

Estimates for singular integrals on homogeneous groups

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ESTIMATES FOR SINGULAR INTEGRALS ON HOMOGENEOUS GROUPS

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ABSTRACT. We consider singular integral operators and maximal singular integral operators with rough kernels on homogeneous groups. We prove certain estimates for the operators that imply L^p boundedness of them by an extrapolation argument under a sharp condition for the kernels. Also, we prove some weighted L^p inequalities for the operators.

1. INTRODUCTION

Let \mathbb{R}^n , $n \geq 2$, be the n dimensional Euclidean space. We also regard \mathbb{R}^n as a homogeneous group with multiplication given by a polynomial mapping. So, we have a dilation family $\{A_t\}_{t>0}$ on \mathbb{R}^n such that each A_t is an automorphism of the group structure, where A_t is of the form

$$A_t x = (t^{a_1} x_1, t^{a_2} x_2, \dots, t^{a_n} x_n), \quad x = (x_1, \dots, x_n),$$

with some real numbers a_1, \dots, a_n satisfying $0 < a_1 \leq a_2 \leq \dots \leq a_n$ (see [28] and [15, Section 2 of Chapter 1]). We also write $\mathbb{R}^n = \mathbb{H}$. In addition to the Euclidean structure, \mathbb{H} is equipped with a homogeneous nilpotent Lie group structure, where Lebesgue measure is a bi-invariant Haar measure, the identity is the origin 0, $x^{-1} = -x$ and multiplication xy , $x, y \in \mathbb{H}$, satisfies

- (1) $(ux)(vx) = ux + vx$, $x \in \mathbb{H}$, $u, v \in \mathbb{R}$;
- (2) $A_t(xy) = (A_t x)(A_t y)$, $x, y \in \mathbb{H}$, $t > 0$;
- (3) if $z = xy$, then $z_k = P_k(x, y)$, where $P_1(x, y) = x_1 + y_1$ and $P_k(x, y) = x_k + y_k + R_k(x, y)$ for $k \geq 2$ with a polynomial $R_k(x, y)$ depending only on $x_1, \dots, x_{k-1}, y_1, \dots, y_{k-1}$.

We denote by $|x|$ the Euclidean norm for $x \in \mathbb{R}^n$. Also, we have a norm function $r(x)$ satisfying $r(A_t x) = tr(x)$ for $t > 0$ and $x \in \mathbb{R}^n$. We assume the following:

- (4) the function r is continuous on \mathbb{R}^n and smooth in $\mathbb{R}^n \setminus \{0\}$;
- (5) $r(x+y) \leq C_0(r(x) + r(y))$, $r(xy) \leq C_0(r(x) + r(y))$ for some constant $C_0 \geq 1$, $r(x^{-1}) = r(x)$;

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(6) there are positive constants $c_1, c_2, c_3, c_4, \alpha_1, \alpha_2, \beta_1$ and β_2 such that

$$\begin{aligned} c_1|x|^{\alpha_1} &\leq r(x) \leq c_2|x|^{\alpha_2} & \text{if } r(x) \geq 1, \\ c_3|x|^{\beta_1} &\leq r(x) \leq c_4|x|^{\beta_2} & \text{if } r(x) \leq 1; \end{aligned}$$

(7) if $\Sigma = \{x \in \mathbb{R}^n : r(x) = 1\}$, then $\Sigma = S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$.

Let $\gamma = a_1 + \cdots + a_n$. Then, $dx = t^{\gamma-1} dS dt$, that is,

$$\int_{\mathbb{R}^n} f(x) dx = \int_0^\infty \int_\Sigma f(A_t \theta) t^{\gamma-1} dS(\theta) dt$$

for an appropriate function f with $dS = \omega dS_0$, where ω is a strictly positive C^∞ function on Σ and dS_0 is the Lebesgue surface measure on Σ . For appropriate functions f, g on \mathbb{H} , the convolution $f * g$ is defined by

$$f * g(x) = \int_{\mathbb{R}^n} f(y)g(y^{-1}x) dy.$$

The space \mathbb{H} with a left invariant quasi-metric $d(x, y) = r(x^{-1}y)$ can be regarded as a space of homogeneous type (see [2, 8, 11, 15, 18, 26, 27, 28] for more details).

The Heisenberg group \mathbb{H}_1 is an example of a homogeneous group. If we define the multiplication

$$(x, y, u)(x', y', u') = (x + x', y + y', u + u' + (xy' - yx')/2),$$

$(x, y, u), (x', y', u') \in \mathbb{R}^3$, then \mathbb{R}^3 with this group law is the Heisenberg group \mathbb{H}_1 ; a dilation is defined by $A_t(x, y, u) = (tx, ty, t^2u)$.

Let Ω be locally integrable in $\mathbb{R}^n \setminus \{0\}$ and homogeneous of degree 0 with respect to the dilation group $\{A_t\}$, that is, $\Omega(A_t x) = \Omega(x)$ for $x \neq 0, t > 0$. We assume that

$$\int_\Sigma \Omega(\theta) dS(\theta) = 0.$$

Let $K(x) = \Omega(x')r(x)^{-\gamma}$, $x' = A_{r(x)^{-1}}x$ for $x \neq 0$. For $s \geq 1$, let d_s denote the collection of measurable functions h on $\mathbb{R}_+ = \{t \in \mathbb{R} : t > 0\}$ satisfying

$$\|h\|_{d_s} = \sup_{j \in \mathbb{Z}} \left(\int_{2^j}^{2^{j+1}} |h(t)|^s dt/t \right)^{1/s} < \infty,$$

where \mathbb{Z} denotes the set of integers. We define $\|h\|_{d_\infty} = \|h\|_{L^\infty(\mathbb{R}_+)}$. Note that $d_s \subset d_u$ if $s \geq u$. Also, put for $t \in (0, 1]$,

$$\omega(h, t) = \sup_{|s| < tR/2} \int_R^{2R} |h(r-s) - h(r)| dr/r,$$

where the supremum is taken over all s and R such that $|s| < tR/2$ (see [12, 25]). For $\eta > 0$, let Λ^η denote the family of functions h such that

$$\|h\|_{\Lambda^\eta} = \sup_{t \in (0, 1]} t^{-\eta} \omega(h, t) < \infty.$$

Define a space $\Lambda_s^\eta = d_s \cap \Lambda^\eta$ and set $\|h\|_{\Lambda_s^\eta} = \|h\|_{d_s} + \|h\|_{\Lambda^\eta}$ for $h \in \Lambda_s^\eta$. Note that $\Lambda_s^{\eta_1} \subset \Lambda_s^{\eta_2}$ if $\eta_2 \leq \eta_1$, and $\Lambda_{s_1}^\eta \subset \Lambda_{s_2}^\eta$ if $s_2 \leq s_1$.

Let

$$(1.1) \quad Tf(x) = \text{p.v.} f * L(x) = \text{p.v.} \int_{\mathbb{R}^n} f(y)L(y^{-1}x) dy,$$

where $L(x) = h(r(x))K(x)$, $h \in d_1$. We consider $L^q(\Sigma)$ spaces and write $\|F\|_q = (\int_\Sigma |F(\theta)|^q dS(\theta))^{1/q}$ for $F \in L^q(\Sigma)$ ($\|F\|_\infty$ is defined as usual). Let $s' = s/(s-1)$ denote the conjugate exponent to s . We shall prove L^p estimates for Tf with $h \in \Lambda_s^{\eta/s'}$ and $\Omega \in L^s(\Sigma)$, $s > 1$, as s approaches 1.

Theorem 1. *Let $s > 1$. Suppose that $\Omega \in L^s(\Sigma)$ and $h \in \Lambda_s^{\eta/s'}$ for some fixed positive number η . Then, if $1 < p < \infty$,*

$$\|Tf\|_p \leq C_p s(s-1)^{-1} \|h\|_{\Lambda_s^{\eta/s'}} \|\Omega\|_s \|f\|_p,$$

where the constant C_p is independent of s , Ω and h .

We denote by $L \log L(\Sigma)$ the Zygmund class of all those functions F on Σ which satisfy

$$\int_\Sigma |F(\theta)| \log(2 + |F(\theta)|) dS(\theta) < \infty.$$

Let Λ denote the collection of functions h on \mathbb{R}_+ such that there exist a sequence $\{h_k\}_{k=1}^\infty$ of functions on \mathbb{R}_+ and a sequence $\{a_k\}_{k=1}^\infty$ of non-negative real numbers satisfying $h = \sum_{k=1}^\infty a_k h_k$, $h_k \in \Lambda_{1+1/k}^{1/(k+1)}$, $\sup_{k \geq 1} \|h_k\|_{\Lambda_{1+1/k}^{1/(k+1)}} \leq 1$ and $\sum_{k=1}^\infty k a_k < \infty$.

Theorem 1 implies the following result.

Theorem 2. *Let Tf be as in (1.1). Suppose that $h \in \Lambda$ and $\Omega \in L \log L(\Sigma)$. Then, T is bounded on $L^p(\mathbb{R}^n)$ for all $p \in (1, \infty)$.*

When $h = 1$ (a constant function), this is due to [28]. See [3, 4, 14, 16, 17, 18] for relevant results and also [23, 25, 28] for weak $(1, 1)$ boundedness.

We also consider the maximal singular integral operator

$$(1.2) \quad T_* f(x) = \sup_{N, \epsilon > 0} \left| \int_{\epsilon < r(y) < N} f(xy^{-1})L(y) dy \right|.$$

We shall prove analogs of Theorems 1 and 2 for the operator T_* .

Theorem 3. *Let a number s and functions h , Ω be as in Theorem 1. Then we have*

$$\|T_* f\|_p \leq C_p s(s-1)^{-1} \|h\|_{\Lambda_s^{\eta/s'}} \|\Omega\|_s \|f\|_p$$

for all $p \in (1, \infty)$, where C_p is independent of s , h and Ω .

By Theorem 3 we have the following result.

Theorem 4. *Suppose that $\Omega \in L \log L(\Sigma)$ and $h \in \Lambda$. Let $T_* f$ be defined as in (1.2) by using the functions Ω and h . Then, T_* is bounded on $L^p(\mathbb{R}^n)$ for $p \in (1, \infty)$.*

This seems to be novel even in the case when $h = 1$. If $h = 1$, Theorem 2 can be proved by interpolation between L^2 estimates and weak $(1, 1)$ estimates, both of which are given in [28]. For T_* with $\Omega \in L \log L$, weak $(1, 1)$ boundedness is yet to be proved even in the case $h = 1$.

In this note we shall show that results of Tao [28] can be used to obtain an analog of a theory of Duoandikoetxea and Rubio de Francia [10] for homogeneous groups which can prove Theorems 1 and 3. In our situation, Littlewood-Paley theory (see Lemma 6 in Section 4) and interpolation arguments are available as in [10], although we cannot apply Fourier transform estimates as effectively as in [10]. We shall show that L^2 estimates of Lemma 1 in Section 3 can be used as a substitute for Fourier transform estimates if we apply Cotlar's lemma instead of Plancherel's theorem. Our methods may extend to the study of some other interesting operators in harmonic analysis (see [5], [10]).

Let $\{B_t\}_{t>0}$, $B_t = t^P = \exp((\log t)P)$, be a dilation group on \mathbb{R}^n , where P is an $n \times n$ real matrix whose eigenvalues have positive real parts. Let N be a locally integrable function on $\mathbb{R}^n \setminus \{0\}$ such that $N(B_t x) = t^{-\gamma} N(x)$, $\gamma = \text{trace } P$, for $t > 0$ and $x \in \mathbb{R}^n \setminus \{0\}$. Let $J(x) = h(r(x))N(x)$ with an appropriate norm function $r(x)$ for $\{B_t\}_{t>0}$. If we define

$$Sf(x) = \text{p.v.} \int_{\mathbb{R}^n} f(y)J(x-y) dy,$$

using Euclidean convolution, assuming an appropriate cancellation condition for J , then we can apply methods of Duoandikoetxea and Rubio de Francia [10] via Fourier transform estimates to prove L^p boundedness, $p \in (1, \infty)$, of S under an $L \log L$ condition on $\{r(x) = 1\}$ for N and the condition

$$\sup_{j \in \mathbb{Z}} \int_{2^j}^{2^{j+1}} |h(r)| (\log(2 + |h(r)|))^a dr/r < \infty$$

for h with some $a > 2$. Also, a similar result for maximal singular integrals holds (see [21, 22]).

