11, there exists a constant $M > 0$ such that

$$-M \le \frac{\partial G}{\partial h}(y', h) \le M \tag{A.2.9}$$

for all $y' \in B(\alpha')$ and $0 < h < \alpha - \alpha'$. Let

$$\delta := \min\left\{\frac{1}{2M}\left(\frac{\beta}{\beta'} - 1\right), \frac{\alpha - \alpha'}{\beta'}\right\}.$$

It is easy to see that $\delta\beta' \leq \alpha - \alpha'$, hence, T is well-defined. Since G is infinitely differentiable on $B(\alpha') \times (0, \alpha - \alpha')$, T is also infinitely differentiable on $K(\alpha', \beta')$.

For $y = (y', y_n) \in K(\alpha', \beta')$, by equation (A.2.9), it follows that

$$1 = -\delta M + 1 + \delta M \leq \frac{\partial G}{\partial y_n}(y', \delta y_n) + 1 + \delta M \leq \delta M + 1 + \delta M = 1 + 2\delta M.$$

Hence, $y_n \mapsto T(y', y_n)$ is strictly increasing for all $y' \in B(\alpha')$;

$$1 \leq \frac{\partial}{\partial y_n}\{G(y', \delta y_n) + (1 + \delta M)y_n\} \leq 1 + 2\delta M \quad \text{for all } (y', y_n) \in K(\alpha', \beta'). \tag{A.2.10}$$

Therefore, T is a bijective mapping from $K(\alpha', \beta')$ to $V(\alpha', \beta') = T(K(\alpha', \beta'))$. Moreover, $T : K(\alpha', \beta') \to V(\alpha', \beta')$ is C^∞-diffeomorphism.

Integrating with respect to y_n from 0 to β', we get

$$\beta' \leq G(y', \delta\beta') - g(y') + (1 + \delta M)\beta' \leq (1 + 2\delta M)\beta' \leq \beta$$

for all $y' \in B(\alpha')$. Hence, we obtain $U_g^+(\alpha', \beta') \subset V(\alpha', \beta') \subset U_g^+(\alpha', \beta)$.

Let $(x_1, \cdots, x_n) = T(y_1, \cdots, y_n)$ for $(y', y_n) = (y_1, \cdots, y_n) \in K(\alpha', \beta')$. For $i = 1, \cdots, n-1$ and $j = 1, \cdots, n$, it follows that

$$\begin{aligned} \frac{\partial x_i}{\partial y_j}(y', y_n) &= \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases} \\ \frac{\partial x_n}{\partial y_i}(y', y_n) &= \frac{\partial G}{\partial y_i}(y', \delta y_n), \\ \frac{\partial x_n}{\partial y_n}(y', y_n) &= \frac{\partial G}{\partial y_n}(y', \delta y_n) + 1 + \delta M. \end{aligned} \tag{A.2.11}$$

Thus, the Jacobian of T;

$$\mathrm{Jac}(T)(y', y_n) = \frac{\partial G}{\partial y_n}(y', \delta y_n) + 1 + \delta M$$

satisfies

$$0 < 1 \leq \mathrm{Jac}(T)(y', y_n) \leq 1 + 2\delta M.$$

□

We recall the following theorem.

Theorem A.2.14. [66, Lemma 3.2] *Let $U, V \subset \mathbb{R}^n$ be two bounded open sets. If a Lipschitz continuous mapping $\Phi : U \to V$ is bijective and satisfies that Φ^{-1} is also a Lipschitz continuous mapping, then the mapping $H^1(V) \ni f \mapsto f \circ \Phi \in H^1(U)$ is a homeomorphism between Banach spaces. Furthermore, the mapping $H_0^1(V) \ni f \mapsto f \circ \Phi \in H_0^1(U)$ is also a homeomorphism between Banach spaces.*

By Lemma A.2.13 and Theorem A.2.14, we obtain the following lemma.

Lemma A.2.15. *For the mapping $T : K(\alpha', \beta') \to V(\alpha', \beta')$ defined in Lemma A.2.13, the mapping $H_0^1(V(\alpha', \beta')) \ni \chi \mapsto \chi \circ T \in H_0^1(K(\alpha', \beta'))$ is isomorphic between Banach spaces. In particular, there exists a constant $c = c(\alpha, \beta, \alpha', \beta', g) > 0$ such that*

$$\frac{1}{c}\|\chi\|_{H^1(V(\alpha', \beta'))} \leq \|\psi\|_{H^1(K(\alpha', \beta'))} \leq c\|\chi\|_{H^1(V(\alpha', \beta'))}$$

for all $\psi \in H_0^1(K(\alpha', \beta'))$ and $\chi = \psi \circ T^{-1}$.

Proof. By (A.2.11), $\frac{\partial x_i}{\partial y_j}$ is bounded in $K(\alpha', \beta')$ and $\frac{\partial y_i}{\partial x_j}$ is bounded in $V(\alpha', \beta')$ for all $i, j = 1, \cdots, n$. By Theorem A.2.14, $H_0^1(V(\alpha', \beta')) \ni \chi \mapsto \chi \circ T \in H_0^1(K(\alpha', \beta'))$ is isomorphic. □

We give the proof of a Hardy-type inequality.