We can also prove some weighted norm estimates for T and T_* . Let B be a subset of \mathbb{H} such that

$$B = \{x \in \mathbb{H} : r(a^{-1}x) < s\}$$

for some $a \in \mathbb{H}$ and $s > 0$. Then we call B a ball in \mathbb{H} with center a and radius s and write $B = B(a, s)$. Note that $|B(a, s)| = cs^\gamma$ with $c = |B(0, 1)|$, where $|S|$ denotes the Lebesgue measure of a set S . Let \mathcal{A}_p , $1 < p < \infty$, be the weight class of Muckenhoupt on \mathbb{H} defined to be the collection of all

weight functions w on \mathbb{H} satisfying

$$\sup_B \left(|B|^{-1} \int_B w(x) dx \right) \left(|B|^{-1} \int_B w(x)^{-1/(p-1)} dx \right)^{p-1} < \infty,$$

where the supremum is taken over all balls B in \mathbb{H} (see [1, 13]). Also, the class \mathcal{A}_1 is defined to be the family of all weight functions w on \mathbb{H} satisfying the pointwise inequality $Mw \leq Cw$ almost everywhere, where M denotes the Hardy-Littlewood maximal operator

$$Mf(x) = \sup_{x \in B} |B|^{-1} \int_B |f(y)| dy;$$

the supremum is taken over all balls B in \mathbb{H} containing x (see [1, 8, 13]). We can prove the following weighted estimates.

Theorem 5. *Let $q > 1$. Suppose that $\Omega \in L^q(\Sigma)$ and $h \in \Lambda_q^\eta$ for some $\eta > 0$. Let $1 < p < \infty$. Then,*

- (1) *T and T_* are bounded on $L^p(w)$ if $q' \leq p < \infty$ and $w \in \mathcal{A}_{p/q'}$;*
- (2) *if $1 < p \leq q$ and $w \in \mathcal{A}_{p'/q'}$, T and T_* are bounded on $L^p(w^{1-p})$.*

See [9, 29] for the case of rough singular integrals defined by Euclidean convolution.

In Section 2, we shall give some preliminary results from [28] for calculation on homogeneous groups. Basic L^2 estimates (Lemma 1) will be proved in Section 3 by applying methods of [28]. Using the L^2 estimate, we shall prove Theorem 1 in Section 4 by means of a process of [10, 21, 22]. In Section 5, we shall prove Theorem 3 by adapting arguments of [10] for the present situation. Theorem 5 will be proved in Section 6 by applying arguments of [9] and using results of Sections 3–5. Finally, we shall prove Theorem 2 from Theorem 1 in Section 7 by an extrapolation argument. Theorem 4 can be proved in the same way from Theorem 3. In what follows, even when we consider functions that may assume general complex values, we deal with real valued functions only to simplify our arguments. The letters C, c will be used to denote positive constants which may be different in different occurrences.

2. PRELIMINARY RESULTS

In this section we recall several results from [28]. Let $f : \mathbb{R} \rightarrow \mathbb{H}$ be smooth. Then the Euclidean derivative $\partial_t f(t)$ is defined by

$$f(t + \epsilon) = f(t) + \epsilon \partial_t f(t) + \epsilon^2 O(1) \quad \text{for } \epsilon \in (0, 1].$$

We define the left invariant derivative $\partial_t^L f(t)$ by

$$f(t + \epsilon) = f(t)(\epsilon \partial_t^L f(t)) + \epsilon^2 O(1) \quad \text{for } \epsilon \in (0, 1].$$

Fix $x \in \mathbb{H}$ and consider $G_x : \mathbb{R}^n \rightarrow \mathbb{R}^n$ defined by $G_x(y) = xy$. Let $JG_x(y)$ be the Jacobian matrix of G_x at y . Then $JG_x(y)$ is a lower triangular matrix. The components of $JG_x(y)$ are polynomials in x, y and each diagonal component is equal to 1 (see (3) in Section 1). We can see that $\partial_t^L f(t) = JG_{f(t)^{-1}}(f(t))\partial_t f(t)$.

We have the product rule

$$(2.1) \quad \partial_t^L (f(t)g(t)) = \partial_t^L g(t) + C[g(t)]\partial_t^L f(t),$$

where $C[x] : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a linear mapping defined by

$$x^{-1}(\epsilon v)x = \epsilon C[x]v + \epsilon^2 O(1) \quad \text{for } \epsilon \in (0, 1].$$

We note that

$$(2.2) \quad \begin{aligned} C[x^{-1}] &= C[x]^{-1}, \quad C[A_t x](A_t v) = A_t(C[x]v), \\ |C[x]v| &\sim |v| \quad \text{if } |x| \leq 1. \end{aligned}$$

Define a polynomial mapping $X : \mathbb{R}^n \rightarrow \mathbb{R}^n$ by

$$A_{1+\epsilon}x = x(\epsilon X(x)) + \epsilon^2 O(1) \quad \text{for } \epsilon \in (0, 1].$$

Then

$$(2.3) \quad X(A_t x) = A_t X(x), \quad r(X(x)) \sim r(x)$$

and

$$(2.4) \quad \partial_t^L (A_{s(t)} f(t)) = A_{s(t)} \partial_t^L f(t) + s'(t)s(t)^{-1} (A_{s(t)} X(f(t))),$$

where $s(t)$ is a strictly positive, smooth function on \mathbb{R}_+ . Also, X is a diffeomorphism with Jacobian comparable to 1.

3. L^2 ESTIMATES

Let ϕ be a C^∞ function with compact support in $B(0, 1) \setminus B(0, 1/2)$ satisfying $\int \phi = 1$, $\phi(x) = \tilde{\phi}(x)$, $\phi(x) \geq 0$ for all $x \in \mathbb{R}^n$, where $\tilde{\phi}(x) = \phi(x^{-1})$. Define

$$\Delta_k = \delta_{\rho^{k-1}}\phi - \delta_{\rho^k}\phi, \quad k \in \mathbb{Z},$$

where $\delta_t \phi(x) = t^{-\gamma} \phi(A_t^{-1}x)$ and $\rho \geq 2$. Note that $\text{supp}(\Delta_k) \subset B(0, \rho^k) \setminus B(0, \rho^{k-1}/2)$, $\Delta_k = \tilde{\Delta}_k$ and $\sum_k \Delta_k = \delta$, where δ is the delta function. Let $\psi_j \in C_0^\infty(\mathbb{R})$, $j \in \mathbb{Z}$, be such that

$$\text{supp}(\psi_j) \subset \{t \in \mathbb{R} : \rho^j \leq t \leq \rho^{j+2}\}, \quad \psi_j \geq 0,$$

$$\sum_{j \in \mathbb{Z}} \psi_j(t) = 1 \quad \text{for } t \neq 0,$$

$$|(d/dt)^m \psi_j(t)| \leq c_m |t|^{-m} \quad \text{for } m = 0, 1, 2, \dots,$$

where c_m is a constant independent of ρ (this is possible since $\rho \geq 2$).

Let

$$S_j L(x) = (\log 2)^{-1} h(r(x)) \int_0^\infty \psi_j(t) \delta_t K_0(x) dt/t,$$

where

$$K_0(x) = K(x) \chi_{D_0}(x), \quad D_0 = \{x \in \mathbb{R}^n : 1 \leq r(x) \leq 2\}.$$

Here χ_E denotes the characteristic function of a set E . Then $\sum_{j \in \mathbb{Z}} S_j L = L$. Furthermore, let

$$(3.1) \quad S_j(F, \ell)(x) = (\log 2)^{-1} \ell(r(x)) \int_0^\infty \psi_j(t) \delta_t F(x) dt/t,$$

where $F \in L^1(\mathbb{R}^n)$, $\text{supp}(F) \subset D_0$ and $\ell \in d_1$. Let Φ be a non-negative smooth function such that $\int \Phi(x) dx = 1$, $\Phi(x^{-1}) = \Phi(x)$, $\text{supp}(\Phi) \subset B(0, 1)$. Define

$$(3.2) \quad U_\sigma f = U_\sigma(F, \ell)(f) = \sum_j \sigma_j f * \nu_j,$$

where

$$\begin{aligned} \nu_j(x) &= \nu_j(F, \ell)(x) = S_j(F, \ell)(x) - \Phi_j(x), \\ \Phi_j(x) &= \Phi_j(F, \ell)(x) = \left(\int S_j(F, \ell) dx \right) \delta_{\rho^j} \Phi(x), \end{aligned}$$

and $\sigma = \{\sigma_j\}$ is an arbitrary sequence such that $\sigma_j = 1$ or -1 . We note that $\int \nu_j(x) dx = 0$, $S_j L = \nu_j(K_0, h) = S_j(K_0, h)$ and $U_\sigma(K_0, h)(f) = Tf$ if $\sigma_j = 1$ for all j . We prove the following L^2 estimates.

Lemma 1. *Suppose that $s > 1$, $F \in L^s(D_0)$ and $\ell \in \Lambda_s^{\eta/s'}$ for some fixed positive number η , where we write $F \in L^s(D_0)$ if $F \in L^s(\mathbb{R}^n)$ and $\text{supp}(F) \subset D_0$. Let $\nu_j = \nu_j(F, \ell)$. Then, for $j, k \in \mathbb{Z}$ we have*

$$(3.3) \quad \|f * \nu_j * \Delta_k\|_2 \leq C(\log \rho) \min(1, \rho^{-\epsilon(|j-k|-c)/s'}) \|\ell\|_{\Lambda_s^{\eta/s'}} \|F\|_s \|f\|_2$$

for some positive constants C, ϵ and c independent of ρ, s, ℓ and F .

Proof. It suffices to prove Lemma 1 with ν in place of ν_j , assuming $j = 0$ on the right hand side of (3.3), where

$$\nu(x) = (\log 2)^{-1} \ell(\rho^j r(x)) \int_0^\infty \psi_j(\rho^j t) \delta_t F(x) dt/t - \left(\int S_j(F, \ell) dx \right) \Phi(x).$$

This can be seen from change of variables and the formulas: $\delta_t(f * g) = (\delta_t f) * (\delta_t g)$, $\delta_{\rho^{-j}} \nu_j = \nu$, $\delta_{\rho^{-j}} \Delta_k = \Delta_{k-j}$.

If $k \geq 0$, then from the cancellation condition for ν and the smoothness of Δ_k we have

$$(3.4) \quad \|\nu * \Delta_k\|_1 \leq C(\log \rho) \min(1, \rho^{-\epsilon k + \tau}) \|\ell\|_{d_1} \|F\|_1$$

for some $\epsilon, \tau > 0$, which implies the conclusion by Young's inequality, if the constant c is large enough.

The following result is useful.

Lemma 2. *Suppose that $\ell \in d_q$, $F \in L^q(D_0)$ for some $q \geq 1$. Put $S = \delta_{\rho^{-j}} S_j(F, \ell)$. Then*

$$\|S\|_q \leq C(\log \rho) \|\ell\|_{d_q} \|F\|_q,$$

where the constant C is independent of ρ and q .

Proof. Suppose that $q < \infty$. Since $\int_0^\infty \psi_j(\rho^j t) dt/t \leq 2 \log \rho$, Hölder's inequality implies

$$\begin{aligned} \|S\|_q^q &\leq (\log 2)^{-q} (2 \log \rho)^{q/q'} \int \int_0^\infty |\ell(\rho^j r(x))|^q \psi_j(\rho^j t) |\delta_t F(x)|^q dt/t dx \\ &= (\log 2)^{-q} (2 \log \rho)^{q/q'} \int \int_0^\infty |\ell(\rho^j tr(x))|^q \psi_j(\rho^j t) |F(x)|^q t^{\gamma(1-q)} dt/t dx. \end{aligned}$$

Let N be a positive integer such that $\rho^2 \leq 2^{N+1} < 2\rho^2$. Then

$$\begin{aligned} &\int_0^\infty |\ell(\rho^j tr(x))|^q \psi_j(\rho^j t) t^{\gamma(1-q)} dt/t \\ &\leq \sum_{m=0}^N \int_{2^m \rho^j r(x)}^{2^{m+1} \rho^j r(x)} |\ell(t)|^q (\rho^{-j} r(x)^{-1} t)^{\gamma(1-q)} dt/t. \\ &\leq \sum_{m=0}^N 2^{m\gamma(1-q)} \int_{2^m \rho^j r(x)}^{2^{m+1} \rho^j r(x)} |\ell(t)|^q dt/t \\ &\leq C(\log \rho) \|\ell\|_{d_q}^q, \end{aligned}$$

where $C \leq 12$. Collecting results, we get the conclusion for $q < \infty$. Also, we easily see that $\|S\|_\infty \leq C \|\ell\|_{d_\infty} \|F\|_\infty$, which implies the conclusion for $q = \infty$. \square

The estimate (3.4) can be shown as follows. First, by Lemma 2 with $q = 1$

$$(3.5) \quad \|\nu * \Delta_k\|_1 \leq C(\log \rho) \|\ell\|_{d_1} \|F\|_1.$$

Suppose that $k \geq 1$. Let $t = \rho^{k-1}$. Then $\Delta_k = \delta_t \Delta_1$. Since $\int \nu = 0$,

$$(3.6) \quad \begin{aligned} \nu * \Delta_k(x) &= \int t^{-\gamma} (\Delta_1(A_t^{-1}(y^{-1}x)) - \Delta_1(A_t^{-1}x)) \nu(y) dy \\ &= \int t^{-\gamma} \left(\int_0^1 W'(u) du \right) \nu(y) dy, \end{aligned}$$

where

$$W(u) = \Delta_1((uY)^{-1}X) = \Delta_1(P_1(-uY, X), \dots, P_n(-uY, X)),$$

with $Y = A_{t^{-1}}y$, $X = A_{t^{-1}}x$ (see (3) in Section 1). Note that

$$\begin{aligned} W'(u) &= \langle (\nabla \Delta_1)(P_1(-uY, X), \dots, P_n(-uY, X)), \\ &\quad (\partial u P_1(-uY, X), \dots, \partial u P_n(-uY, X)) \rangle, \end{aligned}$$

where $\nabla\Delta_1 = (\partial x_1\Delta_1, \partial x_2\Delta_1, \dots, \partial x_n\Delta_1)$ and $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product in \mathbb{R}^n . Also, note that

$$\partial u P_i(-uY, X) = \langle -Y, \nabla_x P_i(-uY, X) \rangle,$$

where

$$\nabla_x P_i(x, y) = (\partial x_1 P_i(x, y), \dots, \partial x_n P_i(x, y)).$$

We may assume that $r(Y) \leq C\rho^2$, $r(X) \leq C\rho^2$ in (3.6) by checking the support condition. Therefore

$$\sup_{u \in [0, 1]} |\partial u P_i(-uY, X)| \leq C|Y|\rho^M$$

for some $M > 0$ and hence

$$|W'(u)| \leq C|Y|\rho^M |(\nabla\Delta_1)((uY)^{-1}X)|.$$

Note that $\|\nabla\Delta_1\|_1 \leq C$, $\|A_t^{-1}\|$ (a norm for A_t^{-1} as a linear transformation on \mathbb{R}^n) is less than $Ct^{-\beta}$ and $|y| \leq C\rho^{2\alpha}$ on the support of ν , with $\beta = 1/\beta_1$, $\alpha = 1/\alpha_1$ (see (6) of Section 1). Therefore,

$$\begin{aligned} (3.7) \quad \|\nu * \Delta_k\|_1 &\leq C\rho^M \|\nabla\Delta_1\|_1 \int |A_t^{-1}(y)| |\nu(y)| dy \\ &\leq C\rho^M t^{-\beta} \int |y| |\nu(y)| dy \\ &\leq C\rho^M t^{-\beta} \rho^{2\alpha} \|\nu\|_1 \leq C(\log \rho) \rho^{-k\beta} \rho^{2\alpha+\beta+M} \|\ell\|_{d_1} \|F\|_1. \end{aligned}$$

By (3.5) and (3.7), we have (3.4) for $k \geq 0$.

We next assume that $k \leq -1$. Since $\int \Delta_0(x) dx = 0$, as in the proof of (3.4) we have

$$\|\Phi * \Delta_k\|_1 = \|\delta_{\rho^{-k}} \Phi * \Delta_0\|_1 \leq C\rho^{-\epsilon|k|}.$$

Therefore, separately estimating $\|f * S * \Delta_k\|_2$ and $\|f * (\int S_j(F, \ell)) \Phi * \Delta_k\|_2$, it suffices to prove

$$(3.8) \quad \|f * S * \Delta_k\|_2 \leq C(\log \rho) \min(1, \rho^{-\epsilon(|k|-c)/s'}) \|\ell\|_{\Lambda_s^{\eta/s'}} \|F\|_s \|f\|_2,$$

where $S = \delta_{\rho^{-j}} S_j(F, \ell)$ as above.

By the estimate

$$\|S * \Delta_k\|_1 \leq C(\log \rho) \|\ell\|_{d_1} \|F\|_1$$

and the T^*T method, to prove (3.8) it suffices to show that

$$(3.9) \quad \left\| f * \Delta_k * \tilde{S} * S * \Delta_k \right\|_2 \leq C(\log \rho)^2 \rho^{\epsilon(k+c)/s'} \|\ell\|_{\Lambda_s^{\eta/s'}}^2 \|F\|_s^2 \|f\|_2.$$

Since $\|T^*T\| = \|(T^*T)^n\|^{1/n}$, (3.9) follows from

$$\left\| f * \left(\Delta_k * \tilde{S} * S * \Delta_k \right)_*^n \right\|_2 \leq C(\log \rho)^{2n} \rho^{\epsilon(k+c)/s'} \|\ell\|_{\Lambda_s^{\eta/s'}}^{2n} \|F\|_s^{2n} \|f\|_2$$

for some $\epsilon, c > 0$, where $\left(\Delta_k * \tilde{S} * S * \Delta_k\right)_*^n$ denotes the convolution product of n factors of $\Delta_k * \tilde{S} * S * \Delta_k$. By Young's inequality, this follows from the L^1 estimate

$$(3.10) \quad \left\| \left(\Delta_k * \tilde{S} * S * \Delta_k\right)_*^n \right\|_1 \leq C(\log \rho)^{2n} \rho^{\epsilon(k+c)/s'} \|\ell\|_{\Lambda_s^{\eta/s'}}^{2n} \|F\|_s^{2n}.$$

Note that $\left(\Delta_k * \tilde{S} * S * \Delta_k\right)_*^n = \Delta_k * \tilde{S} * S * \left(\Delta_k * \Delta_k * \tilde{S} * S\right)_*^{n-1} * \Delta_k$. Since $\|\Delta'_k * \tilde{S}\|_1 \leq C(\log \rho) \|\ell\|_{d_1} \|F\|_1$ and $\Delta'_k * \tilde{S}(x) = \int \delta_y(x) \Delta'_k * \tilde{S}(y) dy$, where $\delta_y(x)$ is the delta function concentrated at y and Δ'_k is either Δ_k or $\Delta_k * \Delta_k$, (3.10) follows from

$$\|\delta_{w_1} * S * \cdots * \delta_{w_n} * S * \Delta_k\|_1 \leq C(\log \rho)^n \rho^{\epsilon(k+c)/s'} \|\ell\|_{\Lambda_s^{\eta/s'}}^n \|F\|_s^n$$

uniformly for $w_1, \dots, w_n \in B(0, C\rho^2)$. To get this, it suffices to prove

$$(3.11) \quad |\langle \delta_{w_1} * S * \cdots * \delta_{w_n} * S * \Delta_k, g \rangle| \leq C(\log \rho)^n \rho^{\epsilon(k+c)/s'} \|\ell\|_{\Lambda_s^{\eta/s'}}^n \|F\|_s^n$$

uniformly in $w_1, \dots, w_n \in B(0, C\rho^2)$, for all smooth g with compact support satisfying $\|g\|_\infty \leq 1$.

Fix g . Then, the inner product on the left hand side of (3.11) is equal to

$$(3.12) \quad \iiint \Delta_k(x) g(H(y, t) x) \prod_{i=1}^n (\ell(t_i, y_i) F(y_i) \psi(t_i)) dy \bar{d}t dx,$$

where $\ell(t_i, y_i) = \ell(t_i \rho^j r(y_i))$, $\psi(t_i) = (\log 2)^{-1} \psi_j(\rho^j t_i)$, $t = (t_1, \dots, t_n)$, $y = (y_1, \dots, y_n) \in D_0^n$, $\bar{d}t = (dt_1/t_1) \cdots (dt_n/t_n)$, $dy = dy_1 \cdots dy_n$ and

$$H(y, t) = \prod_{i=1}^n w_i A_{t_i} y_i = w_1 A_{t_1} y_1 \cdots w_n A_{t_n} y_n.$$

This is valid since

$$\begin{aligned} & \langle \delta_{w_1} * S * \cdots * \delta_{w_n} * S * \Delta_k, g \rangle \\ &= \int S(y_1) \cdots S(y_n) \Delta_k(x) g \left(\left(\prod_{i=1}^n w_i y_i \right) x \right) dy dx \\ &= \int \prod_{i=1}^n [\psi(t_i) \ell(\rho^j r(y_i)) \delta_{t_i} F(y_i)] \Delta_k(x) g \left(\left(\prod_{i=1}^n w_i y_i \right) x \right) dy dx \bar{d}t, \end{aligned}$$

which will coincide with the integral in (3.12) after a change of variables.

Let $DH(y, t)$ be the $n \times n$ matrix whose i th column vector is $\partial_{t_i}^L H(y, t)$:

$$DH(y, t) = (\partial_{t_1}^L H(y, t), \dots, \partial_{t_n}^L H(y, t)).$$

Then, (3.11) follows from the two estimates:

$$(3.13) \quad \left| \iiint \Delta_k(x) G_1(y, t) g(H(y, t)x) \prod_{i=1}^n (\psi(t_i) \ell(t_i, y_i) F(y_i)) \, dy \, \bar{d}t \, dx \right| \\ \leq C (\log \rho)^n \rho^{\epsilon(k+c)/s'} \|\ell\|_{\Lambda_s^{\eta/s'}}^n \|F\|_s^n,$$

$$(3.14) \quad \left| \iiint \Delta_k(x) G_2(y, t) g(H(y, t)x) \prod_{i=1}^n (\psi(t_i) \ell(t_i, y_i) F(y_i)) \, dy \, \bar{d}t \, dx \right| \\ \leq C (\log \rho)^n \rho^{\delta\epsilon(k+c)/s'} \|\ell\|_{\Lambda_s^{\eta/s'}}^n \|F\|_1^n$$

with

$$G_1(y, t) = \zeta_1 (\rho^{-n\epsilon k} \det(DH(y, t))), \\ G_2(y, t) = \zeta_2 (\rho^{-n\epsilon k} \det(DH(y, t))),$$

where ζ_1 is a function in $C_0^\infty(\mathbb{R})$ such that $0 \leq \zeta_1 \leq 1$, $\text{supp}(\zeta_1) \subset [-1, 1]$, $\zeta_1(t) = 1$ for $t \in [-1/2, 1/2]$, $\zeta_2 = 1 - \zeta_1$, and δ, ϵ are small positive numbers.