Lemma A.2.16. *Let $\alpha, \beta > 0$ and $\varphi \in H_0^1(K(\alpha, \beta))$. Then, $\varphi(x_1, \cdots, x_n)/x_n \in L^2(K(\alpha, \beta))$ and*

$$\left\| \frac{\varphi}{x_n} \right\|_{L^2(K(\alpha,\beta))} \leq 2 \left\| \frac{\partial \varphi}{\partial x_n} \right\|_{L^2(K(\alpha,\beta))}.$$

Proof. For $\varphi \in C_0^\infty(K(\alpha, \beta))$, we have

$$\begin{aligned}
\left\| \frac{\varphi}{x_n} \right\|^2_{L^2(K(\alpha,\beta))} &= \int_{B(\alpha)} dx' \int_0^\beta dx_n \left| \frac{\varphi(x', x_n)}{x_n} \right|^2 \\
&= -\int_{B(\alpha)} dx' \int_0^\beta dx_n |\varphi(x', x_n)|^2 \frac{d}{dx_n} \left(\frac{1}{x_n} \right) \\
&= 2 \int_{B(\alpha)} dx' \int_0^\beta dx_n \frac{\partial \varphi}{\partial x_n}(x', x_n) \frac{\varphi(x', x_n)}{x_n} \\
&= 2 \left\| \frac{\partial \varphi}{\partial x_n} \right\|_{L^2(K(\alpha,\beta))} \left\| \frac{\varphi}{x_n} \right\|_{L^2(K(\alpha,\beta))}.
\end{aligned}$$

Hence,

$$\left\| \frac{\varphi}{x_n} \right\|_{L^2(K(\alpha,\beta))} \leq 2 \left\| \frac{\partial \varphi}{\partial x_n} \right\|_{L^2(K(\alpha,\beta))} \qquad \text{for all } \varphi \in C_0^\infty(K(\alpha, \beta)).$$

Since $C_0^\infty(K(\alpha, \beta))$ is dense in $H_0^1(K(\alpha, \beta))$, we obtain the result. □

We use the following lemmas.

Lemma A.2.17. *For the mapping $T : K(\alpha', \beta') \ni (y_1, \cdots, y_n) \mapsto (x_1, \cdots, x_n) \in V(\alpha', \beta')$ defined in Lemma A.2.13, there exists a constant $c = c(\alpha, \beta, \alpha', \beta', g) > 0$ such that*

$$\left\| \frac{\partial x_n}{\partial y_i} \frac{\psi}{\mathrm{Jac}(T)} \right\|_{H^1(K)} \leq c \left\| \frac{\partial \psi}{\partial y_n} \right\|_{L^2(K)}, \qquad \left\| \frac{\psi}{\mathrm{Jac}(T)} \right\|_{H^1(K)} \leq c \left\| \frac{\partial \psi}{\partial y_n} \right\|_{L^2(K)}$$

for all $\psi \in H_0^1(K)$ and $i = 1, \cdots, n-1$, where $K := K(\alpha', \beta')$.

Proof. We compute

$$\begin{aligned}
&\left\| \frac{\partial x_n}{\partial y_i} \frac{\psi}{\mathrm{Jac}(T)} \right\|_{H^1(K)} \\
\leq\ & c_1 \left(\left\| \frac{\partial x_n}{\partial y_i} \frac{\psi}{\mathrm{Jac}(T)} \right\|_{L^2(K)} + \sum_{j=1}^n \left\| \frac{\partial}{\partial y_j} \left(\frac{\partial x_n}{\partial y_i} \frac{\psi}{\mathrm{Jac}(T)} \right) \right\|_{L^2(K)} \right) \\
\leq\ & c_1 \left(c_2 \|\psi\|_{L^2(K)} + \sum_{j=1}^n \left(\left\| \frac{\partial}{\partial y_j} \left(\frac{\partial x_n / \partial y_i}{\mathrm{Jac}(T)} \right) \psi \right\|_{L^2(K)} + \left\| \frac{\partial x_n / \partial y_i}{\mathrm{Jac}(T)} \frac{\partial \psi}{\partial y_j} \right\|_{L^2(K)} \right) \right) \\
\leq\ & c_1 \left(c_2 \|\psi\|_{H^1(K)} + \sum_{j=1}^n \left(\left\| \frac{\partial}{\partial y_j} \left(\frac{\partial x_n / \partial y_i}{\partial x_n / \partial y_n} \right) \psi \right\|_{L^2(K)} + c_2 \|\psi\|_{H^1(K)} \right) \right) \\
\leq\ & c_1 \left(c_2 (n+1) \|\psi\|_{H^1(K)} + \sum_{j=1}^n \left\| \frac{\partial}{\partial y_j} \left(\frac{\partial x_n / \partial y_i}{\partial x_n / \partial y_n} \right) \psi \right\|_{L^2(K)} \right)
\end{aligned}$$