Proof of (3.13). Since $\|\Delta_k\|_1 \leq C$, $\int \psi(t_i) dt_i/t_i \leq C \log \rho$ and

$$\int |F(y_i) \ell(t_i, y_i)|^s \psi(t_i) \, dy_i \, dt_i/t_i \leq C (\log \rho) \|\ell\|_{d_s}^s \|F\|_s^s$$

(see the proof of Lemma 2), by Hölder's inequality, it suffices to show that

$$(3.15) \quad \int_{D_0^n} \chi_{[0,1]} (\rho^{-kn\epsilon} |\det(DH(y, t))|) \, dy \leq C \rho^{\epsilon(k+c)}$$

uniformly in $t \in [1, \rho^2]^n$ and $w_1, \dots, w_n \in B(0, C\rho^2)$. By (2.1) and (2.4)

$$\partial_{t_i}^L H(y, t) = t_i^{-1} C[Q_i](A_{t_i} X(y_i)), 1 \leq i \leq n-1, \quad \partial_{t_n}^L H(y, t) = t_n^{-1} A_{t_n} X(y_n)$$

where $Q_i = \prod_{j=i+1}^n w_j A_{t_j} y_j$ for $1 \leq i \leq n-1$. Fixing y_2, \dots, y_n and changing variables with respect to y_1 , we see that the integral in (3.15) is majorized by

$$C t_1^{-\gamma} \int_{\tilde{D}_0 \times D_0^{n-1}} \chi_{[0,1]} (c \rho^{-kn\epsilon} t_1^{-1} |\det(J(y, t))|) \, dy_1 \, d\bar{y},$$

where $\tilde{D}_0 = \{x \in \mathbb{R}^n : |x| \leq C\rho^M\}$, $M, C > 0$, $d\bar{y} = dy_2 \dots dy_n$ and

$$J(y, t) = (y_1, \partial_{t_2}^L H(y, t), \dots, \partial_{t_n}^L H(y, t)).$$

To see this, it may be convenient to write

$$\partial_{t_1}^L H(y, t) = t_1^{-1} A_{\rho^2} C[A_{\rho^{-2}} Q_1](A_{\rho^{-2}} A_{t_1} X(y_1)),$$

and to note that $|A_{\rho^{-2}} Q_1| \leq C$ (see (2.2)). Repeating this argument successively for y_2, \dots, y_n , we can see that (3.15) follows from

$$(3.16) \quad (t_1 \dots t_n)^{-\gamma} \int_{\tilde{D}_0^n} \chi_{[0,1]} (c \rho^{-kn\epsilon} (t_1 \dots t_n)^{-1} |\det Y|) \, dy \leq C \rho^{\epsilon(k+c)},$$

where Y denotes the $n \times n$ matrix whose i th column vector is y_i . Write $y_i = (y_i^1, \dots, y_i^n)$.

To prove (3.16), expand $\det Y = \sum_{m=1}^n y_1^m \Delta_{m1}$, where Δ_{m1} denotes the $(m, 1)$ cofactor of Y . Then, using this and applying a rotation in y_1 variable, we see that

$$\begin{aligned} & \int_{\tilde{D}_0^n} \chi_{[0,1]} (c\rho^{-kn\epsilon} (t_1 \dots t_n)^{-1} |\det Y|) dy \\ &= \int_{\tilde{D}_0^n} \chi_{[0,1]} \left(c\rho^{-kn\epsilon} (t_1 \dots t_n)^{-1} \left| y_1^1 \left(\sum_{m=1}^n \Delta_{m1}^2 \right)^{1/2} \right| \right) dy \\ &\leq \int_{\tilde{D}_0^n} \chi_{[0,1]} (c\rho^{-kn\epsilon} (t_1 \dots t_n)^{-1} |y_1^1 \Delta_{11}|) dy. \end{aligned}$$

Let $\tilde{D}_{01} = \{y_1 \in \tilde{D}_0 : |y_1^1| < t_1 \rho^{k\epsilon}\}$, $\tilde{D}_{02} = \{y_1 \in \tilde{D}_0 : |y_1^1| \geq t_1 \rho^{k\epsilon}\}$. Then we have

$$\begin{aligned} & \int_{\tilde{D}_0} \chi_{[0,1]} (c\rho^{-kn\epsilon} (t_1 \dots t_n)^{-1} |y_1^1 \Delta_{11}|) dy_1 \\ &= \sum_{i=1}^2 \int_{\tilde{D}_{0i}} \chi_{[0,1]} (c\rho^{-kn\epsilon} (t_1 \dots t_n)^{-1} |y_1^1 \Delta_{11}|) dy_1 \\ &\leq C t_1 \rho^{k\epsilon} \rho^{M(n-1)} + C \rho^{Mn} \chi_{[0,1]} \left(c\rho^{-k(n-1)\epsilon} (t_2 \dots t_n)^{-1} |\Delta_{11}| \right), \end{aligned}$$

and hence

$$\begin{aligned} & \int_{\tilde{D}_0^n} \chi_{[0,1]} (c\rho^{-kn\epsilon} (t_1 \dots t_n)^{-1} |\det Y|) dy \\ &\leq C \rho^{k\epsilon} \rho^b + C \rho^{Mn} \int_{\tilde{D}_0^{n-1}} \chi_{[0,1]} \left(c\rho^{-k(n-1)\epsilon} (t_2 \dots t_n)^{-1} |\Delta_{11}| \right) d\tilde{y} \end{aligned}$$

for some $b > 0$. Repeating a procedure similar to this $n - 1$ times, we reach the estimate

$$\begin{aligned} & \int_{\tilde{D}_0^n} \chi_{[0,1]} (c\rho^{-kn\epsilon} (t_1 \dots t_n)^{-1} |\det Y|) dy \\ &\leq C \rho^{k\epsilon} \rho^b + C \rho^b \int_{\tilde{D}_0} \chi_{[0,1]} (c\rho^{-k\epsilon} t_n^{-1} |y_n^n|) dy_n \leq C \rho^{k\epsilon} \rho^\tau \end{aligned}$$

for some $\tau > 0$. This proves (3.16).

Proof of (3.14). Let

$$\tilde{\ell}(t_i, y_i) = \int_{s_i < t_i/2} \ell(t_i - s_i, y_i) \varphi_{\rho^{k\epsilon}}(s_i) ds_i,$$

where $\varphi_u(s_i) = u^{-1}\varphi(u^{-1}s_i)$, $u > 0$, with $\varphi \in C^\infty(\mathbb{R})$ satisfying $\text{supp}(\varphi) \subset (0, 1/8)$, $\varphi \geq 0$, $\int \varphi(s) ds = 1$. Then

$$\begin{aligned} \int \psi(t_i)|\ell(t_i, y_i)| dt_i/t_i &\leq C(\log \rho)\|\ell\|_{d_1}, \\ \int \psi(t_i)|\tilde{\ell}(t_i, y_i)| dt_i/t_i &\leq C(\log \rho)\|\ell\|_{d_1}, \\ \int \psi(t_i)|\ell(t_i, y_i) - \tilde{\ell}(t_i, y_i)| dt_i/t_i &\leq C(\log \rho)\omega(\ell, \rho^{\epsilon k}). \end{aligned}$$

Therefore, writing

$$\begin{aligned} &\ell(t_1, y_1) \dots \ell(t_n, y_n) - \tilde{\ell}(t_1, y_1) \dots \tilde{\ell}(t_n, y_n) \\ &= (\ell(t_1, y_1) - \tilde{\ell}(t_1, y_1))\ell(t_2, y_2) \dots \ell(t_n, y_n) \\ &\quad + \tilde{\ell}(t_1, y_1)(\ell(t_2, y_2) - \tilde{\ell}(t_2, y_2))\ell(t_3, y_3) \dots \ell(t_n, y_n) \\ &\quad + \dots + \tilde{\ell}(t_1, y_1) \dots \tilde{\ell}(t_{n-1}, y_{n-1})(\ell(t_n, y_n) - \tilde{\ell}(t_n, y_n)), \end{aligned}$$

and applying the inequality $\omega(\ell, t) \leq \|\ell\|_{\Lambda_s^{\eta/s'}, t^{\eta/s'}}$, we see that to get (3.14) it suffices to prove a variant of (3.14) where each $\ell(t_i, y_i)$ is replaced by $\tilde{\ell}(t_i, y_i)$ for $i = 1, 2, \dots, n$. To show the estimate, it suffices to prove

$$(3.17) \quad \left| \iint \Delta_k(x) G_2(y, t) g(H(y, t)x) \prod_{i=1}^n \left(\psi(t_i) \tilde{\ell}(t_i, y_i) \right) \bar{d}t dx \right| \leq C \rho^{\delta \epsilon k} \rho^\tau \|\ell\|_{d_1}^n$$

uniformly in $y \in D_0^n$ and $w_1, \dots, w_n \in B(0, C\rho^2)$ with some $\tau > 0$, since the quantity on the left hand side of (3.17) is also bounded by $C(\log \rho)^n \|\ell\|_{d_1}^n$.

To prove (3.17) we need the following three lemmas.

Lemma 3. *Let f be a continuous function on \mathbb{R}^n such that*

$$\text{supp}(f) \subset B(0, C_1), \quad \int f(x) dx = 0, \quad \|f\|_1 \leq C_2.$$

Then there exist functions f_1, f_2, \dots, f_n such that

$$\begin{aligned} f(x) &= \sum_{i=1}^n \partial_{x_i} f_i(x), \\ \text{supp}(f_i) &\subset B(0, C'_1), \quad \|f_i\|_1 \leq C'_2 \quad \text{for } i = 1, 2, \dots, n, \end{aligned}$$

with some constants C'_1 and C'_2 .

This is from Lemma 7.1 in [28].

Lemma 4. *Let Δ_k be as in (3.14). Then, there exist functions F_j , $j = 1, 2, \dots, n$, such that $\text{supp}(F_j) \subset B(0, C\rho^k)$, $\|F_j\|_1 \leq C\rho^{k\alpha}$ for some $\alpha > 0$ and*

$$\Delta_k(x) = \sum_{j=1}^n \partial_{x_j} F_j(x).$$

This follows from Lemma 3.

Lemma 5. *Suppose that $\det(DH(y, t)x) \neq 0$, where $DH(y, t)x$ is defined in the same way as $DH(y, t)$ with $H(y, t)x$ in place of $H(y, t)$. Then*

$$\partial_{x_i} g(H(y, t)x) = \langle \nabla_t g(H(y, t)x), (DH(y, t)x)^{-1} (\partial_{x_i}^L (H(y, t)x)) \rangle.$$

(see Lemma 7.2 of [28]).

By Lemma 4, (3.17) follows from the estimate

$$(3.18) \quad \left| \iint \partial_{x_m} F_m(x) g(H(y, t)x) a(t) \prod_{i=1}^n \tilde{\ell}(t_i, y_i) \bar{d}t dx \right| \leq C\rho^{k\delta\epsilon} \rho^\tau \|\ell\|_{d_1}^n$$

for each m , $1 \leq m \leq n$, where $a(t) = G_2(y, t) \prod_{i=1}^n \psi(t_i)$. Applying integration by parts and using the L^1 norm estimate for F_m in Lemma 4, to prove (3.18) it suffices to show that

$$(3.19) \quad \left| \int \partial_{x_m} g(H(y, t)x) a(t) \prod_{i=1}^n \tilde{\ell}(t_i, y_i) \bar{d}t \right| \leq C\rho^{-2nk\epsilon} \rho^\tau \|\ell\|_{d_1}^n$$

for all $x \in B(0, C\rho^k)$ with a sufficiently small $\epsilon > 0$. By Lemma 5, the estimate (3.19) follows from

$$(3.20) \quad \left| \int \langle \nabla_t g(H(y, t)x), (DH(y, t)x)^{-1} (\partial_{x_m}^L x) \rangle a(t) \prod_{i=1}^n \tilde{\ell}(t_i, y_i) \bar{d}t \right| \leq C\rho^{-2nk\epsilon} \rho^\tau \|\ell\|_{d_1}^n,$$

since $\partial_{x_m}^L (H(y, t)x) = \partial_{x_m}^L x$ (see Section 2). Note that $|\nabla_t a(t)| \leq C\rho^{-kn\epsilon} \rho^\tau$ and $|\det(DH(y, t)x)| \geq C\rho^{kn\epsilon}$ on the support of a , since

$$|\det(DH(y, t)x)| = |\det \bar{C}[x] \det DH(y, t)| \geq C |\det DH(y, t)|,$$

where $\bar{C}[x]$ denotes the matrix expression for the linear transformation $C[x]$ (see (2.1), (2.2)). Thus, taking into account Cramer's formula, we have

$$\begin{aligned} & \left| \partial_{t_u} \left[a(t) \prod_{i=1}^n \tilde{\ell}(t_i, y_i) (DH(y, t)x)^{-1} (\partial_{x_m}^L x) \right] \right| \\ & \leq C\rho^{-2nk\epsilon} \rho^\tau \prod_{i=1}^n |\tilde{\ell}(t_i, y_i)| + C\rho^{-nk\epsilon} \rho^\tau |\partial_{t_u} \tilde{\ell}(t_u, y_u)| \prod_{i \neq u} |\tilde{\ell}(t_i, y_i)| \end{aligned}$$

for some $\tau > 0$. Also, note that

$$\int_1^{\rho^2} |\tilde{\ell}(t_i, y_i)| dt_i/t_i \leq C(\log \rho) \|\ell\|_{d_1}$$

and

$$\int_1^{\rho^2} |\partial_{t_i} \tilde{\ell}(t_i, y_i)| dt_i/t_i \leq C(\log \rho) \rho^{-k\epsilon} \|\ell\|_{d_1},$$

which follows from

$$|\partial_{t_i} \tilde{\ell}(t_i, y_i)| \leq C \rho^{-k\epsilon} \int |\ell(t_i - s_i, y_i)| |\varphi'_{\rho^{\epsilon k}}(s_i)| ds_i.$$

These estimates along with integration by parts imply (3.20). This completes the proof of (3.14) and hence that of Lemma 1. \square

4. PROOF OF THEOREM 1

We use the following weighted Littlewood-Paley inequalities.