for two constants $c_1, c_2 > 0$. Here, we have

$$
\begin{aligned}
& \left\| \frac{\partial}{\partial y_j}\left(\frac{\partial x_n/\partial y_i}{\partial x_n/\partial y_n}\right)\psi \right\|_{L^2(K)} \\
= & \left\| \frac{\dfrac{\partial^2 x_n}{\partial y_j \partial y_i}\dfrac{\partial x_n}{\partial y_n} - \dfrac{\partial x_n}{\partial y_i}\dfrac{\partial^2 x_n}{\partial y_j \partial y_n}}{\left(\dfrac{\partial x_n}{\partial y_n}\right)^2}\psi \right\|_{L^2(K)} \\
\leq & \; c_3 \left(\left\| \frac{\partial^2 x_n}{\partial y_j \partial y_i}\psi \right\|_{L^2(K)} + \left\| \frac{\partial^2 x_n}{\partial y_j \partial y_n}\psi \right\|_{L^2(K)} \right) \\
= & \; c_3 \left(\left\| y_n \frac{\partial^2 x_n}{\partial y_j \partial y_i}\frac{\psi}{y_n} \right\|_{L^2(K)} + \left\| y_n \frac{\partial^2 x_n}{\partial y_j \partial y_n}\frac{\psi}{y_n} \right\|_{L^2(K)} \right) \\
\leq & \; c_3 \left(\left\| y_n \frac{\partial^2 x_n}{\partial y_j \partial y_i} \right\|_{L^\infty(K)} \left\| \frac{\psi}{y_n} \right\|_{L^2(K)} + \left\| y_n \frac{\partial^2 x_n}{\partial y_j \partial y_n} \right\|_{L^\infty(K)} \left\| \frac{\psi}{y_n} \right\|_{L^2(K)} \right)
\end{aligned}
$$

for a constant $c_3 > 0$. By Lemma A.2.12, there exists a constant $c_4 > 0$ such that

$$
\left\| y_n \frac{\partial^2 x_n}{\partial y_j \partial y_i} \right\|_{L^\infty(K)} \leq c_4
$$

for all $i, j = 1, \cdots, n$. By Lemma A.2.16, it holds that

$$
\left\| \frac{\psi}{y_n} \right\|_{L^2(K)} \leq 2 \left\| \frac{\partial \psi}{\partial y_n} \right\|_{L^2(K)} \leq 2\|\psi\|_{H^1(K)}.
$$

Hence, we obtain that

$$
\begin{aligned}
\left\| \frac{\partial x_n}{\partial y_i}\frac{\psi}{\mathrm{Jac}(T)} \right\|_{H^1(K)} \leq & \; c_1 \left(c_2(n+1)\|\psi\|_{H^1(K)} + \sum_{j=1}^{n} \left\| \frac{\partial}{\partial y_j}\left(\frac{\partial x_n/\partial y_i}{\partial x_n/\partial y_n}\right)\psi \right\|_{L^2(K)} \right) \\
\leq & \; c_1 \left(c_2(n+1)\|\psi\|_{H^1(K)} + \sum_{j=1}^{n} (2c_4\|\psi\|_{H^1(K)} + 2c_4\|\psi\|_{H^1(K)}) \right) \\
= & \; c_1(c_2(n+1) + 4c_4 n)\|\psi\|_{H^1(K)}
\end{aligned}
$$

for all $\psi \in H_0^1(K)$ and $i = 1, \cdots, n-1$. The following inequality can be proven in the same way:

$$
\left\| \frac{\psi}{\mathrm{Jac}(T)} \right\|_{H^1(K)} \leq c_5\|\psi\|_{H^1(K)}
$$

for all $\psi \in H_0^1(K)$. □

Lemma A.2.18. *For the mapping $T : K(\alpha', \beta') \to V(\alpha', \beta')$ defined in Lemma A.2.13, there exists a constant $c = c(\alpha, \beta, \alpha', \beta', g) > 0$ such that*

$$
\|q\|_{H^{-1}(K)} + \|\nabla q\|_{H^{-1}(K)^n} \leq c\left(\|p\|_{H^{-1}(V)} + \|\nabla p\|_{H^{-1}(V)^n}\right)
$$

for all $p \in C_0^\infty(V)$, where $q := p \circ T (\in C_0^\infty(K))$, $K := K(\alpha', \beta')$ and $V := V(\alpha', \beta')$.