Lemma 6. *Let $w \in \mathcal{A}_p$, $1 < p < \infty$, and let the functions Δ_k be as in Section 3. Then*

$$(4.1) \quad \left\| \sum_k f_k * \Delta_k \right\|_{L^p(w)} \leq C_{p,w} \left\| \left(\sum_k |f_k|^2 \right)^{1/2} \right\|_{L^p(w)},$$

$$(4.2) \quad \left\| \left(\sum_k |f * \Delta_k|^2 \right)^{1/2} \right\|_{L^p(w)} \leq C_{p,w} \|f\|_{L^p(w)},$$

where the constant $C_{p,w}$ is independent of $\rho \geq 2$.

Proof. Let $K(x) = \sum_{k \in \mathcal{J}} \sigma_k \Delta_k(x)$, where \mathcal{J} is an arbitrary finite subset of \mathbb{Z} and $\{\sigma_k\}$ is an arbitrary sequence such that $\sigma_k = 1$ or -1 . Let $Sf(x) = f * K(x)$. Then

- (1) S is bounded on L^2 with the operator norm bounded by a constant independent of ρ , \mathcal{J} and $\{\sigma_k\}$;
- (2) $|K(x)| \leq Cr(x)^{-\gamma}$;
- (3) there are positive constants C_1 and ϵ such that $r(x) > C_1 r(y)$ implies

$$|K(y^{-1}x) - K(x)| \leq Cr(y)^\epsilon r(x)^{-\gamma-\epsilon}.$$

The proof of (3.4) applies to show

$$\|\Delta_k * \Delta_j\|_1 \leq C \min(1, \rho^{-\epsilon|j-k|+c}).$$

By this and the Cotlar-Knapp-Stein lemma we get (1). The estimates in (2) and (3) can be shown by a straightforward computation. We note that, to prove the estimate in (3), it suffices to show that if $r(x) > C_1 r(y)$, then

$$|\delta_k \phi(y^{-1}x) - \delta_k \phi(x)| \leq Cr(y)^\epsilon r(x)^{-\gamma-\epsilon}$$

for each k . By application of dilation, this follows from the case $k = 0$, which can be easily proved.

Using (1), (2), (3) and applying methods of [7, Chapitre IV] and the proof of Theorem III in [6], we have $\|Sf\|_{L^p(w)} \leq C\|f\|_{L^p(w)}$ for $w \in \mathcal{A}_p$, $1 < p < \infty$, with a constant C independent of \mathfrak{J} , $\{\sigma_k\}$ and ρ . From this and the Khintchine inequality, (4.2) follows. A duality argument and (4.2) imply (4.1). \square

Let

$$M_{F,\ell}f(x) = \sup_j |f * S_j(|F|, |\ell|)(x)|,$$

where $S_j(F, \ell)$ is as in Section 3 (see (3.1)). Put $\mu^*f = M_{F,\ell}f$. Let $\theta \in (0, 1)$. We prove the following result along with Theorem 1.

Lemma 7. *Let $s > 1$, $F \in L^s(D_0)$ and $\ell \in \Lambda_s^{\eta/s'}$ for some fixed $\eta > 0$. Then, there exist positive constants ϵ, C independent of ρ and s such that*

$$\|\mu^*f\|_p \leq C(\log \rho)(1 - \rho^{-\theta\epsilon/(2s')})^{-4/p} \|\ell\|_{\Lambda_s^{\eta/s'}} \|F\|_s \|f\|_p$$

for $p > 1 + \theta$.

In Lemmas 1 and 7, we can have the same value of ϵ .

Proof of Lemma 7. Let $U_\sigma = U_\sigma(F, \ell)$ (see (3.2)) and write $U_\sigma f = \sum_{k_1, k_2} U_{k_1, k_2} f$, where

$$U_{k_1, k_2} f = \sum_j \sigma_j f * \Delta_{k_1+j} * \nu_j * \Delta_{k_2+j}, \quad \nu_j = \nu_j(F, \ell).$$

Fix integers k_1, k_2 . By Lemma 1 and duality we have

$$\|f * \Delta_k * \nu_j\|_2 \leq C(\log \rho) \min(1, \rho^{-\epsilon(|k-j|-c)/s'}) \|\ell\|_{\Lambda_s^{\eta/s'}} \|F\|_s \|f\|_2.$$

Using this along with Lemma 1, for ν_j and $\tilde{\nu}_j$, and noting that $\|\Delta_{k_2+j} * \Delta_{k_2+j'}\|_1 \leq C \min(1, \rho^{-\epsilon(|j-j'|-c)})$, where we may assume that the number ϵ is equal to the value of ϵ in Lemma 1, $\|\Delta_k\|_1 \leq C$, we have

$$(4.3) \quad \|f * (\Delta_{k_1+j} * \nu_j) * (\Delta_{k_2+j} * \Delta_{k_2+j'}) * (\tilde{\nu}_{j'} * \Delta_{k_1+j'})\|_2 \\ \leq CA^2 \min(1, \rho^{-2\epsilon(|k_1|-c)/s'}) \min(1, \rho^{-\epsilon(|j-j'|-c)}) \|f\|_2,$$

where $A = (\log \rho) \|\ell\|_{\Lambda_s^{\eta/s'}} \|F\|_s$, and also

$$(4.4) \quad \|f * \Delta_{k_1+j} * (\nu_j * \Delta_{k_2+j}) * (\Delta_{k_2+j'} * \tilde{\nu}_{j'}) * \Delta_{k_1+j'}\|_2 \\ \leq CA^2 \min(1, \rho^{-2\epsilon(|k_2|-c)/s'}) \|f\|_2.$$

By (4.3) and (4.4), taking the geometric mean we have

$$\|f * \Delta_{k_1+j} * \nu_j * \Delta_{k_2+j} * \Delta_{k_2+j'} * \tilde{\nu}_{j'} * \Delta_{k_1+j'}\|_2 \\ \leq CA^2 \prod_{i=1}^2 \min(1, \rho^{-\epsilon(|k_i|-c)/s'}) \min(1, \rho^{-\epsilon(|j-j'|-c)/2}) \|f\|_2.$$

We can obtain a similar estimate for

$$\|f * \Delta_{k_2+j'} * \tilde{\nu}_{j'} * \Delta_{k_1+j'} * \Delta_{k_1+j} * \nu_j * \Delta_{k_2+j}\|_2.$$

Therefore, by the Cotlar-Knapp-Stein lemma we see that

$$(4.5) \quad \|U_{k_1, k_2} f\|_2 \leq CA \prod_{i=1}^2 \min(1, \rho^{-\epsilon(|k_i|-c)/(2s')}) \|f\|_2$$

uniformly in σ . By (4.5) we have

$$(4.6) \quad \|U_\sigma f\|_2 \leq \sum_{k_1, k_2} \|U_{k_1, k_2} f\|_2 \leq CA(1 - \rho^{-\epsilon/(2s')})^{-2} \|f\|_2 \leq CAB \|f\|_2,$$

where $B = (1 - \rho^{-\theta\epsilon/(2s')})^{-2}$.

We define a sequence $\{p_j\}_1^\infty$ by $p_1 = 2$ and $1/p_{j+1} = 1/2 + (1-\theta)/(2p_j)$ for $j \geq 1$. Then, $1/p_j = (1-a^j)/(1+\theta)$, where $a = (1-\theta)/2$, so $\{p_j\}$ decreasingly converges to $1+\theta$. For $m \geq 1$ we show that

$$(4.7) \quad \|U_\sigma f\|_{p_m} \leq C_m AB^{2/p_m} \|f\|_{p_m}$$

uniformly in σ , for all F and ℓ satisfying the assumptions of Lemma 7. For $m = 1$, this is a consequence of (4.6). Fix $m \geq 1$ and assume (4.7) for this m . Then, using it for $U_\sigma(|F|, |\ell|)$ and applying the Khintchine inequality, we see that

$$(4.8) \quad \|g(f)\|_{p_m} \leq CAB^{2/p_m} \|f\|_{p_m},$$

where

$$g(f) = \left(\sum_j |f * \nu_j(|F|, |\ell|)|^2 \right)^{1/2}$$

(note that $\omega(|\ell|, t) \leq C\omega(\ell, t)$). Let $\nu^*(f) = \sup_j |f * \nu_j|$ and $\Phi^*(f) = \sup_j |f * \Phi_j(|F|, |\ell|)|$, where $\nu_j = \nu_j(F, \ell)$ as above. Note that

$$\begin{aligned} \nu^*(f) &\leq \mu^*(|f|) + \Phi^*(|f|) \leq g(|f|) + 2\Phi^*(|f|), \\ \Phi^*(|f|) &\leq C(\log \rho) \|\ell\|_{d_1} \|F\|_1 M f. \end{aligned}$$

These estimates and (4.8) along with the Hardy-Littlewood maximal theorem (see [1, 8, 13]) imply

$$(4.9) \quad \|\nu^*(f)\|_{p_m} \leq CAB^{2/p_m} \|f\|_{p_m}.$$

Define r_m by $1/r_m - 1/2 = 1/(2p_m)$. Then by (4.9) and the estimate $\|\nu_j\|_1 \leq CA$ we have the vector valued inequality (see [10] and also [21, 22])

$$(4.10) \quad \left\| \left(\sum |g_k * \nu_k|^2 \right)^{1/2} \right\|_{r_m} \leq CAB^{1/p_m} \left\| \left(\sum |g_k|^2 \right)^{1/2} \right\|_{r_m}.$$

By the Littlewood-Paley theory (see Lemma 6) and (4.10) we have

$$\begin{aligned}
(4.11) \quad \|U_{k_1, k_2} f\|_{r_m} &\leq C \left\| \left(\sum_j |f * \Delta_{k_1+j} * \nu_j|^2 \right)^{1/2} \right\|_{r_m} \\
&\leq CAB^{1/p_m} \left\| \left(\sum_j |f * \Delta_{k_1+j}|^2 \right)^{1/2} \right\|_{r_m} \\
&\leq CAB^{1/p_m} \|f\|_{r_m}.
\end{aligned}$$

Interpolating between (4.5) and (4.11), since $1/p_{m+1} = (1-\theta)/r_m + \theta/2$, we see that

$$(4.12) \quad \|U_{k_1, k_2} f\|_{p_{m+1}} \leq CAB^{(1-\theta)/p_m} \prod_{i=1}^2 \min(1, \rho^{-\theta\epsilon(|k_i|-c)/(2s')}) \|f\|_{p_{m+1}}.$$

Thus

$$\begin{aligned}
\|U_\sigma f\|_{p_{m+1}} &\leq \sum_{k_1, k_2} \|U_{k_1, k_2} f\|_{p_{m+1}} \\
&\leq CAB^{(1-\theta)/p_m} (1 - \rho^{-\theta\epsilon/(2s')})^{-2} \|f\|_{p_{m+1}} \\
&\leq CAB^{2/p_{m+1}} \|f\|_{p_{m+1}}.
\end{aligned}$$

This proves (4.7) for all m by induction. For any $p \in (1+\theta, 2]$ there exists a positive integer j such that $p_{j+1} < p \leq p_j$. So, interpolating between the estimates (4.7) with $m = j$ and $m = j+1$, we have

$$(4.13) \quad \|U_\sigma f\|_p \leq CAB^{2/p} \|f\|_p.$$

Let $g(f)$ be as in (4.8). The estimate (4.13) implies $\|g(f)\|_p \leq CAB^{2/p} \|f\|_p$ for $p \in (1+\theta, 2]$, from which Lemma 7 for $p \in (1+\theta, 2]$ follows, since $\mu^*(f) \leq g(f) + \Phi^*(f)$. For $p > 2$ Lemma 7 follows from interpolation between the estimate for $p = 2$ of Lemma 7 and the estimate

$$\|\mu^*(f)\|_\infty \leq C(\log \rho) \|\ell\|_{d_1} \|F\|_1 \|f\|_\infty.$$

This completes the proof of Lemma 7. \square

Theorem 1 is an immediate consequence of the following result.