Proof. For $\psi \in H_0^1(K)$ and $\chi = \psi \circ T^{-1}$, we obtain

$$
\begin{aligned}
\int_K q(y)\psi(y)dy &= \int_V p(x)\chi(x)\frac{1}{(\mathrm{Jac}(T))(T^{-1}(x))}dx, \\
\int_K \frac{\partial q}{\partial y_i}(y)\psi(y)dy &= \int_V \sum_{j=1}^n \frac{\partial p}{\partial x_j}(x)\frac{\partial x_j}{\partial y_i}(T^{-1}(x))\ \chi(x)\frac{1}{(\mathrm{Jac}(T))(T^{-1}(x))}dx \\
&= \int_V \left\{\frac{\partial p}{\partial x_i}(x) + \frac{\partial p}{\partial x_n}(x)\frac{\partial x_n}{\partial y_i}(T^{-1}(x))\right\}\frac{\chi(x)}{(\mathrm{Jac}(T))(T^{-1}(x))}dx, \\
\int_K \frac{\partial q}{\partial y_n}(y)\psi(y)dy &= \int_V \sum_{j=1}^n \frac{\partial p}{\partial x_j}(x)\frac{\partial x_j}{\partial y_n}(T^{-1}(x))\ \chi(x)\frac{1}{(\mathrm{Jac}(T))(T^{-1}(x))}dx \\
&= \int_V \frac{\partial p}{\partial x_n}(x)\frac{\partial x_n}{\partial y_n}(T^{-1}(x))\frac{\chi(x)}{\frac{\partial x_n}{\partial y_n}(T^{-1}(x))}dx \\
&= \int_V \frac{\partial p}{\partial x_n}(x)\chi(x)dx
\end{aligned}
$$

for $i = 1, \cdots, n-1$. By Lemmas A.2.15 and A.2.17, there exist three constants $c_1, c_2, c_3 > 0$ such that

$$
\begin{aligned}
\|\chi\|_{H^1(V)} &\leq c_1\|\psi\|_{H^1(K)} \\
\left\|\frac{\partial x_n}{\partial y_i} \circ T^{-1}\frac{\chi}{(\mathrm{Jac}(T)) \circ T^{-1}}\right\|_{H^1(V)} &\leq c_2\|\chi\|_{H^1(V)}, \\
\left\|\frac{\chi}{(\mathrm{Jac}(T)) \circ T^{-1}}\right\|_{H^1(V)} &\leq c_3\|\chi\|_{H^1(V)}
\end{aligned}
\tag{A.2.12}
$$

for all $\chi \in H_0^1(V)$ and $i = 1, \cdots, n-1$. Thus we have

$$
\begin{aligned}
\|q\|_{H^{-1}(K)} &\leq \sup_{0 \neq \psi \in H_0^1(K)} \frac{\int_K q(y)\psi(y)dy}{\|\psi\|_{H^1(K)}} \\
&\leq c_1 \sup_{0 \neq \chi \in H_0^1(V)} \frac{\int_V p(x)\chi(x)\frac{1}{(\mathrm{Jac}(T))(T^{-1}(x))}dx}{\|\chi\|_{H^1(V)}} \\
&\leq c_1 c_3 \sup_{0 \neq \chi \in H_0^1(V)} \frac{\int_V p(x)\chi(x)\frac{1}{(\mathrm{Jac}(T))(T^{-1}(x))}dx}{\left\|\frac{\chi}{(\mathrm{Jac}(T)) \circ T^{-1}}\right\|_{H^1(V)}} \\
&\leq c_1 c_3\|p\|_{H^{-1}(V)},
\end{aligned}
$$

$$
\begin{aligned}
\left\|\frac{\partial q}{\partial y_i}\right\|_{H^{-1}(K)} &\le \sup_{\psi\in H_0^1(K)} \frac{\displaystyle\int_K \frac{\partial q}{\partial y_i}(y)\psi(y)dy}{\|\psi\|_{H^1(K)}} \\
&\le c_1 \sup_{0\neq\chi\in H_0^1(V)} \frac{\displaystyle\int_V \left\{\frac{\partial p}{\partial x_i}(x)+\frac{\partial p}{\partial x_n}(x)\frac{\partial x_n}{\partial y_i}(T^{-1}(x))\right\}\frac{\chi(x)}{(\mathrm{Jac}(T))(T^{-1}(x))}dx}{\|\chi\|_{H^1(V)}} \\
&\le c_1 \sup_{0\neq\chi\in H_0^1(V)} \frac{\displaystyle\int_V \frac{\partial p}{\partial x_i}(x)\frac{\chi(x)}{(\mathrm{Jac}(T))(T^{-1}(x))}dx}{\|\chi\|_{H^1(V)}} \\
&\quad + c_1 \sup_{0\neq\chi\in H_0^1(V)} \frac{\displaystyle\int_V \frac{\partial p}{\partial x_n}(x)\frac{\partial x_n}{\partial y_i}(T^{-1}(x))\frac{\chi(x)}{(\mathrm{Jac}(T))(T^{-1}(x))}dx}{\|\chi\|_{H^1(V)}} \\
&\le c_1c_3 \sup_{0\neq\chi\in H_0^1(V)} \frac{\displaystyle\int_V \frac{\partial p}{\partial x_i}(x)\frac{\chi(x)}{(\mathrm{Jac}(T))(T^{-1}(x))}dx}{\left\|\frac{\chi}{(\mathrm{Jac}(T))\circ T^{-1}}\right\|_{H^1(V)}} \\
&\quad + c_1c_2 \sup_{0\neq\chi\in H_0^1(V)} \frac{\displaystyle\int_V \frac{\partial p}{\partial x_n}(x)\frac{\partial x_n}{\partial y_i}(T^{-1}(x))\frac{\chi(x)}{(\mathrm{Jac}(T))(T^{-1}(x))}dx}{\left\|\frac{\chi}{(\mathrm{Jac}(T))\circ T^{-1}}\frac{\partial x_n}{\partial y_i}\circ T^{-1}\right\|_{H^1(V)}} \\
&\le c_4\left(\left\|\frac{\partial p}{\partial x_i}\right\|_{H^{-1}(V)}+\left\|\frac{\partial p}{\partial x_n}\right\|_{H^{-1}(V)}\right),
\end{aligned}
$$

$$
\begin{aligned}
\left\|\frac{\partial q}{\partial y_n}\right\|_{H^{-1}(K)} &\le \sup_{0\neq\psi\in H_0^1(K)} \frac{\displaystyle\int_K \frac{\partial q}{\partial y_n}(y)\psi(y)dy}{\|\psi\|_{H^1(K)}} \\
&\le c_1 \sup_{0\neq\chi\in H_0^1(V)} \frac{\displaystyle\int_V \frac{\partial p}{\partial x_n}(x)\chi(x)dx}{\|\chi\|_{H^1(V)}} \\
&= c_1\left\|\frac{\partial p}{\partial x_n}\right\|_{H^{-1}(V),}
\end{aligned}
$$

for $i = 1, \cdots, n-1$, where $c_4 := c_1 \max\{c_2, c_3\}$. Finally, by Lemma A.2.1, it follows that

$$
\begin{aligned}
\|q\|_{H^{-1}(K)} + \|\nabla q\|_{H^{-1}(K)^n} &\le \|q\|_{H^{-1}(K)} + \sum_{i=1}^n \left\|\frac{\partial q}{\partial y_i}\right\|_{H^{-1}(K)} \\
&\le c_5\left(\|p\|_{H^{-1}(V)} + \sum_{i=1}^n \left\|\frac{\partial p}{\partial x_i}\right\|_{H^{-1}(V)}\right) \\
&\le \sqrt{n}c_5\left(\|p\|_{H^{-1}(V)} + \|\nabla p\|_{H^{-1}(V)^n}\right)
\end{aligned}
$$

for all $p \in C_0^\infty(V)$ and $q := p \circ T (\in C_0^\infty(K))$, where $c_5 := c_1 + (n-1)c_4$. □

Lemma A.2.19. *Let $0 < \alpha' < \alpha$, $0 < \beta' < \beta$ and let $g : \overline{B(\alpha)} \to \mathbb{R}$ be a Lipschitz continuous function. There exists a constant $c = c(\alpha, \beta, \alpha', \beta', g) > 0$ such that*

$$\|p\|_{L^2(U)} \le c(\|p\|_{H^{-1}(U)} + \|\nabla p\|_{H^{-1}(U)^n}) \qquad \text{for all } p \in C_0^\infty(U_g^+(\alpha', \beta')),$$

where $U = U_g^+(\alpha, \beta)$.

Proof. Let $\alpha_1', \alpha_2', \beta_1'$ and β_2' satisfy $\alpha_1' := \alpha' < \alpha_2' < \alpha, \beta_1' := \beta' < \beta_2' < \beta$ and let $K := K(\alpha_2', \beta_2')$. By Lemma A.2.11, there exists a C^∞-diffeomorphic map $T : K \to V := V(\alpha_2', \beta_2')$. Let us denote $q = p \circ T$ for $p \in C_0^\infty(U_g^+(\alpha_1', \beta_1'))$. We have $\operatorname{supp}(p) \subset U_g^+(\alpha_2', \beta_2') \subset V$, thus,

$$\|p\|_{L^2(U)} = \|p\|_{L^2(V)}. \tag{A.2.13}$$

Since $L^2(V) \ni p \mapsto p \circ T \in L^2(K)$ is isomorphic between Banach spaces, we get the following inequality:

$$\|p\|_{L^2(V)} \leq c_1 \|q\|_{L^2(K)}. \tag{A.2.14}$$

By Lemma A.2.13, we have $U_g^+(\alpha_1', \beta_1') \subset T(K(\alpha_1', \beta_1'))$, hence, $\operatorname{supp}(q) \subset K(\alpha_1', \beta_1')$. By Lemma A.2.9, there exists a constant $c_2 > 0$ such that

$$\|q\|_{L^2(K)} \leq c_2(\|q\|_{H^{-1}(K)} + \|\nabla q\|_{H^{-1}(K)^n}). \tag{A.2.15}$$

Moreover, by Lemma A.2.18, there exists a constant $c_3 > 0$ such that

$$\|q\|_{H^{-1}(K)} + \|\nabla q\|_{H^{-1}(K)^n} \leq c_3(\|p\|_{H^{-1}(V)} + \|\nabla p\|_{H^{-1}(V)^n}). \tag{A.2.16}$$