Lemma 8. *Let the functions h, Ω be as in Theorem 1 and put $\delta(p) = |1/p - 1/p'|$. Suppose that $p \in (1+\theta, (1+\theta)/\theta)$. Let $A = (\log \rho) \|h\|_{\Lambda_3^{2/s'}} \|\Omega\|_s$ and let B be as above: $B = (1 - \rho^{-\theta\epsilon/(2s')})^{-2}$. Then*

$$\|Tf\|_p \leq CAB^{1+\delta(p)} \|f\|_p,$$

where the constant C is independent of $s > 1, \Omega, h$ and $\rho \geq 2$.

Proof. Since $Tf = U_{\sigma^*}(K_0, h)(f)$, where $\sigma^* = \{\sigma_j\}$ with $\sigma_j = 1$ for all j , by (4.13) we have

$$\|Tf\|_p \leq CAB^{2/p}\|f\|_p \quad \text{for } p \in (1 + \theta, 2].$$

Now, a duality argument using an estimate similar to this one for $T^*f = U_{\sigma^*}(\tilde{K}_0, h)(f)$ will imply the conclusion for all $p \in (1 + \theta, (1 + \theta)/\theta)$. \square

Proof of Theorem 1. Take $\rho = 2^{s'}$ in Lemma 8. Then

$$\|Tf\|_p \leq Cs'(1 - 2^{-\theta\epsilon/2})^{-2(1+\delta(p))} \|h\|_{\Lambda_s^{\eta/s'}} \|\Omega\|_s \|f\|_p$$

for $p \in (1 + \theta, (1 + \theta)/\theta)$ and $s > 1$. Since $(1 + \theta, (1 + \theta)/\theta) \rightarrow (1, \infty)$ as $\theta \rightarrow 0$, Theorem 1 follows from this estimate. \square

5. PROOF OF THEOREM 3

We need the following result to prove Theorem 3.

Lemma 9. *Let h, Ω be as in Theorem 3. Let $\theta \in (0, 1)$ and $A = (\log \rho) \|h\|_{\Lambda_s^{\eta/s'}} \|\Omega\|_s$. We define*

$$R(f)(x) = \sup_{k \in \mathbb{Z}} \left| \sum_{j=k}^{\infty} f * S_j L(x) \right|,$$

where $S_j L$ is as in Section 3. Let $I_\theta = (2(1 + \theta)/(\theta^2 - \theta + 2), (1 + \theta)/\theta)$. Then, for $p \in I_\theta$ we have

$$\|R(f)\|_p \leq CA \left((1 - \rho^{-\theta\delta/s'})^{-2(1+\delta(p))} + (1 - \rho^{-\theta\delta/s'})^{-4/p-1-\theta} \right) \|f\|_p$$

with some $\delta > 0$, where C is independent of $s > 1$, $h \in \Lambda_s^{\eta/s'}$, $\Omega \in L^s(\Sigma)$ and ρ .

Proof. Let $\varphi_k = \sum_{m \geq k+2} \Delta_m = \delta_{\rho^{k+1}} \phi$. Using the decomposition

$$\sum_{j=k}^{\infty} f * S_j L = T(f) * \varphi_k - \left(\sum_{j=-\infty}^{k-1} f * S_j L \right) * \varphi_k + \left(\sum_{j=k}^{\infty} f * S_j L \right) * (\delta - \varphi_k),$$

we have

$$(5.1) \quad R(f) \leq \sup_k |T(f) * \varphi_k| + \sup_k \left| \left(\sum_{j=-\infty}^{k-1} f * S_j L \right) * \varphi_k \right| + \sum_{j=0}^{\infty} N_j(f),$$

where $N_j(f) = \sup_k |(f * S_{j+k} L) * (\delta - \varphi_k)|$. Lemma 8 and the Hardy-Littlewood maximal theorem imply that

$$(5.2) \quad \left\| \sup_k |T(f) * \varphi_k| \right\|_p \leq C \|M(Tf)\|_p \leq CA(1 - \rho^{-\theta\epsilon/(2s')})^{-2(1+\delta(p))} \|f\|_p$$

for $p \in (1 + \theta, (1 + \theta)/\theta)$. Also, Lemma 7 and the Hardy-Littlewood maximal theorem imply that

$$(5.3) \quad \|N_j(f)\|_u \leq CA(1 - \rho^{-\theta\epsilon/(2s')})^{-4/u} \|f\|_u \quad \text{for } u > 1 + \theta.$$

On the other hand,

$$(5.4) \quad N_j(f) \leq \left(\sum_k |f * S_{j+k} L * (\delta - \varphi_k)|^2 \right)^{1/2}.$$

Let

$$V_\sigma f = \sum_k \sigma_k f * S_{j+k} L * (\delta - \varphi_k),$$

where $\sigma = \{\sigma_k\}$, $\sigma_k = 1$ or -1 . We prove

$$(5.5) \quad \|V_\sigma f\|_2 \leq CA(1 - \rho^{-\delta/s'})^{-3} \min(1, \rho^{-\delta(j-c)/s'}) \|f\|_2$$

for some $\delta, c > 0$, uniformly in σ . Estimates (5.4) and (5.5) with the Khintchine inequality imply

$$(5.6) \quad \|N_j(f)\|_2 \leq CA(1 - \rho^{-\delta/s'})^{-3} \min(1, \rho^{-\delta(j-c)/s'}) \|f\|_2.$$

To prove (5.5), we apply an argument similar to the one used to prove (4.5). We prove the estimates

$$(5.7) \quad \|f * S_{j+k} L * (\delta - \varphi_k) * (\delta - \varphi_{k'}) * S_{j+k'} \tilde{L}\|_2 \\ \leq CA^2(1 - \rho^{-\delta/s'})^{-2} \min(1, \rho^{-\delta(j-c)/s'}) \min(1, \rho^{-\delta(|k-k'|-c)/s'}) \|f\|_2,$$

$$(5.8) \quad \|f * (\delta - \varphi_{k'}) * S_{j+k'} \tilde{L} * S_{j+k} L * (\delta - \varphi_k)\|_2 \\ \leq CA^2(1 - \rho^{-\delta/s'})^{-4} \min(1, \rho^{-\delta(|k-k'|-c)/s'}) \min(1, \rho^{-\delta(j-c)/s'}) \|f\|_2.$$

for some $\delta, c > 0$, where $S_{j+k'} \tilde{L} = S_{j+k'}(\tilde{K}_0, h)$. By the Cotlar-Knapp-Stein lemma, the estimates (5.7) and (5.8) imply (5.5).

To prove (5.7), note that $\delta - \varphi_k = \sum_{m \leq k+1} \Delta_m$. Therefore,

$$(5.9) \quad \|f * S_{j+k} L * (\delta - \varphi_k) * (\delta - \varphi_{k'}) * S_{j+k'} \tilde{L}\|_2 \\ \leq \sum_{m \leq k+1, m' \leq k'+1} \|f * S_{j+k} L * \Delta_m * \Delta_{m'} * S_{j+k'} \tilde{L}\|_2.$$

By Lemma 1 we see that

$$(5.10) \quad \|f * (S_{j+k} L * \Delta_m) * (\Delta_{m'} * S_{j+k'} \tilde{L})\|_2 \\ \leq CA^2 \min(1, \rho^{-\epsilon(|j+k-m|-c)/s'}) \min(1, \rho^{-\epsilon(|j+k'-m'|-c)/s'}) \|f\|_2.$$

Also, we have

$$(5.11) \quad \|f * S_{j+k}L * (\Delta_m * \Delta_{m'}) * S_{j+k'}\tilde{L}\|_2 \leq CA^2 \min(1, \rho^{-\epsilon(|m-m'|-c)})\|f\|_2.$$

The estimates (5.10) and (5.11) imply

$$(5.12) \quad \|f * S_{j+k}L * \Delta_m * \Delta_{m'} * S_{j+k'}\tilde{L}\|_2 \leq CA^2 \min(1, \rho^{-\epsilon(|j+k-m|-c)/(2s')}) \min(1, \rho^{-\epsilon(|j+k'-m'|-c)/(2s')}) \times \min(1, \rho^{-\epsilon(|m-m'|-c)/2})\|f\|_2.$$

By (5.9) and (5.12) we have (5.7).

Similarly,

$$(5.13) \quad \|f * (\delta - \varphi_{k'}) * S_{j+k'}\tilde{L} * S_{j+k}L * (\delta - \varphi_k)\|_2 \leq \sum_{m \leq k+1, m' \leq k'+1} \|f * \Delta_{m'} * S_{j+k'}\tilde{L} * S_{j+k}L * \Delta_m\|_2 \leq \sum_{m \leq k+1, m' \leq k'+1} \sum_{\ell, \ell'} \|f * \Delta_{m'} * S_{j+k'}\tilde{L} * \Delta_\ell * \Delta_{\ell'} * S_{j+k}L * \Delta_m\|_2.$$

(See [28, p. 1555] for the idea of interposing $\Delta_\ell * \Delta_{\ell'}$ in the convolution product.) By Lemma 1 we have

$$(5.14) \quad \|f * (\Delta_{m'} * S_{j+k'}\tilde{L}) * (\Delta_\ell * \Delta_{\ell'}) * (S_{j+k}L * \Delta_m)\|_2 \leq CA^2 \min(1, \rho^{-\epsilon(|j+k'-m'|-c)/s'}) \min(1, \rho^{-\epsilon(|j+k-m|-c)/s'}) \times \min(1, \rho^{-\epsilon(|\ell-\ell'|-c)})\|f\|_2.$$

Also,

$$(5.15) \quad \|f * \Delta_{m'} * (S_{j+k'}\tilde{L} * \Delta_\ell) * (\Delta_{\ell'} * S_{j+k}L) * \Delta_m\|_2 \leq CA^2 \min(1, \rho^{-\epsilon(|j+k'-\ell|-c)/s'}) \min(1, \rho^{-\epsilon(|j+k-\ell'|-c)/s'})\|f\|_2.$$

By (5.14) and (5.15),

$$(5.16) \quad \|f * \Delta_{m'} * S_{j+k'}\tilde{L} * \Delta_\ell * \Delta_{\ell'} * S_{j+k}L * \Delta_m\|_2 \leq CA^2 \min(1, \rho^{-\epsilon(|j+k'-m'|-c)/(2s')}) \min(1, \rho^{-\epsilon(|j+k-m|-c)/(2s')}) \times \min(1, \rho^{-\epsilon(|\ell-\ell'|-c)/2}) \min(1, \rho^{-\epsilon(|j+k'-\ell|-c)/(2s')}) \min(1, \rho^{-\epsilon(|j+k-\ell'|-c)/(2s')})\|f\|_2.$$

Summation with respect to ℓ, ℓ' in (5.16) implies

$$(5.17) \quad \|f * \Delta_{m'} * S_{j+k'}\tilde{L} * S_{j+k}L * \Delta_m\|_2 \leq CA^2(1 - \rho^{-\delta/s'})^{-2} \min(1, \rho^{-\delta(|k-k'|-c)/s'}) \times \min(1, \rho^{-\epsilon(|j+k'-m'|-c)/(2s')}) \min(1, \rho^{-\epsilon(|j+k-m|-c)/(2s')})\|f\|_2$$

for some $\delta, c > 0$. By (5.13) and (5.17) we obtain (5.8).

For $p \in I_\theta$ we can find $u \in (1+\theta, 2(1+\theta)/\theta)$ such that $1/p = (1-\theta)/u + \theta/2$, so an interpolation between (5.3) and (5.6) implies that

$$(5.18) \quad \|N_j(f)\|_p \leq CA(1 - \rho^{-\theta\delta/s'})^{-4(1-\theta)/u - 3\theta} \min(1, \rho^{-\theta\delta(j-c)/s'}) \|f\|_p$$

for some $\delta, c > 0$.

Also, we need the following result.