Therefore, by (A.2.13), (A.2.14), (A.2.15) and (A.2.16), it follows that

$$\begin{aligned} \|p\|_{L^2(U)} &\leq c_1 c_2 c_3 (\|p\|_{H^{-1}(V)} + \|\nabla p\|_{H^{-1}(V)^n}) \\ &\leq c_1 c_2 c_3 (\|p\|_{H^{-1}(U)} + \|\nabla p\|_{H^{-1}(U)^n}) \end{aligned}$$

for all $p \in C_0^\infty(U_g^+(\alpha_1', \beta_1'))$. □

A.2.5 Original Nečas inequality

We prove Lemma 2.2.13 which is the goal of this appendix.

Theorem A.2.20 (Reshown, see Lemma 2.2.13)**.** *If* Ω *is a bounded Lipschitz domain, then there exists a constant* $c = c(\Omega) > 0$ *such that*

$$\|p\|_{L^2(\Omega)} \leq c(\|p\|_{H^{-1}(\Omega)} + \|\nabla p\|_{H^{-1}(\Omega)^n}) \qquad \textit{for all } p \in L^2(\Omega).$$

Proof. By Lemma A.2.2, it is sufficient to prove that there exists a constant $c_1 > 0$ such that

$$\|p\|_{L^2(\Omega)} \leq c_1(\|p\|_{H^{-1}(\Omega)} + \|\nabla p\|_{H^{-1}(\Omega)^n}) \qquad \text{for all } p \in C_0^\infty(\Omega).$$

By Definition A.1.1, we have $\Gamma \subset \bigcup_{r=1}^m U_r(\alpha, \beta)$. We can choose two real numbers $0 < \alpha' < \alpha, 0 < \beta' < \beta$ and an open subset $U_0(\alpha', \beta')$ ($\overline{U_0(\alpha', \beta')} \subset \Omega$) such that $\overline{\Omega} \subset \bigcup_{r=0}^m U_r(\alpha', \beta')$. By Lemma A.1.1, there exist functions $\eta_0, \cdots, \eta_m \in C^\infty(\mathbb{R}^n)$ such that

$$\begin{aligned} &\eta_r \in C_0^\infty(U_r(\alpha', \beta')) && \text{for all } r = 0, 1, \cdots, m, \\ &0 \leq \eta_r(x) \leq 1 && \text{for all } r = 0, 1, \cdots, m, x \in U_r(\alpha', \beta'), \\ &\sum_{r=0}^m \eta_r(x) = 1 && x \in \overline{\Omega}. \end{aligned}$$

Let $p_r := p\eta_r \in C_0^\infty(U_r^+(\alpha', \beta'))$ for $r = 0, 1, \cdots, m$. Then, it follows that

$$\|p\|_{L^2(\Omega)} = \left\| \sum_{r=0}^{m} p_r \right\|_{L^2(\Omega)} \le \sum_{r=0}^{m} \|p_r\|_{L^2(\Omega)}$$

for all $p \in C_0^\infty(\Omega)$. On the other hand, for $\psi \in H_0^1(\Omega)$, we have

$$\begin{aligned} \int_\Omega p\eta_r \psi \, dx &\le \|p\|_{H^{-1}(\Omega)} \|\eta_r \psi\|_{H^1(\Omega)} \\ &= c_2 \|p\|_{H^{-1}(\Omega)} \|\eta_r\|_{H^1(\Omega)} \|\psi\|_{H^1(\Omega)}, \end{aligned}$$

and

$$\begin{aligned} &\int_\Omega \frac{\partial}{\partial x_i}(p\eta_r)\psi \, dx \\ \le\ & \left| \int_\Omega \frac{\partial p}{\partial x_i} \eta_r \psi \, dx \right| + \left| \int_\Omega p \frac{\partial \eta_r}{\partial x_i} \psi \, dx \right| \\ \le\ & \left\| \frac{\partial p}{\partial x_i} \right\|_{H^{-1}(\Omega)} \|\eta_r \psi\|_{H^1(\Omega)} + \|p\|_{H^{-1}(\Omega)} \left\| \frac{\partial \eta_r}{\partial x_i} \psi \right\|_{H^1(\Omega)} \\ \le\ & c_2 \left(\left\| \frac{\partial p}{\partial x_i} \right\|_{H^{-1}(\Omega)} \|\eta_r\|_{H^1(\Omega)} + \|p\|_{H^{-1}(\Omega)} \left\| \frac{\partial \eta_r}{\partial x_i} \right\|_{H^1(\Omega)} \right) \|\psi\|_{H^1(\Omega)} \end{aligned}$$