Lemma 10. *There exist positive constants C, C_1 independent of ρ such that*

$$\left| \left(\sum_{j=-\infty}^{k-1} S_j L \right) * \varphi_k(x) \right| \leq C(\log \rho) \|h\|_{d_1} \|K_0\|_1 \rho^{-(k+1)\gamma} \chi_{[0, C_1]}(\rho^{-k-1} r(x)).$$

Proof. Since $\int S_j L = 0$, for $j \leq k-1$ we have

$$S_j L * \varphi_k(x) = \rho^{-(k+1)\gamma} \int (\phi(A_{\rho^{-k-1}} y^{-1} A_{\rho^{-k-1}} x) - \phi(A_{\rho^{-k-1}} x)) S_j L(y) dy.$$

Also, since $\text{supp}(S_j L) \subset \{r(x) \leq 2\rho^{j+2}\}$ and $\text{supp}(\varphi_k) \subset \{r(x) \leq \rho^{k+1}\}$, it follows that $\text{supp}(S_j L * \varphi_k) \subset \{r(x) \leq C_1 \rho^{k+1}\}$. Therefore

$$\begin{aligned} |S_j L * \varphi_k(x)| &\leq C \rho^{-(k+1)\gamma} \chi_{[0, C_1]}(\rho^{-k-1} r(x)) \int |A_{\rho^{-k-1}} y| |S_j L(y)| dy \\ &\leq C \rho^{-(k+1)\gamma} \chi_{[0, C_1]}(\rho^{-k-1} r(x)) \rho^{(-k-1+j+2)/\beta_1} (\log \rho) \|h\|_{d_1} \|K_0\|_1. \end{aligned}$$

Thus summing over $j \leq k-1$, we get the conclusion. \square

By Lemma 10

$$(5.19) \quad \sup_k \left| f * \left(\sum_{j=-\infty}^{k-1} S_j L \right) * \varphi_k \right| \leq C(\log \rho) \|h\|_{d_1} \|K_0\|_1 Mf.$$

So, to estimate the maximal function on the left hand side of (5.19), we can use the Hardy-Littlewood maximal theorem.

By (5.1), (5.2), (5.18) and (5.19), for $p \in I_\theta$ we have

$$\|R(f)\|_p \leq CA \left((1 - \rho^{-\theta\delta/s'})^{-2(1+\delta(p))} + (1 - \rho^{-\theta\delta/s'})^{-4(1-\theta)/u - 3\theta - 1} \right) \|f\|_p$$

for some $\delta > 0$. This implies the conclusion of Lemma 9, since $4(1-\theta)/u + 3\theta + 1 = 4/p + 1 + \theta$. \square

Proof of Theorem 3. Note that $T_*(f) \leq 2R(f) + CM_{K_0, h}(|f|)$. Therefore, Lemma 7 and Lemma 9 imply that

$$\|T_*(f)\|_p \leq C(\log \rho) \left(1 - \rho^{-\theta\delta/s'}\right)^{-6} \|h\|_{\Lambda_s^{\eta/s'}} \|\Omega\|_s \|f\|_p$$

for $p \in I_\theta$ with some $\delta > 0$. Using this with $\rho = 2^{s'}$ and noting that $I_\theta \rightarrow (1, \infty)$ as $\theta \rightarrow 0$, we can get the conclusion of Theorem 3. \square

6. PROOF OF THEOREM 5

Let $M_{F,\ell}$ be as in Section 4 with $\rho = 2$. We prove Theorem 5 along with the following result.

Proposition 1. *Let $F \in L^q(D_0)$ and $\ell \in \Lambda_q^\eta$ for some $q > 1$ and $\eta > 0$. Let $1 < p < \infty$. Then, we have the following:*

- (1) *if $q' \leq p < \infty$ and $w \in \mathcal{A}_{p/q'}$, the operator $M_{F,\ell}$ is bounded on $L^p(w)$;*
- (2) *the operator $M_{F,\ell}$ is bounded on $L^p(w^{1-p})$ if $1 < p \leq q$ and $w \in \mathcal{A}_{p'/q'}$.*

We use results of Sections 3, 4 and 5 with $\rho = 2$. We also write $\|f\|_{L^p(w)} = \|f\|_{p,w}$. First, we prove results of Theorem 5 for T .

Proof of Proposition 1(1). Since $\|S_j(|F|, |\ell|)\|_q \leq C2^{-j\gamma/q'} \|\ell\|_{d_q} \|F\|_q$, by the proof of Lemma 2, and $\text{supp}(S_j(|F|, |\ell|)) \subset \{2^j \leq r(x) \leq 2^{j+3}\}$, Hölder's inequality implies that

$$M_{F,\ell}(f) \leq C \|F\|_q \|\ell\|_{d_q} M_{q'} f,$$

where $M_{q'} f = \left(M(|f|^{q'})\right)^{1/q'}$. From this and the Hardy-Littlewood maximal theorem it follows that

$$\|M_{F,\ell}(f)\|_{p,w} \leq C \|F\|_q \|\ell\|_{d_q} \|M_{q'} f\|_{p,w} \leq C_{p,w} \|f\|_{p,w}$$

if $q' < p$ and $w \in \mathcal{A}_{p/q'}$.

Next, we handle the case $p = q' > 1$. Let $w \in \mathcal{A}_1$. If $s > q'$, then $w \in \mathcal{A}_1 \subset \mathcal{A}_{s/q'}$ and hence what we have already proved implies

$$(6.1) \quad \|M_{F,\ell}(f)\|_{s,w} \leq C_{s,w} \|f\|_{s,w}.$$

If $1 < r < q'$, then by Lemma 7

$$(6.2) \quad \|M_{F,\ell}(f)\|_r \leq C_r \|f\|_r.$$

Interpolating with change of measure between (6.2) and (6.1) with w replaced by $w^{1+\tau}$ for sufficiently small $\tau > 0$, we get

$$\|M_{F,\ell}(f)\|_{q',w} \leq C_{q',w} \|f\|_{q',w}.$$

This proves Proposition 1 (1). \square

Remark 1. If $q' < p$ in Proposition 1(1), then the assumption $\ell \in \Lambda^\eta$ is not needed. Also, we can replace the assumption for ℓ of Proposition 1 with the condition that there exists $\ell^* \in d_q$, $q > 1$, such that $|\ell| \leq \ell^*$ and $\ell^* \in \Lambda^\eta$ for some $\eta > 0$, keeping the conclusion unchanged, since $M_{F,\ell}(f) \leq M_{F,\ell^*}(|f|)$. In particular, if $\ell \in d_\infty$, we can take a constant function as ℓ^* .

Lemma 11. *Let $B_j f(x) = f * \nu_j(x)$, where $\nu_j = \nu_j(F, \ell)$, $F \in L^1(D_0)$, $\ell \in d_1$ (see (3.2)). Consider the inequality*

$$(6.3) \quad \left\| \left(\sum_{j=-\infty}^{\infty} |B_j f_j|^2 \right)^{1/2} \right\|_{p,w} \leq C_{p,w} \left\| \left(\sum_{j=-\infty}^{\infty} |f_j|^2 \right)^{1/2} \right\|_{p,w}.$$

- (1) *Suppose that F and ℓ are as in Proposition 1. Let $\delta \in (0, 1)$. If (6.3) holds for some $p \in (1, \infty)$ and $w \in \mathcal{A}_p$, then $U_\sigma = U_\sigma(F, \ell)$ is bounded on $L^p(w^{1-\delta})$ uniformly in σ (see (3.2)).*
- (2) *If $M_{F,\ell}$ is bounded on $L^p(w)$ for some $1 < p \leq 2$ and $w \in \mathcal{A}_p$, then (6.3) holds with these p and w .*

Proof. As in Section 4, we decompose U_σ of (1) as $U_\sigma f = \sum_{k_1, k_2} U_{k_1, k_2} f$. By (6.3) and Lemma 6 we have

$$(6.4) \quad \begin{aligned} \|U_{k_1, k_2} f\|_{p,w} &\leq C \left\| \left(\sum_j |f * \Delta_{k_1+j} * \nu_j|^2 \right)^{1/2} \right\|_{p,w} \\ &\leq C \left\| \left(\sum_j |f * \Delta_{k_1+j}|^2 \right)^{1/2} \right\|_{p,w} \\ &\leq C \|f\|_{p,w}. \end{aligned}$$

On the other hand, by the proof of Lemma 7 (see (4.12)) and duality we have

$$(6.5) \quad \|U_{k_1, k_2} f\|_p \leq C 2^{-\epsilon(|k_1|+|k_2|)} \|f\|_p$$

for some $\epsilon > 0$. Interpolating with change of measure between (6.5) and (6.4), we see that

$$\|U_{k_1, k_2} f\|_{p, w^{1-\delta}} \leq C 2^{-\delta\epsilon(|k_1|+|k_2|)} \|f\|_{p, w^{1-\delta}}$$

for all $\delta \in (0, 1)$. This implies that

$$\|U_\sigma f\|_{p, w^{1-\delta}} \leq \sum_{k_1, k_2} \|U_{k_1, k_2} f\|_{p, w^{1-\delta}} \leq C \|f\|_{p, w^{1-\delta}},$$

which proves part (1).

Suppose that $M_{F,\ell}$ is bounded on $L^p(w)$ for $1 < p \leq 2$. Then

$$(6.6) \quad \left\| \left(\sum_j |M_{F,\ell} f_j|^p \right)^{1/p} \right\|_{p,w} \leq C \left\| \left(\sum_j |f_j|^p \right)^{1/p} \right\|_{p,w}.$$

Also, we have

$$(6.7) \quad \left\| \sup_j |M_{F,\ell} f_j| \right\|_{p,w} \leq C \left\| \sup_j |f_j| \right\|_{p,w}.$$

Interpolating between (6.6) and (6.7),

$$\left\| \left(\sum_j |M_{F,\ell} f_j|^2 \right)^{1/2} \right\|_{p,w} \leq C \left\| \left(\sum_j |f_j|^2 \right)^{1/2} \right\|_{p,w}.$$

Now, (6.3) follows from this estimate and a vector valued inequality for the Hardy-Littlewood maximal operator (see [13, pp. 265–267], [20]). This proves part (2). \square

Let $q \geq 2$. If $q' \leq p \leq 2$, $p > 1$, by Proposition 1 (1) and Lemma 11, U_σ is bounded on $L^p(w^{1-\delta})$ for $w \in \mathcal{A}_{p/q'}$, where $U_\sigma = U_\sigma(F, \ell)$ and F, ℓ satisfy the assumptions of Proposition 1. Replacing w by $w^{1+\tau}$ for sufficiently small $\tau > 0$ and taking δ suitably, we see that U_σ is bounded on $L^p(w)$. This boundedness also holds for $p \in (2, \infty)$ by the extrapolation theorem of Rubio de Francia [19]. If $1 < p \leq q$, $w \in \mathcal{A}_{p'/q'}$, then this implies that U_σ is bounded on $L^{p'}(w)$. Obviously, this is also valid for $U_\sigma^* = U_\sigma(\tilde{F}, \ell)$. Therefore, by duality we can see that U_σ is bounded on $L^p(w^{1-p})$. Let Ω, h be as in Theorem 5. By taking $F = K_0$, $\ell = h$, $\sigma_j = 1$ for all j in the definition of U_σ , now we can see that Theorem 5 holds for T when $q \geq 2$.

Also, from a result of previous paragraph it follows that if $q \geq 2$, $1 < p \leq q$, $w \in \mathcal{A}_{p'/q'}$ and F, ℓ are as in Proposition 1, then $M_{F,\ell}$ is bounded on $L^p(w^{1-p})$, since $M_{F,\ell} f \leq g(f) + CMf$ by the proof of Lemma 7 and the boundedness of g follows from the uniform boundedness in σ of $U_\sigma = U_\sigma(|F|, |\ell|)$, where

$$g(f) = \left(\sum_j |f * \nu_j(|F|, |\ell|)|^2 \right)^{1/2}.$$

Here we recall that $\omega(|\ell|, t) \leq \omega(\ell, t)$. This proves Proposition 1 (2) for $q \geq 2$.

It remains to prove Theorem 5 (for T) and Proposition 1 (2) when $1 < q < 2$.