for all $i = 1, 2, \cdots, n$ and a constant $c_2 > 0$. Hence, we obtain

$$\begin{aligned} &\|p_r\|_{H^{-1}(\Omega)} + \|\nabla p_r\|_{H^{-1}(\Omega)^n} \\ \le\ & \|p\eta_r\|_{H^{-1}(\Omega)} + \sum_{i=1}^{n} \left\| \frac{\partial}{\partial x_i}(p\eta_r) \right\|_{H^{-1}(\Omega)} \\ =\ & \sup_{\psi \in H_0^1(\Omega),\ \|\psi\|_{H^1(\Omega)}=1} \int_\Omega p\eta_r \psi \, dx + \sum_{i=1}^{n} \sup_{\psi \in H_0^1(\Omega),\ \|\psi\|_{H^1(\Omega)}=1} \int_\Omega \frac{\partial}{\partial x_i}(p\eta_r)\psi \, dx \\ \le\ & c_{2,r} \left(\|p\|_{H^{-1}(\Omega)} + \sum_{i=1}^{n} \left\| \frac{\partial p}{\partial x_i} \right\|_{H^{-1}(\Omega)} \right) \\ \le\ & \sqrt{n} c_{2,r} (\|p\|_{H^{-1}(\Omega)} + \|\nabla p\|_{H^{-1}(\Omega)^n}), \end{aligned}$$

where $c_{2,r} := c_2 \left(\|\eta_r\|_{H^1(\Omega)} + n \sum_{i=1}^{n} \left\| \frac{\partial \eta_r}{\partial x_i} \right\|_{H^1(\Omega)} \right)$. Thus it suffices to show that for $r = 0, 1, \cdots, m$,

$$\|p_r\|_{L^2(\Omega)} \le c_3 (\|p_r\|_{H^{-1}(\Omega)} + \|\nabla p_r\|_{H^{-1}(\Omega)^n}) \qquad \text{for all } p_r \in C_0^\infty(U_r^+(\alpha', \beta')). \quad \text{(A.2.17)}$$

(i) The case $r = 0$.

We have $\mathrm{supp}(p_0) \subset U_0(\alpha', \beta')$ and $\overline{U_0(\alpha', \beta')} \subset \Omega$. By Lemma A.2.7, the inequality (A.2.17) holds with $r = 0$.

(i) The case $r = 1, 2, \cdots, m$.

Let $U_r := U_r^+(\alpha, \beta)$. By Lemma A.2.19, we obtain (A.2.17);

$$\begin{aligned} \|p_r\|_{L^2(\Omega)} &= \|p_r\|_{L^2(U_r)} \\ &\le c_4 (\|p_r\|_{H^{-1}(U_r)} + \|\nabla p_r\|_{H^{-1}(U_r)^n}) \\ &\le c_4 (\|p_r\|_{H^{-1}(\Omega)} + \|\nabla p_r\|_{H^{-1}(\Omega)^n}) \end{aligned}$$

for all $p_r \in C_0^\infty(U_r^+(\alpha', \beta'))$. □

A.3 Corollaries of the Nečas inequality

Using Theorem 2.2.13, we obtain the following corollary which is important for existence and uniqueness of the solution to the Stokes problem.

Corollary A.3.1 (Reshown, see Theorem 2.2.14). *If Ω is a bounded Lipschitz domain, then there exists a constant $c = c(\Omega) > 0$ such that*

$$\|p\|_{L^2(\Omega)/\mathbb{R}} \leq c\|\nabla p\|_{H^{-1}(\Omega)^n} \qquad \text{for all } p \in L^2(\Omega).$$

Proof. This proof is based on [2, Theorem 3.1] It suffices to show that there exists a constant $c > 0$ such that

$$\|p\|_{L^2(\Omega)} \leq c\|\nabla p\|_{H^{-1}(\Omega)^n} \qquad \text{for all } p \in L^2(\Omega)/\mathbb{R}.$$

Assume that this property does not hold. Then, there exists a sequence of functions $(p_k)_{k\in\mathbb{N}} \subset L^2(\Omega)/\mathbb{R}$ such that $\|p_k\|_{L^2(\Omega)} = 1$ for all $k \in \mathbb{N}$ and

$$\|\nabla p_k\|_{H^{-1}(\Omega)^n} \to 0 \qquad \text{as } k \to \infty.$$

Since the sequence $(p_k)_{k\in\mathbb{N}}$ is bounded in $L^2(\Omega)$, there exists a subsequence $(p_l)_{l\in\mathbb{N}}$ that converges weakly in $L^2(\Omega)$. Let $\varphi \in L^2(\Omega)$ and $\delta > 0$ be arbitrary. Then, there exists a constant $N_{\varphi,\delta} \in \mathbb{N}$ such that

$$l_1, l_2 \geq N_{\varphi,\delta} \Rightarrow |(\varphi, p_{l_1} - p_{l_2})_{L^2(\Omega)}| < \delta.$$

On the other hand, since $S := \{\varphi \in H^1(\Omega) \mid \|\varphi\|_{H^1(\Omega)} = 1\}$ is bounded in $H^1(\Omega)$, for every fixed $\delta' > 0$, there exist functions $\varphi_1, \cdots, \varphi_m \in S$ such that

$$S \subset \bigcup_{i=1}^{m} \left\{\varphi \in L^2(\Omega) \mid \|\varphi - \varphi_i\|_{L^2(\Omega)} < \delta'\right\}$$

by the Rellich–Kondrashov Theorem, i.e., there exists a number $i \in \{1, \cdots, m\}$ such that

$$\|\varphi - \varphi_i\|_{L^2(\Omega)} < \delta'$$

for all $\varphi \in S$. Thus we obtain

$$\begin{aligned} |(\varphi, p_{l_1} - p_{l_2})_{L^2(\Omega)}| &\leq |(\varphi - \varphi_i, p_{l_1} - p_{l_2})_{L^2(\Omega)}| + |(\varphi_i, p_{l_1} - p_{l_2})_{L^2(\Omega)}| \\ &\leq \|\varphi - \varphi_i\|_{L^2(\Omega)}\|p_{l_1} - p_{l_2}\|_{L^2(\Omega)} + |(\varphi_i, p_{l_1} - p_{l_2})_{L^2(\Omega)}| \\ &\leq 2\delta' + \delta \end{aligned}$$

for all $\varphi \in S$ and $l_1, l_2 \geq N_\delta := \max\{N_{\varphi_1,\delta}, \cdots, N_{\varphi_m,\delta}\}$, and then

$$\|p_{l_1} - p_{l_2}\|_{H^{-1}(\Omega)} = \sup_{\varphi\in S} |(\varphi, p_{l_1} - p_{l_2})_{L^2(\Omega)}| \leq 2\delta' + \delta.$$

It satisfies that

$$\limsup_{l_1,l_2\to\infty} \|p_{l_1} - p_{l_2}\|_{H^{-1}(\Omega)} \leq 2\delta'$$

for every $\delta' > 0$, which implies that $(p_l)_{l\in\mathbb{N}}$ is a Cauchy sequence in $H^{-1}(\Omega)$. Besides, by the assumption, $(\nabla p_l)_{l\in\mathbb{N}}$ is also a Cauchy sequence in $H^{-1}(\Omega)$.

Theorem 2.2.13 leads that $(p_l)_{l\in\mathbb{N}}$ is a Cauchy sequence in $L^2(\Omega)$. Thus, there exists a function $p \in L^2(\Omega)$ such that

$$\|p_l - p\|_{L^2(\Omega)} \to 0 \qquad \text{as } l \to \infty.$$

We have

$$\|\nabla p_l - \nabla p\|_{H^{-1}(\Omega)^n} \to 0 \qquad \text{as } l \to \infty,$$

since the operator $\nabla : L^2(\Omega) \to H^{-1}(\Omega)^n$ is continuous. Indeed, it holds that

$$\begin{aligned}
|\langle \nabla\omega, u\rangle| = |(\omega, -\operatorname{div} u)_{L^2(\Omega)}| \le \|\omega\|_{L^2(\Omega)} \|\operatorname{div} u\|_{L^2(\Omega)} &\quad \text{for all } \omega \in L^2(\Omega), u \in H^1(\Omega)^n,\\
|\langle \nabla\omega, u\rangle| \le \sqrt{n}\|\omega\|_{L^2(\Omega)}\|u\|_{H^1(\Omega)^n} &\quad \text{for all } \omega \in L^2(\Omega), u \in H^1(\Omega)^n,\\
\|\nabla\omega\|_{H^{-1}(\Omega)^n} \le \sqrt{n}\|\omega\|_{L^2(\Omega)} &\quad \text{for all } \omega \in L^2(\Omega),\\
\|\nabla * \|_{\mathcal{L}(L^2(\Omega),H^{-1}(\Omega))} \le \sqrt{n}. &
\end{aligned}$$

By connectivity of the open set Ω, the function p is a constant, and this constant is 0 since $\int_\Omega p_l \, dx = 0$ for all $l \in \mathbb{N}$. But this contradicts the relation $\|p_l\|_{L^2(\Omega)} = 1$ for all $l \in \mathbb{N}$. □

Corollary A.3.2 (Reshown, see Theorem 2.2.15)**.** *If Ω is a bounded Lipschitz domain, then the divergence operator* div *maps $H_0^1(\Omega)^n$ onto $L^2(\Omega)/\mathbb{R}$.*

Proof. The operator $\nabla : L^2(\Omega) \to H^{-1}(\Omega)^n$ satisfies

$$\langle \nabla\omega, u\rangle = (\omega, -\operatorname{div} u)_{L^2(\Omega)} \quad \text{for all } \omega \in H^1(\Omega), u \in H^1(\Omega)^n, \tag{A.3.18}$$

and thus $-\operatorname{div} : H^1(\Omega)^n \to L^2(\Omega)/\mathbb{R}$ is the dual operator of ∇. By the proof of Theorem 2.2.14, ∇ is continuous and thus closed. By Theorem 2.2.14, we deduce that the image of ∇ is closed, and so, the image of div is $(\text{Ker}\nabla)^\perp = \mathbb{R}^\perp = L^2(\Omega)/\mathbb{R}$. □