Lemma 12. *Let $1 < q < 2$, $2 < p < \infty$. Let $F \in L^q(D_0)$, $\ell \in d_q$. If $M_{|\tilde{F}|^{2-q}, |\ell|^{2-q}}$ is bounded on $L^{(p/2)'}(w^{-(p/2-1)^{-1}})$ and $w \in \mathcal{A}_p$, then*

$$\left\| \left(\sum_{j=-\infty}^{\infty} |B_j f_j|^2 \right)^{1/2} \right\|_{p,w} \leq C_{p,w} \left\| \left(\sum_{j=-\infty}^{\infty} |f_j|^2 \right)^{1/2} \right\|_{p,w},$$

where B_j is defined as in Lemma 11 by the functions F, ℓ .

Proof. It suffices to prove the conclusion for B'_j in place of B_j , where $B'_j f = f * S_j(F, \ell)$, on account of a vector valued inequality for the Hardy-Littlewood maximal operator. Take a non-negative function u in $L^{(p/2)'}(w)$ with norm 1 such that

$$(6.8) \quad \left\| \left(\sum_j |B'_j f_j|^2 \right)^{1/2} \right\|_{p,w}^2 = \int \left(\sum_j |B'_j f_j|^2 \right) u(x) w(x) dx.$$

We see that

$$(6.9) \quad |B'_j f(x)|^2 \leq C \|\ell\|_{d_q}^q \|F\|_q^q (|f|^2 * S_j(|F|^{2-q}, |\ell|^{2-q}))(x).$$

This can be proved as follows. First, the Schwarz inequality implies that

$$\begin{aligned} & |S_j(F, \ell)(x)|^2 \\ & \leq C \int_0^\infty \psi_j(t) |\ell(r(x)) \delta_t F(x)|^q dt/t \int_0^\infty \psi_j(t) |\ell(r(x)) \delta_t F(x)|^{2-q} dt/t. \end{aligned}$$

Therefore, using

$$\int \int_0^\infty \psi_j(t) |\ell(r(x)) \delta_t F(x)|^q dt/t dx \leq C 2^{j\gamma(1-q)} \|\ell\|_{d_q}^q \|F\|_q^q,$$

again by the Schwarz inequality, we have

$$\begin{aligned} & |f * S_j(F, \ell)(x)|^2 \\ & \leq C 2^{j\gamma(1-q)} \|\ell\|_{d_q}^q \|F\|_q^q \int \int_0^\infty \psi_j(t) |f(y)|^2 |\ell(r(y^{-1}x)) \delta_t F(y^{-1}x)|^{2-q} dt/t dy. \end{aligned}$$

This implies (6.9). Therefore, the integral in (6.8) is majorized by

$$(6.10) \quad C \|\ell\|_{d_q}^q \|F\|_q^q \int \left(\sum_j |f_j(y)|^2 \right) M_{|\bar{F}|^{2-q}, |\ell|^{2-q}}(uw)(y) dy.$$

By Hölder's inequality, the integral in (6.10) is bounded by

$$\left\| \left(\sum_j |f_j|^2 \right)^{1/2} \right\|_{p,w}^2 \left(\int |M_{|\bar{F}|^{2-q}, |\ell|^{2-q}}(uw)(y)|^{(p/2)'} w^{-(2/p)(p/2)'}(y) dy \right)^{1/(p/2)'}$$

Since $-(2/p)(p/2)' = -(p/2 - 1)^{-1}$, from the boundedness of $M_{|\bar{F}|^{2-q}, |\ell|^{2-q}}$ the last integral is majorized, up to a constant factor, by

$$\int |u(y)w(y)|^{(p/2)'} w^{-(2/p)(p/2)'}(y) dy = \|u\|_{(p/2)',w}^{(p/2)'} = 1.$$

Collecting results, we get the conclusion. \square

Let $c_n = 1 - (1/2)^n$, $n = 0, 1, 2, \dots$. Suppose that $q^{-1} \in (c_n, c_{n+1}]$, $n \geq 1$. Put $r = q/(2 - q)$. Then $(2r)^{-1} = q^{-1} - 1/2$, $r^{-1} \in (c_{n-1}, c_n]$. For $n \geq 1$, we consider the following.

Assertion $A(n)$. *Theorem 5 for T and Proposition 1 (2) hold when $q^{-1} \in (c_{n-1}, c_n]$.*

Assuming Proposition 1 (2) when $q^{-1} \in (c_{n-1}, c_n]$, we prove $A(n+1)$ ($n \geq 1$). This will prove Theorem 5 for T and Proposition 1 (2) when $1 < q < 2$, since we have already proved $A(1)$.

Suppose that $q^{-1} \in (c_n, c_{n+1}]$, $w \in \mathcal{A}_{p/q'}$, $q' \leq p < \infty$. Then, $(p/2)' \leq (q'/2)' = q/(2 - q) = r$. Let F, ℓ satisfy the assumptions of Proposition 1. Since $r^{-1} \in (c_{n-1}, c_n]$, $p/q' = (p/2)/r'$, $-(p/2 - 1)^{-1} = 1 - (p/2)'$ and $|\ell|^{2-q} \in d_r$, $\omega(|\ell|^{2-q}, t) \leq C\omega(\ell, t)^{2-q}$, $|F|^{2-q} \in L^r(D_0)$, by what we assume ($A(n)$ for Proposition 1 (2)), $M_{|\bar{F}|^{2-q}, |\ell|^{2-q}}$ is bounded on $L^{(p/2)'}(w^{-(p/2-1)^{-1}})$. By Lemmas 11 and 12, $U_\sigma = U_\sigma(F, \ell)$ is bounded on $L^p(w^{1-\delta})$. From this, boundedness of U_σ on $L^p(w)$ follows as before. This implies $A(n+1)$ for Theorem 5 (1) concerning T as in the case when $q \geq 2$.

Suppose that $1 < p \leq q$, $w \in \mathcal{A}_{p'/q'}$. Then, since $q' \leq p'$, by a result in the previous paragraph, U_σ is bounded on $L^{p'}(w)$. We can see that the same is true for U_σ^* . By duality U_σ is bounded on $L^p(w^{1-p})$. This implies the boundedness on $L^p(w^{1-p})$ of T and $M_{F, \ell}$ as in the case for $q \geq 2$. This finishes proving $A(n+1)$, and hence completes the proof of Theorem 5 for T and Proposition 1.

Next, we prove Theorem 5 for T_* . Let Ω, h, p, q, w be as in Theorem 5 (1). By (5.1), Lemma 10 and Theorem 5 for T , we have

$$(6.11) \quad \begin{aligned} \|R(f)\|_{p,w} &\leq C\|M(Tf)\|_{p,w} + C\|Mf\|_{p,w} + C \sum_{j=0}^{\infty} \|N_j(f)\|_{p,w} \\ &\leq C\|f\|_{p,w} + C \sum_{j=0}^{\infty} \|N_j(f)\|_{p,w}. \end{aligned}$$

Since $N_j(f) \leq CMM_{K_0, h}(|f|)$,

$$(6.12) \quad \|N_j(f)\|_{p,w} \leq C\|f\|_{p,w}$$

by Proposition 1. By (5.3) and (5.6)

$$(6.13) \quad \|N_j(f)\|_p \leq C2^{-\epsilon j} \|f\|_p$$

for some $\epsilon > 0$. So, interpolating with change of measure between (6.13) and (6.12) with $w^{1+\tau}$ in place of w for sufficiently small $\tau > 0$, we have

$$(6.14) \quad \|N_j(f)\|_{p,w} \leq C2^{-\epsilon j} \|f\|_{p,w}$$

for some $\epsilon > 0$. Since $T_*(f) \leq CR(f) + CM_{K_0, h}(|f|)$, by (6.11), (6.14) and Proposition 1 we have the $L^p(w)$ boundedness of T_* . This proves Theorem 5 (1). Theorem 5 (2) can be proved similarly.

7. PROOF OF THEOREM 2

We give a proof of Theorem 2 by applying Theorem 1. Define

$$E_m = \{\theta \in \Sigma : 2^{m-1} < |\Omega(\theta)| \leq 2^m\}$$

for $m = 2, 3, \dots$ and

$$E_1 = \{\theta \in \Sigma : |\Omega(\theta)| \leq 2\}.$$

Let $\Omega_m = \Omega \chi_{E_m} - S(\Sigma)^{-1} \int_{E_m} \Omega dS$. Then $\int_{\Sigma} \Omega_m dS = 0$, $\Omega = \sum_{m=1}^{\infty} \Omega_m$.

Fix $p \in (1, \infty)$ and an appropriate function f with $\|f\|_p \leq 1$. Write $U(h, \Omega) = \|Tf\|_p$, where $Tf = \text{p.v. } f * L$, $L(x) = h(r(x))\Omega(x')r(x)^{-\gamma}$. Since $h \in \Lambda$, we can write $h = \sum_{k=1}^{\infty} a_k h_k$, where $\{a_k\}$ and h_k are as in the definition of the space Λ . Then, we decompose

$$(7.1) \quad h\Omega = \sum_{m=1}^{\infty} \left(\sum_{k=m+1}^{\infty} a_k h_k \Omega_m + \sum_{k=1}^m a_k h_k \Omega_m \right).$$

By the sublinearity of U and Theorem 1 we have

$$(7.2) \quad \begin{aligned} U \left(\sum_{k=m+1}^{\infty} a_k h_k, \Omega_m \right) &\leq \sum_{k=m+1}^{\infty} a_k U(h_k, \Omega_m) \\ &\leq C \sum_{k=m+1}^{\infty} k a_k \|h_k\|_{\Lambda_{1+1/k}^{1/(1+k)}} \|\Omega_m\|_{1+1/k} \\ &\leq C \sum_{k=m+1}^{\infty} k a_k \|\Omega_m\|_{1+1/m} \leq C \|\Omega_m\|_{1+1/m}, \end{aligned}$$

since $\|\Omega_m\|_{1+1/k} \leq C \|\Omega_m\|_{1+1/m}$ if $k > m$. On the other hand,

$$(7.3) \quad \begin{aligned} U \left(\sum_{k=1}^m a_k h_k, \Omega_m \right) &\leq \sum_{k=1}^m a_k U(h_k, \Omega_m) \\ &\leq C \sum_{k=1}^m a_k m \|h_k\|_{\Lambda_{1+1/m}^{1/(1+m)}} \|\Omega_m\|_{1+1/m} \\ &\leq C \sum_{k=1}^m a_k m \|\Omega_m\|_{1+1/m} \leq C m \|\Omega_m\|_{1+1/m}, \end{aligned}$$

since $\|h_k\|_{\Lambda_{1+1/m}^{1/(1+m)}} \leq C \|h_k\|_{\Lambda_{1+1/k}^{1/(1+k)}} \leq C$ if $k \leq m$.

Note that

$$\|\Omega_m\|_u \leq C 2^m e_m^{1/u}, \quad 1 \leq u < \infty,$$

where $e_m = S(E_m)$. Using this and applying Young's inequality, we see that

$$\begin{aligned}
 (7.4) \quad & \sum_{m \geq 1} m \|\Omega_m\|_{1+1/m} \leq C \sum_{m \geq 1} m 2^m e_m^{m/(m+1)} \\
 & \leq 2C \sum_{m \geq 1} (m/(m+1)) m 2^{m(1+1/m)} e_m + 2C \sum_{m \geq 1} m 2^{-m-1}/(m+1) \\
 & \leq C \sum_{m \geq 1} m 2^m e_m + C \leq C \int_{\Sigma} |\Omega(\theta)| \log(2 + |\Omega(\theta)|) dS(\theta) + C.
 \end{aligned}$$

Theorem 2 follows from (7.1), (7.2), (7.3) and (7.4).

Remark 2. Let \mathcal{M}_a , $a > 0$, be the family of functions h on \mathbb{R}_+ such that there exist a sequence $\{h_k\}_{k=1}^{\infty}$ of functions on \mathbb{R}_+ and a sequence $\{a_k\}_{k=1}^{\infty}$ of non-negative real numbers satisfying $h = \sum_{k=1}^{\infty} a_k h_k$, $h_k \in d_{1+1/k}$, $\|h_k\|_{d_{1+1/k}} \leq 1$ uniformly in $k \geq 1$ and $\sum_{k=1}^{\infty} k^a a_k < \infty$. Then, the space \mathcal{M}_1 can be used to form kernels of singular integrals with a minimum size condition that allows us to get L^p boundedness of singular integrals defined by the kernels from results of [21, 22] (see [24]).

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